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Noncommutative topological theories of gravity

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The possibility of noncommutative topological gravity arising in the same manner as Yang-Mills theory is explored. We use the Seiberg-Witten map to construct such a theory based on a $SL(2,\mathbf{C})$ complex connection, from which the Euler characteristic and the signature invariant are obtained. Finally, we speculate on the description of noncommutative gravitational instantons, as well as noncommutative local gravitational anomalies.

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I. INTRODUCTION

The idea of the noncommutative nature of space-time coordinates is quite old [1]. Many authors have studied it extensively from the mathematical [2] as well as the field theoretical point of view (for a review, see, for instance, [3,4]).

Recently, noncommutative gauge theory has attracted a lot of attention, especially in connection with M(atrix) [5] and string theory [6]. In particular, Seiberg and Witten [6] have found noncommutativity in the description of the low energy excitations of open strings (possibly attached to D-branes) in the presence of a Neveu-Schwarz constant background B field. Moreover, they have observed that, depending on the regularization scheme of the two-dimensional correlation functions, Pauli-Villars or point splitting, ordinary and noncommutative gauge fields can be induced from the same worldsheet action. Thus, this procedure tells us that there is a relation of the resulting theory of noncommutative gauge fields, deformed by the Moyal star product or Kontsevich star product for systems with general covariance, with a gauge theory in terms of the usual commutative fields. This relation is the so-called Seiberg-Witten map.

In string theory, gravity and gauge theories are realized in very different ways. The gravitational interaction is associated with a massless mode of closed strings, while Yang-Mills theories are more naturally described in open strings or in heterotic string theory. Furthermore, as mentioned, noncommutative Yang-Mills theories should arise from string theory. Thus the question emerges of whether a noncommutative description of gravity would arise from it. This is a difficult question and it will not be addressed here. However, in a recent paper [7], gravitation on noncommutative D-branes has been discussed.

In this context, recently Chamseddine has made several proposals for noncommutative formulations of Einstein's gravity [8–10], where a Moyal deformation is done. Moreover, in [9,10], he gives a Seiberg-Witten map for the vierbein and the Lorentz connection, which is obtained starting from the gauge transformations, of $SO(4,1)$ in the first work and of $U(2,2)$ in the second one. However, in both cases the actions are not invariant under the full noncommutative transformations; namely, in [9] the action does not have a definite noncommutative symmetry, and in [10] the Seiberg-Witten map is obtained for $U(2,2)$, but the action is invariant under the subgroup $U(1,1)\times U(1,1)$. These actions deformed by the Moyal product, with a constant noncommutativity parameter, are not diffeomorphism invariant. However, as pointed out in this work, [9,10], they could be made diffeomorphism invariant, substituting the Moyal $*_M$ product by the Kontsevich $*_K$ product. A more recent proposal for a noncommutative deformation of the Einstein-Hilbert Lagrangian in four dimensions is given in [11]. For other proposals for noncommutative gravity, see [12–20].

Further, as shown in [21–25], starting from the Seiberg-Witten map, noncommutative gauge theories with matter fields based on any gauge group can be constructed. In this way, a proposal for the noncommutative standard model based on the gauge group product $SU(3)\times SU(2)\times U(1)$ has been constructed [26]. In these developments, the key argument is that no additional degrees of freedom have to be introduced in order to formulate noncommutative gauge theories. That is, although the explicit symmetry of the noncommutative action corresponds to the enveloping algebra of the limiting symmetry group of the commutative theory, it is also invariant with respect to the proper group of this commutative theory, a fact made manifest by the Seiberg-Witten map.

In this paper, following these results, we present a pro-

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posal for a noncommutative topological quadratic theory of gravity in four dimensions, from which the noncommutative topological invariants of Riemannian manifolds, corresponding to the Euler characteristic and the signature, can be obtained. We then explore, in this context of noncommutative gravity, the notion of the gravitational instanton, which is expected to be classified by these invariants, as in the commutative case. Other possible global aspects of noncommutative gravity like gravitational anomalies will be briefly addressed as well. It is important to note that these latter considerations of Sec. V, on noncommutative instantons and anomalies, are of rather a speculative character, so they are not on the same footing with our main proposal and results of Sec. IV.

The paper is organized as follows. In Sec. II we quickly review noncommutative gauge theories. In Sec. III the main features of topological quadratic gravity are introduced, for the $SO(3,1)$ gauge group, by means of a complex formulation based on self-dual topological quadratic gravity. In Sec. IV we present noncommutative topological gravity, with explicit results up to order θ^3. In Sec. V, based on a study of the global properties of the noncommutative version of the Lorentz and diffeomorphism groups, we explore the possibility of a definition of noncommutative gravitational instantons, as well as local gravitational anomalies for a theory of gravity. Finally, Sec. VI contains our concluding remarks.

II. NONCOMMUTATIVE GAUGE SYMMETRY AND THE SEIBERG-WITTEN MAP

We start this section with the conventions and properties of noncommutative spaces for future reference. For a recent review, see, e.g., [27]. Noncommutative spaces can be understood as generalizations of the usual quantum mechanical commutation relations, by the introduction of a linear operator algebra $\mathcal{A}$, with a noncommutative associative product

$$[\hat{x}^{\mu},\hat{x}^{\nu}]=i\,\theta^{\mu\nu}, \tag{1}$$

where $\hat{x}^{\mu}$ are linear operators acting on the Hilbert space $L^2(\mathbf{R}^n)$ and $\theta^{\mu\nu}=-\theta^{\nu\mu}$ are real numbers. The Weyl-Wigner-Moyal correspondence establishes (under certain conditions) an isomorphic relation between $\mathcal{A}$ and the algebra of functions on $\mathbf{R}^n$, the last with an associative and noncommutative $\star$ product, the Moyal product, given by

$$f(x)\star g(x)\equiv\left[\exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\varepsilon^{\mu}}\frac{\partial}{\partial\eta^{\nu}}\right)f(x+\varepsilon)\right.\\ \left.\times g(x+\eta)\right]_{\varepsilon=\eta=0}. \tag{2}$$

In order to avoid causality problems we will take $\theta^{0\nu}=0$.

Due to the fact that we will be working with non-Abelian groups, we must also include matrix multiplication, so a $*$ product will be used as the matrix multiplication with the $\star$ product. Inside integrals, this product has the property $\mathrm{Tr}\int f_1*f_2*f_3*\cdots*f_n=\mathrm{Tr}\int f_n*f_1*f_2*f_3*\cdots*f_{n-1}$. In particular, the trace of the integral of the product of two functions has the property that $\mathrm{Tr}\int f_1*f_2=\mathrm{Tr}\int f_1 f_2$.

Let us consider a gauge theory with a Hermitian connection, invariant under the symmetry Lie group G, with gauge fields A_{μ},

$$\delta_{\lambda}A_{\mu}=\partial_{\mu}\lambda+i[\lambda,A_{\mu}], \tag{3}$$

where $\lambda=\lambda^i T_i$, and T_i are the generators of the Lie algebra $\mathcal{G}$ of the group G, in the adjoint representation. These transformations are generalized for the noncommutative theory as

$$\delta_{\lambda}\hat{A}_{\mu}=\partial_{\mu}\hat{\Lambda}+i[\hat{\Lambda}\overset{*}{,}\hat{A}_{\mu}], \tag{4}$$

where the noncommutative parameters $\hat{\Lambda}$ have some dependence on λ and the connection A. The commutators $[A\overset{*}{,}B]\equiv A*B-B*A$ have the correct derivative properties when acting on products of noncommutative fields.

Due to noncommutativity, commutators like $[\hat{\Lambda}\overset{*}{,}\hat{A}_{\mu}]$ take values in the enveloping algebra of $\mathcal{G}$ in the adjoint representation $\mathcal{U}(\mathcal{G},\mathrm{ad})$. Therefore, $\hat{\Lambda}$ and the gauge fields $\hat{A}_{\mu}$ will also take values in this algebra. In general, for some representation R, we will denote as $\mathcal{U}(\mathcal{G},R)$ the corresponding section of the enveloping algebra $\mathcal{U}(\mathcal{G})$. Let us write, for instance, as $\hat{\Lambda}=\hat{\Lambda}^I T_I$ and $\hat{A}=\hat{A}^I T_I$; then

$$[\hat{\Lambda}\overset{*}{,}\hat{A}_{\mu}]=\{\hat{\Lambda}^I\overset{*}{,}\hat{A}^J_{\mu}\}[T_I,T_J]+[\hat{\Lambda}^I\overset{*}{,}\hat{A}^J_{\mu}]\{T_I,T_J\}, \tag{5}$$

where $\{A\overset{*}{,}B\}\equiv A*B+B*A$ is the noncommutative anticommutator. Thus all the products of the generators T_I will be needed in order to close the algebra $\mathcal{U}(\mathcal{G},\mathrm{ad})$. Its structure can be obtained by successive computation of commutators and anticommutators starting from the generators of $\mathcal{G}$, until it closes,

$$[T_I,T_J]=if^K_{IJ}T_K,\quad \{T_I,T_J\}=d^K_{IJ}T_K.$$

The field strength is defined as $\hat{F}_{\mu\nu}=\partial_{\mu}\hat{A}_{\nu}-\partial_{\nu}\hat{A}_{\mu}-i[\hat{A}_{\mu}\overset{*}{,}\hat{A}_{\nu}]$; hence it also takes values in $\mathcal{U}(\mathcal{G},\mathrm{ad})$. From Eq. (4) it turns out that

$$\delta_{\lambda}\hat{F}_{\mu\nu}=i(\hat{\Lambda}*\hat{F}_{\mu\nu}-\hat{F}^{*}_{\mu\nu}\hat{\Lambda}). \tag{6}$$

We see that these transformation rules can be obtained from the commutative ones, just by replacing the ordinary product of smooth functions by the Moyal product, with a suitable product ordering. This allows construction of invariant quantities in a simple way.

If the components of the noncommutativity parameter θ are constant, then Lorentz invariance is spoiled. In order to recover it [9,10,23] one should change the Moyal star product to the Kontsevich star product $*_K$ [28]. However, as a result of the diffeomorphism invariance, for an even dimensional (symplectic) space-time X, there exists a local coordinate system (which coincides with Darboux's coordinate system) in which $\theta^{\mu\nu}$ is constant. Therefore, without loss of generality, the Kontsevich product can be reduced to the Moyal one, which will be used from now on.

The fact that the observed world is (up to the present experimental evidence) commutative means that it must be possible to obtain it from the noncommutative one by taking the limit $\theta \to 0$. Thus the noncommutative fields $\hat{A}$ are given by a power series expansion in θ, starting from the commutative ones A,

$$\hat{A} = A + \theta^{\mu\nu} A^{(1)}_{\mu\nu} + \theta^{\mu\nu}\theta^{\rho\sigma} A^{(2)}_{\mu\nu\rho\sigma} + \cdots . \tag{7}$$

The terms of this expansion are determined by the Seiberg-Witten map, which states that the symmetry transformations of Eq. (7), given by Eq. (4) are induced by the symmetry transformations of the commutative fields (3). In order that these transformations be consistent, the transformation parameter $\hat{\Lambda}$ must satisfy [22]

$$\delta_\lambda \hat{\Lambda}(\eta) - \delta_\eta \hat{\Lambda}(\lambda) - i[\hat{\Lambda}(\lambda) \overset{*}{,} \hat{\Lambda}(\eta)] = \hat{\Lambda}(-i[\lambda,\eta]). \tag{8}$$

Similarly, the terms in Eq. (7) are functions of the commutative fields and their derivatives, and are determined by the requirement that $\hat{A}$ transforms as Eq. (4) [25].

The fact that the noncommutative gauge fields take values in the enveloping algebra has the consequence that they have a bigger number of components than the commutative ones, unless the enveloping algebra coincides with the Lie algebra of the commutative theory, as is the case for $U(N)$. However, the physical degrees of freedom of the noncommutative fields can be related one to one to the physical degrees of freedom of the commutative fields by the Seiberg-Witten map [6], a fact used in Refs. [21–25] to construct noncommutative gauge theories, for any Lie group in principle.

In order to obtain the Seiberg-Witten map to first order, the noncommutative parameters are first obtained from Eq. (8) [6,21–25],

$$\hat{\Lambda}(\lambda,A) = \lambda + \frac{1}{4}\theta^{\mu\nu}\{\partial_\mu \lambda, A_\nu\} + \mathcal{O}(\theta^2). \tag{9}$$

Then, from Eqs. (4) and (7), the following solution is obtained:

$$\hat{A}_\mu(A) = A_\mu - \frac{1}{4}\theta^{\rho\sigma}\{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\} + \mathcal{O}(\theta^2), \tag{10}$$

and for the field strength it turns out that

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{4}\theta^{\rho\sigma}(2\{F_{\mu\rho}, F_{\nu\sigma}\} - \{A_\rho, D_\sigma F_{\mu\nu} + \partial_\sigma F_{\mu\nu}\}) + \mathcal{O}(\theta^2). \tag{11}$$

The higher terms in Eq. (7) can be obtained from the observation that the Seiberg-Witten map preserves the operations of the commutative function algebra; hence the following differential equation can be written [6]:

$$\delta\theta^{\mu\nu}\frac{\partial}{\partial\theta^{\mu\nu}}\hat{A}(\theta) = \delta\theta^{\mu\nu}\hat{A}^{(1)}_{\mu\nu}(\theta), \tag{12}$$

where $\hat{A}^{(1)}_{\mu\nu}$ is obtained from $A^{(1)}_{\mu\nu}$ in Eq. (7), by substituting for the commutative fields with the noncommutative ones under the * product.

Let us take the generators T^i of the Lie algebra $\mathcal{G}$ to be Hermitian; then the generators T^I of the corresponding enveloping algebra can be chosen to be also Hermitian, for instance, if they are given by the symmetrized products $:T^{i_1}T^{i_2}\cdots T^{i_n}:$. Further, the noncommutative transformation parameters $\hat{\Lambda}(\lambda,A)$ are functions whose arguments are matrices. Let us now substitute the matrix products inside $\hat{\Lambda}(\lambda,A)$, by $MN \to \frac{1}{2}\{M,N\} - i/2(i[M,N])$, for any two matrices M and N. Hence $\hat{\Lambda}(\lambda,A)$ can be understood as a function whose nonlinear part of depends polynomially, with complex numerical coefficients, on anticommutators $\{\cdot,\cdot\}$ and commutators $i[\cdot,\cdot]$, of λ, A, and their derivatives. With this understanding, we will continue to write it as $\hat{\Lambda}(\lambda,A)$, and we have

$$[\hat{\Lambda}(\lambda,A)]^\dagger = \hat{\Lambda}^\dagger(\lambda^\dagger, A^\dagger), \tag{13}$$

where $\hat{\Lambda}^\dagger$ is obtained by complex-conjugating the mentioned numerical coefficients.

Let us now consider the Hermitian conjugation of the transformation law (3), $(\delta_\lambda A_\mu)^\dagger = \partial_\mu \lambda^\dagger + i[\lambda^\dagger, A^\dagger_\mu]$. From it and Eq. (8), taking into account Eq. (13), we get,

$$\delta_{\lambda^\dagger}\hat{\Lambda}^\dagger(\lambda^\dagger,A^\dagger) - \delta_{\eta^\dagger}\hat{\Lambda}^\dagger(\lambda^\dagger,A^\dagger) - i[\hat{\Lambda}^\dagger(\lambda^\dagger,A^\dagger) \overset{*}{,} \hat{\Lambda}^\dagger(\eta^\dagger,A^\dagger)] = \hat{\Lambda}^\dagger(-i[\lambda^\dagger,\eta^\dagger],A^\dagger). \tag{14}$$

Comparing this equation with Eq. (8), with the mentioned convention, it can be seen that the noncommutative parameters satisfy $[\hat{\Lambda}(\lambda,A)]^\dagger = \hat{\Lambda}(\lambda^\dagger,A^\dagger)$. From the transformation law (4), a similar conclusion can be obtained for the noncommutative connection $[\hat{A}_\mu(A)]^\dagger = \hat{A}_\mu(A^\dagger)$, as well for the field strength $[\hat{F}_{\mu\nu}(A)]^\dagger = \hat{F}_{\mu\nu}(A^\dagger)$. By this means, if we have a group with real parameters and Hermitian generators, with a Hermitian connection, then the noncommutative connection and the noncommutative field strength will also be Hermitian.

III. TOPOLOGICAL GRAVITY

In this section we briefly shortly review four-dimensional topological gravity. Let R be the field strength corresponding to a $SO(3,1)$ connection ω,

$$R^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^{ac}_\mu \omega^{b}_{\nu c} - \omega^{bc}_\mu \omega^{a}_{\nu c}, \tag{15}$$

and let $\tilde{R}$ be the dual of R with respect to the group (not with respect to space-time) given by

$$\tilde{R}^{ab}_{\mu\nu} = -\frac{i}{2}\varepsilon^{ab}{}_{cd} R^{cd}_{\mu\nu}. \tag{16}$$

We start from the following $SO(3,1)$ invariant action:

$$I_{TOP}=\frac{\Theta_G^P}{2\pi}\mathrm{Tr}\int_X R\wedge R+i\frac{\Theta_G^E}{2\pi}\mathrm{Tr}\int_X R\wedge\tilde{R}, \tag{17}$$

where X is a four-dimensional closed pseudo-Riemannian manifold and the coefficients are the gravitational analogues of the Θ vacuum in QCD [29–31].

In this action, the connection satisfies the first Cartan structure equation, which relates it to a given tetrad. This action can be written as the integral of a divergence, and the variation of it with respect to the tetrad vanishes; hence it is metric independent and therefore topological.

The action (17) arises naturally from the MacDowell-Mansouri type action [32]. A similar construction can be done for $(2+1)$-dimensional Chern-Simons gravity [33]. Keeping this philosophy in mind, the action (17) can be rewritten in terms of the self-dual and anti-self-dual parts, $R^{\pm}=\frac{1}{2}(R\pm\tilde{R})$ of the Riemann tensor as follows:

$$\begin{aligned}I_{TOP}&=\mathrm{Tr}\int_X(\tau R^+\wedge R^+ +\bar{\tau}R^-\wedge R^-)\\&=\mathrm{Tr}\int_X(\tau R^+\wedge R^+ +\overline{\tau R^+}\wedge\overline{R^+}),\end{aligned} \tag{18}$$

where $\tau=(1/2\pi)(\Theta_G^E+i\Theta_G^P)$, and the overbar denotes complex conjugation. In local coordinates on X, this action can be rewritten as

$$I_{TOP}=2\mathrm{Re}\left(\tau\int_X d^4x\varepsilon^{\mu\nu\rho\sigma}R^{+\ ab}_{\mu\nu}R^+_{\rho\sigma ab}\right). \tag{19}$$

Therefore, it is enough to study the complex action,

$$I=\int_X d^4x\ \varepsilon^{\mu\nu\rho\sigma}R^{+\ ab}_{\mu\nu}R^+_{\rho\sigma ab}. \tag{20}$$

Further, the self-dual Riemann tensor satisfies $\varepsilon^{ab}{}_{cd}R^{+\ cd}_{\mu\nu}=2iR^{+\ ab}_{\mu\nu}$. This tensor has the useful property that it can be written as a usual Riemann tensor, but in terms of the self-dual components of the spin connection, $\omega^{+ab}_{\mu}=\frac{1}{2}[\omega^{ab}_{\mu}-(i/2)\varepsilon^{ab}{}_{cd}\omega^{cd}_{\mu}]$, as

$$R^{+ab}_{\mu\nu}=\partial_\mu\omega^{+ab}_{\nu}-\partial_\nu\omega^{+ab}_{\mu}+\omega^{+ac}_{\mu}\omega^{+b}_{\nu c}-\omega^{+bc}_{\mu}\omega^{+a}_{\nu c}. \tag{21}$$

In this case, the action (19) can be rewritten as

$$\begin{aligned}I=\int_X d^4x\varepsilon^{\mu\nu\rho\sigma}[&2R^{0i}_{\mu\nu}(\omega^+)R_{\rho\sigma 0i}(\omega^+)\\&+R^{ij}_{\mu\nu}(\omega^+)R_{\rho\sigma ij}(\omega^+)].\end{aligned} \tag{22}$$

Now, we define $\omega^i_\mu=i\omega^{+0i}_\mu$, from which we obtain, by means of the self-duality properties, $\omega^{+ij}_\mu=-\varepsilon^{ij}{}_k\omega^k_\mu$. Then it turns out that

$$R^{oi}_{\mu\nu}(\omega^+)=-i(\partial_\mu\omega^i_\nu-\partial_\nu\omega^i_\mu+2\varepsilon^i_{jk}\omega^j_\mu\omega^k_{\nu c})=-i\mathcal{R}^{\ i}_{\mu\nu}(\omega) \tag{23}$$

$$\begin{aligned}R^{ij}_{\mu\nu}(\omega^+)&=\partial_\mu\omega^{+ij}_\nu-\partial_\nu\omega^{+ij}_\mu-2(\omega_\mu{}^i\omega^j_\nu-\omega_\nu{}^i\omega^j_\mu)\\&=-\varepsilon^{ij}_k\mathcal{R}^{\ k}_{\mu\nu}(\omega).\end{aligned} \tag{24}$$

This amounts to a decomposition between the real orthogonal Lie group $SO(3,1)$ and the product of two complex Lie groups $SL(2,\mathbf{C})$ given by the isomorphism $SO(3,1)\cong SL(2,\mathbf{C})\times SL(2,\mathbf{C})$, such that ω^i_μ is a complex $SL(2,\mathbf{C})$ connection. If we choose the algebra $s\ell(2,\mathbf{C})$ to satisfy $[T_i,T_j]=2i\varepsilon_{ij}{}^kT_k$ and $\mathrm{Tr}(T_iT_j)=2\delta_{ij}$, then we can write

$$I=\mathrm{Tr}\int_X\tilde{\mathcal{R}}\wedge\star\tilde{\mathcal{R}}=\mathrm{Tr}\int_X d^4x\varepsilon^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu}(\omega)\mathcal{R}_{\rho\sigma}(\omega), \tag{25}$$

where $\mathcal{R}_{\mu\nu}=\partial_\mu\omega_\nu-\partial_\nu\omega_\mu-i[\omega_\mu,\omega_\nu]$ is the field strength, $\star$ is the usual Hodge star operation with respect to the underlying space-time metric, $\mathcal{R}$ is the two-form field strength, and $\tilde{\mathcal{R}}$ is the dual of $\mathcal{R}$ with respect to the group. This action is invariant under the $SL(2,\mathbf{C})$ transformations $\delta_\lambda\omega_\mu=\partial_\mu\lambda+i[\lambda,\omega_\mu]$.

In the case of a Riemannian manifold X, the signature and the Euler topological invariants of X, are the real and imaginary parts of Eq. (25):

$$\sigma(X)=-\frac{1}{24\pi^2}\mathrm{Re}\left(\mathrm{Tr}\int_X d^4x\varepsilon^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu}(\omega)\mathcal{R}_{\rho\sigma}(\omega)\right), \tag{26}$$

$$\chi(X)=\frac{1}{32\pi^2}\mathrm{Im}\left(\mathrm{Tr}\int_X d^4x\varepsilon^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu}(\omega)\mathcal{R}_{\rho\sigma}(\omega)\right). \tag{27}$$

IV. NONCOMMUTATIVE TOPOLOGICAL GRAVITY

We wish to have a noncommutative formulation of the $SO(3,1)$ action (17). Its first term can be straightforwardly made noncommutative, in the same way as for the usual Yang-Mills theory,

$$\mathrm{Tr}\int_X\hat{R}\wedge\hat{R}. \tag{28}$$

If the $SO(3,1)$ generators are chosen to be Hermitian, for example, in the spin $\frac{1}{2}$ representation given by $\gamma^{\mu\nu}$, then from the discussion at the end of Sec. II, it turns out that $\hat{R}_{\mu\nu}$ is Hermitian and consequently Eq. (28) is real.

If we now turn to the second term of Eq. (17), such an action cannot be written, because it involves the Levi-Cività symbol, an invariant Lorentz tensor, but which is not invariant under the full enveloping algebra. However, as mentioned at the end of the preceding section, this term can be obtained from Eq. (25).

Thus, in general we will consider as the noncommutative topological action of gravity the $SL(2,\mathbf{C})$ invariant action

$$\hat{I}=\mathrm{Tr}\int_X d^4x\varepsilon^{\mu\nu\rho\sigma}\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}_{\rho\sigma}, \tag{29}$$

where $\hat{\mathcal{R}}_{\mu\nu}=\partial_\mu\hat{\omega}_\nu-\partial_\nu\hat{\omega}_\mu-i[\hat{\omega}_\mu \overset{*}{,} \hat{\omega}_\nu]$, is the $SL(2,\mathbf{C})$ noncommutative field strength. This action does not depend on the metric of X. Indeed, as well as the commutative one, it is given by the divergence

$$\hat{I}=\mathrm{Tr}\int_X d^4x\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\left(\hat{\omega}_\nu^*\partial_\rho\hat{\omega}_\sigma+\frac{2}{3}\hat{\omega}_\nu^*\hat{\omega}_\rho^*\hat{\omega}_\sigma\right). \quad (30)$$

Thus, a variation of Eq. (29) with respect to the noncommutative connection will vanish identically because of the noncommutative Bianchi identities,

$$\delta_{\hat{\omega}}\hat{I}=8\,\mathrm{Tr}\int \varepsilon^{\mu\nu\rho\sigma}\delta\hat{\omega}_\mu^*\hat{D}_\nu\hat{R}_{\rho\sigma}\equiv 0, \quad (31)$$

where $\hat{D}_\mu$ is the noncommutative covariant derivative.

At this stage, we can make use of the first Cartan structure equation; then the $SO(3,1)$ connection, and thus its $SL(2,\mathbf{C})$ projection ω_μ^i, can be written in terms of the tetrad and the torsion. Furthermore, from the Seiberg-Witten map, the noncommutative connection can be written as well as $\hat{\omega}(e)$. Therefore, a variation of the action (29) with respect to the tetrad of the action can be written as

$$\delta_e\hat{I}=8\,\mathrm{Tr}\int \varepsilon^{\mu\nu\rho\sigma}\delta_e\hat{\omega}_\mu(e)^*\hat{D}_\nu\hat{R}_{\rho\sigma}\equiv 0; \quad (32)$$

hence it is topological, like the commutative one.

As we will show later, the explicit expansion of the action (29) in the noncommutative parameter θ gives terms that one does not expect to vanish identically. Thus, we see from Eq. (30) that, in a θ power expansion of the action, each one of the resulting terms will be independent of the metric and they will be given by a divergence. Therefore, these terms will be topological. (For the case of the Euler characteristic, compare with the noncommutative nontrivial generalization of it given by Connes in pp. 64–69 of Ref. [2].)

Furthermore, the whole noncommutative action, expressed in terms of the commutative fields by the Seiberg-Witten map, is invariant under the $SO(3,1)$ transformations. Thus, each term of the expansion will also be invariant and these terms will be topological invariants.

The action (29) is not real, nor is the limiting commutative action. Hence, it is not obvious that the signature (28) will be precisely its real part. In this case we could not say that $\hat{\chi}(X)$ is given by its imaginary part. In fact we could only say that $\hat{\chi}(X)$ could be obtained from the difference of Eqs. (29) and (28). However, the real and the imaginary parts of Eq. (29) are invariant under $SL(2,\mathbf{C})$ and consequently under $SO(3,1)$, and thus they are the natural candidates for $\hat{\sigma}(X)$ and $\hat{\chi}(X)$, as in Eqs. (26) and (27). In order to write down these noncommutative actions as an expansion in θ, we will take as generators for the algebra of $SL(2,\mathbf{C})$ the Pauli matrices. In this case, to second order in θ, the Seiberg-Witten map for the Lie algebra valued commutative field strength $\mathcal{R}_{\mu\nu}=\mathcal{R}^{\,i}_{\mu\nu}(\omega)\sigma_i$ is given by

$$\hat{\mathcal{R}}_{\mu\nu}=\mathcal{R}_{\mu\nu}+\theta^{\alpha\beta}\mathcal{R}^{(1)}_{\mu\nu\alpha\beta}+\theta^{\alpha\beta}\theta^{\gamma\delta}\mathcal{R}^{(2)}_{\mu\nu\alpha\beta\gamma\delta}+\cdots, \quad (33)$$

where, from Eq. (11) we get

$$\theta^{\rho\sigma}\mathcal{R}^{(1)}_{\mu\nu\rho\sigma}=\frac{1}{2}\theta^{\rho\sigma}[2\mathcal{R}^{\,i}_{\mu\rho}\mathcal{R}_{\nu\sigma i}-\omega^i_\rho(\partial_\sigma\mathcal{R}_{\mu\nu i}+D_\sigma\mathcal{R}_{\mu\nu i})]\mathbf{1}, \quad (34)$$

where $\mathbf{1}$ is the unity 2×2 matrix. Further, by means of Eq. (12), we get

$$\begin{aligned}\theta^{\rho\sigma}\theta^{\tau\theta}\mathcal{R}^{(2)}_{\mu\nu\rho\sigma\tau\theta}=\frac{1}{4}\theta^{\rho\sigma}\theta^{\tau\theta}\Big(&\varepsilon^i_{jk}[i\partial_\tau\mathcal{R}^{\,j}_{\mu\rho}\partial_\theta\mathcal{R}^{\,k}_{\nu\sigma}+\partial_\tau\omega^j_\rho\partial_\theta(\partial_\sigma+D_\sigma)\mathcal{R}^{\,k}_{\mu\nu}]-\omega^i_\rho\partial_\tau\omega^j_\sigma\partial_\theta\mathcal{R}_{\mu\nu j}\\&+\mathcal{R}^{\,i}_{\mu\rho}[2\mathcal{R}^{\,j}_{\nu\tau}\mathcal{R}_{\sigma\theta j}-\omega^j_\tau(\partial_\theta+D_\theta)\mathcal{R}_{\nu\sigma j}]-\mathcal{R}^{\,i}_{\nu\rho}[2\mathcal{R}^{\,j}_{\mu\tau}\mathcal{R}_{\sigma\theta j}-\omega^j_\tau(\partial_\theta+D_\theta)\mathcal{R}_{\mu\sigma j}]\\&+\frac{1}{2}\omega^j_\tau(\partial_\theta\omega_{\rho j}+\mathcal{R}_{\theta\rho j})(\partial_\sigma+D_\sigma)\mathcal{R}^{\,i}_{\mu\nu}-2\omega^i_\rho\{2\partial_\sigma\mathcal{R}^{\,j}_{\mu\tau}\mathcal{R}_{\nu\theta j}-\partial_\sigma[\omega^j_\tau(\partial_\theta+D_\theta)\mathcal{R}_{\mu\nu j}]\}\Big)\sigma_i. \quad (35)\end{aligned}$$

Therefore, to second order in θ, the action (29) will be given by

$$\hat{I}=\mathrm{Tr}\int_X d^4x\varepsilon^{\mu\nu\rho\sigma}[\mathcal{R}_{\mu\nu}\mathcal{R}_{\rho\sigma}+2\theta^{\tau\vartheta}\mathcal{R}_{\mu\nu}\mathcal{R}^{(1)}_{\rho\sigma\tau\vartheta}+\theta^{\tau\theta}\theta^{\vartheta\zeta}(2\mathcal{R}_{\mu\nu}\mathcal{R}^{(2)}_{\rho\sigma\tau\theta\vartheta\zeta}+\mathcal{R}^{(1)}_{\mu\nu\tau\theta}\mathcal{R}^{(1)}_{\rho\sigma\vartheta\zeta})]. \quad (36)$$

Taking into account Eq. (34), we get that the first order term is proportional to $\mathrm{Tr}(\sigma_i)$ and thus vanishes identically. Further, using Eq. (35), we finally get

$$\hat{I}=\int_X d^4x \;\; \varepsilon^{\mu\nu\rho\sigma}\Bigg\{2\mathcal{R}^{\;i}_{\mu\nu}\mathcal{R}_{\rho\sigma i}+\frac{1}{4}\theta^{\tau\theta}\theta^{\vartheta\zeta}\Bigg(-\varepsilon_{ijk}R^i_{\mu\nu}[\partial_\vartheta R^j_{\rho\tau}\partial_\zeta R^k_{\sigma\theta}-\partial_\vartheta\omega^j_\tau\partial_\zeta(\partial_\theta+D_\theta)R^k_{\rho\sigma}]+\Bigg[R^i_{\mu\tau}R_{\nu\theta i}-\frac{1}{2}\omega^i_\tau(\partial_\theta+D_\theta)R_{i\mu\nu}\Bigg]$$
$$\times\Bigg[R^j_{\rho\vartheta}R_{\sigma\zeta j}-\frac{1}{2}\omega^j_\vartheta(\partial_\zeta+D_\zeta)R_{\rho\sigma j}\Bigg]+R^i_{\mu\nu}\Bigg\{R_{i\sigma\theta}[2R^j_{\rho\vartheta}R_{\tau\zeta j}-\omega^j_\vartheta(\partial_\zeta+D_\zeta)R_{\rho\tau j}]+\frac{1}{4}(\partial_\theta+D_\theta)R_{\rho\sigma i}\omega^j_\vartheta(\partial_\zeta\omega_{\tau j}+R_{\zeta\tau j})$$
$$+\omega_{\theta i}\Bigg[\partial_\tau(R^j_{\rho\vartheta}R_{\sigma\zeta j})-\frac{1}{2}\partial_\tau\omega^j_\vartheta(\partial_\zeta+D_\zeta)R_{\rho\sigma j}\Bigg]\Bigg\}-\frac{1}{2}R^i_{\mu\nu}\omega_{\tau i}\partial_\vartheta\omega^j_\theta\partial_\zeta R_{\rho\sigma j}\Bigg)\Bigg\}, \tag{37}$$

where the second order correction does not identically vanish.

Similarly to the second order term (35), the third order term for $\hat{\mathcal{R}}$ can be computed by means of Eq. (12). The result is given by a rather long expression, which, however, is proportional to the unity matrix **1**, like Eq. (34). Thus the third order term in Eq. (36), given by

$$2\theta^{\tau_1\theta_1}\theta^{\tau_2\theta_2}\theta^{\tau_3\theta_3}\mathrm{Tr}\int_X \varepsilon^{\mu\nu\rho\sigma}(\mathcal{R}_{\mu\nu}\mathcal{R}^{(3)}_{\rho\sigma\tau_1\theta_1\tau_2\theta_2\tau_3\theta_3}$$
$$+\mathcal{R}^{(1)}_{\mu\nu\tau_1\theta_1}\mathcal{R}^{(2)}_{\rho\sigma\tau_2\theta_2\tau_3\theta_3}), \tag{38}$$

vanishes identically, because $\mathcal{R}^{(2)}$ is proportional to σ_i. Thus, Eq. (37) is valid to third order. In fact, it seems that all its odd order terms vanish.

V. TOWARD NONCOMMUTATIVE GRAVITATIONAL INSTANTONS AND ANOMALIES

A. Toward noncommutative gravitational instantons

In the Euclidean signature, the action (17), with local Lorentz group $SO(4)$, is proportional to a linear combination of integer valued topological invariants, the Euler $\chi(X)$ and the signature $\sigma(X)$, which characterize the gravitational instantons. In fact, $\sigma(X)$ and $\chi(X)$ are the analogue of the instanton number k of $SU(2)$ Yang-Mills instantons, which is a manifestation of the gauge group topology, through $k \in \pi_3(SU(2))$. These topological invariants χ and σ should of course include the corresponding boundary and η-invariant terms. Gravitational instantons are finite action solutions of the self-dual Einstein equations, which are asymptotically Euclidean [34], or asymptotically locally Euclidean (ALE) [35], at infinity (for a review, see [36]). Then one would ask about the possibility of getting gravitational instanton solutions in noncommutative gravity. The first natural step would be to analyze the positive action conjecture [37], in the context of noncommutative gravity, although it would require a more complete version of noncommutative gravity. However, it is possible to give some generic arguments, and we will focus on the description of the global aspects by analyzing the invariants χ and σ in the noncommutative context. In order to do that, we concentrate on the spin connection dependence, leaving the explicit metrics for later analysis.

In the previous section, from explicit computations of the noncommutative corrections (in the noncommutative parameter θ) of the topological invariants [see Eq. (37)], we got that they do not vanish at $\mathcal{O}(\theta^2)$; hence the classical topological invariants are clearly modified. Thus, the use of the Seiberg-Witten map for the Lorentz group leads to essentially modified invariants $\hat{\chi}$ and $\hat{\sigma}$, which would characterize "noncommutative gravitational instantons." Further, the corresponding deformed equation under the Seiberg-Witten map $\hat{R}^+_{\mu\nu}=0$, does admit an expansion in θ with the term at the zero order being $R^+_{\mu\nu}$. Thus these corrections should be associated with the θ corrections of the self-duality equation $R^+_{\mu\nu}=0$. Furthermore, we could expect for the gravitational instantons similar effects as for the case of Yang-Mills instantons [6,38], where the singularities of moduli space are resolved by the noncommutative deformations. We already know from models of the minisuperspace in quantum cosmology that noncommutative gravity leads to a version of noncommutative minisuperspace [39]. Thus, one would expect some new physical effects from the moduli space of metrics of a noncommutative gravity theory, which may help to resolve space-time singularities.

This description of noncommutative gravitational instantons is, of course, not conclusive. They deserve further study, also in the context of a noncommutative dynamical theory of gravity (for a proposal, see [40]).

B. Comments on gravitational anomalies in noncommutative spaces

1. A brief survey of gravitational anomalies

The study of topological invariants leads us also to other nontrivial topological effects, like anomalies, in our gravitational case. Gravitational anomalies, as well as gauge anomalies, are classified into local and global anomalies. In this paper we will mainly focus on local anomalies, whereas global anomalies will be mentioned as a reference for future work.

Local anomalies are associated with the lack of invariance of the quantum one-loop effective action, under infinitesimal local transformations. There are different types of local gravitational anomalies, depending on the type of transformation, like the Lorentz (or automorphism) anomaly and the diffeomorphism anomaly.

Let $\mathcal{G}^L_0$ be the group of vertical automorphisms of the frame bundle over the space-time X. In a local trivialization, the frame bundle $\mathcal{G}^L_0$ can be identified with the set of continuous maps from X to $SO(4)$, which approach the identity at infinity, i.e. $\mathcal{G}_0^L \equiv \text{Map}_0(X, SO(4)) \equiv \{g: X \to SO(4), g \text{ continuous}\}$. Let $\mathcal{W}$ be the space of gauge field configurations, which consists of all spin connections $\omega_\mu^{ab}(x)$ with appropriate boundary conditions, and let $\mathcal{B} = \mathcal{W}/\mathcal{G}_0^L$. The automorphism group $\mathcal{G}_0^L$ acts on $\mathcal{W}$ in such a way that one can construct the gauge bundle: $\mathcal{G}_0^L \to \mathcal{W} \xrightarrow{\pi} \mathcal{B}$. For the case of the real n-sphere, i.e., $X = S^n$ of $n = \dim X = 2m$ dimensions, the existence of the local Lorentz gravitational anomaly is associated with the nontriviality of the nontorsion part of the homotopy of $\mathcal{B}$, i.e., $\pi_2(\mathcal{B}) \cong \pi_1(\mathcal{G}_0^L) = \pi_{2m+1}(SO(2m)) \neq 1$. For the specific case of S^4, we get the pure topological torsion $\pi_1(\mathcal{G}_0^L) \cong \pi_5(SO(4)) = \pi_5(SU(2) \times SU(2)) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Thus, in four dimensions there is no local Lorentz anomaly. However, in $n = 4k+2$ dimensions, for $k = 0, 1, \ldots$, it certainly exists.

For local diffeomorphism transformations, the moduli space involves a richer phase space structure, given by the quotient space of a generalized Teichmüller space and the generalized mapping class group. These anomalies can exist only for $n = 4k+2$ dimensions for $k = 0, 1, 2, \ldots$. However, mixed local Lorentz and diffeomorphism anomalies can exist in $2k+2$ dimensions [41].

Global gravitational Lorentz anomalies arise from the fact that Lorentz transformations are disconnected, which is related to the nontrivial topology of the group $\mathcal{G}_\infty^L = \mathcal{G}^L/\mathcal{G}_0^L$, where $\mathcal{G}^L$ is the set of local Lorentz transformations that have a limit at infinity. In particular for $X = S^4$, $\pi_0(\mathcal{G}_\infty^L) \cong \pi_4(SO(4)) = \pi_4(SU(2) \times SU(2)) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$, and a nontrivial global Lorentz anomaly arises. Similarly, the global gravitational diffeomorphism anomalies are related to the disconnectedness of the mapping class group Γ_∞^+, i.e., $\pi_0(\Gamma_\infty^+) \neq 1$ [42].

2. Noncommutative local Lorentz anomalies

Let us turn to the noncommutative side. The noncommutative version of the Lorentz group will be denoted by $\widehat{SO(4)}$, and it is defined in terms of some suitable operator algebra on a *real* Hilbert space. Here and in the following, unless otherwise stated, the noncommutative spaces and groups corresponding to the ones in the preceding section, will be denoted by carets. Following [43], we propose that $\widehat{SO(4)}$ will be given by the set of compact orthogonal operators $\mathbf{O}_{cpt}(\mathcal{H})$, defined on the separable real Hilbert space $\mathcal{H}$.[1] The compactness property avoids the Kuiper theorem, which states that the set of pure orthogonal operators $\mathbf{O}(\mathcal{H})$ has trivial homotopy groups [44]. However, the restriction to subalgebras of normed orthogonal operators $\mathbf{O}_p(\mathcal{H}) = \{\alpha | \alpha = \mathbf{1} + K\}$ has very important consequences. Here K stands for a compact, finite rank, trace class, and Hilbert-Schmidt operator. By a mathematical result [45], the family of normed operator algebras $(\mathbf{O}_p(\mathcal{H}), ||\cdot||_p)$, with the L^p norm given by $||D||_p = (\text{Tr}|D|^p)^{1/p}$, together with the set $(\mathbf{O}_{cpt}(\mathcal{H}), ||\cdot||_\infty)$, have exactly the same stable homotopy groups as $SO(\infty)$ (defined through the Bott periodicity theorem). Further, the stable homotopy groups of $SO(\infty)$, $\pi_j(SO(\infty))$, are given by $\mathbf{Z}_2$ for $j=0$, $\mathbf{Z}_2$ for $j=1$, $\mathbf{Z}$ for $j=3$, and 1 otherwise. Also these groups have Bott periodicity mod 8, i.e., $\pi_n(SO(\infty)) = \pi_{n+8}(SO(\infty))$. Thus, the stable homotopy groups of $\widehat{SO(4)} = \mathbf{O}_{cpt}(\mathcal{H})$ are in general nontrivial, and new topological effects in noncommutative gravity theories are possible.

Let us turn now to the noncommutative analogue of the local Lorentz anomaly. It is determined by the nontrivial nontorsion part of homotopy groups of a suitable noncommutative version of the Lorentz group $\hat{\mathcal{G}}_0^L$, which could be defined as the set $\hat{\mathcal{G}}_0^L \equiv Map_0(X, \mathbf{O}_{cpt}(\mathcal{H}))$. The noncommutative local Lorentz anomaly is detected by the homotopy group $\pi_2(\hat{\mathcal{B}}) = \pi_1(\hat{\mathcal{G}}_0^L) = \pi_j(\mathbf{O}_{cpt}(\mathcal{H})) \neq 1$ for $j = 0,1,3$ mod 8. For $j = 0,1$ we have $\pi_j(\mathbf{O}_{cpt}(\mathcal{H})) = \mathbf{Z}_2$, while for $j=3$, $\pi_j(\mathbf{O}_{cpt}(\mathcal{H})) = \mathbf{Z}$. Thus for $j=3$ a nontorsion part is detected, and therefore the existence of a local Lorentz anomaly.

Finally, in the global perspective, the Seiberg-Witten map can be regarded as a map $SW: \mathcal{B} \to \hat{\mathcal{B}}$, which preserves the infinitesimal Lorentz transformation (the gauge equivalence relation), and thus the locally Lorentz invariant observables of the theory. The Seiberg-Witten map is not well defined globally since both spaces $\mathcal{B}$ and $\hat{\mathcal{B}}$ are different, and their corresponding topologies can be different as well. However, in some specific cases the operator representation of the Seiberg-Witten map is quite useful to define the Seiberg-Witten map globally [46].

Finally, it is important to emphasize that the considerations in the present section are not conclusive and they deserve further study in order to clarify some of them.

VI. CONCLUDING REMARKS

In this section we summarize the main results of the paper and separately we make further comments and remarks of a more speculative character.

In this paper, we propose a noncommutative version of topological gravity with quadratic actions. Our proposal is based on the complex action (29), in terms of the self-dual and anti-self-dual connections, and from which, we found in Sec. IV, that the noncommutative natural generalization of the signature (26) and Euler (27) topological invariants can be extracted. More precisely, it is shown that the corresponding noncommutative versions of the signature and Euler topological invariants are given by the real and imaginary parts of Eq. (29), respectively. This proposed action can be written as an $SL(2,\mathbf{C})$ action, whose noncommutative counterpart can be obtained in the same way as in the Yang-Mills case, by means of the Seiberg-Witten map. We compute this action up to third θ order, and we obtain that the first and the third order vanish, but the second order is different from zero. The action to this order is given by Eq. (37). It seems that all odd θ orders vanish identically. Thus we found that these natural generalizations for the topological invariants are modified nontrivially by the noncommutative deformation.

[1] We are aware that this proposal is not necessarily the natural one and various candidates are possible; for instance, one of them would be the consideration of a suitable Hopf algebra.

Now, some comments are in order. On a (commutative) Riemannian manifold, the signature and Euler topological invariants characterize gravitational instantons. Thus the study of noncommutative topological invariants should allow us, through the Seiberg-Witten map, to deform gravitational instantons into noncommutative versions of them. In order to make explicit computations, specific gravitational (noncommutative) metrics have to be chosen. In this context, it would be very interesting to give a noncommutative formulation for dynamical gravity, following the lines of this work. This analysis was reported in [40], where based on the self-dual formulation of gravity we obtain a noncommutative deformation of Einstein gravity in four dimensions.

Similarly to the gauge theory case, one could speculate on a definition of noncommutative local gravitational Lorentz anomaly, by a suitable definition of the noncommutative Lorentz group $\widehat{SO(4)}$ in compact space-time of an Euclidean signature. The application of these ideas to the diffeomorphism transformations connected with the identity might predict new nontrivial noncommutative gravitational effects, which should be computed explicitly as a noncommutative correction to the gravitational contribution to the chiral anomaly. The usual gravitational correction was computed for the standard commutative case in Refs. [41,47]. Moreover, this effect can also be regarded as a noncommutative gravitational correction of the local chiral anomaly in noncommutative gauge theory. This latter case of the pure noncommutative gauge field was discussed recently in Refs. [48]. It would be very interesting to pursue this method and compare it with the results given recently by Perrot [49].

Regarding noncommutative global Lorentz anomalies, in order to understand them, we would need to specify the connected components of the corresponding group $\hat{\mathcal{G}}^L_\infty$. In this case one would have to compute $\pi_1(\hat{\mathcal{W}}/\hat{\mathcal{G}}^L_\infty)=\pi_0(\hat{\mathcal{G}}^L_\infty)\neq 1$. Of course, to get it, a suitable operator definition of $\hat{\mathcal{G}}^L_\infty$ is necessary, as in the case of the local Lorentz anomaly. This is a difficult open problem.

Finally, the ALE gravitational instantons are an important case of gravitational instantons. They can be obtained as smooth resolutions of **A**-**D**-**E** orbifold singularities $\mathbf{C}^2/\Gamma$, with Γ being an **A**-**D**-**E** finite subgroup of $SU(2)$. These gravitational instantons are classified through the Kronheimer construction [50], which is the analogue of the Atiyah, Drinfeld, Hitchin, and Manin (ADHM) construction of Yang-Mills instantons. There is a proposal to extend the ADHM construction to the noncommutative case [38]. Thus, it would be interesting to give the noncommutative analogue of the Kronheimer construction of ALE instantons.

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