Reduction of the QCD string to a time component vector potential

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We demonstrate the equivalence of the relativistic flux tube model of mesons to a simple potential model in the regime of large radial excitation. We make no restriction on the quark masses; either quark may have a zero or finite mass. Our primary result shows that for fixed angular momentum and large radial excitation, the flux tube or QCD string meson with a short-range Coulomb interaction is described by a spinless Salpeter equation with a time component vector potential V(r) = ar - k/r.

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I. INTRODUCTION

Although quantum chromodynamics (QCD) is almost surely the correct theory of strong interactions, it remains difficult to explore its predictions in the nonperturbative regime. For hadron states the nonperturbative, or confinement, regime corresponds to large distances. For mesons confinement dominates the dynamics of even the heaviest quark states. It has long been suspected that when the color sources are widely separated, the color electric fields collapse into relatively thin configurations known as flux tubes, or QCD strings. The evidence for such stringlike configurations is primarily as follows:

- (i) Universal linear Regge trajectories, reflecting a linear confining potential and relativistic kinematics [1].
- (ii) Lattice simulation of the energy density [2].
- (iii) Relativistic corrections of the flux tube model [3] agree with those of Wilson loop QCD [4].
- (iv) For heavy onia, the flux tube model reduces to the very successful linear confinement potential model.
- (v) Lattice simulations of excited QCD states with fixed sources [5,6].

The relativistic string and quark model for mesons can be exactly solved numerically for arbitrary quark masses [7].

An excellent approximation for the bound state energy E of a meson consisting of one massive and one massless quark is the spectroscopic relation

$$\frac{E^2}{\pi a} = L + 2n + \frac{3}{2}.$$
 (1.1)

Here the angular momentum quantum number L and radial quantum number n take values $0,1,2,\ldots$. The constant a is the tension of the QCD string. For a meson with two massless quarks, the spectroscopic relationship becomes [8]

$$\frac{E^2}{2\pi a} = L + 2n + \frac{3}{2}.$$
 (1.2)

Recently we have shown semiclassically that in the limit of large radial excitation, Eq. (1.1) and Eq. (1.2) follow in both the flux tube model and in linear time component vector confinement [8].

In this paper we demonstrate that the wave equation for the flux tube meson with a short-range Coulomb interaction can be reduced to a spinless Salpeter equation with a pure time component vector (TCV) potential interaction. This reduction holds rigorously for large radial excitation but is accurate for all states. This result establishes the close connection between flux tube dynamics and the TCV potential model for spinless quarks.

In Sec. II we discuss the dynamics of a flux tube with quarks of arbitrary masses. The critical approximation that the string can be assumed to be nonrelativistic is shown to be accurate for $n \ge L$ in Sec. III. In Sec. IV we establish the TCV spinless Salpeter equation from the QCD string. We conclude in Sec. V. We include the detailed algebraic steps for Secs. III and IV in the Appendix.

II. THE QCD STRING WITH ARBITRARY QUARK MASSES

There are two equivalent methods of extracting the conserved quantities of the spinless quark-string system. The momentum-energy approach considers a straight color electric tube of energy a per unit length. From Lorentz boosting a string element perpendicular to its orientation, the momentum, angular momentum, and energy of the string are easily obtained [3,7]. This intuitive construction is appealing for its simplicity.

One can also extract the conserved quantities more formally by using Noether's theorem and an action formalism. We take an action [9] consisting of two pieces: one piece, the Nambu-Goto action, is proportional to the invariant surface area swept out by the string connecting the two quarks, the other piece is a sum of two terms, each proportional to the invariant length of a quark worldline:

$$S = -m_1 \int d\tau \sqrt{-\dot{x}_1^2} - m_2 \int d\tau \sqrt{-\dot{x}_2^2}$$
$$-a \int d\tau \int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2},$$
$$= \int d\tau \left(L_q + \int_{\sigma_1}^{\sigma_2} d\sigma \mathcal{L}_s \right).$$
(2.1)

In this expression x_1^{μ} and x_2^{μ} are the positions of the quarks, and $X^{\mu}(\sigma, \tau)$ is the position of the string. The quark masses are m_1 and m_2 , a is the string tension, a dot denotes a derivative with respect to τ , and a prime denotes a derivative with respect to σ . In our notation, the dot product between two four-vectors A and B, uses a metric of signature +2; $A \cdot B = A^{\mu}B^{\nu}\eta_{\mu\nu} = -A^0B^0 + \mathbf{A} \cdot \mathbf{B}$. The string terminates on a quark at each end

$$X^{\mu}(\sigma_1) = x_2^{\mu}, \quad X^{\mu}(\sigma_2) = x_1^{\mu}.$$
(2.2)

The momentum and angular momentum of the quarkstring system can be found from Noether's theorem. The action is invariant under the combination of a boost and a translation

$$\delta x_i^{\mu} = a^{\mu} + \omega_{\nu}^{\mu} x_i^{\nu},$$

$$\delta X^{\mu}(\sigma) = a^{\mu} + \omega_{\nu}^{\mu} X^{\nu}(\sigma). \qquad (2.3)$$

The variation of the finite time action under Eqs. (2.3) is the sum of the translation times momentum change and rotation times angular momentum change,

$$0 = \delta \int_{\tau_1}^{\tau_2} d\tau \left(L_q + \int d\sigma \mathcal{L}_s \right)$$

= $a^{\mu} [P_{\mu}(\tau_2) - P_{\mu}(\tau_1)]$
+ $\frac{1}{2} \omega^{\mu\nu} [L_{\nu\mu}(\tau_2) - L_{\nu\mu}(\tau_1)].$ (2.4)

The resulting momentum and angular momentum are the canonical ones,

$$P_{\mu} = \frac{\partial L_{q}}{\partial \dot{x}_{1}^{\mu}} + \frac{\partial L_{q}}{\partial \dot{x}_{2}^{\mu}} + \int_{\sigma_{1}}^{\sigma_{2}} d\sigma \frac{\partial \mathcal{L}_{s}}{\partial \dot{X}^{\mu}}$$

$$= m_{1} \frac{\dot{x}_{1\mu}}{\sqrt{-\dot{x}_{1}^{2}}} + m_{2} \frac{\dot{x}_{2\mu}}{\sqrt{-\dot{x}_{2}^{2}}}$$

$$+ a \int_{\sigma_{1}}^{\sigma_{2}} d\sigma \frac{\dot{X}_{\mu}(X')^{2} - X'_{\mu}(\dot{X} \cdot X')}{\sqrt{(\dot{X} \cdot X')^{2} - (\dot{X})^{2}(X')^{2}}}, \qquad (2.5)$$

$$L_{\mu\nu} = x_{1[\mu} \frac{\partial L_{q}}{\partial \dot{x}_{1}^{\nu]}} + x_{2[\mu} \frac{\partial L_{q}}{\partial \dot{x}_{2}^{\nu]}} + \int_{\sigma_{1}}^{\sigma_{2}} d\sigma X_{[\mu}(\sigma) \frac{\partial \mathcal{L}_{s}}{\partial \dot{X}^{\nu]}}$$
$$= m_{1} \frac{x_{1[\mu} \dot{x}_{1\nu]}}{\sqrt{-\dot{x}_{1}^{2}}} + m_{2} \frac{x_{2[\mu} \dot{x}_{2\nu]}}{\sqrt{-\dot{x}_{2}^{2}}}$$
$$+ a \int_{\sigma_{1}}^{\sigma_{2}} d\sigma \frac{X_{[\mu} \dot{X}_{\nu]}(X')^{2} - X_{[\mu} X'_{\nu]}(\dot{X} \cdot X')}{\sqrt{(\dot{X} \cdot X')^{2} - (\dot{X})^{2}(X')^{2}}}. \quad (2.6)$$

The action (2.1), momentum (2.5), and angular momentum (2.6) are invariant under changes of variables in σ and τ ,

$$(\sigma,\tau) \mapsto (\tilde{\sigma}(\sigma,\tau), \tilde{\tau}(\sigma,\tau)), \tag{2.7}$$

as long as the boundaries remain at values of $\tilde{\sigma}$ that are independent of $\tilde{\tau}$.

We use this coordinate invariance to choose the parameter τ to be the laboratory time, and the end points to be at particular values of σ_1 and σ_2 ,

$$x_1^0 = x_2^0 = X^0(\sigma, \tau) = \tau = t, \qquad (2.8)$$

$$\sigma_1 = 0, \quad \sigma_2 = 1.$$
 (2.9)

The straight string approximation to the equations of motion is exact for uniform circular motion of the quarks and is a good approximation in realistic mesons [10]. In this approximation, the position of the string at any time τ lies along a straight line between the quarks, which we can parametrize linearly as

$$\mathbf{X}(\sigma) = (1 - \sigma)\mathbf{x}_2 + \sigma \mathbf{x}_1. \tag{2.10}$$

We further denote the separation between the quarks at a particular instant as

$$\mathbf{r} \equiv \mathbf{x}_1 - \mathbf{x}_2. \tag{2.11}$$

Under the approximation (2.10), the energy of the system becomes

$$P^{0} = \frac{m_{1}}{\sqrt{1 - v_{1}^{2}}} + \frac{m_{2}}{\sqrt{1 - v_{2}^{2}}} + a \int_{0}^{1} d\sigma \frac{r^{2}}{\sqrt{[\mathbf{r} \cdot \mathbf{v}(\sigma)]^{2} - r^{2}[-1 + v^{2}(\sigma)]}} = m_{1}\gamma_{1} + m_{2}\gamma_{2} + ar \int_{0}^{1} d\sigma \frac{1}{\sqrt{1 - v_{\perp}^{2}(\sigma)}}, \quad (2.12)$$

where $\mathbf{v}_1 = d\mathbf{x}_1/dt$ and $\mathbf{v}_2 = d\mathbf{x}_2/dt$ are the quark velocities and we have defined

$$\mathbf{v}(\sigma) = \dot{\mathbf{X}}(\sigma) = (1 - \sigma)\mathbf{v}_2 + \sigma\mathbf{v}_1,$$

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$$\mathbf{v}_{\perp}(\sigma) = \mathbf{v}(\sigma) - \frac{\mathbf{r} \cdot \mathbf{v}(\sigma)}{r} \frac{\mathbf{r}}{r}.$$
 (2.13)

The momentum of the system with the straight string approximation becomes

$$\mathbf{P} = \frac{m_1 \mathbf{v}_1}{\sqrt{1 - v_1^2}} + \frac{m_2 \mathbf{v}_2}{\sqrt{1 - v_2^2}} + a \int_0^1 d\sigma \frac{\mathbf{v}(\sigma) r^2 - \mathbf{r}[\mathbf{v}(\sigma) \cdot \mathbf{r}]}{\sqrt{[\mathbf{r} \cdot \mathbf{v}(\sigma)]^2 - r^2[-1 + v^2(\sigma)]}} = m_1 \gamma_1 \mathbf{v}_1 + m_2 \gamma_2 \mathbf{v}_2 + ar \int_0^1 d\sigma \frac{\mathbf{v}_{\perp}(\sigma)}{\sqrt{1 - v_{\perp}^2(\sigma)}}.$$
(2.14)

The vector angular momentum is obtained from $L_{\mu\nu}$ as $L^i = \frac{1}{2} \epsilon^{ijk} L_{jk}$, which, for a straight string, becomes

$$\mathbf{L} = m_1 \frac{\mathbf{x}_1 \times \mathbf{v}_1}{\sqrt{1 - v_1^2}} + m_2 \frac{\mathbf{x}_2 \times \mathbf{v}_2}{\sqrt{1 - v_2^2}} + a \int_0^1 d\sigma \frac{r^2 \mathbf{X}(\sigma) \times \mathbf{v}(\sigma) - \mathbf{X}(\sigma) \times \mathbf{r}[\mathbf{r} \cdot \mathbf{v}(\sigma)]}{\sqrt{[\mathbf{r} \cdot \mathbf{v}(\sigma)]^2 - r^2[-1 + v^2(\sigma)]}} = m_1 \gamma_1 \mathbf{x}_1 \times \mathbf{v}_1 + m_2 \gamma_2 \mathbf{x}_2 \times \mathbf{v}_2 + ar \int_0^1 d\sigma \frac{\mathbf{X}(\sigma) \times \mathbf{v}_{\perp}(\sigma)}{\sqrt{1 - v_{\perp}^2(\sigma)}}.$$
(2.15)

We evaluate the integrals in Eqs. (2.12), (2.14), and (2.15) for a string rotating about the *z*-axis and instantaneously lying along the *x*-axis from $-r_2$ to r_1 . The string's perpendicular velocity runs from $-v_{\perp 2}$ to $v_{\perp 1}$,

$$X(\sigma) = \sigma r_1 + (\sigma - 1)r_2,$$

$$v_{\perp}(\sigma) = \sigma v_{\perp 1} + (\sigma - 1)v_{\perp 2}.$$
 (2.16)

The total length of the string is $r=r_1+r_2$. Because we assume that the string stays straight as it rotates, we have

$$\frac{v_{\perp 1}}{r_1} = \frac{v_{\perp 2}}{r_2}.$$
 (2.17)

Equation (2.17) implies

$$\frac{r}{v_{\perp 1} + v_{\perp 2}} = \frac{r_1}{v_{\perp 1}} = \frac{r_2}{v_{\perp 2}}.$$
 (2.18)

We denote the nonquark pieces of the energy, momentum, and angular momentum by λ_M , λ_P , and λ_L respectively. The only contribution to λ_P and λ_L is from the string. The energy λ_M has contributions from the Coulomb piece as well as the string. We find the string energy in Eq. (2.12) to be

$$ar \int_{0}^{1} d\sigma \frac{1}{\sqrt{1 - v_{\perp}^{2}}} = \frac{ar(\sqrt{1 - v_{\perp 2}^{2}} - \sqrt{1 - v_{\perp 1}^{2}})}{v_{\perp 1} + v_{\perp 2}}$$
$$= \frac{ar}{v_{\perp 1} + v_{\perp 2}} (\gamma_{\perp 2}^{-1} - \gamma_{\perp 1}^{-1})$$
$$\equiv \lambda_{M} + \frac{k}{r}.$$
(2.19)

The perpendicular momentum of the string from Eq. (2.14) is

$$ar \int_{0}^{1} d\sigma \frac{v_{\perp}}{\sqrt{1 - v_{\perp}^{2}}} = \frac{ar[\arcsin(v_{\perp 1}) + \arcsin(v_{\perp 2})]}{v_{\perp 1} + v_{\perp 2}}$$
$$\equiv \lambda_{P}. \qquad (2.20)$$

The angular momentum of the string from Eq. (2.15) becomes

$$ar \int_{0}^{1} d\sigma \frac{Xv_{\perp}}{\sqrt{1-v_{\perp}^{2}}} = \frac{ar}{2(v_{\perp 1}+v_{\perp 2})} \bigg[-r_{1}\gamma_{\perp 1}^{-1} - r_{2}\gamma_{\perp 2}^{-1} + \frac{ar}{v_{\perp 1}+v_{\perp 2}} [\arcsin(v_{\perp 1}) + \arcsin(v_{\perp 2})] + (\gamma_{\perp 2}^{-1} - \gamma_{\perp 1}^{-1}) \frac{v_{\perp 2}r_{1} - v_{\perp 1}r_{2}}{v_{\perp 1}+v_{\perp 2}} \bigg] = \lambda_{L}.$$
(2.21)

We can simplify the quark pieces of the conserved quantities through use of the identity

$$\frac{p_r^2 + m^2}{m^2} = \gamma^2 v_r^2 + 1 = \frac{v_r^2}{1 - v_r^2 - v_\perp^2} + 1$$
$$= \frac{1 - v_\perp^2}{1 - v_r^2 - v_\perp^2} = \frac{\gamma^2}{\gamma_\perp^2}, \qquad (2.22)$$

or

$$m \gamma = \sqrt{p_r^2 + m^2} \gamma_\perp \equiv W_r \gamma_\perp . \qquad (2.23)$$

Using Eqs. (2.19), (2.20), (2.21), and (2.23), we obtain our final expressions for the conserved quantities

$$P^{0} = W_{r1} \gamma_{\perp 1} + W_{r2} \gamma_{\perp 2} + ar \left(\frac{r_{1}}{v_{\perp 1}} \arcsin(v_{\perp 1}) + \frac{r_{2}}{v_{\perp 2}} \arcsin(v_{\perp 2}) \right),$$
(2.24)

$$P_{\perp} = W_{r1} \gamma_{\perp 1} v_{\perp 1} - W_{r2} \gamma_{\perp 2} v_{\perp 2} + a \left(\frac{r_2}{v_{\perp 2}} \gamma_{\perp 2}^{-1} - \frac{r_1}{v_{\perp 1}} \gamma_{\perp 1}^{-1} \right), \qquad (2.25)$$

$$L = W_{r1} \gamma_{\perp 1} v_{\perp 1} r_1 + W_{r2} \gamma_{\perp 2} v_{\perp 2} r_2 + \frac{a}{2} \frac{r_1^2}{v_{\perp 1}} \left(\frac{\arcsin(v_{\perp 1})}{v_{\perp 1}} - \gamma_{\perp 1}^{-1} \right) + \frac{a}{2} \frac{r_2^2}{v_{\perp 2}} \left(\frac{\arcsin(v_{\perp 2})}{v_{\perp 2}} - \gamma_{\perp 2}^{-1} \right).$$
(2.26)

These results are exactly the starting point in the momentumenergy approach [3,7].

Finally, we note that the Nambu-Goto Lagrangian for a straight string reduces to

$$L_{\text{string}} = -ar \int_{0}^{1} d\sigma \sqrt{1 - v_{\perp}^{2}(\sigma)}$$

= $-\frac{a}{2} \left(\frac{r_{1}}{v_{\perp 1}} \arcsin(v_{\perp 1}) + \frac{r_{2}}{v_{\perp 2}} \arcsin(v_{\perp 2}) + r_{1} \gamma_{\perp 1}^{-1} + r_{2} \gamma_{\perp 2}^{-1} \right).$ (2.27)

III. THE HIGH RADIAL EXCITATION REGIME

We now make the crucial observation that, for fixed angular momentum and large radial excitation, we may assume the string velocity is small without changing the meson dynamics. This is because v_{\perp} only becomes large near the inner turning point, where it reaches $v_{\perp} = 1$ in the case of a massless quark. However, near the inner turning point the string is short for very radial orbits so it carries little angular momentum or energy. Henceforth, for notational convenience, we generally suppress the \perp subscript, and let v and γ denote v_{\perp} and γ_{\perp} .

To demonstrate our approximation, we consider the equal quark mass case $m_1 = m_2$, from which follows $v_1 = v_2 \equiv v$, and $r_1 = r_2 = r/2$. We define

$$S(v) = \frac{\arcsin(v)}{v},\tag{3.1}$$

$$f(v) = \frac{1}{2v} (S - \sqrt{1 - v^2}), \qquad (3.2)$$

and hence Eqs. (2.19), (2.20) and (2.21) [or Eqs. (A5), (A7), and (A9)] can be expressed as

$$\lambda_P = 0, \tag{3.3}$$

$$\lambda_L = \frac{1}{2}ar^2f \tag{3.4}$$

$$\lambda_M = arS - \frac{k}{r}.\tag{3.5}$$

The total energy E and angular momentum L of the system are related to the quark separation and transverse velocities by Eq. (A15). In this notation the relation (A15) becomes



FIG. 1. Solution to Eq. (3.8) for the perpendicular velocity of the string end as a function of distance x = ar/E for two values of $\beta = aL/E^2$ with no Coulomb potential; $\kappa = 0$. For any attractive Coulomb interaction ($\kappa > 0$) the velocities are lower, making the approximation $v \ll 1$ better. We observe that for radially excited states $\beta \ll 1$ and $v \rightarrow 0$ except at the inner turning point.

$$v\left(E - arS + \frac{k}{r}\right) + arf = \frac{2L}{r}.$$
(3.6)

To reach reasonably general numerical conclusions we define the dimensionless quantities

$$x = \frac{ar}{E}, \quad \beta = \frac{aL}{E^2}, \quad \kappa = \frac{ak}{E^2}.$$
 (3.7)

In these dimensionless variables, Eq. (3.6) becomes

$$xv\left(1-xS+\frac{\kappa}{x}\right)+x^2f=2\beta.$$
(3.8)

For fixed *L* and increasing *E*, we expect both β and κ to be small and $xv \approx 2\beta$. We thus expect *v* to be small unless *x* is small. In Fig. 1 we show the exact numerical solution of Eq. (3.8) for two values of β . For simplicity we consider no short range interaction (κ =0).

It remains to demonstrate that the ratio of string to quark angular momentum is only appreciable when v is small. From the total angular momentum relation (A6) we find

$$R = \frac{\text{string angular momentum}}{\text{quark angular momentum}} = \frac{\lambda_L}{2\Omega v \frac{r}{2}}, \qquad (3.9)$$

where $\Omega \equiv W_r \gamma_{\perp}$ is the quark kinetic energy. The total system energy, Eq. (A8), becomes

$$E = 2\Omega + arS - \frac{k}{r}.$$
 (3.10)

In terms of dimensionless parameters, again in the case $k = 0 = \kappa$, by using Eq. (3.10) we have



FIG. 2. The ratio *R* of string to quark angular momentum as a function of dimensionless distance *x* as in Eq. (3.11) with $\beta = 0.01$. The string angular momentum is negligible for small quark separation.

$$R = \frac{xf(v)}{v(1-xS)}.$$
(3.11)

We show *R* as a function of *x* in Fig. 2 for $\beta = 0.01$ by using Eq. (3.8). As we stated at the beginning of this section, the string angular momentum becomes small compared to either quark's near the inner turning point. This tells us that, in the radially excited regime, we are justified in assuming that v_{\perp} is small for the string. In the regions in which this is not true, the string contributes negligibly to the energy and angular momentum. Our approach will be to keep full relativistic kinematics for the quarks and to assume that the string velocity is small.

IV. REDUCTION TO THE SPINLESS SALPETER WAVE EQUATION

In the preceding section we examined the exact numerical solution of the QCD string equations for the equal mass case, $m_1 = m_2$. We saw that if $aL \ll E^2$ the string perpendicular velocity can be assumed to be small without changing any dynamical result. In the Appendix we examine the detailed algebraic consequences of this approximation. We begin this section by outlining the main results found there.

In Sec. II we established the total perpendicular momentum, angular momentum, and energy for arbitrary masses m_1 and m_2 at the ends of a straight QCD string. The results are expressed in terms of the quark perpendicular velocities and distances from the center of momentum point. This point is defined by the condition $P_{\perp} = 0$.

Our strategy will be first to set up the relations defining $v_1 \equiv v_{1\perp}$ and $v_2 \equiv v_{2\perp}$ in terms of $r = r_1 + r_2$, the state mass M, and the quark masses. We obtain an explicit solution for v_1 and v_2 in the desired limit of small velocities. The result confirms the numerical result of the preceding section in that $v_i \rightarrow 0$ for large excitation energy E assuming the angular momentum L is fixed. It then becomes evident that in this limit the string does not enter the dynamics except for its static energy. The TCV wave equation for arbitrary masses then follows.

Referring to the Appendix, we use Eqs. (A1) to (A12) to

obtain the radial energy factors, Eqs. (A13) and (A14),

$$v_i \Omega_i = \frac{L}{r} + \alpha_i, \quad i = 1, 2, \tag{4.1}$$

where $\Omega_i \equiv W_{ri} \gamma_{\perp i}$. Most of the remainder of the Appendix is devoted to finding the quantities α_1 and α_2 .

We use Eq. (4.1) to eliminate the Ω_i in the energy equation (A8) and obtain Eq. (A15),

$$M - \lambda_{M} = \frac{L}{r} \left(\frac{1}{v_{1}} + \frac{1}{v_{2}} \right) - \frac{\lambda_{L}}{r} \left(\frac{1}{v_{1}} + \frac{1}{v_{2}} \right) + \lambda_{P} \left(\frac{1}{v_{2}} - \frac{1}{v_{1}} \right).$$
(4.2)

This is our first relation between v_1 and v_2 . Next, we use the definition of $\Omega_i = W_{ri} \gamma_{\perp i}$ given in Eq. (A16) to eliminate p_r and obtain our second relation between v_1 and v_2 , Eq. (A19),

$$\frac{(M - \lambda_M)}{v_1 v_2} \left[\frac{L}{r} - \frac{\lambda_L}{r} + \frac{\lambda_P (v_1^2 + v_2^2)}{v_1^2 - v_2^2} \right] - \frac{\lambda_P}{v_1 - v_2} \left[\frac{2L}{r} - \frac{2\lambda_L}{r} + \lambda_P \frac{v_1 - v_2}{v_1 + v_2} \right] = \frac{m_2^2 - m_1^2}{v_1 - v_2}.$$
(4.3)

By eliminating L/r between the two relations (4.2) and (4.3), we obtain the very useful relation given in Eq. (A20),

$$(M - \lambda_M)^2 + \frac{2\lambda_P}{v_1 - v_2} (1 - v_1 v_2) (M - \lambda_M) + \lambda_P^2$$
$$= \left(\frac{m_2^2 - m_1^2}{v_1 - v_2}\right) (v_1 + v_2).$$
(4.4)

Up to this point, we have made no approximations concerning the velocities v_i . In the auxiliary relation (4.4) the terms involving v_1v_2 and λ_p^2 are higher order in quark velocities. We now make the approximation of small quark velocities, so we drop these terms and use the approximations of Eq. (A21) to set $\lambda_p/(v_1-v_2) \approx ar/2$. In the small quark velocity limit, the string energy, λ_M , has no dependence on quark velocities, so we may approximate Eq. (4.4) as an equation for $(v_1+v_2)/(v_1-v_2)$ alone,

$$(m_2^2 - m_1^2) \left(\frac{v_1 + v_2}{v_1 - v_2} \right) = (M - \lambda_M)(M - \lambda_M + ar), \quad (4.5)$$

which is Eq. (A22). From Eq. (4.5) we can find the ratio of the quark velocities,

$$\frac{v_2}{v_1} = \frac{A-B}{A+B},\tag{4.6}$$

$$A = (M - \lambda_M)(M - \lambda_M + ar), \qquad (4.7)$$

$$B = (m_2^2 - m_1^2). \tag{4.8}$$

We now use the small v_i approximation of Eq. (A21) to rework Eq. (4.3) as

$$\frac{m_2^2 - m_1^2}{M - \lambda_M} \frac{v_1 v_2}{v_1 - v_2} = \frac{L}{r} + \frac{ar}{6} (v_1 + v_2).$$
(4.9)

Finally, we use Eq. (4.6) for v_2/v_1 to obtain v_1 as a function of r,

$$\frac{2L}{rv_1} = \frac{A-B}{M-\lambda_M} - \frac{2ar}{3} \frac{A}{(A+B)}.$$
(4.10)

A similar expression obtains for v_2 by replacing *B* by -B in Eq. (4.10).

Equation (4.10) is the key to obtaining our final goal of reducing the string equations to a TCV wave equation. We can now evaluate α_1 from Eq. (4.1). From the expression for α_1 given in Eq. (A13) and the small v_i expansions of Eq. (A21), we find

$$\alpha_1 = -\frac{ar}{6}(2v_1 - v_2). \tag{4.11}$$

Elimination of v_2 through the use of Eq. (4.6) yields

$$\alpha_1 = -\frac{arv_1}{6} \left(\frac{A+3B}{A+B} \right). \tag{4.12}$$

The corresponding α_2 follows by replacing v_1 by v_2 and B by -B.

Finally, we use Eq. (4.10) to eliminate v_1 and find

$$\alpha_1 = -\frac{aL}{3} \left(\frac{(A+3B)}{\frac{A^2 - B^2}{(M - \lambda_M)} - \frac{2}{3}arA} \right).$$
(4.13)

The angular momentum relation (4.1) can then be expressed as

$$v_1 \gamma_1 W_{r1} = \frac{L}{r} (1 - F_1), \qquad (4.14)$$

$$F_1 = \frac{(A+3B)ar/3}{\frac{A^2 - B^2}{(M - \lambda_M)} - \frac{2}{3}arA}.$$

The corresponding quantity, F_2 is obtained from Eq. (4.14) by the replacement of B by -B,

$$F_2 = \frac{(A-3B)ar/3}{\frac{A^2 - B^2}{(M-\lambda_M)} - \frac{2}{3}arA}.$$
 (4.15)

We examine the two important limits of a "light-light" and a "heavy-light" meson.

A. Light-light limit

In the light-light limit, we take $m_1 = m_2 = 0$, so that B = 0 and M = E. In this case we have

$$F_1^{LL} = F_2^{LL} = \frac{ar/3}{E + \frac{k}{r} - \frac{2}{3}ar}.$$
 (4.16)

B. Heavy-light limit

The heavy-light limit has $m_1=0$ and $m_2 \rightarrow \infty$. In this case we take $M=m_2+E$ and find

$$F_1^{HL} = \frac{ar/3}{E + \frac{k}{r} - \frac{2}{3}ar},$$
(4.17)

$$F_2^{HL} = -\frac{1}{2}F_1^{HL}.$$
 (4.18)

The function F_1 varies by at most a factor of 2 over the whole mass range $0 < m_2 < \infty$, while F_2 varies at most by a factor of 2, but can change sign over that region.

We can solve for γ_1 from Eq. (4.14) using $v_1\gamma_1 = \sqrt{\gamma_1^2 - 1}$,

$$\Omega_1 = W_{r1} \gamma_1 = \sqrt{W_{r1}^2 + \frac{L^2}{r^2} (1 - F_1)^2}.$$
 (4.19)

This result is added to the corresponding Ω_2 , using $W_{ri} = \sqrt{p_r^2 + m_i^2}$, to yield the Hamiltonian (A8)

$$H = \Omega_1 + \Omega_2 + \lambda_M$$

= $\sqrt{p_r^2 + \frac{L^2}{r^2}(1 - F_1)^2 + m_1^2} + \sqrt{p_r^2 + \frac{L^2}{r^2}(1 - F_2)^2 + m_2^2}$
+ $ar - \frac{k}{r}$. (4.20)

The spinless Salpeter equation,

$$H\psi = M\psi, \qquad (4.21)$$

follows from Eq. (4.20).

To complete the argument, we now observe that the angular momentum term L^2/r^2 is large only near the inner turning point which in turn is small for highly radially excited states. Near this inner turning point, the F_i factors are small since they are proportional to r. Where the F_i become larger, their effect is negligible since their factors multiply L^2/r^2 . Thus, for highly excited radial states, dropping the F_i will not change any dynamical results. The subsequent Hamiltonian (4.20) then becomes

$$H = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} + ar - \frac{k}{r}, \qquad (4.22)$$

which is the TCV Hamiltonian with a TCV potential



FIG. 3. The transition from a time component vector (TCV) confinement to string dynamics. The lines represent exact numerical solutions to the string equations in the case $m_1=0$, $m_2=\infty$. The dots are exact numerical solutions to linear TCV confinement. The squared excitation energies ($E=M-m_2$) of the TCV system converge to those of the string for large radial excitations with small angular momentum *L*.

$$V(r) = ar - \frac{k}{r}.$$
(4.23)

We note that at every step we have imposed the center of momentum rest condition and hence the Hamiltonian, Eq. (4.22), satisfies all conservation laws, even for arbitrary masses.

V. DISCUSSION AND SUMMARY

We have shown that a relativistic QCD string meson model, with a short range Coulomb part, reduces to a time component vector (TCV) potential model for large radial excitations. Our result follows for spinless quarks with any masses from zero to infinite. The result is remarkable since the relativistic string generally dominates dynamically, carrying both angular momentum and rotational energy. However, for fixed angular momentum and eccentric orbits, where $E^2 \ge aL$, the quarks act as if they were moving in a static TCV potential that is linear at large distances and Coulombic at short ones.

The coincidence of the two systems in the radially excited regime results from a confluence of effects. For large radial excitation, the radial velocities dominate over the perpendicular velocity, except near the inner turning point, where the motion is rotation and the transverse velocities reach light velocity in the extreme limit of a massless quark. However, for large radial excitation, the inner turning point occurs at small radius, so the string carries little angular momentum. Thus we may assume that the transverse velocity is small everywhere and approaches zero as the radial excitation increases.

Conversely, the quark's radial energy is large and relativistic and satisfies the spinless Salpeter equation given in Eqs. (4.21) and (4.22). To show that the final result is correct, we observe in Fig. 3 two numerically exact sets of solutions. The dots represent the TCV solution with k=0 and a linear confinement potential V(r) = ar. The solid lines are interpolations of the QCD string solutions. The latter are numerically exact quantized solutions of the string equations (A1) to (A12). For s-waves (L=0) there is no transverse motion ($v_{\perp}=0$) and the string equations are exactly the s-wave TCV equation and the curves for each radial state pass through the L=0 dots. The nontrivial result of this paper is seen for L= 1, 2, or 3, where the curves come closer to the dots as the radial excitation increases.

In previous work [8] we have approached this same result from a different route. If one quantizes the string semiclassically, one reproduces the (numerically) exact results of the string spectroscopy for highly radially excited states. Conversely, if one semiclassically quantizes the TCV spinless Salpeter equation with a linear confining potential, again the string spectroscopy emerges for highly radially excited states. The work of this paper shows directly from the conserved string quantities that TCV dynamics result for the highly radially excited states.

Finally, the situation can be clarified by going back to the string action. As we saw from Eq. (2.27), the straight string Lagrangian can be written as

$$L_{\text{string}} = -ar \int_0^1 d\sigma \sqrt{1 - v_{\perp}^2(\sigma)}.$$
 (5.1)

For small v_{\perp} , the string Lagrangian above expands to

$$L_{\text{string}} \simeq -ar \left[1 + \frac{1}{6} (v_{\perp 1} v_{\perp 2} - v_{\perp 1}^2 - v_{\perp 2}^2) + \cdots \right].$$
(5.2)

If one immediately sets $v_{\perp} \rightarrow 0$, the string action becomes the linear piece of the TCV interaction Lagrangian. One might worry whether this is justifiable. The present paper shows that it is.

In this work we began with the justification of the small v_{\perp} string approximation. From the exact expressions for the straight string momentum, angular momentum, and energy, we systematically approximate these quantities for low string velocities. We are then able to recast the energy equation into the spinless Salpeter form and finally, to show that this equation must be dynamically identical to the TCV equation in the large radial excitation regime.

We might remind the reader that the strict constraint of a straight string can also be relaxed. Small deviations from straightness do not change the conservation relations to first order and in ordinary hadrons the deviations from straightness never become large [10].

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APPENDIX

In this appendix we give the detailed algebraic steps involved in writing the energy of the quark/string system in terms of conserved quantities and the quark separation. We begin by finding the center of mass of the system. After several steps, we can express the quark velocities in terms of the quark separation, the quark masses, and conserved quantities.

1. Notation

We use the straight string conserved quantities given in Sec. II and generally drop the \perp subscripts on the quark velocities and γ s for notational simplicity. The coordinates r_i and velocities v_i are relative to the, as yet unknown, center of momentum point. The instantaneous positions of the quarks are taken to be along the *x*-axis, with quark 1 at $x = r_1$ and quark 2 at $x = -r_2$. Their transverse velocities are v_1 and $-v_2$ respectively. We also define

$$\Omega_i = W_{ri} \gamma_{\perp i}, \qquad (A1)$$

$$W_{ri} = \sqrt{p_r^2 + m_i^2},\tag{A2}$$

$$\gamma_{\perp i} = \frac{1}{\sqrt{1 - v_{\perp i}^2}}.\tag{A3}$$

2. Conserved quantities

In Sec. II we found the conserved quantities for the quark and straight string system. We gather these results together here. The transverse momentum (2.14) of the system is

$$\Omega_1 v_1 - \Omega_2 v_2 + \lambda_P = 0, \tag{A4}$$

where

$$\lambda_P = \frac{ar_2}{v_2} \gamma_2^{-1} - \frac{ar_1}{v_1} \gamma_1^{-1}.$$
 (A5)

The angular momentum (2.15) consists of the contribution of the quarks plus that of the string,

$$L = \Omega_1 v_1 r_1 + \Omega_2 v_2 r_2 + \lambda_L, \qquad (A6)$$

with

$$\lambda_{L} = \frac{a}{2} \frac{r_{1}^{2}}{v_{1}} \left(\frac{\arcsin(v_{1})}{v_{1}} - \gamma_{1}^{-1} \right) \\ + \frac{a}{2} \frac{r_{2}^{2}}{v_{2}} \left(\frac{\arcsin(v_{2})}{v_{2}} - \gamma_{2}^{-1} \right).$$
(A7)

The energy of the system (2.12) is the sum of the quark energies and the string energies

$$M = \Omega_1 + \Omega_2 + \lambda_M, \qquad (A8)$$

where the string plus Coulomb contribution is

$$\lambda_M = ar_1 \frac{\arcsin(v_1)}{v_1} + ar_2 \frac{\arcsin(v_2)}{v_2} - \frac{k}{r}.$$
 (A9)

3. Straight string yields uniform angular velocity

The straight string condition (2.10) leads to equal quark angular velocities,

$$\frac{v_1}{r_1} = \frac{v_2}{r_2}.$$
 (A10)

From Eq. (A10) and the definition of the total quark separation

$$r = r_1 + r_2,$$
 (A11)

we have

$$r_i = r \frac{v_i}{v_1 + v_2}.\tag{A12}$$

4. Quark kinetic energies

The first step in finding the total energy of the system in terms of the conserved quantities and the relative positions of the quarks is to write the quark kinetic energies Ω_i in terms of the system quantities L, r, and the quark perpendicular velocities v_i . We use Eq. (A4) and Eq. (A12) to eliminate Ω_2 in Eq. (A6). We find

$$v_1\Omega_1 = \frac{L}{r} + \alpha_1, \quad \alpha_1 = -\frac{\lambda_L}{r} - \frac{v_2}{v_1 + v_2}\lambda_P, \quad (A13)$$

and

$$v_2\Omega_2 = \frac{L}{r} + \alpha_2, \quad \alpha_2 = -\frac{\lambda_L}{r} + \frac{v_1}{v_1 + v_2}\lambda_P. \quad (A14)$$

5. Total energy

Next we express the total energy (A8) in terms of the rewritten Ω_i 's above. We find

$$M - \lambda_{M} = \frac{L}{r} \left(\frac{1}{v_{1}} + \frac{1}{v_{2}} \right) - \frac{\lambda_{L}}{r} \left(\frac{1}{v_{1}} + \frac{1}{v_{2}} \right) + \lambda_{P} \left(\frac{1}{v_{2}} - \frac{1}{v_{1}} \right).$$
(A15)

We can find another relation without Ω_i 's by going back to the definition of Ω_i and W_{ri}

$$\Omega_i = W_{ri} \gamma_i = \sqrt{\frac{p_r^2 + m_i^2}{1 - v_i^2}}.$$
 (A16)

Because the string has no radial momentum, the quarks have equal and opposite radial momenta. We thus expand

$$m_2^2 - m_1^2 = \Omega_2^2 (1 - v_2^2) - \Omega_1^2 (1 - v_1^2), \qquad (A17)$$

using Eq. (A4) and Eq. (A8), to find

$$(\Omega_2 - \Omega_1)(M - \lambda_M) - \lambda_P (\Omega_2 v_2 + \Omega_1 v_1)$$

= $m_2^2 - m_1^2$. (A18)

Then, we use Eqs. (A13) and (A14) to obtain

$$\frac{(M-\lambda_M)}{v_1 v_2} \left[\frac{L}{r} - \frac{\lambda_L}{r} + \frac{\lambda_P (v_1^2 + v_2^2)}{v_1^2 - v_2^2} \right] - \frac{\lambda_P}{v_1 - v_2} \left[\frac{2L}{r} - \frac{2\lambda_L}{r} + \lambda_P \frac{v_1 - v_2}{v_1 + v_2} \right] = \frac{m_2^2 - m_1^2}{v_1 - v_2}.$$
(A19)

By combining Eqs. (A15) and (A19), we can eliminate L/r in Eq. (A19). After a bit of algebra we find

$$(M - \lambda_M)^2 + \frac{2\lambda_P}{v_1 - v_2} (1 - v_1 v_2) (M - \lambda_M) + \lambda_P^2$$
$$= \left(\frac{m_2^2 - m_1^2}{v_1 - v_2}\right) (v_1 + v_2).$$
(A20)

So far, we have made no approximations other than the straight string assumption.

6. Low transverse velocity approximation

Now we use the low velocity approximation to solve for the ratio of the quark velocities v_2/v_1 . As we have seen in Sec. III, $v_i \equiv v_{\perp i}$ is small if $n \ge L$, except near the inner radial turning point. Near this point, the string angular momentum and energy are small compared to either quark's because the string is short. As far as the string is concerned, we can assume that $v_i \ll 1$ everywhere without changing the dynamics. From Eqs. (A5), (A7), and (A9), we may make the approximations

$$\begin{split} \lambda_{P} &\simeq \frac{1}{2} ar(v_{1} - v_{2}), \\ \lambda_{L} &\simeq \frac{1}{3} ar^{2} \bigg(\frac{v_{1}^{2} - v_{1}v_{2} + v_{2}^{2}}{v_{1} + v_{2}} \bigg), \end{split} \tag{A21} \\ \lambda_{M} &\simeq ar - \frac{k}{r}. \end{split}$$

We note that in Eq. (A20) we may drop the small quantities v_1v_2 and λ_P^2 , whereupon Eq. (A20) reduces to

$$(m_2^2 - m_1^2) \left(\frac{v_1 + v_2}{v_1 - v_2} \right) = (M - \lambda_M)(M - \lambda_M + ar).$$
(A22)

We can solve Eq. (A22) for v_2/v_1 ,

$$\frac{v_2}{v_1} = \frac{(M - \lambda_M)(M - \lambda_M + ar) - (m_2^2 - m_1^2)}{(M - \lambda_M)(M - \lambda_M + ar) + (m_2^2 - m_1^2)}.$$
 (A23)

We note that if $m_1 = m_2$, then $v_2 = v_1$, as expected. Also, if $m_2 \gg m_1$, then, since $M = m_2 + E$ and $E \ll m_2$, we find $v_2 \ll v_1$, as one might expect.

7. Solving for quark velocities

We are now in a position to solve for the individual v_i 's. We use the small v_i approximation of Eq. (A21) in Eq. (A19) and again drop small terms. These small terms are the second bracket of Eq. (A19), whose origin is in the γ_{\perp}^2 term from Eq. (A16). The result is

$$\frac{m_2^2 - m_1^2}{M - \lambda_M} \frac{v_1 v_2}{v_1 - v_2} = \frac{L}{r} + \frac{ar}{6} (v_1 + v_2).$$
(A24)

From Eq. (A23) in the form of

$$\frac{v_2}{v_1} = \frac{A - B}{A + B},$$
(A25)

$$A = (M - \lambda_M)(M - \lambda_M + ar),$$

$$B = (m_2^2 - m_1^2),$$

we can substitute v_2 from Eq. (A25) into Eq. (A24) to find

$$\frac{2L}{rv_1} = \frac{A-B}{M-\lambda_M} - \frac{2ar}{3}\frac{A}{(A+B)}.$$
 (A26)

A corresponding expression for v_2 follows upon replacing *B* by -B.

There are two important special cases of Eq. (A26).

(i) In the equal mass case $m_1 = m_2$ and so B = 0. In this case we define E = M and find

$$\frac{2L}{rv_1} = E + \frac{k}{r} - \frac{2}{3}ar.$$
 (A27)

(ii) In the heavy-light case, $m_2 \ge m_1$. The total energy is $M = m_2 + E$, where the excitation energy *E* is small compared to m_2 . The leading terms are

$$\frac{2L}{rv_1} = 2\left(E + \frac{k}{r} - \frac{2}{3}ar\right) - \frac{1}{m_2}\left[\left(E + \frac{k}{r}\right)^2 - \frac{5}{3}ar\left(E + \frac{k}{r}\right) + \frac{5}{6}(ar)^2\right].$$
(A28)

We can estimate the region of validity of the heavy-light approximation by noting that the $1/m_2$ correction is roughly E^2/m_2 . We compare this with *E* of the first term and conclude that if $E \ll m_2$, we may neglect the $1/m_2$ correction.

8. String corrections to quark kinetic energies

The last step is to find the string corrections to Ω_i . The same type of small velocity approximation used for the quarks, together with Eq. (A26) can be used with the α_i of Eqs. (A13) and (A14). We find

$$\alpha_1 = -\frac{aL}{3} \left(\frac{(A+3B)(M-\lambda_M)}{A^2 - B^2 - \frac{2}{3}arA(M-\lambda_M)} \right).$$
(A29)

We find a similar expression for α_2 , which is obtained by replacing *B* by -B (i.e., $m_2 \leftrightarrow m_1$). In the light equal mass limit we find

$$\alpha_1^{LL} = -\frac{aL}{3\left(E + \frac{k}{r} - \frac{2}{3}ar\right)}.$$
 (A30)

In the heavy-light limit we obtain the same expression for α_1 to lowest order, and find the $\mathcal{O}(1/m_2)$ corrections,

$$\alpha_{1}^{HL} = -\frac{aL}{3\left(E + \frac{k}{r} - \frac{2}{3}ar\right)} + \frac{1}{m_{2}}\frac{a^{2}rL}{12}\frac{\left(E + \frac{k}{r} - ar\right)}{\left(E + \frac{k}{r} - \frac{2}{3}ar\right)^{2}}.$$
(A31)

In either case, as in the general case, if $E \gg L$

$$e \gg L$$

 $\alpha_i \to 0,$ (A32)

which is the high radial excitation regime. In this limit of $\Omega_i \rightarrow L/(rv_i)$ the relativistic TCV wave equation follows.

- [1] M.G. Olsson, Phys. Rev. D 55, 5479 (1997).
- [2] G. Bali, Phys. Rep. **343**, 1 (2001).
- [3] C. Olson, M.G. Olsson, and Ken Williams, Phys. Rev. D 45, 4307 (1992).
- [4] N. Brambilla and G.M. Prosperi, Phys. Lett. B 236, 69 (1990);
 Phys. Rev. D 46, 1096 (1992).
- [5] K.J. Juge, J. Kuti, and C.J. Morningstar, in Proceedings of the XVth International Symposium on Lattice Field Theory [Nucl. Phys. B (Proc. Suppl.) 63, 326 (1998)].
- [6] T.J. Allen, M.G. Olsson, and S. Veseli, Phys. Lett. B 434, 110 (1998).
- [7] D. La Course and M.G. Olsson, Phys. Rev. D 39, 2751 (1989);
 M.G. Olsson and S. Veseli, *ibid.* 51, 3578 (1995).

- [8] T.J. Allen, C. Goebel, M.G. Olsson, and S. Veseli, Phys. Rev. D 64, 094011 (2001); T.J. Allen, Todd Coleman, M.G. Olsson, and S. Veseli, *ibid.* 67, 054016 (2003).
- [9] Y. Nambu, Quark Model and the Factorization of the Veneziano Amplitude, in Symmetries and Quark Models, edited by R. Chand (Gordon and Breach, New York, 1970); T. Goto, Prog. Theor. Phys. 46, 1560 (1971); L. Susskind, Nuovo Cimento A 69A, 457 (1970); A.M. Polyakov, Phys. Lett. 103B, 207 (1981); B.M. Barbashov and V.V. Nesterenko, Introduction to the Relativistic String Theory (World Scientific, Singapore, 1990).
- [10] T.J. Allen, M.G. Olsson, and S. Veseli, Phys. Rev. D 60, 074026 (1999); 59, 094011 (1999).