Quantum corrections to the mass of the supersymmetric vortex

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We calculate quantum corrections to the mass of the vortex in the $N=2$ supersymmetric Abelian Higgs model in $2+1$ dimensions. We put the system in a box and apply zeta function regularization. The boundary conditions inevitably violate a part of the supersymmetries. The remaining supersymmetry is, however, enough to ensure isospectrality of relevant operators in bosonic and fermionic sectors. A nonzero correction to the mass of the vortex comes from finite renormalization of couplings.

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I. INTRODUCTION

The Abrikosov-Nielsen-Olesen (ANO) vortices [1] play an important role in modern particle physics [2]. In particular, supersymmetric ANO vortices are essential for understanding monopole condensation (see, e.g., [3] and references therein). In $2+1$ dimensions the relation between extended $N=2$ supersymmetry and the Bogomol'nyi-Prasad-Sommerfield (BPS) bound has been demonstrated in [4] following a more general discussion of [5].

Quantum corrections to the mass of the supersymmetric ANO vortex[1] in $2+1$ dimensions were calculated in [7]; the Chern-Simons terms were included in [8]. Both papers [7,8] give a zero result for the mass shift. The authors used arguments similar to that of Imbimbo and Mukhi [9][2] based on the nonlocal index theorem of [10] and its generalization by Weinberg [11]. Roughly speaking, the line of reasoning in [7,8] was as follows. The index theorem was used to show that

$$\rho_B(\omega)-\rho_F(\omega)\propto\delta(\omega), \tag{1}$$

where $\rho_{B,F}$ are the spectral densities in the bosonic and fermionic sectors, respectively. Then the mass shift was identified as

$$\Delta E\propto\int d\omega\,\omega[\rho_B(\omega)-\rho_F(\omega)]. \tag{2}$$

Due to Eq. (1) the mass shift (2) should be zero. In this way, the authors [7,8] avoided explicit use of any regularization. There is, however, a loophole in this kind of argument. First of all, it is assumed that there is a regularization which supports the mode-by-mode cancellations needed to apply Eq. (2). Such regularizations are indeed available. One of them is the zeta function regularization [12]. However, it requires transition to the discrete spectrum at least at intermediate steps. In other words, one has to put the system in a box and impose some boundary conditions. There is no *a priori* guarantee that these boundary conditions can be chosen in such a way to preserve the index theorem arguments. Besides, without regularizing the whole theory with arbitrary, not only BPS background, fields one cannot control finite renormalization of charges which are present in the model.

Quantum corrections to $(2+1)$-dimensional solitons should have been reconsidered already some time ago. Recent years have seen a considerable increase of interest in quantum effects around supersymmetric solitons in $1+1$ dimensions, initiated by the papers [13], which resulted in some very interesting developments in this field (see [14] for a literature survey).

In this paper we recalculate quantum corrections to the mass of the supersymmetric vortex using the method [15] applied previously to the supersymmetric kink. We put the vortex in a box with a circular boundary and impose the boundary condition which preserves as many symmetries as possible. We find that one-half of the supersymmetries of the vortex is inevitably broken at the boundary.[3] This is, however, enough to ensure coincidence of the eigenfrequencies of the bosonic and fermionic fluctuations. We then conclude that the total energy of the vortex and the boundaries is zero. At the next step, we define the energy associated with the boundaries and find that it is also zero. Therefore the whole mass shift of the vortex is due to the finite renormalization of couplings.[4] It is not zero and is given by Eq. (70) below.

Formally the zero point energy can be represented as

$$\Delta E=\Delta E_B-\Delta E_F, \quad \Delta E_{B,F}=\frac{1}{2}\sum_{\omega_{B,F}}\omega_{B,F}, \tag{3}$$

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[1]Since no analytic form for the profile functions of the ANO vortex is available, calculations of the mass shift in a nonsupersymmetric case is a rather complicated problem. Recently the fermionic contribution to the vacuum energy was calculated in a toy model closely resembling the Abelian Higgs model [6].

[2]Note that the authors [9] used these arguments to show saturation of the Bogomol'nyi bound (i.e., to estimate the difference between quantum corrections to the mass and to the central charge) rather than to calculate the mass shift itself.

[3]BPS states preserve half of the supersymmetries of the theory. Since boundaries break another half, we have a quarter of the original $N=2$ supersymmetry.

[4]I am grateful to R. Wimmer for pointing out the importance of the finite renormalization effects.

where $\omega_{B,F}$ are eigenfrequencies of bosonic and fermionic fluctuations. The sums in Eq. (3) are divergent and must be regularized. We use the zeta function regularization [12]:

$$\Delta E_{B,F}^{\rm reg}=\frac{1}{2}\sum_{\omega_{B,F}\neq 0}\omega_{B,F}^{1-2s}, \tag{4}$$

where s is the regularization parameter. Note that zero frequencies [which do not contribute to Eq. (3) anyhow] should be explicitly excluded.

This paper is organized as follows. In the next section we describe properties of classical solutions and define the operators acting on quantum fluctuation in the Abelian Higgs model without boundaries. In Sec. III we define gauge invariant boundary conditions which ensure that all nonzero eigenfrequencies in fermionic and bosonic sectors coincide. In Sec. IV we analyze the supersymmetry of these boundary conditions and find that one-half of the superinvariancies of the vortex are broken. Section V is devoted to the calculation of the mass shift. Some concluding remarks are given in Sec. VI. Technical details of the calculations related to the boundary supersymmetries are presented in the Appendix.

II. THE MODEL

This section is devoted to some known properties of the supersymmetric vortices on manifolds without boundaries. Here we mostly follow [8,16].

A. Classical theory

The Lagrangian of a $N=2$ supersymmetric Abelian Higgs model in $2+1$ dimensions reads:

$$\mathcal{L}=\mathcal{L}_B+\mathcal{L}_F, \tag{5}$$

$$\mathcal{L}_B=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-|D_\mu\phi|^2-\frac{1}{2}(\partial_\mu w)^2-\frac{e^2}{2}(|\phi|^2-v^2)^2 -e^2w^2|\phi|^2, \tag{6}$$

$$\mathcal{L}_F=i\bar{\psi}\gamma^\mu D_\mu\psi+i\bar{\chi}\gamma^\mu\partial_\mu\chi-i\sqrt{2}e(\bar{\psi}\chi\phi-\bar{\chi}\psi\phi^*) +ew\bar{\psi}\psi, \tag{7}$$

where w (ϕ) is a real (complex) scalar, ψ and χ are two-component complex spinors. v is a constant. The signature of the metric $g^{\mu\nu}$ is $(-++)$. As usual, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the field strength. D_μ is gauge covariant derivative, $D_\mu\phi=(\partial_\mu-ieA_\mu)\phi$. The action (5) is invariant under the supersymmetry transformations

$$\delta A_\mu=i(\bar{\eta}\gamma_\mu\chi-\bar{\chi}\gamma_\mu\eta),$$

$$\delta\phi=\sqrt{2}\bar{\eta}\psi,\quad \delta w=i(\bar{\chi}\eta-\bar{\eta}\chi),$$

$$\delta\chi=\gamma^\mu\eta\left(\partial_\mu w+\frac{i}{2}\epsilon_{\mu\nu\lambda}F^{\nu\lambda}\right)+i\eta(e|\phi|^2-ev^2),$$

$$\delta\psi=-\sqrt{2}(i\gamma^\mu\eta D_\mu\phi-\eta ew\phi) \tag{8}$$

with complex constant spinor parameter η. $\epsilon^{\mu\nu\rho}$ is the Levi-Civita tensor, $\epsilon^{012}=1$. The gamma matrices

$$\gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \gamma^2=\begin{pmatrix}0&i\\i&0\end{pmatrix} \tag{9}$$

satisfy the equation

$$\gamma^\mu\gamma^\nu=-g^{\mu\nu}-i\epsilon^{\mu\nu\rho}\gamma_\rho. \tag{10}$$

We shall mark the upper (lower) components of all spinors with the subscript "+" ("−"), so that

$$\eta=\begin{pmatrix}\eta_+\\ \eta_-\end{pmatrix}, \tag{11}$$

for example.

Consider now static bosonic field configurations such that $A_0=w=0$. Such configurations are invariant with respect to one-half of the supersymmetry transformations (8) corresponding to $\eta_+=0$ if and only if

$$(D_1+iD_2)\phi=0, \tag{12}$$

$$F_{12}+e(|\phi|^2-v^2)=0. \tag{13}$$

These are just the Bogomol'nyi [17] self-duality equations.

The classical vortices

$$\phi=f(r)e^{in\theta},\quad eA_j=\epsilon_{jk}\frac{x^k}{r^2}[a(r)-n] \tag{14}$$

satisfy Eqs. (12) and (13) if

$$\frac{1}{r}\frac{d}{dr}a(r)=e^2(f^2(r)-v^2),$$

$$r\frac{d}{dr}\ln f(r)=a(r). \tag{15}$$

In these equations $n\in\mathbb{N}$ is vorticity (which we assume to be positive), $j,k\in\{1,2\}$, $\epsilon_{12}=1$, and r, θ are usual polar coordinates on the plane. The functions $f(r)$ and $a(r)$ satisfy the conditions

$$f(0)=0,\quad f(\infty)=v, \tag{16}$$

$$a(0)=n,\quad a(\infty)=0. \tag{17}$$

The classical energy of this configuration reads (see, e.g., [8]):

$$E^{\rm cl}=2\pi nv^2. \tag{18}$$

B. Quantum fluctuations

Let us now turn to quantum fluctuations about the background (14). We shift $\phi\to\phi+\varphi$ and $A_\mu\to A_\mu+\alpha_\mu$, where φ and α_μ are the fluctuations. Since all other fields are zero on the background, we do not need to introduce more notations. It is convenient to use the background gauge fixing term[5]

$$\mathcal{L}_{\rm gf}=-\frac{1}{2}[\partial_\mu\alpha^\mu-ie(\varphi^*\phi-\varphi\phi^*)]^2, \tag{19}$$

which generates the following action for the complex ghosts σ:

$$\mathcal{L}_{\rm ghost}=\sigma^*(\partial_\mu\partial^\mu-2e^2\phi\phi^*)\sigma. \tag{20}$$

Next we expand the action (6) about the classical background. The terms linear in fluctuations vanish due to the equations of motion. In the next, quadratic, order we have in the bosonic sector:

$$\mathcal{L}_B^2+\mathcal{L}_{\rm gf}=-\frac{1}{2}\alpha_\mu(\Box-2e^2\phi^*\phi)\alpha^\mu-(D_\mu\varphi)(D^\mu\varphi)^* -e^2\varphi\varphi^*(3\phi\phi^*-v^2)-2ie\alpha^\mu(\varphi^*D_\mu\phi -\varphi D_\mu\phi^*)-\frac{1}{2}(\partial_\mu w)^2-e^2w^2|\phi|^2, \tag{21}$$

where the covariant derivative D_μ depends on the background gauge potential A_μ; $\Box=\partial_\mu\partial^\mu$.

The quadratic part of the fermionic action coincides with Eq. (7) where all bosonic fields take their background values (so that $w=0$, for example). Therefore, the equation which defines eigenfrequencies ω_F in the fermionic sector reads:

$$\omega_F\begin{pmatrix}\psi\\ \chi\end{pmatrix}:=i\partial_0\begin{pmatrix}\psi\\ \chi\end{pmatrix}=-i\gamma^0\begin{pmatrix}\gamma^jD_j, & -\sqrt{2}e\phi\\ \sqrt{2}e\phi^*, & \gamma^k\partial_k\end{pmatrix}\begin{pmatrix}\psi\\ \chi\end{pmatrix}. \tag{22}$$

By taking the square of this equation one obtains [8]:

$$\omega_F^2\begin{pmatrix}U\\ V\end{pmatrix}=\begin{pmatrix}D_FD_F^\dagger, & 0\\ 0, & D_F^\dagger D_F\end{pmatrix}\begin{pmatrix}U\\ V\end{pmatrix}, \tag{23}$$

where

$$U=\begin{pmatrix}\psi_+\\ \chi_-\end{pmatrix},\quad V=\begin{pmatrix}\psi_-\\ \chi_+\end{pmatrix} \tag{24}$$

and

$$D_F=\begin{pmatrix}D_+, & -\sqrt{2}e\phi\\ -\sqrt{2}e\phi^*, & \partial_-\end{pmatrix},$$

$$-D_F^\dagger=\begin{pmatrix}D_-, & \sqrt{2}e\phi\\ \sqrt{2}e\phi^*, & \partial_+\end{pmatrix}. \tag{25}$$

In this equation we have used holomorphic and antiholomorphic components of two-dimensional differential operators:

$$D_\pm:=D_1\pm iD_2,\quad \partial_\pm:=\partial_1\pm i\partial_2. \tag{26}$$

Let us now return to the bosonic fluctuation. As one can see from Eq. (21), the equations for α_0 and for w decouple from the rest of the bosonic modes. Moreover, the squared eigenfrequencies of these fields are given by the eigenvalues of the operator

$$\Delta_w=-\partial_j\partial_j+2e^2|\phi|^2. \tag{27}$$

The same operator also defines the ghost eigenfrequencies. Therefore contributions of all these fields to the vacuum energy cancel (provided they all satisfy the same boundary conditions).

A very important observation regarding the rest of the bosonic perturbations was made by Lee and Min [8]. They demonstrated that the eigenfrequencies for φ and α_j follow from the equation

$$\omega_B^2\begin{pmatrix}\varphi\\ i\alpha_+\sqrt{2}\end{pmatrix}:=-\partial_0^2\begin{pmatrix}\varphi\\ i\alpha_+\sqrt{2}\end{pmatrix}=D_F^\dagger D\begin{pmatrix}\varphi\\ i\alpha_+\sqrt{2}\end{pmatrix}, \tag{28}$$

where $\alpha_+=\alpha_1+i\alpha_2$. One can check this statement by a direct calculation using the Bogomol'nyi equations (12) and (13) for the background fields.

For the sake of completeness we give here explicit expressions for $D_FD_F^\dagger$ and $D_F^\dagger D_F$:

$$D_F^\dagger D_F=-\begin{pmatrix}D_j^2-e^2(3|\phi|^2-v^2), & -\sqrt{2}e(D_-\phi)\\ -\sqrt{2}e(D_+\phi^*), & \partial_j^2-2e^2|\phi|^2\end{pmatrix}, \tag{29}$$

$$D_FD_F^\dagger=-\begin{pmatrix}D_j^2-e^2(|\phi|^2+v^2), & 0\\ 0, & \partial_j^2-2e^2|\phi|^2\end{pmatrix}. \tag{30}$$

We stress that these formulas are valid only if the background satisfies the Bogomol'nyi equations.

III. BOUNDARY CONDITIONS

The aim of this section is to define the boundary conditions which support the factorization properties of the eigenfrequency equations (23) and (28). We like to keep as much symmetry between the bosonic and fermionic fluctuation as possible.

Let us put the system in a spherical shell with the boundary at $r=R$ (the time coordinate x^0 remains, of course, unrestricted). The relation

$$u_\pm=e^{\pm i\theta}\left(u_r\pm\frac{i}{r}u_\theta\right) \tag{31}$$

between complex and angular representations of an arbitrary two-vector u_j will be useful in this and subsequent sections.

We start with gauge invariant boundary conditions for α_μ and σ. By gauge invariance we understand the following property [19]. Let $\mathcal{B}^{[\alpha]}$ and $\mathcal{B}^{[\sigma]}$ be boundary operators which define boundary conditions for α and σ, respectively:

[5]This gauge condition belongs to the so-called R_ξ family [18].

$$\mathcal{B}^{[\alpha]}\alpha_\mu|_{\partial M}=0,\quad \mathcal{B}^{[\sigma]}\sigma|_{\partial M}=0, \tag{32}$$

where ∂M is the boundary of the manifold.[6] This system is gauge invariant if

$$\mathcal{B}^{[\alpha]}\partial_\mu\sigma|_{\partial M}=0. \tag{33}$$

This property simply means that space defined by Eq. (32) is invariant under the gauge transformations.

There are only two sets of gauge-invariant local boundary conditions for the Maxwell field.[7] Let us take one of them:[8]

$$\alpha_0|_{\partial M}=0,\quad \alpha_\theta|_{\partial M}=0,\quad \left.\left(\partial_r+\frac{1}{r}\right)\alpha_r\right|_{\partial M}=0,\quad \sigma|_{\partial M}=0. \tag{34}$$

Obviously, if σ satisfies Dirichlet boundary conditions, $\partial_0\sigma$ and $\partial_\theta\sigma$ also satisfy Dirichlet boundary conditions since ∂_0 and ∂_θ act in tangential directions to the boundary. A bit more work is needed to show that the condition for α_r is also gauge invariant. Gauge transformation of the boundary condition (34) for α_r reads:

$$\left(\partial_r+\frac{1}{r}\right)\partial_r\sigma=[-\Delta_w\sigma]+\left[-\frac{1}{r^2}\partial_\theta^2+2\phi\phi^*\right]\sigma, \tag{35}$$

where we added and subtracted several terms such that the first bracket contains the operator (27) which defines eigenfrequencies in the ghost sector. We can expand σ in a sum over eigenfrequencies: $\sigma=\Sigma_k\sigma_k$ so that $\Delta_w\sigma_k=\omega_k^2\sigma_k$ and each σ_k satisfies Dirichlet boundary conditions as required by Eq. (34). Therefore

$$[-\Delta_w\sigma]|_{\partial M}=-\sum_k \left.\omega_k^2\sigma_k\right|_{\partial M}=0. \tag{36}$$

This proves that the first term on the right-hand side of Eq. (35) vanishes on the boundary. The second term there is also zero on the boundary since it does not contain normal derivatives acting on σ. We conclude that the boundary conditions (34) are indeed gauge invariant.

Eigenfrequencies of σ, α_0, and w are defined by the same operator Δ_w. Therefore it is natural to impose on w the same (Dirichlet) boundary conditions:

$$w|_{\partial M}=0. \tag{37}$$

Radial and angular components of α can be expressed through α_+:

$$\alpha_r=\Re(e^{-i\theta}\alpha_+),\quad \alpha_\theta=r\Im(e^{-i\theta}\alpha_+). \tag{38}$$

The operator $D_F^\dagger D_F$ acts on the bosonic fluctuations $(\varphi,i\alpha_+/\sqrt{2})$ as well as on the fermionic components V [cf. Eq. (24)]. Hence we impose the same boundary conditions on the lower component $V_2=\chi_+$ as we have already defined for $i\alpha_+/\sqrt{2}$. Namely,[9]

$$\Re(e^{-i\theta}\chi_+)|_{\partial M}=0,\quad \left.\left(\partial_r+\frac{1}{r}\right)\Im(e^{-i\theta}\chi_+)\right|_{\partial M}=0. \tag{39}$$

To fix boundary conditions on the rest of the fields we shall use intertwining relations between $D_F^\dagger D_F$ and $D_F D_F^\dagger$. Let $U(\omega)$ and $V(\omega)$ be solutions of Eq. (23) with $\omega_F=\omega$. We can write formally:

$$V(\omega)=\omega^{-2}D_F^\dagger U(\omega), \tag{40}$$

$$U(\omega)=\omega^{-2}D_F V(\omega) \tag{41}$$

for $\omega\neq 0$. We are looking for boundary conditions compatible with Eqs. (40) and (41). Such boundary conditions will ensure that the operators $D_F^\dagger D_F$ and $D_F D_F^\dagger$ have coinciding nonzero eigenvalues.

Let us consider the first line in Eq. (41) which reads:

$$U_1(\omega)=\omega^{-2}(D_+V_1(\omega)-\sqrt{2}e\phi V_2(\omega)). \tag{42}$$

Let us suppose that the boundary conditions for all components U_1, U_2, V_1, V_2 are mutually independent. This technical requirement will simplify the calculations below, but will not affect our main result. Let us take $V_1=0$ first. Then the first equation in Eq. (39) yields

$$\Re(e^{-i\theta}\phi^*U_1)|_{\partial M}=\Re(e^{-i\theta}\phi^*\psi_+)|_{\partial M}=0. \tag{43}$$

Note that we are not allowed to take the normal derivative of Eq. (42) after we have put $V_1=0$ in order to get further conditions on U_1 since $\partial_r^2V_1$ is related to V_2 by the equations of motion, and, therefore, cannot be considered as an independent quantity on the boundary. Instead, we take the other component of Eq. (41):

$$U_2(\omega)=\omega^2(\partial_-V_2-\sqrt{2}e\phi^*V_1). \tag{44}$$

The boundary conditions (39) immediately give

$$\Im(U_2)|_{\partial M}=0,\quad \Im(\phi^*V_1)|_{\partial M}=0. \tag{45}$$

Next we return to Eq. (42) and put there $V_2=0$ to see that

$$[\partial_r-2(\partial_r\ln\phi^*)]\Re(\phi^*V_1)|_{\partial M}=0 \tag{46}$$

[6] For Dirichlet boundary conditions the operator $\mathcal{B}$ is just the identity operator, so that $\mathcal{B}\phi|_{\partial M}=0$ simply means $\phi|_{\partial M}=0$. For Neumann boundary conditions $\mathcal{B}$ contains a normal derivative (∂_r in our case). More complicated boundary operators will be introduced below.

[7] This point is discussed in the monographs [20,21], see also [22].

[8] Calculations for the other (dual) set of boundary conditions go in a similar manner.

[9] Strictly speaking, eigenfrequencies of $(\varphi,i\alpha_+/\sqrt{2})$ and (V_1,V_2) are the same even if we identify respective boundary conditions up to a common constant phase factor. This freedom will be discussed in Sec. IV.

TABLE I. Summary of the boundary conditions for ghosts and bosons.

Field	σ	α	w	φ
Equation	(34)	(34)	(37)	(47)

as a consequence of Eqs. (43) and (45). Again, we identify the boundary conditions for V_1 with those for the first component of the boson doublet $(\varphi, i\alpha_+/\sqrt{2})$:

$$\Im(\phi^*\varphi)|_{\partial M}=0,\ \ [\partial_r-2(\partial_r \ln \phi^*)]\Re(\phi^*\varphi)|_{\partial M}=0. \tag{47}$$

Similarly, we use Eq. (40) to fix the boundary conditions for $U_1=\psi_+$ and $U_2=\chi_-$:

$$\partial_r\Re(\chi_-)|_{\partial M}=0, \quad \left(\partial_r+\frac{1}{r}\right)\Im(e^{-i\theta}\phi^*\psi_+)\Big|_{\partial M}=0. \tag{48}$$

We have found a set of the boundary conditions which guarantees coincidence of nonzero eigenfrequencies for bosons and for fermions. We summarize the results of this section in Tables I and II.

IV. SUPERSYMMETRY BREAKING AT THE BOUNDARY

In the previous section we have constructed boundary conditions which support isospectrality of the operators acting in the bosonic and fermionic sectors. This suggests that a certain degree of supersymmetry still remains in the problem even in the presence of boundaries. Due to the vortex, initial $N=2$ supersymmetry (8) is broken to the transformations with $\eta_+=0$. However, the other complex component η_- of the parameter η remains unrestricted. In this section we show that in the presence of boundaries supersymmetry is broken to a real subgroup.

Let us consider the η_- transformation of α_+:

$$\delta\alpha_+=2i\eta_-^*\chi_+ . \tag{49}$$

From this equation we see that if η_- is an arbitrary complex parameter, it is not possible to impose different supersymmetric boundary conditions on real and imaginary parts of α_+. For example, if $\Im(re^{-i\theta}\alpha_+)=\alpha_\theta$ satisfies Dirichlet boundary conditions (as in our case), then because of Eq. (49) both real and imaginary parts of χ_+ should also satisfy Dirichlet boundary conditions. This, in turn, yields Dirichlet boundary conditions for $\Re(e^{-i\theta}\alpha_+)=\alpha_r$ contradicting gauge invariance of the boundary value problem.

However, if we require

$$\Re(\eta_-)=0 \tag{50}$$

TABLE II. Summary of the boundary conditions for spinors.

Field	$\psi_+=U_1$	$\psi_-=V_1$	$\chi_+=V_2$	$\chi_-=U_2$
Equation	(43), (48)	(45), (46)	(39)	(45), (48)

the boundary conditions obtained in the previous section become invariant under the supersymmetry transformations of the boson fields [i.e., boundary conditions are compatible with first three variations in Eq. (8)]. This statement can be checked by direct and rather elementary calculations.[10] For example, compatibility of Eq. (49) is obvious since α_+ and $i\chi_+$ satisfy the same boundary conditions.

Supertransformations (8) of the fermions are also compatible with our boundary conditions if $\Re(\eta_-)=0$. Proof of this statement (which is more involved than in the case of the bosons) is sketched in the Appendix.

One can change the residual supersymmetry by using the freedom mentioned above in footnote III. Since multiplication by a constant phase factor commutes with all operators and preserves normalization of the eigenfunctions, one can replace the spinor field $\mathcal{F}=(U,V)$ by $\mathcal{F}_\kappa=e^{i\kappa}\mathcal{F}$ in the boundary conditions derived in Sec. III. However, this phase factor can be absorbed in a redefinition of the supersymmetry transformation parameter: $\eta\to\eta_\kappa=e^{i\kappa}\eta$. Then the supersymmetry transformations (8) remain the same in terms of $\mathcal{F}_\kappa,\eta_\kappa$. Supersymmetry of the new transformed boundary condition would therefore require $\Re\eta_{\kappa-}=0$.

Let us stress that the remaining supersymmetry is enough to achieve isospectrality of relevant operators in the bosonic and fermionic sectors. Of course, there is no guarantee that such cancellations will occur at higher loops as well. To understand the situation from the nonperturbative point of view one has to modify the Witten–Olive construction [24] accordingly.

V. QUANTUM CORRECTIONS TO THE MASS OF THE VORTEX

In the one-loop approximation the renormalized mass shift of the vortex consists of three terms:

$$\Delta E^{\text{ren}}=\Delta E(V+B)^{\text{ren}}-\Delta E(B)^{\text{ren}}+\Delta E^{\text{f.r.}}, \tag{51}$$

where the first term is the zero point energy in for the vortex in the spherical box, the second term is the energy associated with the boundaries of the box, and the third term is a contribution from finite renormalization of charges in the classical expression for the mass of the vortex.

In Sec. III we have found such boundary conditions that all nonzero eigenfrequencies in the bosonic sector coincide with nonzero eigenfrequencies in the fermionic sector. Therefore for a sufficiently large s [cf. Eq. (4)],

$$\Delta E_B^{\text{reg}}=\frac{1}{2}\sum_{\omega_B\neq 0}\omega_B^{1-2s}=\frac{1}{2}\sum_{\omega_F\neq 0}\omega_F^{1-2s}=\Delta E_F^{\text{reg}}. \tag{52}$$

[10] One has to take into account that complex conjugation of the Grassmann variables also changes order in their products. For example, $(\eta_-^*\chi_-)^*=\chi_-^*\eta_-$. Therefore the product of two real Grassmann variables is imaginary. Forgetting this property one would get $\Im(\eta_-)=0$ instead of Eq. (50) and a contradiction with superinvariance of the boundary conditions for fermions.

If now we analytically continue Eq. (52) to $s=0$ we find that both divergent and finite parts of the vacuum energy for the vortex in the box are zero,

$$\Delta E(V+B)^{\rm ren}=0. \tag{53}$$

This equation is, of course, valid for arbitrary radius R of the box.

A. Quantum energy of the boundaries

Here we calculate the vacuum energy of the boundary of the box in the limit $R\to\infty$. First we have to show that between some characteristic radius R_1 (which is defined essentially by the size of the vortex) and R the theory may be approximated by free massive fields.

As r goes to infinity both profile functions of the vortex f and a go exponentially fast to their asymptotic values (16), (17). Therefore near the boundary we can assume that a and f are constants and neglect their derivatives. Consequently, the operator Δ_w which defines the eigenfrequencies of w, α_0 and of the ghosts σ can be approximated by

$$\tilde{\Delta}=-\partial_j^2+2e^2v^2. \tag{54}$$

To understand what happens with the rest of the fields as $r\to\infty$ one has to analyze the operators (29) and (30). The Bogomol'nyi equation (12) yields

$$\partial_r\phi=-\frac{i}{r}D_\theta\phi. \tag{55}$$

Consequently,

$$(D_-\phi)=e^{-i\theta}2\partial_r\phi\to 0 \tag{56}$$

as $r\to\infty$. The same is true for $(D_+\phi^*)$, and both functions are approaching zero exponentially fast. This means that for large r the off-diagonal terms in Eq. (29) can be neglected. The operators (29) and (30) contain the background vector potential (14) which does not vanish sufficiently fast at the infinity. This potential can be, however, transformed away by the following unitary change of variables for charged quantum fluctuations:

$$\tilde{\varphi}=e^{i\beta(r)\theta}\varphi,\quad \tilde{\psi}=e^{i\beta(r)\theta}\psi, \tag{57}$$

where the phase $\beta(r)$ is chosen in such a way that $\beta(r)=-n$ for $r>R_1$ and $\beta(r)\to 0$ inside the vortex. It is easy to see that in terms of new fields $\tilde{\varphi}$ and $\tilde{\psi}$ in the asymptotic region the eigenfrequencies are defined by the free operator (54) up to exponentially small terms.

One can easily show that not only the operators, but also the boundary conditions, are identical in the bosonic and fermionic sectors up to exponentially small terms. Indeed, the fields w, α_0, and σ satisfy Dirichlet boundary conditions. Therefore their contributions to the vacuum energy cancel also in the effective theory near the boundary. The fields $i\alpha_+$, χ_+, and $\tilde{\psi}_+$ satisfy

$$\left(\partial_r+\frac{1}{r}\right)\Im(e^{-i\theta}(i\alpha_+,\chi_+,\tilde{\psi}_+))\Big|_{\partial M}=0,$$

$$\Re(e^{-i\theta}(i\alpha_+,\chi_+,\tilde{\psi}_+))|_{\partial M}=0. \tag{58}$$

To derive the effective boundary conditions for $\tilde{\psi}_+$ we have used that $e^{-i\beta(r)\theta}\phi^*$ goes exponentially fast to a constant when $r\to\infty$. Similarly we have

$$\partial_r\Re((\tilde{\varphi},\tilde{\psi}_-,\chi_-))|_{\partial M}=0,$$

$$\Im((\tilde{\varphi},\tilde{\psi}_-,\chi_-))|_{\partial M}=0. \tag{59}$$

Taking into account a relative factor of 1/2 in the contributions of spinors to the vacuum energy, we see that the total quantum energy associated with the effective field theory near the boundary is zero. This is true for arbitrary values of the regularization parameter, and, therefore

$$\Delta E(B)^{\rm ren}=0. \tag{60}$$

B. Finite renormalization

As usual the renormalization is performed in the topologically trivial sector. We put $\phi=\text{const}$ and calculate the effective potential. We shall not need other background fields. We use again the zeta function regularization as in Eq. (4). A real bosonic field with the mass m contributes to the regularized effective potential

$$W_m(s)=\frac{1}{2}\sum\ \omega(m)^{1-2s}=\frac{1}{2}\zeta_m\left(s-\frac{1}{2}\right), \tag{61}$$

where ζ_m is the zeta function for the operator $\Delta_m=-\partial_j^2+m^2$. It can be expressed through the corresponding heat kernel:

$$\zeta_m\left(s-\frac{1}{2}\right)=\Gamma\left(s-\frac{1}{2}\right)^{-1}\int d^2x\int_0^\infty dt\, t^{s-1/2-1}K(t,x). \tag{62}$$

The heat kernel reads

$$K(t,x)=\langle x|e^{-t\Delta_m}|x\rangle=(4\pi t)^{-1}e^{-m^2t}. \tag{63}$$

The integral over x in Eq. (62) is divergent due to the translational invariance of the background. Therefore it is convenient to consider the density $\mathcal{W}$: $\int d^2x\mathcal{W}=W$. The integration over t can be easily performed. The subsequent analytic continuation to $s=0$ yields a finite result,

$$\mathcal{W}_m=-\frac{m^3}{12\pi}. \tag{64}$$

By collecting the contributions from all elementary excitations on this background we obtain

$$\mathcal{W}^{\text{1-loop}}=-\frac{e^3}{6\pi}[(3|\phi|^2-v^2)^{3/2}-(2|\phi|^2)^{3/2}]. \tag{65}$$

Although the effective potential (65) is convergent in 2 $+1$ dimensions, there are finite renormalization effects which shift classical values of e and v. To fix these shifts we consider

$$\mathcal{W}^{\text{tot}}=\mathcal{W}^{\text{cl}}(e+\hbar\,\delta e,v+\hbar\,\delta v)+\hbar\mathcal{W}^{\text{1-loop}}, \tag{66}$$

where we have reinserted the $\hbar$ dependence. The first term on the right-hand side is just the classical potential

$$\mathcal{W}^{\text{cl}}(e,v)=\frac{e^2}{2}(|\phi|^2-v^2) \tag{67}$$

with shifted values of e and v. We require that to the first order in $\hbar$ the potential $\mathcal{W}^{\text{tot}}$ has a minimum at $|\phi|=v$ ("no tadpole" condition). This condition yields

$$\delta v=-\frac{e}{4\sqrt{2}\pi}. \tag{68}$$

To fix δe one also needs another normalization condition, but for our purposes Eq. (68) is already enough.

The shift (68) induces a shift in the vacuum energy:

$$\Delta E^{\text{f.r.}}=\hbar(\delta v)\frac{dE^{\text{cl}}}{dv}=-\frac{evn\hbar}{\sqrt{2}}. \tag{69}$$

Since other contributions (53) and (60) vanish,

$$\Delta E^{\text{ren}}=-\frac{evn\hbar}{\sqrt{2}}. \tag{70}$$

This completes the calculation of the mass shift of the supersymmetric vortex.

VI. CONCLUSIONS

In this paper we have recalculated one-loop quantum corrections to the mass of the supersymmetric ANO vortex. We put the system into a box with a circular boundary and applied the zeta function regularization. We have demonstrated that boundaries violate a part of the supersymmetries, but the remaining invariances are enough to guarantee coincidence of the eigenfrequencies in the bosonic and fermionic sectors. Therefore contributions from the bosons and the fermions to the vacuum energy cancel each other both in the full theory (vortex in a box) and in the effective theory near the boundary. Up to this point we agree with the previous works [7,8] (though our conclusion is based on somewhat more reliable grounds). There is, however, a contribution (70) to the vacuum energy which comes from finite renormalization of the couplings in the classical mass of the vortex.[11] Such a contribution was neglected in the approach of [7,8].[12] To see what happens with the BPS bound one has to calculate also quantum corrections to the central charge.

Let us now give some comments on the vortex mass corrections in a pure bosonic theory. These comments are motivated by the discussion [26] on renormalization of the Casimir energy. In the supersymmetric case it was essential that the bosonic and fermionic contributions are cancelled mode-by-mode. In purely bosonic theory no such cancellation may appear and the vacuum energy will be, in general, divergent.[13] There are two types of divergences which are given by volume or by boundary integrals. Normally, boundary divergences are the same in the full theory and in the effective theory defined near the boundary when $R\to\infty$. Therefore $[\Delta E(V+B)-\Delta E(B)]$ will contain volume divergences only which can be removed by some standard renormalization procedure. However, to define $\Delta E(V+B)$ or $\Delta E(B)$ separately one has to introduce new surface counterterms which are absent in the original model.

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APPENDIX: SUPERTRANSFORMATION AND BOUNDARY CONDITIONS

In this appendix we show how one can prove compatibility of our boundary conditions with the supersymmetry transformations of the spinor fields with pure imaginary η_-. Let us consider the supertransformation of χ_-:

$$\delta\chi_-=-\eta_-(\partial_0 w+i\epsilon_{ojk}\partial^j\alpha^k-2ie\Re(\phi^*\varphi)). \tag{A1}$$

We are going to prove that $\delta\chi_-$ satisfies the same boundary conditions as χ_- if $\Re(\eta_-)=0$. The condition (45) on $U_2=\chi_-$ can be checked easily:

$$\Im(\delta\chi_-)|_{\partial M}\sim-\partial_0 w|_{\partial M}=0, \tag{A2}$$

where we have used the boundary condition (37). Let us now check the boundary condition (48):

$$0=\partial_r\Re(\delta\chi_-)|_{\partial M}\sim\partial_r(-\epsilon_{ojk}\partial^j\alpha^k+2e\Re(\phi^*\varphi))|_{\partial M}. \tag{A3}$$

Consider the term on the right-hand side of Eq. (A3) which contains α:

[11] This situation is similar to the BPS black hole mass shift discussed in [25].

[12] It was pointed out to the present author by R. Wimmer that finite renormalizations will lead to a nonvanishing correction.

[13] In the zeta function regularization the one-loop divergences are defined by the heat kernel coefficients. For the (mixed) boundary conditions used in this work the heat kernel expansion can be found in [27].

$$-\partial_r \epsilon_{ojk} \partial^j \alpha^k|_{\partial M} = \frac{1}{r}\left[\left(\partial_r - \frac{1}{r} \right) \partial_r \alpha_\theta + \frac{1}{r} \partial_\theta \alpha_r - \partial_r \partial_\theta \alpha_r \right] \Bigg|_{\partial M}$$
$$= \frac{1}{r}\left[-(\Delta\alpha)_\theta - \left(\partial_r + \frac{1}{r} \right) \partial_\theta \alpha_r - \frac{1}{r^2} \partial_\theta^2 \alpha_\theta \right] \Bigg|_{\partial M}$$
$$= -\frac{1}{r}(\Delta\alpha)_\theta \Bigg|_{\partial M}, \qquad \text{(A4)}$$

where we first reexpressed the left-hand side through α_r and α_θ, then we used the vector Laplacian in the polar coordinates (cf., e.g., [23]):

$$-(\Delta\alpha)_r = \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 - \frac{1}{r^2} \right) \alpha_r - \frac{2}{r^3} \partial_\theta \alpha_\theta ,$$
$$-(\Delta\alpha)_\theta = \left(\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 \right) \alpha_\theta + \frac{2}{r} \partial_\theta \alpha_r . \qquad \text{(A5)}$$

Finally, to obtain the last line of Eq. (A4) we made use of the boundary conditions (34). The equations of motion for α_θ yield:[14]

$$-\frac{1}{r}(\Delta\alpha)_\theta = -\frac{1}{r}\omega^2 \alpha_\theta - 2e(\varphi^* \partial_r \phi + \varphi \partial_r \phi^*). \qquad \text{(A6)}$$

Now we collect all contributions to see

$$\partial_r \Re(\delta\chi_-)|_{\partial M} \sim \left[-\frac{1}{r}\omega^2 \alpha_\theta + e \Re(\phi^* \partial_r \varphi - \varphi \partial_r \phi^*) \right] \Bigg|_{\partial M} = 0 \qquad \text{(A7)}$$

due to Eqs. (34) and (47).

Calculations for other components of the spinor fields can be done in a similar manner.

[14]More precisely, the equation to follow is obtained by varying Eq. (21) with respect to α_θ and then using the Bogomol'nyi equation (12) for the background ϕ.

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