Axial-vector–vector amplitude and neutrino effective charge in a magnetized medium

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(Received 21 May 2003; published 30 September 2003)

To one loop the effective neutrino photon interaction takes place through vector-vector-type and axial-vector–vector–type amplitudes. In this work we explicitly write down the form of the axial-vector–vector amplitude to all orders in the external magnetic field in a medium. We then infer its zero external momentum limit which contributes to the effective charge of the neutrinos inside a magnetized medium. We further show its gauge invariance properties.

DOI: 10.1103/PhysRevD.68.053011 PACS number(s): 13.15.+g, 13.66.−a

I. INTRODUCTION

Neutrino mediated processes are of great importance in cosmology and astrophysics [1,2]. Various interesting possibilities involving neutrinos have been looked into in the context of cosmology, e.g., large scale structure formation in the universe, to name one [3]. In this note we consider the induced effect on the neutrinos by a magnetized medium, using quantum field theoretic techniques. It is usually conjectured, taking into account the conservation of the surface magnetic field of a protoneutron star, that during a supernova collapse the magnetic field strength in some regions inside the nascent star can reach up to $\mathcal{B}\sim m^2/e$ or more. Here m denotes the mass of the electron. This conjecture makes it worthwhile to investigate the role of the magnetic field in the effective neutrino-photon vertex.

Neutrinos do not couple to photons at the tree level in the standard model of particle physics, and this coupling can take place only at the loop level, mediated by the fermions and gauge bosons. This coupling can give birth to scattering process such as $\gamma\gamma\rightarrow\nu\nu$. The cross section of neutrino-photon scattering is highly suppressed in the standard model due to Yang's theorem [4], which makes the scattering cross section vanish to orders of the Fermi coupling G_F. In the presence of a magnetic field, neutrino-photon scattering can occur and to orders of G_F the cross section has been calculated [5]. There can be neutrino processes in a medium or a magnetic field or both which involve only one photon, such as $\nu\rightarrow\nu\gamma$ and $\gamma\rightarrow\nu\bar{\nu}$. In vacuum these reactions are restrained kinematically. Only in the presence of a medium or a magnetic field or both can all the particles be on shell as there the dispersion relation of the photon changes, giving the much required phase space for the reactions. Intuitively, when a neutrino moves inside a thermal medium composed of electrons and positrons, they interact with these background particles. The background electrons and positrons themselves have interactions with the electromagnetic fields, and this fact gives rise to an effective coupling of the neutrinos to the photons. Under these circumstances the neutrinos may acquire an "effective electric charge" through which they interact with the ambient plasma.

The effective charge of the neutrino in a medium has been calculated previously by many authors [8–10]. All of these works were concentrated on the vector-vector part of the interaction. In this paper we concentrate upon the effective neutrino-photon interaction vertex coming from the axial-vector–vector part $\Pi^5_{\mu\nu}$ of the interaction. Some work on the axial-vector–vector part in a time independent background electromagnetic field has been done previously [6,7] where in some cases the authors were able to obtain a gauge invariant expression for the axial-vector–vector contribution to the neutrino-photon effective vertex. We are interested in the zero momentum limit of the axial-vector–vector amplitude, as it contributes to the effective charge of the neutrinos inside a magnetized plasma. We discuss the physical situations where the axial-vector–vector amplitude arises, and then show how it affects the physical processes.

The plan of the paper is as follows. We start with Sec. II which deals with the formalism through which the physical importance of $\Pi^5_{\mu\nu}(k)$ is appreciated. In Sec. III the general form factor analysis of the second rank tensor on the basis of symmetry arguments is provided. In Sect. IV we show the fermion propagator in a magnetized medium, and using it explicitly write down $\Pi^5_{\mu\nu}(k)$ in the rest frame of the medium. In Sec. V we calculate the effective electric charge from the expression of the axial-vector–vector amplitude. In Sec. VI we discuss our results and conclude by touching upon the physical relevance of our work. A formal proof of the gauge invariance of $\Pi^5_{\mu\nu}$ is attached in the Appendix.

II. FORMALISM

In this work we consider the background temperature and neutrino momenta which are small compared to the masses of the W and Z bosons. We can, therefore, neglect the momentum dependence in the W and Z propagators, which is justified if we are performing a calculation to the leading order in the Fermi constant G_F. In this limit the four-

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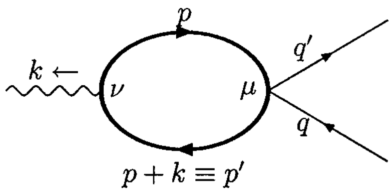

FIG. 1. One-loop diagram for the effective electromagnetic vertex of the neutrino in the limit of infinitely heavy W and Z masses.

fermion interaction is given by the following effective Lagrangian:

$$\mathcal{L}_{\rm eff}=-\frac{1}{\sqrt{2}}G_F\overline{\nu}\gamma^\mu(1-\gamma_5)\nu\overline{l}\gamma_\mu(g_{\rm V}+g_{\rm A}\gamma_5)l, \quad (1)$$

where ν and l are the neutrino and corresponding lepton fields. For electron neutrinos,

$$g_{\rm V}=\frac{1}{2}+2\sin^2\theta_{\rm W},$$

$$g_{\rm A}=-\frac{1}{2},$$

where the first terms in $g_{\rm V}$ and $g_{\rm A}$ are the contributions from the W exchange diagram and the second one from the Z exchange diagram. $\theta_{\rm W}$ is the Weinberg angle. For muon and tau neutrinos

$$g_{\rm V}=-\frac{1}{2}+2\sin^2\theta_{\rm W},$$

$$g_{\rm A}=\frac{1}{2}.$$

With this interaction Lagrangian the $\nu\nu\gamma$ vertex, as shown in Fig. 1, can be written in terms of two tensors, the vector-vector amplitude and the axial-vector–vector amplitude. The vector-vector amplitude term $\Pi_{\mu\nu}(k)$ is defined as

$$i\Pi_{\mu\nu}(k)=(-ie)^2(-1)\int\frac{d^4p}{(2\pi)^4}{\rm Tr}[\gamma_\mu iS(p)\gamma_\nu iS(p')]. \quad (2)$$

Here and henceforward $p'=p+k$. The above equation looks exactly like the photon polarization tensor, but does not have the same interpretation here. The Feynman diagram associated with the neutrino-photon interaction to one loop is depicted in Fig. 1. The axial-vector–vector amplitude $\Pi^5_{\mu\nu}(k)$ is defined as

$$i\Pi^5_{\mu\nu}(k)=(-ie)^2(-1)\int\frac{d^4p}{(2\pi)^4}{\rm Tr}[\gamma_\mu\gamma_5 iS(p)\gamma_\nu iS(p')]. \quad (3)$$

Both tensors are obtained by calculating the Feynman diagram given in Fig. 1.

While discussing $\Pi^5_{\mu\nu}(k)$ it should be remembered that for the electromagnetic vertex we have the current conservation relation

$$k^\nu\Pi^5_{\mu\nu}(k)=0, \quad (4)$$

which is the gauge invariance condition.

In order to calculate the effective charge of the neutrinos inside a medium, we have to calculate $\Pi^5_{\mu\nu}(k)$. The formalism so discussed is a general one, and we extend the calculations previously done based upon this formalism to the case where we have a constant background magnetic field in addition to a thermal medium.

III. GENERAL FORM OF $\Pi^5_{\mu\nu}(K)$ IN VARIOUS CASES

A. The vacuum case

We start this section with a discussion of the possible tensor structure and form factor analysis of $\Pi^5_{\mu\nu}(k)$, based on the symmetry of the interaction. To begin with we note that $\Pi^5_{\mu\nu}(k)$ in vacuum should vanish. This follows from the arguments below. In vacuum the available vectors and tensors at hand are the following:

$$k_\mu, g_{\mu\nu}, \text{ and } \epsilon_{\mu\nu\lambda\sigma}. \quad (5)$$

The two-point axial-vector–vector amplitude $\Pi^5_{\mu\nu}$ can be expanded in a basis constructed out of the above tensors. Given the parity structure of the theory the only relevant tensor at hand is $\epsilon_{\mu\nu\lambda\sigma}k_\lambda k_\sigma$, which is zero.

B. In medium

On the other hand, in a medium in the absence of any magnetic field, we have an additional vector u^μ, i.e., the velocity of the center of mass of the medium. Therefore the axial-vector–vector amplitude can be expanded in terms of form factors along with the new tensors constructed out of u^μ and the ones we already had in the absence of a medium. A second rank tensor constructed out of them might be $\varepsilon_{\mu\nu\alpha\beta}u^\alpha k^\beta$ [8], which would satisfy the current conservation condition for the two-point function. In a medium an interesting thing happens. In the tensor $\Pi^5_{\mu\nu}(k)$ one of the tensor indices refers to a vector-type vertex and another one to an axial-vector type. The electromagnetic current conservation condition is supposed to hold for the vector-type vertex. But due to the tensor structure of $\Pi^5_{\mu\nu}(k)$ in a medium, as discussed,

$$k^\mu\Pi^5_{\mu\nu}(k)=0=\Pi^5_{\mu\nu}(k)k^\nu \quad (6)$$

to all orders in the Fermi and electromagnetic coupling.

C. In a background magnetic field

As in our case neither C or P but CP is a symmetry, first we look at the CP transformation properties of the axial-vector–vector amplitude. Under a CP transformation the various components of the tensor transform as

$$CP:\ \Pi^5_{00}\to\Pi^5_{00}, \tag{7}$$

$$CP:\ \Pi^5_{ij}\to\Pi^5_{ij}, \tag{8}$$

$$CP:\ \Pi^5_{0i}\to-\Pi^5_{0i}. \tag{9}$$

The above transformation properties are important because the basis tensors must also satisfy the same transformation properties. The functions that multiply the basis tensors to build up $\Pi^5_{\mu\nu}(k)$, called the form factors, must be even functions of the magnetic field and so their *CP* transformations are trivial. This fact directly implies that the basis tensors must vanish in the $\mathcal{B}\to 0$ limit, as $\Pi^5_{\mu\nu}(k)$ vanishes in vacuum. In general, $\Pi^5_{\mu\nu}(k)$ will be an odd function of the external field so that it vanishes when the external field goes to the zero limit. As a result the basis tensors must also be odd in the external fields.

In a uniform background magnetic field, the vectors and tensors at hand are

$$F_{\mu\nu},\quad \tilde{F}_{\mu\nu},\quad k^\mu_\parallel,\quad k^\mu_\perp. \tag{10}$$

Here $F_{\mu\nu}$ is the electromagnetic field strength tensor and $\tilde{F}^{\mu\nu}=\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. Here we stick to the fact that the magnetic field is in the z direction, and so

$$F_{12}=-F_{21}=\mathcal{B}, \tag{11}$$

with all other components of $F_{\mu\nu}$ zero.

With these building blocks we can build four vectors,

$$b^\mu_1=(Fk)^\mu, \tag{12}$$

$$b^\mu_2=(\tilde{F}k)^\mu, \tag{13}$$

$$b^\mu_3=k^\mu_\parallel, \tag{14}$$

$$b^\mu_4=k^\mu_\perp. \tag{15}$$

The expressions $(Fk)_\mu$ and $(\tilde{F}k)_\mu$ stand for

$$(Fk)_\mu=F_{\mu\nu}k^\nu,$$

$$(\tilde{F}k)_\mu=\tilde{F}_{\mu\nu}k^\nu. \tag{16}$$

Here $k^\mu_\parallel=(k^0,0,0,k^3)$ and $k^\mu_\perp=(0,k^1,k^2,0)$, so that $k^\mu=k^\mu_\parallel+k^\mu_\perp$. Also, in our convention

$$g_{\mu\nu}=g^\parallel_{\mu\nu}+g^\perp_{\mu\nu},$$

where

$$g^\parallel_{\mu\nu}=(1,0,0,-1),$$

$$g^\perp_{\mu\nu}=(0,-1,-1,0).$$

We also use

$$k^2_\parallel=k^2_0-k^2_3,$$

$$k^2_\perp=k^2_1+k^2_2.$$

The set of four vectors b^μ_i, $i=1,2,3,4$, are mutually orthogonal to each other and can serve as the basis vectors to build up the tensor basis of $\Pi^5_{\mu\nu}(k)$.

Next the *CP* transformation properties of these vectors are summarized:

$$CP:\ b^0_1\to b^0_1, \tag{17}$$

$$CP:\ b^i_1\to b^i_1. \tag{18}$$

The other three vectors have similar transformation properties as

$$CP:\ b^0_{1,2,3}\to b^0_{1,2,3}, \tag{19}$$

$$CP:\ b^i_{1,2,3}\to -b^i_{1,2,3}. \tag{20}$$

From Eq. (7), Eq. (8), and Eq. (9) we can see that a suitable tensor basis can be built up from vectors b^μ_i where $i=2,3,4$. The *CP* transformation of the axial-vector–vector amplitude compels us to disregard b^μ_1 as a basis vector.

Now we can list the possible candidates which can serve as basis tensors of $\Pi^5_{\mu\nu}(k)$. There are nine of them. For later usage we explicitly write them down as

$$\begin{aligned}\mathrm{B}^{\mu\nu}_1&=b^\mu_2 b^\nu_2\\&=(\tilde{F}k)^\mu(\tilde{F}k)^\nu,\end{aligned} \tag{21}$$

$$\begin{aligned}\mathrm{B}^{\mu\nu}_2&=b^\mu_3 b^\nu_3\\&=k^\mu_\parallel k^\nu_\parallel,\end{aligned} \tag{22}$$

$$\begin{aligned}\mathrm{B}^{\mu\nu}_3&=b^\mu_4 b^\nu_4\\&=k^\mu_\perp k^\nu_\perp,\end{aligned} \tag{23}$$

$$\begin{aligned}\mathrm{B}^{\mu\nu}_4&=b^\mu_2 b^\nu_3\\&=(\tilde{F}k)^\mu k^\nu_\parallel,\end{aligned} \tag{24}$$

$$\begin{aligned}\mathrm{B}^{\mu\nu}_5&=b^\mu_3 b^\nu_2\\&=(\tilde{F}k)^\nu k^\mu_\parallel,\end{aligned} \tag{25}$$

$$\begin{aligned}\mathrm{B}^{\mu\nu}_6&=b^\mu_2 b^\nu_4\\&=(\tilde{F}k)^\mu k^\nu_\perp,\end{aligned} \tag{26}$$

$$\begin{aligned}\mathrm{B}^{\mu\nu}_7&=b^\mu_4 b^\nu_2\\&=(\tilde{F}k)^\nu k^\mu_\perp,\end{aligned} \tag{27}$$

$$\mathrm{B}_8^{\mu\nu} = b_3^{\mu} b_4^{\nu} = k_{\parallel}^{\mu} k_{\perp}^{\nu}, \tag{28}$$

$$\mathrm{B}_9^{\mu\nu} = b_4^{\mu} b_3^{\nu} = k_{\parallel}^{\nu} k_{\perp}^{\mu}. \tag{29}$$

This basis gives nine second rank mutually orthogonal tensors. Any second rank tensor containing higher field dependence can be represented by suitable linear combinations of these tensors.

Out of these nine basis tensors some are useless. To explain the point we focus our attention on $\mathrm{B}_2^{\mu\nu}$, $\mathrm{B}_3^{\mu\nu}$, $\mathrm{B}_8^{\mu\nu}$, and $\mathrm{B}_9^{\mu\nu}$. None of these four vanish in the $\mathcal{B}\to 0$ limit and so they are redundant. Also $\mathrm{B}_1^{\mu\nu}$ is even in the external fields, and as discussed previously is not a suitable candidate for the basis of $\Pi_{\mu\nu}^5(k)$.

Only four basis tensors qualify successfully as the building blocks of the axial-vector–vector amplitude. They are $\mathrm{B}_4^{\mu\nu}$, $\mathrm{B}_5^{\mu\nu}$, $\mathrm{B}_6^{\mu\nu}$, and $\mathrm{B}_7^{\mu\nu}$. The result as given in the papers by Hari Dass and Raffelt verifies this choice [11,12]:[1].

$$\Pi_{\mu\nu}^5(k) = \frac{e^3}{(4\pi)^2 m^2}[-C_{\parallel} k_{\nu_{\parallel}} (\tilde{F}k)_{\mu} + C_{\perp}\{k_{\nu_{\perp}}(k\tilde{F})_{\mu} + k_{\mu_{\perp}}(k\tilde{F})_{\nu} - k_{\perp}^2 \tilde{F}_{\mu\nu}\}]. \tag{30}$$

In the results we find $\tilde{F}^{\mu\nu}$ which we have not listed in our tensor basis, but that is not a fault because it can be made up from the basis supplied. $\tilde{F}^{\mu\nu}$ can be written

$$\tilde{F}^{\mu\nu} = \frac{1}{k_{\parallel}^2}[(\tilde{F}k)^{\mu} k_{\parallel}^{\nu} - (\tilde{F}k)^{\nu} k_{\parallel}^{\mu}]. \tag{31}$$

So to build up the tensorial basis of the axial-vector–vector amplitude, the number of independent tensors required is four.

The four tensors at hand are still not suitable to be the basis tensors of the axial-vector–vector amplitude. As all of them are not transverse to k^{μ}, which is a requirement from electromagnetic current conservation. Furthermore we have to make linear combinations of these tensors which will ultimately give two tensors orthogonal to each other and to k^{μ}, which will serve as the right tensor basis of $\Pi_{\mu\nu}^5(k)$.

D. In a magnetized medium

In the presence of a magnetized medium the situation is complicated. In this analysis we are not going into an in-depth study of the tensorial basis as was done in the case where there is no medium. Here we outline the strategy by which we can build up the tensorial basis, which in reality is similar to the previous case but contains more building blocks. To start with we again emphasize the *CP* transformation properties of the axial-vector–vector amplitude. Unlike the vacuum case now the theory may not be *CP* invariant. This can arise if the background does not respect *CP*. We will discuss here only those cases where the background does not break *CP*. Moreover, now the form factors can be functions of odd powers of the magnetic field, as now new scalars like $(Fk)u$ and $(\tilde{F}k)u$ are also available. These scalars change sign under *CP* transformation. Some of the form factors containing odd orders of the external fields may be accompanied by equal powers of the chemical potential of the background charged fermions, and they will not change sign. So in a magnetized medium there can be basis tensors with different *CP* transformation properties as the form factors which multiply them can also have different transformation properties.

In the presence of a medium, we can have two sets of orthogonal vectors. The first set is as supplied in Eq. (12), Eq. (13), Eq. (14), and Eq. (15). They are all included now. b_1^{μ} is not excluded as in vacuum because the *CP* transformation property of the basis tensors has changed. The other set of orthogonal vectors useful in a medium is

$$b'^{\mu}_1 = (\tilde{F}u)^{\mu}, \tag{32}$$

$$b'^{\mu}_2 = u_{\parallel}^{\mu}. \tag{33}$$

In listing the above vectors we have omitted two vectors, $(Fu)^{\mu}$ and $u_{\perp}^{\mu}$. The reason they are omitted is that ultimately we are interested in the rest frame of the medium. In the medium rest frame there is no electric field. Also in the medium rest frame no contribution will come with $u_{\perp}^{\mu}$.

This above set of vectors has similar *CP* transformation properties with those of b_2^{μ} and b_3^{μ}. But the two sets of vectors are not linearly independent and as such cannot serve as basis vectors to build up the tensorial basis of $\Pi_{\mu\nu}^5$. Only a linear combination of them can make an orthogonal vector basis. Now we list the set of orthogonal basis vectors that can be made from the two sets of vectors, they are

$$b''^{\mu}_1 = (Fk)^{\mu}, \tag{34}$$

$$b''^{\mu}_2 = (\tilde{F}u)^{\mu} + (\tilde{F}k)^{\mu}, \tag{35}$$

$$b''^{\mu}_3 = k_{\perp}^{\mu}, \tag{36}$$

$$b''^{\mu}_4 = k_{\parallel}^{\mu} + u_{\parallel}^{\mu}. \tag{37}$$

In a magnetized medium we have these four basis vectors which serve as the building blocks of the axial-vector–vector amplitude. The basis tensors in this case will be the direct products of these basis vectors. There will be 16 of them but all of them will not be useful.

As was the case in vacuum, all the 16 basis tensors here are not useful because we have the electromagnetic current conservation condition. This constraint will reduce the num-

[1]However, the metric used by the authors in the references mentioned is different from ours, and in their calculation the μ vertex is the vector-type vertex.

ber of basis tensors. The axial-vector–vector amplitude is not orthogonal to k^μ in our case, as the μ vertex is the axial-vector vertex, but then still

$$k^\mu \Pi^5_{\mu\nu} = \mathcal{C}_\nu \tag{38}$$

for some $\mathcal{C}_\nu$, which depends on the mass of the looping fermions. This condition also restrains the number of basis tensors of the axial-vector–vector amplitude in a magnetized medium. The exact calculation of the number of useful elements as basis now goes in the same way as in the absence of a medium.

IV. ONE-LOOP CALCULATION OF THE AXIAL-VECTOR–VECTOR AMPLITUDE

Since we investigate the case with a uniform background magnetic field, without any loss of generality it can be taken to be in the z direction. We denote the magnitude of this field by $\mathcal{B}$, which can be incorporated in various gauges with $A_0 = 0$ and the other components of A being time independent. First ignoring the presence of the medium, the electron propagator in such a field can be written down following Schwinger's approach [13–16]:

$$iS^V_B(p) = \int_0^\infty ds\, e^{\Phi(p,s)} G(p,s), \tag{39}$$

where Φ and G are as given below:

$$\Phi(p,s) \equiv is\left(p_\parallel^2 - \frac{\tan(e\mathcal{B}s)}{e\mathcal{B}s} p_\perp^2 - m^2 \right) - \epsilon|s|, \tag{40}$$

$$\begin{aligned} G(p,s) &\equiv \frac{e^{ie\mathcal{B}s\sigma_z}}{\cos(e\mathcal{B}s)} \left(\not{p}_\parallel + \frac{e^{-ie\mathcal{B}s\sigma_z}}{\cos(e\mathcal{B}s)} \not{p}_\perp + m \right) \\ &= \{ [1 + i\sigma_z \tan(e\mathcal{B}s)](\not{p}_\parallel + m) \\ &\quad + \sec^2(e\mathcal{B}s)\not{p}_\perp \}, \end{aligned} \tag{41}$$

where

$$\sigma_z = i\gamma_1\gamma_2 = -\gamma_0\gamma_3\gamma_5, \tag{42}$$

and we have used

$$e^{ie\mathcal{B}s\sigma_z} = \cos(e\mathcal{B}s) + i\sigma_z \sin(e\mathcal{B}s). \tag{43}$$

To make the expressions transparent we specify our conventions in the following way:

$$\begin{aligned} \not{p}_\parallel &= \gamma_0 p^0 + \gamma_3 p^3, \\ \not{p}_\perp &= \gamma_1 p^1 + \gamma_2 p^2. \end{aligned} \tag{44}$$

Of course, in the range of integration indicated in Eq. (39) s is never negative and hence $|s|$ equals s. In the presence of a background medium, the above propagator is now modified to [17,18]

$$iS(p) = iS^V_B(p) + S^\eta_B(p), \tag{45}$$

where

$$S^\eta_B(p) \equiv -\eta_F(p)[iS^V_B(p) - i\overline{S}^V_B(p)] \tag{46}$$

and

$$\overline{S}^V_B(p) \equiv \gamma_0 S^{V\dagger}_B(p)\gamma_0 \tag{47}$$

for a fermion propagator, such that

$$S^\eta_B(p) = -\eta_F(p)\int_{-\infty}^\infty ds\, e^{\Phi(p,s)} G(p,s). \tag{48}$$

Here $\eta_F(p)$ contains the distribution function for the fermions and the antifermions:

$$\eta_F(p) = \Theta(p\cdot u) f_F(p,\mu,\beta) + \Theta(-p\cdot u) f_F(-p,-\mu,\beta). \tag{49}$$

f_F denotes the Fermi-Dirac distribution function

$$f_F(p,\mu,\beta) = \frac{1}{e^{\beta(p\cdot u - \mu)} + 1}, \tag{50}$$

and Θ is the step function given by

$$\Theta(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Here the four-velocity of the medium is u. In the rest frame its components are $u^\mu = (1,0,0,0)$.

A. The expression for $\Pi^5_{\mu\nu}(k)$ in a thermal medium and in the presence of a background uniform magnetic field

The relevant Feynman diagram of the process appears in Fig. 1. Following that diagram the axial-vector–vector amplitude $\Pi^5_{\mu\nu}(k)$ is expressed as

$$i\Pi^5_{\mu\nu}(k) = (-ie)^2(-1)\int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}[\gamma_\mu\gamma_5 iS(p)\gamma_\nu iS(p')]. \tag{51}$$

The vacuum part has already been calculated in [11] and the thermal part with two factors of η_F is related to pure absorption effects in the medium, which we are leaving out for the time being. The remaining terms are

$$\begin{aligned} i\Pi^5_{\mu\nu}(k) = e^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}[&\gamma_\mu\gamma_5 S^\eta_B(p)\gamma_\nu iS^V_B(p') \\ &+ \gamma_\mu\gamma_5 iS^V_B(p)\gamma_\nu S^\eta_B(p')]. \end{aligned} \tag{52}$$

Using the form of the fermion propagator in a magnetic field in the presence of a thermal medium, as given by expressions (39) and (48) we get

$$i\Pi^5_{\mu\nu}(k)=-e^2\int\frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty}ds\,e^{\Phi(p,s)}\int_0^{\infty}ds'\,e^{\Phi(p',s')}$$
$$\times\{\mathrm{Tr}[\gamma_\mu\gamma_5 G(p,s)\gamma_\nu G(p',s')]\eta_F(p)$$
$$+\mathrm{Tr}[\gamma_\mu\gamma_5 G(-p',s')\gamma_\nu G(-p,s)]\eta_F(-p)\}$$
$$=-e^2\int\frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty}ds\,e^{\Phi(p,s)}$$
$$\times\int_0^{\infty}ds'\,e^{\Phi(p',s')}\mathrm{R}_{\mu\nu}(p,p',s,s') \tag{53}$$

where $\mathrm{R}_{\mu\nu}(p,p',s,s')$ contains the traces.

B. $\mathrm{R}_{\mu\nu}$ to even and odd orders in the magnetic field

We calculate $\mathrm{R}_{\mu\nu}(p,p',s,s')$ to even and odd orders in the external magnetic field and call them $\mathrm{R}^{(e)}_{\mu\nu}$ and $\mathrm{R}^{(o)}_{\mu\nu}$. The reason for doing this is that the two contributions have different properties as far as their dependence on the medium is concerned, and the contributions are

$$\mathrm{R}^{(e)}_{\mu\nu}=4i\eta_-(p)\varepsilon_{\mu\nu\alpha\beta}\{p^{\alpha_\parallel}p'^{\beta_\parallel}[1+\tan(e\mathcal{B}s)\tan(e\mathcal{B}s')]$$
$$+p^{\alpha_\parallel}p'^{\beta_\perp}\sec^2(e\mathcal{B}s')+p^{\alpha_\perp}p'^{\beta_\parallel}\sec^2(e\mathcal{B}s)$$
$$+p^{\alpha_\perp}p'^{\beta_\perp}\sec^2(e\mathcal{B}s)\sec^2(e\mathcal{B}s')\} \tag{54}$$

and

$$\mathrm{R}^{(o)}_{\mu\nu}=4i\eta_+(p)(m^2\varepsilon_{\mu\nu12}[\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')]$$
$$+\{(g_{\mu\alpha_\parallel}p^{\widetilde{\alpha_\parallel}}p'_{\nu_\parallel}-g_{\mu\nu}p'_{\alpha_\parallel}p^{\widetilde{\alpha_\parallel}}+g_{\nu\alpha_\parallel}p^{\widetilde{\alpha_\parallel}}p'_{\mu_\parallel})$$
$$+(g_{\mu\alpha_\parallel}p^{\widetilde{\alpha_\parallel}}p'_{\nu_\perp}+g_{\nu\alpha_\parallel}p^{\widetilde{\alpha_\parallel}}p'_{\mu_\perp})\sec^2(e\mathcal{B}s')\}\tan(e\mathcal{B}s)$$
$$+\{(g_{\mu\alpha_\parallel}p'^{\widetilde{\alpha_\parallel}}p_{\nu_\parallel}-g_{\mu\nu}p_{\alpha_\parallel}p'^{\widetilde{\alpha_\parallel}}+g_{\nu\alpha_\parallel}p'^{\widetilde{\alpha_\parallel}}p_{\mu_\parallel})$$
$$+(g_{\mu\alpha_\parallel}p'^{\widetilde{\alpha_\parallel}}p_{\nu_\perp}+g_{\nu\alpha_\parallel}p'^{\widetilde{\alpha_\parallel}}p_{\mu_\perp})\sec^2(e\mathcal{B}s)\}$$
$$\times\tan(e\mathcal{B}s')). \tag{55}$$

Here

$$\eta_+(p)=\eta_F(p)+\eta_F(-p), \tag{56}$$

$$\eta_-(p)=\eta_F(p)-\eta_F(-p), \tag{57}$$

which contain the information about the distribution functions. It should also be noted that, in our convention,

$$a_\mu b^{\widetilde{\mu_\parallel}}=a_0b^3+a_3b^0.$$

As stated, we have split the contributions to $\Pi^5_{\mu\nu}(k)$ into odd and even orders in the external constant magnetic field. The main reason for doing so is the fact that $\Pi^{5(o)}_{\mu\nu}(k)$ and $\Pi^{5(e)}_{\mu\nu}(k)$, the axial-vector–vector amplitude to odd and even powers in $e\mathcal{B}$, have different dependences on the background matter. Pieces proportional to even powers in $\mathcal{B}$ are proportional to $\eta_-(p_0)$, an odd function of the chemical potential. On the other hand, pieces proportional to odd powers in $\mathcal{B}$ depend on $\eta_+(p_0)$, and are even in μ, and as a result survives in the limit $\mu\to 0$. This is a direct consequence of the charge conjugation and parity symmetries of the underlying theory.

From Eq. (54) we notice that $\Pi^5_{\mu\nu}(k)$ to even orders in the magnetic field satisfies the current conservation condition in both the vertices. In Eq. (55) we see that all the terms in the right hand side are symmetric in the μ and ν indices except the first term. This term differentiates between the two vertices in this case, and as $\Pi^5_{\mu\nu}(k)$ to odd orders in magnetic field is gauge invariant in the ν vertex we do not get the same condition for the axial-vector vertex. If in Eq. (55) we put $m=0$ then all the terms on the right will be symmetric in both the tensor indices, and as a result the current conservation condition will hold for both vertices. If the mass of the looping fermion is not zero then from the above analysis we can say that only Eq. (4) will hold. If the looping fermion is massless then Eq. (6) will hold, something which is expected.

If we concentrate on the rest frame of the medium, then $p\cdot u=p_0$. Thus, the distribution function does not depend on the spatial components of p. From the form of Eq. (54) and Eq. (55) we find that in Eq. (53) the integral over the transverse components of p has the following generic structure:

$$\int d^2p_\perp e^{\Phi(p,s)}e^{\Phi(p',s')}\times(p^{\beta_\perp}\text{ or }p'^{\beta_\perp}). \tag{58}$$

Notice now that

$$\frac{\partial}{\partial p_{\beta_\perp}}[e^{\Phi(p,s)}e^{\Phi(p',s')}]$$
$$=\frac{2i}{e\mathcal{B}}[\tan(e\mathcal{B}s)p^{\beta_\perp}+\tan(e\mathcal{B}s')p'^{\beta_\perp}]e^{\Phi(p,s)}e^{\Phi(p',s')}. \tag{59}$$

However, this expression, being a total derivative, should integrate to zero. Thus we obtain that

$$\tan(e\mathcal{B}s)p^{\beta_\perp}\mathrel{\hat{=}}-\tan(e\mathcal{B}s')p'^{\beta_\perp}, \tag{60}$$

where the symbol $\mathrel{\hat{=}}$ means that the expressions on both sides of it, although not necessarily equal algebraically, yield the same integral. This gives

$$p^{\beta_\perp}\mathrel{\hat{=}}-\frac{\tan(e\mathcal{B}s')}{\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')}k^{\beta_\perp},$$

$$p'^{\beta_\perp}\mathrel{\hat{=}}\frac{\tan(e\mathcal{B}s)}{\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')}k^{\beta_\perp}. \tag{61}$$

Similarly, we can derive some other relations which can be used under the momentum integral signs. To write them in a useful form, we turn to Eq. (59) and take another derivative with respect to $p^{\alpha_\perp}$. From the fact that this derivative should also vanish on p integration, we find

$$p_\perp^\alpha p_\perp^\beta \,\hat{=}\, \frac{1}{\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')}\left[\frac{ie\mathcal{B}}{2}g_\perp^{\alpha\beta} + \frac{\tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')}k_\perp^\alpha k_\perp^\beta\right]. \tag{62}$$

In particular, then,

$$p_\perp^2 \,\hat{=}\, \frac{1}{\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')}\times\left[-ie\mathcal{B}+\frac{\tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')}k_\perp^2\right]. \tag{63}$$

It then simply follows that

$$p_\perp'^2 \,\hat{=}\, \frac{1}{\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')}\times\left[-ie\mathcal{B}+\frac{\tan^2(e\mathcal{B}s)}{\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')}k_\perp^2\right]. \tag{64}$$

And finally using the definition of the exponential factor in Eq. (40) we can write

$$m^2 \,\hat{=}\left(i\frac{d}{ds}+[p_\parallel^2-\sec^2(e\mathcal{B}s)p_\perp^2]\right). \tag{65}$$

Using the above relations we get

$$\begin{aligned}\mathrm{R}^{(e)}_{\mu\nu} \,\hat{=}\, 4i\eta_-(p_0)\{&\varepsilon_{\mu\nu\alpha_\parallel\beta_\parallel}p^{\alpha_\parallel}p'^{\beta_\parallel}[1+\tan(e\mathcal{B}s)\tan(e\mathcal{B}s')]\\ &+\varepsilon_{\mu\nu\alpha_\parallel\beta_\perp}p^{\alpha_\parallel}p'^{\beta_\perp}\sec^2(e\mathcal{B}s')\\ &+\varepsilon_{\mu\nu\alpha_\perp\beta_\parallel}p^{\alpha_\perp}p'^{\beta_\parallel}\sec^2(e\mathcal{B}s)\}\end{aligned} \tag{66}$$

and

$$\begin{aligned}\mathrm{R}^{(o)}_{\mu\nu} \,\hat{=}\, 4i\eta_+(p_0)\Bigg[&-\varepsilon_{\mu\nu12}\left\{\frac{\sec^2(e\mathcal{B}s)\tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')}k_\perp^2\right.\\ &\left.+(k\cdot p)_\parallel[\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')]\right\}\\ &+2\varepsilon_{\mu12\alpha_\parallel}[p'_{\nu_\parallel}p^{\alpha_\parallel}\tan(e\mathcal{B}s)+p_{\nu_\parallel}p'^{\alpha_\parallel}\tan(e\mathcal{B}s')]\\ &+g_{\mu\alpha_\parallel}k_{\nu_\perp}\left\{p^{\widetilde{\alpha_\parallel}}[\tan(e\mathcal{B}s)-\tan(e\mathcal{B}s')]\right.\\ &\left.-k^{\widetilde{\alpha_\parallel}}\frac{\sec^2(e\mathcal{B}s)\tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')}\right\}+\{g_{\mu\nu}(p\cdot\widetilde{k})_\parallel\\ &+g_{\nu\alpha_\parallel}p^{\widetilde{\alpha_\parallel}}k_{\mu_\perp}\}[\tan(e\mathcal{B}s)-\tan(e\mathcal{B}s')]\\ &+g_{\nu\alpha_\parallel}k^{\widetilde{\alpha_\parallel}}p_{\mu_\perp}\sec^2(e\mathcal{B}s)\tan(e\mathcal{B}s')\Bigg].\end{aligned} \tag{67}$$

Before going into the next section we comment on the nature of the integral appearing in Eq. (53). The first point to make is that from the form of $\mathrm{R}^{(e)}_{\mu\nu}$ in Eq. (66) we note that the axial-vector–vector amplitude in a magnetized medium to even orders in the magnetic field is antisymmetric, as it was in a medium without any magnetic field. Contrary to this, $\mathrm{R}^{(o)}_{\mu\nu}$ does not have any well defined symmetry property.

Secondly, as the integrals are not done explicitly something must be said about the possible divergences that may appear in evaluating them. In principle, we expect no divergences here. The reasons are as follows. First, we are working in finite temperatures and so an automatic ultraviolet cutoff, the temperature T of the medium, is already present. Second, it must be noted that magnetic fields brings no new divergences into the calculations. The divergence that could have been present would have come from the vacuum contribution of $\Pi^5_{\mu\nu}(k)$ when $\mathcal{B}=0$, but in this case that part does not exist at all, as we saw in Sec. III. In this connection it can be said that in the absence of the medium but in the presence of the background magnetic field another divergent structure could arise, that is, anomaly, due to the presence of the axial-vector vertex. Anomaly is essentially an ultraviolet phenomenon which shows up in nonconservation of some currents, after making the quantum corrections, which were conserved classically. But in the present case these need not worry us because we are working in a thermal medium and as discussed previously the ultraviolet regulators are already present in our theory.

As a result, the integral expression for $\Pi^5_{\mu\nu}(k)$ in our case does not have any singularities. So we can now write the full expression of the axial-vector–vector amplitude as

$$i\Pi^5_{\mu\nu}(k) = -e^2\int\frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty}ds\,e^{\Phi(p,s)}\times\int_0^\infty ds'\,e^{\Phi(p',s')}[\mathrm{R}^{(o)}_{\mu\nu}+\mathrm{R}^{(e)}_{\mu\nu}] \tag{68}$$

where $\mathrm{R}^{(o)}_{\mu\nu}$ and $\mathrm{R}^{(e)}_{\mu\nu}$ are given by Eqs. (67) and (66) in the rest frame of the medium.

V. ZERO MOMENTUM LIMIT AND EFFECTIVE CHARGE

The off-shell electromagnetic vertex function Γ_ν is defined in such a way that, for on-shell neutrinos, the $\nu\nu\gamma$ amplitude is given by

$$\mathcal{M} = -i\overline{u}(q')\Gamma_\nu u(q)A^\nu(k), \tag{69}$$

where k is the photon momentum. Here, $u(q)$ is the neutrino spinor and A^ν stands for the electromagnetic vector potential. In general Γ_ν will depend on k and the characteristics of the medium. With our effective Lagrangian in Eq. (1), Γ_ν is given by

$$\Gamma_\nu = -\frac{1}{\sqrt{2}e}G_F\gamma^\mu(1-\gamma_5)(g_{\mathrm{V}}\Pi_{\mu\nu}+g_{\mathrm{A}}\Pi^5_{\mu\nu}). \tag{70}$$

The effective charge of the neutrinos is defined in terms of the vertex function by the following relation [8]:

$$e_{\mathrm{eff}}=\frac{1}{2q_0}\overline{u}(q)\Gamma_0(k_0=0,\mathbf{k}\to\mathbf{0})u(q). \tag{71}$$

For massless Weyl spinors this definition can be rendered in the form

$$e_{\mathrm{eff}}=\frac{1}{2q_0}\mathrm{Tr}[\Gamma_0(k_0=0,\mathbf{k}\to\mathbf{0})(1+\lambda\gamma^5)\not{q}] \tag{72}$$

where $\lambda=\pm 1$ is the helicity of the spinors.

We remarked earlier in Sec. III that in a medium we have an additional vector u^μ. The axial-vector–vector amplitudes in this case, of the form $\varepsilon_{\mu\nu\alpha\beta}u^\alpha k^\beta$, do not contribute to the effective electric charge of the neutrinos since for charge calculation we have to put the index $\nu=0$. In the rest frame only the time component of the four-vector u exists, which forces the totally antisymmetric tensor to vanish. But the polarization tensor can be expanded in terms of form factors along with the new tensors constructed out of u^μ and the ones we already had in absence of a medium as

$$\Pi_{\mu\nu}(k)=\Pi_T T_{\mu\nu}+\Pi_L L_{\mu\nu}. \tag{73}$$

Here

$$T_{\mu\nu}=\tilde{g}_{\mu\nu}-L_{\mu\nu},$$

$$L_{\mu\nu}=\frac{\tilde{u}_\mu\tilde{u}_\nu}{\tilde{u}^2}$$

with

$$\tilde{g}_{\mu\nu}=g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2},$$

$$\tilde{u}_\mu=\tilde{g}_{\mu\rho}u^\rho.$$

The longitudinal projector $L_{\mu\nu}$ is not zero in the limit $k_0=0,\vec{k}\to 0$, and Π_L is also not zero in the above mentioned limit [3]. This fact is responsible for giving a nonzero contribution to the effective charge of a neutrino in a medium.

From Eq. (30) we see that the axial-vector–vector amplitude in a background magnetic field without any medium does not survive when the momentum of the external photon vanishes, and as a result there cannot be any effective electric charge of the neutrinos in a constant background magnetic field. Actually, this formal statement could have been spoilt by the presence of possible infrared divergence in the loop; i.e., in $C_\parallel$ and $C_\perp$ [12]. Since the particle inside the loop is massive, there is no scope of having infrared divergence; hence it does not contribute to the neutrino effective charge.

Now we concentrate on the zero momentum limit of that part of the axial polarization tensor which is going to contribute to the neutrino effective charge in a magnetized plasma. From the outset it is to be made clear that we are calculating only the axial contribution to the effective charge.

Effective charge to odd orders in external field

Denoting $\Pi^5_{\mu\nu}(k_0=0,\vec{k}\to 0)$ by $\Pi^5_{\mu\nu}$, we obtain

$$\Pi^5_{\mu 0}=\lim_{k_0=0\vec{k}\to 0}4e^2\int\frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty}ds\,e^{\Phi(p,s)}\times\int_0^\infty ds'\,e^{\Phi(p',s')}[\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')]\eta_+(p_0)\times[2p_0^2-(k\cdot p)_\parallel]\varepsilon_{\mu 012}; \tag{74}$$

the other terms turn out to be zero in this limit. The above equation shows that, except for the exponential functions, the integrand is free of the perpendicular components of momenta. It is a peculiarity of this case that the perpendicular excitations of the loop momenta are present only in the phaselike parts of the integrals and in effect decouple from the scene once they are integrated out. Its presence is felt only through a linear dependence of the external field $\mathcal{B}$ when the perpendicular components of k vanish. Upon performing the Gaussian integration over the perpendicular components and taking the limit $k_\perp\to 0$, we obtain

$$\Pi^5_{30}=\lim_{k_0=0,\vec{k}\to 0}\frac{(4ie^3\mathcal{B})}{4\pi}\int\frac{d^2p_\parallel}{(2\pi)^2}\int_{-\infty}^{\infty}ds\,e^{is(p_\parallel^2-m^2)-\varepsilon|s|}\times\int_0^\infty ds'\,e^{is'(p_\parallel'^2-m^2)-\varepsilon|s'|}\eta_+(p_0)[2p_0^2-(k\cdot p)_\parallel]. \tag{75}$$

It is worth noting that the s integral gives

$$\int_\infty^\infty ds\,e^{is(p_\parallel^2-m^2)-\varepsilon|s|}=2\pi\delta(p_\parallel^2-m^2) \tag{76}$$

and the s' integral gives

$$\int_0^\infty ds'\,e^{is'(p_\parallel'^2-m^2)-\varepsilon|s'|}=\frac{i}{(p_\parallel'^2-m^2)+i\varepsilon}. \tag{77}$$

Using the above results in Eq. (75) and using the delta function constraint, we arrive at

$$\Pi^5_{30}=\lim_{k_0=0,\vec{k}\to 0}-2(e^3\mathcal{B})\int\frac{d^2p_\parallel}{(2\pi)^2}\delta(p_\parallel^2-m^2)\eta_+(p_0)\times\left[\frac{2p_0^2}{(k_\parallel^2+2(p.k)_\parallel)}-\frac{1}{2}\right]. \tag{78}$$

In deriving Eq. (78), pieces proportional to $k_\parallel^2$ in the numerator were neglected. Now if one makes the substitution $p_\parallel'\to(p_\parallel+k_\parallel/2)$ and sets $k_0=0$, one arrives at

$$\Pi^5_{30} = \lim_{k_0=0,\vec{k}\to 0} 2(e^3\mathcal{B}) \int \frac{dp_3}{(2\pi)^2}[n_+(E'_p)+n_-(E'_p)] \times \left[\frac{E'_p}{p_3k_3}+\frac{1}{2E'_p}\right]. \tag{79}$$

Here $n_\pm(E'_p)$ are the functions $f_F(E'_p,-\mu,\beta)$ and $f_F(E'_p,\mu,\beta)$, as given in Eq. (50), which are nothing but the Fermi-Dirac distribution functions of the particles and the antiparticles in the medium with a modified energy E'_p. The new term E'_p is defined as follows:

$$E'^2_p = [(p_3-k_3/2)]^2+m^2,$$

and it can be expanded for small external momenta in the following way:

$$E'^2_p \simeq p_3^2+m^2-p_3k_3 = E_p^2-p_3k_3,$$

where $E_p^2=p_3^2+m^2$. Noting that

$$E'_p = E_p - \frac{p_3k_3}{2E_p}+O(k_3^2), \tag{80}$$

one can use this expansion in Eq. (79) to arrive at

$$\Pi^5_{30} = \lim_{k_0=0,\vec{k}\to 0} 2(e^3\mathcal{B}) \int \frac{dp_3}{(2\pi)^2}[n_+(E'_p)+n_-(E'_p)]\left[\frac{E_p}{p_3k_3}\right]. \tag{81}$$

The expression for $\eta_+(E'_p)=n_+(E'_p)+n_-(E'_p)$ when expanded in powers of the external momentum k_3 is given by

$$\eta_+(E'_p) = \left(1+\frac{1}{2}\frac{\beta p_3k_3}{E_p}\right)\eta_+(E_p) \tag{82}$$

up to first order terms in the external momentum k_3.

1. Effective charge for $\mu \ll m$

In the limit, when $\mu \ll m$, one can use the following expansion:

$$\begin{aligned}\eta_+(E'_p) &= [n_+(E'_p)+n_-(E'_p)] \\ &= 2\sum_{n=0}^{\infty}(-1)^n\cosh([n+1]\beta\mu) \\ &\quad \times e^{-(n+1)\beta E_p}\left(1+\frac{\beta p_3k_3}{2E_p}+O(k_3^2)+\cdots\right)\end{aligned} \tag{83}$$

Now using Eq. (83) in Eq. (81) we get

$$\Pi^5_{30} = \lim_{k_0=0,\vec{k}\to 0} (4e^3\mathcal{B})\sum_{n=0}^{\infty}(-1)^n\cosh([n+1]\beta\mu) \times \int \frac{dp_3}{(2\pi)^2}e^{-(n+1)\beta E_p}\left[\frac{E_p}{(p_3k_3)}+\frac{\beta}{2}\right]. \tag{84}$$

The first term vanishes by symmetry of the integral, but the second term is finite and so we get

$$\Pi^5_{30} = \beta\frac{(e^3\mathcal{B})}{2\pi^2}\sum_{n=0}^{\infty}(-1)^n\cosh([n+1]\beta\mu) \times \int dp_3 e^{-(n+1)\beta E_p}. \tag{85}$$

To perform the momentum integration, use of the following integral transform turns out to be extremely convenient:

$$e^{-\alpha\sqrt{s}} = \frac{\alpha}{2\sqrt{\pi}}\int_0^\infty du e^{-us-\alpha^2/4u}u^{-3/2}. \tag{86}$$

Identifying $\sqrt{s}$ with E_p and $[(n+1)\beta]$ with α one can easily perform the Gaussian p_3 integration without any difficulty. The result is

$$\Pi^5_{30} = \beta\frac{(e^3\mathcal{B})}{2\pi^2}\sum_{n=0}^{\infty}(-1)^n\cosh([n+1]\beta\mu)(\beta(n+1)/2) \times \int du e^{-m^2u-[(n+1)\beta/2]^2/u}u^{-2}. \tag{87}$$

Performing the u integration the axial part of the effective charge of neutrino in the limit of $m>\mu$ turns out to be

$$e^{\nu_a}_{\rm eff} = -\sqrt{2}g_A m\beta G_F\frac{e^2\mathcal{B}}{\pi^2}(1-\lambda)\cos(\theta)\sum_{n=0}^{\infty}(-1)^n \times\cosh[(n+1)\beta\mu]K_{-1}[m\beta(n+1)]. \tag{88}$$

Here θ is the angle between the neutrino three-momentum and the background magnetic field. The superscript ν_a on $e^{\nu_a}_{\rm eff}$ denotes that we are calculating the axial contribution of the effective charge. $K_{-1}[m\beta(n+1)]$ is the modified Bessel function (of the second kind) of order 1 [for this function $K_{-1}(x)=K_1(x)$], which falls off sharply as we move away from the origin in the positive direction. As temperature tends to zero Eq. (88) seems to blow up because of the presence of $m\beta$, but $K_{-1}[m\beta(n+1)]$ would damp its growth as $e^{-m\beta}$; hence the result remains finite.

2. Effective charge for $\mu \gg m$

Here we would try to estimate neutrino effective charge when $\mu \gg m$. Using Eqs. (81) and (82) we would obtain

$$\Pi^5_{30} = \frac{e^3\mathcal{B}}{2\pi}\beta\int\frac{dp}{2\pi}\eta_+(E_p). \tag{89}$$

Neglecting m in the expression in E_p we would obtain

$$\Pi^5_{30}=\frac{e^3\mathcal{B}}{2\pi^2}\ln[(1+e^{\beta\mu})(1+e^{-\beta\mu})]. \tag{90}$$

The same can also be written as

$$\Pi^5_{30}=\frac{e^3\mathcal{B}}{\pi^2}\ln\left[2\cosh\left(\frac{\beta\mu}{2}\right)\right]. \tag{91}$$

The expression for the effective charge then turns out to be

$$e^{\nu_a}_{\rm eff}=-\sqrt{2}g_A G_F\frac{e^2\mathcal{B}}{\pi^2}\ln\left[2\cosh\left(\frac{\beta\mu}{2}\right)\right](1-\lambda)\cos(\theta) \tag{92}$$

where λ is the helicity of the neutrino spinors.

Before going to the next section some general discussion about the effective charge expression can be made. In a background magnetic field the field dependence of the form factors, which are usually scalars, can be of the following form:

$$k^\mu F_{\mu\nu}F^{\nu\lambda}k_\lambda \quad \text{and} \quad F_{\mu\nu}F^{\mu\nu} \tag{93}$$

or

$$(\tilde{F}u)^\mu(\tilde{F}u)_\mu. \tag{94}$$

These forms do not exhaust all the possibilities; other terms can also be constructed by the above forms. The thing that must be noted is that when k tends to zero only terms that can survive in the form factors must be an even function of $\mathcal{B}$.

Of all possible tensorial structures for the axial-vector–vector amplitude in a magnetized plasma, there exists one term which is independent of the external momentum k, and is given by

$$\tilde{F}_{\mu\alpha}u^\alpha u^\parallel_\nu.$$

It is worth noting that this term in Eq. (A1), which is odd in the external field, survives in the zero external momentum limit in the rest frame of the medium. We noted earlier that the form factors which exist in the rest frame of the medium and in the zero momentum limit are even in powers of the external field. This tells us directly that the axial polarization tensor must be odd in the external field in the zero external momentum limit, a result which we have verified in this work.

VI. CONCLUSION

In this work we elucidated the physical significance of the axial-vector–vector amplitude in various neutrino mediated processes in a magnetized medium. We analyzed its gauge invariance properties. Its tensor structure was written down, and we showed that the integral expression of the tensor is ultraviolet finite. It has been shown that the part of $\Pi^5_{\mu\nu}(k)$ even in $\mathcal{B}$ does not contribute to the effective electric charge. However, it does contribute to physical processes, e.g., neutrino Cherenkov radiation or neutrino decay in a medium. It is worth noting that in the low density high temperature limit, the magnitude of $e^{\nu_a}_{\rm eff}$ can become greater than the effective charge of the neutrino in the ordinary medium provided $e\mathcal{B}$ is large enough. On the other hand, in the high density limit $e^{\nu_a}_{\rm eff}$ can dominate the effective charge of the neutrino as found in an unmagnetized medium, provided the temperature is low enough. However, in standard astrophysical objects, e.g., for the core of a type II supernova the temperature is of the order of 30–60 MeV with Fermi momentum around 300 MeV, for red giants the values same are 10 keV and 400 keV, and for young white dwarves the temperature is around 0.1–1 keV and the Fermi momentum 500 keV. In these systems one can have a relatively large induced neutrino charge, provided the field strength is large enough.

ACKNOWLEDGMENT

We would like to thank Professor Palash B. Pal and Professor Parthasarathi Majumdar for helpful discussions, comments, and suggestions.

APPENDIX: GAUGE INVARIANCE

Now we concentrate on Eq. (4), which we discussed in Sec. II. The axial-vector–vector amplitude has an electromagnetic vertex and as a result electromagnetic current must be conserved. From Eq. (30) we see that $\Pi^5_{\mu\nu}(k)$ is gauge invarient in the μ vertex, which is the electromagnetic vertex in that case. In our case as discussed the ν vertex is the electromagnetic vertex, and we explicitly show the gauge invariance in that vertex below.

1. Gauge invariance for $\Pi^5_{\mu\nu}(k)$ to even orders in the external field

The axial-vector–vector amplitude even in the external field is given by

$$\Pi^{5(e)}_{\mu\nu}(k)=-(-ie)^2(-1)\int\frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty}ds\,e^{\Phi(p,s)}\times\int_0^\infty ds'\,e^{\Phi(p',s')}\mathrm{R}^{(e)}_{\mu\nu}(p,p',s,s'). \tag{A1}$$

Noting that

$$q^\alpha p_\alpha=q^{\alpha\parallel}p_{\alpha_\parallel}+q^{\alpha_\perp}p_{\alpha_\perp},$$

we can write Eq. (66) as

$$\begin{aligned}\mathrm{R}^{(e)}_{\mu\nu}\triangleq\,&4i\eta_-(p_0)\{(\varepsilon_{\mu\nu\alpha\beta}p^\alpha p'^\beta-\varepsilon_{\mu\nu\alpha\beta_\perp}p^\alpha p'^{\beta_\perp}\\&-\varepsilon_{\mu\nu\alpha_\perp\beta}p^{\alpha_\perp}p'^\beta)[1+\tan(e\mathcal{B}s)\tan(e\mathcal{B}s')]\\&+\varepsilon_{\mu\nu\alpha\beta_\perp}p^\alpha p'^{\beta_\perp}\sec^2(e\mathcal{B}s')\\&+\varepsilon_{\mu\nu\alpha_\perp\beta}p^{\alpha_\perp}p'^\beta\sec^2(e\mathcal{B}s)\}.\end{aligned} \tag{A2}$$

Here throughout we have omitted terms such as $\varepsilon_{\mu\nu\alpha_\perp\beta_\perp}p^{\alpha_\perp}p'^{\beta_\perp}$, since by the application of Eq. (61) we have

$$\varepsilon_{\mu\nu\alpha_\perp\beta_\perp}p^{\alpha_\perp}p'^{\beta_\perp}=\varepsilon_{\mu\nu\alpha_\perp\beta_\perp}p^{\alpha_\perp}p^{\beta_\perp}+\varepsilon_{\mu\nu\alpha_\perp\beta_\perp}p^{\alpha_\perp}k^{\beta_\perp}$$
$$\triangleq-\frac{\tan(e\mathcal{B}s')}{\tan(e\mathcal{B}s')+\tan(e\mathcal{B}s')}\times\varepsilon_{\mu\nu\alpha_\perp\beta_\perp}k^{\alpha_\perp}k^{\beta_\perp},$$

which is zero.

After rearranging the terms appearing in Eq. (A2), and by the application of Eq. (61) we arrive at the expression

$$\mathrm{R}^{(e)}_{\mu\nu}\triangleq4i\,\eta_-(p_0)\Bigg[\varepsilon_{\mu\nu\alpha\beta}p^\alpha k^\beta[1+\tan(e\mathcal{B}s)\tan(e\mathcal{B}s')]+\varepsilon_{\mu\nu\alpha\beta_\perp}k^\alpha k^{\beta_\perp}\tan(e\mathcal{B}s)\times\tan(e\mathcal{B}s')\frac{\tan(e\mathcal{B}s)-\tan(e\mathcal{B}s')}{\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')}\Bigg].\tag{A3}$$

Because of the presence of terms like $\varepsilon_{\mu\nu\alpha\beta}k^\beta$ and $\varepsilon_{\mu\nu\alpha\beta_\perp}k^\alpha$ if we contract $\mathrm{R}^{(e)}_{\mu\nu}$ by k^ν, it vanishes.

2. Gauge invariance for $\Pi^5_{\mu\nu}(k)$ to odd orders in the external field

The axial-vector–vector amplitude odd in the external field is given by

$$\Pi^{5(o)}_{\mu\nu}(k)=-(-ie)^2(-1)\int\frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty}ds\,e^{\Phi(p,s)}\times\int_0^\infty ds'\,e^{\Phi(p',s')}\mathrm{R}^{(o)}_{\mu\nu}(p,p',s,s')\tag{A4}$$

where $\mathrm{R}^{(o)}_{\mu\nu}(p,p',s,s')$ is given by Eq. (67). The general gauge invariance condition in this case,

$$k^\nu\Pi^{5(o)}_{\mu\nu}(k)=0,\tag{A5}$$

can always be written down in terms of the following two equations:

$$k^\nu\Pi^{5(o)}_{\mu_\parallel\nu}(k)=0,\tag{A6}$$

$$k^\nu\Pi^{5(o)}_{\mu_\perp\nu}(k)=0,\tag{A7}$$

where $\Pi^{5(o)}_{\mu_\parallel\nu}(k)$ is that part of $\Pi^{5(o)}_{\mu\nu}(k)$ where the index μ can take the values 0 and 3 only. Similarly, $\Pi^{5(o)}_{\mu_\perp\nu}(k)$ stands for the part of $\Pi^{5(o)}_{\mu\nu}(k)$ where μ can take the values 1 and 2 only. $\Pi^{5(o)}_{\mu_\parallel\nu}(k)$ contains $\mathrm{R}^{(o)}_{\mu_\parallel\nu}(p,p',s,s')$, which from Eq. (67) is as follows:

$$\mathrm{R}^{(o)}_{\mu_\parallel\nu}\triangleq4i\,\eta_+(p_0)\Bigg[-\varepsilon_{\mu_\parallel\nu12}\left\{\frac{\sec^2(e\mathcal{B}s)\tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')}k_\perp^2+(k\cdot p)_\parallel[\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')]\right\}+2\varepsilon_{\mu_\parallel12\alpha_\parallel}[p'_{\nu_\parallel}p^{\alpha_\parallel}\tan(e\mathcal{B}s)+p_{\nu_\parallel}p'^{\alpha_\parallel}\tan(e\mathcal{B}s')]+g_{\mu_\parallel\alpha_\parallel}k_{\nu_\perp}\left\{p^{\widetilde{\alpha_\parallel}}[\tan(e\mathcal{B}s)-\tan(e\mathcal{B}s')]-k^{\widetilde{\alpha_\parallel}}\frac{\sec^2(e\mathcal{B}s)\tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')}\right\}+g_{\mu_\parallel\nu}(p\cdot\tilde{k})_\parallel[\tan(e\mathcal{B}s)-\tan(e\mathcal{B}s')]\Bigg]\tag{A8}$$

and $\Pi^{5(o)}_{\mu_\perp\nu}$ contains $\mathrm{R}^{(o)}_{\mu_\perp\nu}(p,p',s,s')$, which is

$$R^{(o)}_{\mu_\perp\nu}\triangleq4i\,\eta_+(p_0)(\{g_{\mu_\perp\nu}(p\cdot\tilde{k})_\parallel+g_{\nu\alpha_\parallel}p^{\widetilde{\alpha_\parallel}}k_{\mu_\perp}\}[\tan(e\mathcal{B}s)-\tan(e\mathcal{B}s')]+g_{\nu\alpha_\parallel}k^{\widetilde{\alpha_\parallel}}p_{\mu_\perp}\sec^2(e\mathcal{B}s)\tan(e\mathcal{B}s')).\tag{A9}$$

Equations (A6), (A7) imply that one should have the following relations satisfied:

$$k^\nu\int\frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty}ds\,e^{\Phi(p,s)}\int_0^\infty ds'\,e^{\Phi(p',s')}R^{(o)}_{\mu_\perp\nu}=0\tag{A10}$$

and

$$k^\nu\int\frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty}ds\,e^{\Phi(p,s)}\int_0^\infty ds'\,e^{\Phi(p',s')}R^{(o)}_{\mu_\parallel\nu}=0.\tag{A11}$$

Of the two above equations, Eq. (A10) can be verified easily since

$$k^\nu R_{\mu_\perp \nu}=0. \tag{A12}$$

Now we look at Eq. (A11). We explicitly consider the case $\mu_\parallel=3$ (the $\mu_\parallel=0$ case leads to a similar result). For $\mu_\parallel=3$,

$$k^\nu R^{(o)}_{3\nu} \triangleq -p_0\{(p'^2_\parallel - p^2_\parallel)[\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')] - k^2_\perp[\tan(e\mathcal{B}s)-\tan(e\mathcal{B}s')]\}[4i\,\eta_+(p_0)]. \tag{A13}$$

Apart from the small convergence factors,

$$\frac{i}{e\mathcal{B}}[\Phi(p,s)+\Phi(p',s')]=(p'^2_\parallel+p^2_\parallel-2m^2)\xi-(p'^2_\parallel-p^2_\parallel)\zeta - p'^2_\perp\tan(\xi-\zeta)-p^2_\perp\tan(\xi+\zeta), \tag{A14}$$

where we have defined the parameters

$$\xi=\frac{1}{2}e\mathcal{B}(s+s'),$$

$$\zeta=\frac{1}{2}e\mathcal{B}(s-s'). \tag{A15}$$

From the last two equations we can write

$$ie\mathcal{B}\frac{d}{d\zeta}e^{\Phi(p,s)+\Phi(p',s')}=e^{\Phi(p,s)+\Phi(p',s')}[p'^2_\parallel-p^2_\parallel - p'^2_\perp\sec^2(\xi-\zeta)+p^2_\perp\sec^2(\xi+\zeta)], \tag{A16}$$

which implies

$$p'^2_\parallel-p^2_\parallel=ie\mathcal{B}\frac{d}{d\xi}+[p'^2_\perp\sec^2(e\mathcal{B}s')-p^2_\perp\sec^2(e\mathcal{B}s)]. \tag{A17}$$

The equation above is valid in the sense that both sides of it actually act upon $e^{\tilde{\Phi}(p,s,p',s')}$, where

$$\tilde{\Phi}(p,p',s,s')=\Phi(p,s)+\Phi(p',s'). \tag{A18}$$

From Eqs. (A13) and (A17), we have

$$k^\nu R_{3\nu}e^{\tilde{\Phi}} \triangleq -4i\,\eta_+(p_0)p_0\Big[[p'^2_\perp\sec^2(e\mathcal{B}s)-p^2_\perp\sec^2(e\mathcal{B}s)] \times[\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')] - k^2_\perp[\tan(e\mathcal{B}s)-\tan(e\mathcal{B}s')] + ie\mathcal{B}p_0[\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')]\frac{d}{d\xi}\Big]e^{\tilde{\Phi}}. \tag{A19}$$

Now using the expressions for $p^2_\perp$ and $p'^2_\perp$ from Eqs. (63) and (64) we can write

$$k^\nu R_{3\nu}e^{\tilde{\Phi}} \triangleq 4e\mathcal{B}\eta_+(p_0)p_0\Big[[\sec^2(e\mathcal{B}s)-\sec^2(e\mathcal{B}s')] +[\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')]\frac{d}{d\xi}\Big]e^{\tilde{\Phi}}. \tag{A20}$$

The above equation can also be written as

$$k^\nu R_{3\nu}e^{\tilde{\Phi}} \triangleq 4e\mathcal{B}\eta_+(p_0)p_0\frac{d}{d\xi}\{e^{\tilde{\Phi}}[\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')]\}. \tag{A21}$$

Transforming to ξ,ζ variables and using the above equation, we can write the parametric integrations (integrations over s and s') on the left hand side of Eq. (A11) as

$$\int_{-\infty}^{\infty}ds\int_0^\infty ds'\,k^\nu R_{3\nu}e^{\tilde{\Phi}} =\frac{8\,\eta_+(p_0)p_0}{e\mathcal{B}}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\zeta\,\Theta(\xi-\zeta)\frac{d}{d\xi}\mathcal{F}(\xi,\zeta)$$

where

$$\mathcal{F}(\xi,\zeta)=e^{\tilde{\Phi}}[\tan(e\mathcal{B}s)+\tan(e\mathcal{B}s')].$$

The integration over the ξ and ζ variables in the above equation can be represented as

$$\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\zeta\,\Theta(\xi-\zeta)\frac{d}{d\xi}\mathcal{F}(\xi,\zeta) =\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\zeta\Big[\frac{d}{d\xi}\{\Theta(\xi-\zeta)\mathcal{F}(\xi,\zeta)\} - \delta(\xi-\zeta)\mathcal{F}(\xi,\zeta)\Big] =-\int_{-\infty}^{\infty}d\xi\,\mathcal{F}(\xi,\xi). \tag{A22}$$

Here the second step follows from the first one as the first integrand containing the Θ function vanishes at both limits of the integration. The remaining integral is now a function of ξ only and is even in p_0. But in Eq. (A22) we have $\eta_+(p_0)p_0$, which makes the integrand odd under p_0 integration in the left hand side of Eq. (A11), as $\eta_+(p_0)$ is an even function in p_0. So the p_0 integral as it occurs in the left hand side of Eq. (A11) vanishes as expected, yielding the required result shown in Eq. (A6).

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