Asymptotic quasinormal modes of Reissner-Nordström and Kerr black holes

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According to a recent proposal, the so-called Barbero-Immirzi parameter of loop quantum gravity can be fixed, using Bohr's correspondence principle, from a knowledge of highly damped black hole oscillation frequencies. Such frequencies are rather difficult to compute, even for Schwarzschild black holes. However, it is now quite likely that they may provide a fundamental link between classical general relativity and quantum theories of gravity. Here we carry out the first numerical computation of very highly damped quasinormal modes (QNM's) for charged and rotating black holes. In the Reissner-Nordström case QNM frequencies and damping times show an oscillatory behavior as a function of charge. The oscillations become faster as the mode order increases. At fixed mode order, QNM's describe spirals in the complex plane as the charge is increased, tending towards a well defined limit as the hole becomes extremal. Kerr QNM's have a similar oscillatory behavior when the angular index $m=0$. For $l=m=2$ the *real* part of Kerr QNM frequencies tends to 2Ω, Ω being the angular velocity of the black hole horizon, while the asymptotic spacing of the *imaginary* parts is given by $2\pi T_H$.

I. INTRODUCTION

Quasinormal modes (QNM's) play a fundamental role in black hole physics. They are known to "carry the fingerprints" of a black hole, since their frequencies only depend on fundamental black hole parameters such as mass, charge, and angular momentum. They determine the late-time evolution of fields in the black hole exterior. Even more importantly, they may play a fundamental role in the newborn field of gravitational wave astrophysics; indeed, numerical simulations of stellar collapse and black hole collisions have shown that in the final stage of such processes ("ringdown") QNM's dominate the black hole response to any kind of perturbation. Therefore QNM's have been extensively studied for more than 30 years (for comprehensive reviews see Refs. [1,2]) with the aim to shed light on the strong-field behavior of classical general relativity.

Research was mainly focused on modes having small imaginary part for two main reasons: first, weakly damped modes are expected to be dominant in gravitational wave radiation; second, numerical methods used to compute QNM's generally run into trouble when the modes' imaginary part grows, i.e., when the damping is high. Early investigations of highly damped modes had motivations which were radically different from those of the present paper. Highly damped modes were seen as a good benchmark for the reliability of numerical methods. Their study could provide hints (if not a formal proof) to whether or not QNM's are infinite in number. Finally, there was hope that their study could lead to a better understanding of the long-standing issue of mode completeness. Applications of different methods yielded at first puzzling and contradictory results: WKB methods predicted that the asymptotic real part of the frequency, ω_R, should vanish for highly damped modes [3], while the continued-fraction methods developed by Leaver [4] seemed to suggest that ω_R should be finite. An improvement of the continued fraction technique devised by Nollert [5] finally showed that ω_R is indeed finite and determined its

value; these results were confirmed by Andersson [6] building on an improved WKB-type technique, the phase integral method, previously developed in Ref. [7].

More recently QNM's, which are essentially related to the *classical* dynamical properties of a black hole, have become a subject of great interest for the quantum gravity community. This interest stems essentially from Hod's proposal [8] to apply Bohr's correspondence principle to black hole physics. By the Bekenstein-Hawking formula the surface area of a black hole is nothing but its entropy. In a quantum theory of gravity the surface area should have a discrete spectrum, and the eigenvalues of this spectrum are likely to be uniformly spaced. In order to give a prediction on the area spacing, Hod observed that the real parts of the asymptotic (highly damped) quasinormal frequencies of a Schwarzschild black hole of mass M, as numerically computed by Nollert [5], can be written as

$$\omega_R = T_H \ln 3, \qquad (1)$$

where $T_H = (8\pi M)^{-1}$ is the black hole Hawking temperature (here and in the following we use units such that $c = G = 1$). He then exploited Bohr's correspondence principle, requiring that transition frequencies at large quantum numbers should equal classical oscillation frequencies, to predict the spacing in the area spectrum for a Schwarzschild black hole. It is worth stressing that in this quantum gravity context, as opposed to the study of QNM's in the context of gravitational wave emission, relevant modes are those for which the imaginary part tends to infinity—that is, modes which damp infinitely fast and do *not* radiate at all.

We point out that a dynamical interpretation of the Hawking effect through a semiclassical treatment of the quantum-mechanical uncertainty associated to QNM oscillations was proposed long ago by York [9]. York's dynamical treatment could successfully be used to calculate the temperature and entropy of the hole. However, his proposal to relate classical black hole oscillations to the hole's quantum-mechanical be-

havior is essentially different from Hod's, in that the dynamical effects considered in Ref. [9] are dominated by slowly damped QNM's.

Following Hod's suggestion, Bohr's correspondence principle has recently been used by Dreyer to fix a free parameter (the so-called Barbero-Immirzi parameter) appearing in loop quantum gravity [10]. Supposing that transitions of a quantum black hole are characterized by the appearance or disappearance of a puncture with lowest possible spin j_{min}, and that changes ΔM in the black hole mass M, corresponding to such a transition, are related to the asymptotic frequency (1) by

$$\Delta M = \hbar \omega_R, \tag{2}$$

Dreyer found that loop quantum gravity gives a correct prediction for the Bekenstein-Hawking entropy if $j_{min}=1$, consequently fixing the Barbero-Immirzi parameter. Motivated by the occurrence of an integer value for j_{min}, Dreyer went on to suggest that the gauge group of loop quantum gravity should be SO(3), and not SU(2). Such a proposal has recently been questioned in Refs. [11,12]. In the latter paper, Hod's proposal has also been used as an argument in favor of an equidistant black hole area spectrum. Evidence that the earlier formula for black hole entropy in loop quantum gravity still holds when $j_{min}=1$ has been presented in Ref. [13].

Numerical computations of gravitational QNM's for higher-dimensional black holes are still lacking. However, Kunstatter [14] recently generalized Dreyer's argument to give a prediction for the area spacing (and for the asymptotic oscillation frequency) of d-dimensional black holes.

When Hod made his original proposal, the fact that ω_R is proportional to ln 3 was just a curious numerical coincidence. Further support to the aforementioned arguments came from the analytical proof by Motl [15] that highly damped QNM frequencies are indeed proportional to ln 3. More precisely, choosing units such that $2M=1$, high-n mode frequencies satisfy the relation

$$\omega^{Schw} \sim \frac{\ln 3}{4\pi} + \frac{i(n-1/2)}{2} + \mathcal{O}(n^{-1/2}). \tag{3}$$

Such an expression has recently been confirmed through a different analytical approach in Ref. [16], where it has also been generalized to higher-dimensional black holes and to four-dimensional Reissner-Nordström (RN) black holes. The WKB approach used in Ref. [16] has later been exploited to analytically compute reflexion and transmission coefficients of multidimensional Schwarzschild black holes (and of four-dimensional RN black holes) in the limit of large imaginary frequencies [17].

There are many important reasons to try to understand the behavior of highly damped QNM's for general black holes. Let us consider charged, rotating black holes, having angular momentum per unit mass $a=J/M$ and charge Q. The black hole's (event and inner) horizons are given in terms of the black hole parameters by $r_{\pm}=M\pm\sqrt{M^2-a^2-Q^2}$. The hole's temperature $T_H=(r_+-r_-)/A$, where $A=4\pi(2Mr_+ -Q^2)$ is the hole's surface area, related to its entropy S by

the relation $A=S/4$. Let us introduce the so-called "angular velocity of the horizon" $\Omega\equiv4\pi a/A$, and $\Phi\equiv4\pi Qr_+/A$. Applying the first law of black hole thermodynamics,

$$\Delta M = T_H \Delta S + \Omega \Delta J + \Phi \Delta Q, \tag{4}$$

but *dropping without justification the ΔQ term*, and *assuming that the formula for the area spectrum derived for a Schwarzschild black hole still holds in this case*, Hod conjectured [8] that the real parts of the asymptotic frequencies for charged and rotating black holes are given by

$$\omega_R = \tilde{\omega}_R \equiv T_H \ln 3 + m\Omega, \tag{5}$$

where m is the azimuthal eigenvalue of the field. In particular, such a conjecture implies that QNM's of extremal RN black holes would have a *vanishing* asymptotic real part. Unfortunately, numerical studies of asymptotic frequencies of charged and rotating black holes have been lacking until now. Hod [18] recently used the most systematic exploration of Kerr black hole QNM's, which was carried out a few years ago by Onozawa [19], to lend qualitative support to formula (5). However, in the following we will extend Onozawa's numerical calculations to larger imaginary part, showing that the use of low-order frequencies to deduce the asymptotic behavior as $\omega_I\to\infty$ is rather questionable.

Motl and Neitzke [16] recently obtained an analytic formula for the asymptotic frequencies of scalar and electromagnetic-gravitational perturbations of a RN black hole:

$$e^{\beta\omega} + 2 + 3e^{-\beta_I\omega} = 0. \tag{6}$$

For computational convenience, the authors fixed their units in a somewhat unconventional way: they introduced a parameter k related to the black hole charge and mass by $Q/M=2\sqrt{k}/(1+k)$, so that $\beta=4\pi/(1-k)=1/T_H$ is the inverse black hole Hawking temperature and $\beta_I=-k^2\beta$ is the inverse Hawking temperature of the inner horizon. However, some features of their result are particularly puzzling.

(1) As emphasized (and partially justified with plausibility arguments) in Refs. [16,17], the predicted asymptotic RN quasinormal frequencies do not reduce to the Schwarzschild limit as the black hole charge Q tends to zero.

(2) Quasinormal frequencies of a charged black hole, according to formula (6), depend not only on the black hole's Hawking temperature, but also on the Hawking temperature of the (causally disconnected) inner horizon.

(3) The authors suggest that, should the black hole mass and charge acquire a small imaginary part (which in their words "may not be an unreasonable thing to do," since "the black hole eventually evaporates"), their asymptotic RN frequencies would be proportional to ln 2. This is in stark contrast with the Schwarzschild results: should this be true, Dreyer's argument could be used to infer that the gauge group of loop quantum gravity is SU(2).

(4) The result does not seem to agree with the conjectured behavior predicted by formula (5).

Therefore, their result cannot be considered conclusive, and there are many issues to clarify. Furthermore, if

asymptotic frequencies for "generic" black holes depend on the hole's charge and angular momentum, a relevant question is: how should arguments based on Bohr's correspondence principle be modified? A similar question was recently raised by Cardoso and Lemos [20]. They studied the asymptotic spectrum of Schwarzschild black holes in de Sitter space-times and found that, when the black hole radius is comparable to the cosmological radius, the asymptotic spectrum depends not only on the hole's parameters, but also on the angular separation index l.

For all these reasons, a numerical computation of highly damped QNM's is now needed more than ever. Such a numerical computation is technically challenging: even though the first overtones of the QNM spectrum have now been studied for more than 15 years, we present here the first computation of this kind for RN and Kerr black holes.

We will present numerical results that generally support the calculations carried out in Refs. [15–17]. Indeed, a testable prediction of those analytical derivations is that asymptotic frequencies for *scalar* perturbations should have the same value as gravitational frequencies; however, to our knowledge, asymptotic scalar modes have never been shown in the published literature. We will extend Nollert's calculation to scalar modes, confirming the analytical prediction. Furthermore, our scalar mode calculation gives useful hints on leading order corrections to the asymptotic frequencies of Schwarzschild black holes.

The plan of the paper is as follows. In Sec. II we describe our numerical method, extending Nollert's technique to RN and Kerr black holes. In Sec. III we show our numerical results, and put forward some conjectures on their implications for the asymptotic behavior of the modes. The conclusions and a discussion follow.

II. NUMERICAL METHOD

A first comprehensive analysis of the QNM spectrum of RN black holes was first carried out by Gunter [21]; then Kokkotas and Schutz [22] verified and extended his results using numerical integrations and WKB methods. Unfortunately, the standard WKB techniques cannot be applied to our case, since they become inaccurate in estimating the modes' real part as the imaginary part increases, unless one resorts to more sophisticated phase-integral methods [7]. The first few modes of the Schwarzschild and Kerr black holes were studied by Leaver using a continued fraction technique [4], which was then extended to the RN case [23]. This technique is generally rather accurate for modes having $\omega_I \sim \omega_R$, but it eventually loses accuracy when $\omega_I \gg \omega_R$. The error is essentially introduced by a truncation of the power-series solution to the radial equation at some large N. For large N, Leaver has shown that this error can be written as an integral, which is rapidly convergent near the lower quasi-normal frequencies (where $|\omega_R| > |\omega_I|$ and $|\omega| \sim 1$); how-ever, the convergence becomes slower as the overtone index increases [23]. That is why the problem of computing high-order overtones is so numerically challenging. An improvement of the continued fraction method was used by Nollert [5] to find the asymptotic behavior of the QNM frequencies

for Schwarzschild black holes. In the following sections we show how to generalize Nollert's method to the charged and rotating cases.

A. Reissner-Nordström black holes

In this section and the following we briefly describe the computational procedure we used. More details are given, e.g., in Refs. [5,4,23]. Let us introduce a tortoise coordinate r_*, defined in the usual way by the relation

$$\frac{dr}{dr_*} = \frac{\Delta}{r^2}, \tag{7}$$

where $\Delta = r^2 - 2Mr + Q^2$; after a separation of the angular dependence and a Fourier decomposition, axial electromagnetic and gravitational perturbations of a RN metric are described by a couple of wave equations

$$\left(\frac{d^2}{dr_*^2} + \omega^2 \right) Z_i^- = V_i^- Z_i^- . \tag{8}$$

Polar perturbations can be obtained from the axial ones through a Chandrasekhar transformation [24]. In the limit $Q = 0$, the potentials V_1^- and V_2^- describe, respectively, purely electromagnetic and axial gravitational perturbations of a Schwarzschild black hole. Now, the radial equations for the perturbations can be solved using a series expansion around some suitably chosen point. The coefficients a_n of the expansion are then determined by a recursion relation. For Schwarzschild black holes the recursion relation has three terms, i.e., it is of the form

$$\alpha_0 a_1 + \beta_0 a_0 = 0,$$

$$\alpha_1 a_2 + \beta_1 a_1 + \gamma_1 a_0 = 0, \tag{9}$$

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} + \delta_n a_{n-2} = 0, \quad n = 2,3,\ldots$$

where the recursion coefficients α_n, β_n and γ_n are functions of the frequency ω and of l (if we fix units such that $2M = 1$).

For RN black holes we actually have a four-term recursion relation (whose coefficients also depend on the charge Q), but we can reduce it to the previous form using a Gaussian elimination step. It turns out that the QNM boundary conditions are satisfied when the following continued-fraction condition on the recursion coefficients holds:

$$0 = \beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \cdots \tag{10}$$

The nth quasinormal frequency is (numerically) the most stable root of the nth inversion of the continued-fraction relation (10), i.e., it is the root of

$$\beta_n - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1}-}\frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2}-}\cdots\frac{\alpha_0\gamma_1}{\beta_0}$$

$$=\frac{\alpha_n\gamma_{n+1}}{\beta_{n+1}-}\frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2}-}\cdots \quad (n=1,2,\dots). \quad (11)$$

The infinite continued fraction appearing in Eq. (11) can be summed "bottom to top" starting from some large truncation index N. Nollert has shown that the convergence of the procedure improves if such a sum is started using a wise choice for the value of the "rest" of the continued fraction, R_N. This rest can be obtained solving the equation

$$R_N = \frac{\gamma_{N+1}}{\beta_{N+1}-\alpha_{N+1}R_{N+1}}, \quad (12)$$

and assuming that

$$R_N = \sum_{k=0}^{\infty} C_k N^{-k/2}. \quad (13)$$

If we introduce $\rho \equiv -i\omega$, and denote by r_+ the coordinate radius of the black hole's event horizon, it turns out that the first few coefficients in the series are $C_0 = -1$, $C_1 = \pm\sqrt{2\rho(2r_+ - 1)}$, $C_2 = (3/4 - 2\rho r_+)$.

As first shown in Ref. [25] using WKB techniques and then confirmed using different numerical methods in Ref. [26], even for modes with moderate imaginary parts, RN QNM frequencies show a very peculiar behavior as the charge increases. Our numerical results agree with those shown in Ref. [26]. Furthermore, we have checked that our method yields Nollert's asymptotic frequencies in the Schwarzschild limit. Due to convergence reasons, calculations become more and more computationally intensive as the charge is increased. Indeed, because of the merging of the inner and outer horizons, maximally charged black holes yield radial equations for the perturbation variables which have a different singularity structure, and deserve a special treatment [27]. We plan to investigate the behavior of highly damped QNM's for maximally charged black holes in the future.

B. Kerr black holes

As we did for the charged black hole case, here we only briefly sketch our computational procedure, referring the reader to Refs. [5,4,19] for more details. In the Kerr case, the perturbation problem reduces to a couple of differential equations—one for the angular part of the perturbations and the other for the radial part. In Boyer-Lindquist coordinates, defining $u = \cos\theta$, the angular equation reads

$$[(1-u^2)S_{lm,u}]_{,u} + \left[(a\omega u)^2 - 2a\omega su + s + A_{lm}\right.$$
$$\left. - \frac{(m+su)^2}{1-u^2}\right]S_{lm} = 0, \quad (14)$$

and the radial one is

$$\Delta R_{lm,rr} + (s+1)(2r-1)R_{lm,r} + V(r)R_{lm} = 0, \quad (15)$$

where

$$V(r) = \{(r^2+a^2)^2\omega^2 - 2am\omega r + a^2m^2 + is[am(2r-1)$$
$$- \omega(r^2-a^2)]\}\Delta^{-1} + [2is\omega r - a^2\omega^2 - A_{lm}]. \quad (16)$$

In writing down these equation we have adopted, consistently with what we did for the RN case, Leaver's conventions. In particular, we chose units such that $2M = 1$. The parameter $s = 0, -1, -2$ for scalar, electromagnetic and gravitational perturbations, respectively, a is the Kerr rotation parameter ($0 < a < 1/2$), and A_{lm} is an angular separation constant. In the Schwarzschild limit the angular separation constant can be determined analytically, and is given by the relation $A_{lm} = l(l+1) - s(s+1)$.

Boundary conditions for each equation translate into a couple of three-term continued fraction relations of the form (10). Finding QNM frequencies is now a two-step procedure: for assigned values of a, ℓ, m and ω, first find the angular separation constant $A_{lm}(\omega)$ looking for zeros of the *angular* continued fraction; then replace the corresponding eigenvalue into the *radial* continued fraction, and look for its zeros as a function of ω. In principle, the convergence of the procedure for modes with large imaginary parts can be improved, as described earlier, by a wise choice of the rest, R_N, of the radial continued fraction. Expanding this rest as in formula (13) and introducing $b \equiv \sqrt{1-4a^2}$, we get for the first few coefficients: $C_0 = -1$, $C_1 = \pm\sqrt{2\rho b}$, $C_2 = [3/4 - \rho(b+1) - s]$.

As for the RN case, we have checked that our results agree with those shown in Refs. [4,19] for small values of ω_I, and that we get Nollert's asymptotic quasinormal frequencies in the nonrotating limit.

III. RESULTS

As a first step in the computation of asymptotic modes for charged and rotating black holes we have checked that our numerical methods reproduce known results in the Schwarzschild limit [28]. In particular, we have verified the asymptotic behavior found by Nollert [5]. His main result was that the real part of the asymptotic QNM frequencies corresponding to *gravitational* perturbations can be well fitted by a relation of the form

$$\omega_R = \omega_\infty + \frac{\lambda_{s,l}}{\sqrt{n}}. \quad (17)$$

The leading-order fitting coefficient is independent of l and given by $\omega_\infty = 0.0874247$, consistently (within numerical accuracy) with the analytical formula (3). Corrections of order $\sim n^{-1/2}$, however, are l dependent. Furthermore, we will see in a moment that they also depend on the spin s of the perturbing field, and that is why we denoted them by $\lambda_{s,l}$. For gravitational perturbations ($s = -2$) Nollert found (and we verified to the same level of accuracy) that $\lambda_{-2,2} = 0.4850$, $\lambda_{-2,3} = 1.067$, $\lambda_{-2,6} = 3.97$.

Now, an important testable prediction of the recent derivations of the "ln 3" asymptotic behavior is that *scalar* black hole perturbations should lead to the same asymptotic QNM frequency, at leading order in an expansion in powers of $n^{-1/2}$ [15–17]. To check this prediction we have extended our numerical calculations to $s=0$. Using the procedure described in Ref. [5], we found that our scalar QNM data are well fitted by formula (17). As $n \to \infty$, the asymptotic frequency for scalar modes is again given by $\omega_\infty = 0.0874247$, consistently with the analytic calculation. What changes are the numerical values of the leading-order correction coefficients $\lambda_{0,l}$; namely, we find $\lambda_{0,0} = 0.0970$, $\lambda_{0,1} = 0.679$, $\lambda_{0,2} = 1.85$.

Neitzke [17] recently suggested that leading-order corrections to the asymptotic frequency should be proportional to $[(s^2-1)-3l(l+1)]$. As an interesting byproduct of our calculation, we found that our numerical values for $\lambda_{s,l}$'s are consistent with Neitzke's conjecture; that the proportionality constant is dependent on s,

$$\lambda_{s,l} = k_s[(s^2-1)-3l(l+1)], \tag{18}$$

and finally, we determined the proportionality constants k_s to be given by $k_{-2} = -0.0323$, $k_0 = -0.0970$. Recently Maassen van den Brink [29] derived k_{-2} analytically, finding

$$k_{-2} = -\frac{\sqrt{2}\,\Gamma(1/4)^4}{432\pi^{5/2}} \simeq -0.0323356. \tag{19}$$

Notice that, within our numerical accuracy, $k_0 = 3k_{-2}$. This prediction may be helpful to analytically determine k_0. We shall say more about the importance of leading-order corrections to the frequencies in the following; now we turn to a discussion of our numerical results for RN and Kerr black holes.

A. Reissner-Nordström black holes

The numerical behavior of the first few overtones of a RN black hole was studied numerically by Andersson and Onozawa [26], who unveiled a very peculiar behavior as the modes' imaginary part increases. Our numerical codes are in excellent agreement with their results. In comparing with their paper note, however, that Andersson and Onozawa count modes starting from $n=0$, while we label the fundamental mode by $n=1$, following Leaver (see Table 1 in Ref. [4]). The results we display refer to perturbations reducing to pure gravitational perturbations of Schwarzschild in the uncharged limit. The trajectories described by the modes in the complex-ω plane first show "closed loops," as in the top left panel of Fig. 1. Then they get a spiral-like shape, moving out of their Schwarzschild value and "looping in" towards some limiting frequency as Q tends to the extremal value. This kind of behavior is shown in the top right panel of Fig. 1. We have observed that such a spiralling behavior sets in for larger values of the modes' imaginary part (i.e., larger values of n) as the angular index l increases. In other words, in-

creasing l for a given value of the mode index n has the effect of "unwinding" the spirals, as can be observed in the two panels in the second row of Fig. 1. However, for each l the spiralling behavior is eventually observed when n is large enough; a typical example for $l=6$ is shown in the last plot of Fig. 1.

Maybe a clearer picture of the modes' behavior can be obtained looking separately at the real and imaginary parts of the mode frequencies as function of charge. Let us focus first on the real parts. The corresponding numerical results are shown in Fig. 2, together with the predictions of the analytic formula (6). It is quite apparent that, as the mode order grows, the oscillating behavior as a function of charge start earlier and earlier. As n increases, the oscillations become faster, the convergence of the continued fraction method slower, and the required computing time gets longer. Therefore, when the imaginary part increases it becomes more difficult to follow the roots numerically as we approach the extremal value $Q=1/2$. That is why our data for large values of n do not cover the whole range of allowed values for Q. Despite these difficulties, we have many reasons to trust our numerics. We have carefully checked our results, using first double and then quadrupole precision in our Fortran codes (indeed, as n increases, we can obtain results for large values of the charge only using quadrupole precision). As we have shown earlier, our frequencies accurately reproduce Nollert's results in the Schwarzschild limit, so our numerics can be trusted for small values of Q. Furthermore, the predictions of the analytic formula exactly overlap with the oscillations we observe for large values of the charge. Not only this does give support to the asymptotic formula [17], it also gives us faith that the numerics are meaningful also for large charge, where we don't have any published results to confirm our predictions.

Similar considerations apply to the imaginary parts. We present a few plots illustrating the general trend for the imaginary parts in Fig. 3, showing again excellent agreement with the asymptotic formula (6) as n increases. Once again, the analytic formula shows deviations from our "exact" numerical results only for small values of the charge, probably indicating that corrections of order $n^{-1/2}$ should be taken into account for small values of Q. When we look at the mode trajectories in the complex-ω plane, the increasingly oscillating behavior of the real and imaginary parts means that the number of "spirals" described by the mode before reaching the extremal value increases roughly as the mode order n.

Can we deduce something from the agreement of our numerical results with formula (6) at large values of the charge? It would be extremely interesting to draw consequences on the extremal RN case, for various reasons. First of all, the QNM spectrum for extremal RN black holes is characterized by an isospectrality between electromagnetic and gravitational perturbations, which has been motivated in Ref. [30] as a manifestation of supersymmetry. Furthermore, topological arguments have been used to show that the entropy-area relation breaks down for extreme QNM's [31]. Therefore, we believe that some caution is required in claiming that the connection between QNM's and the area spectrum is still valid for extreme black holes, as recently advo-
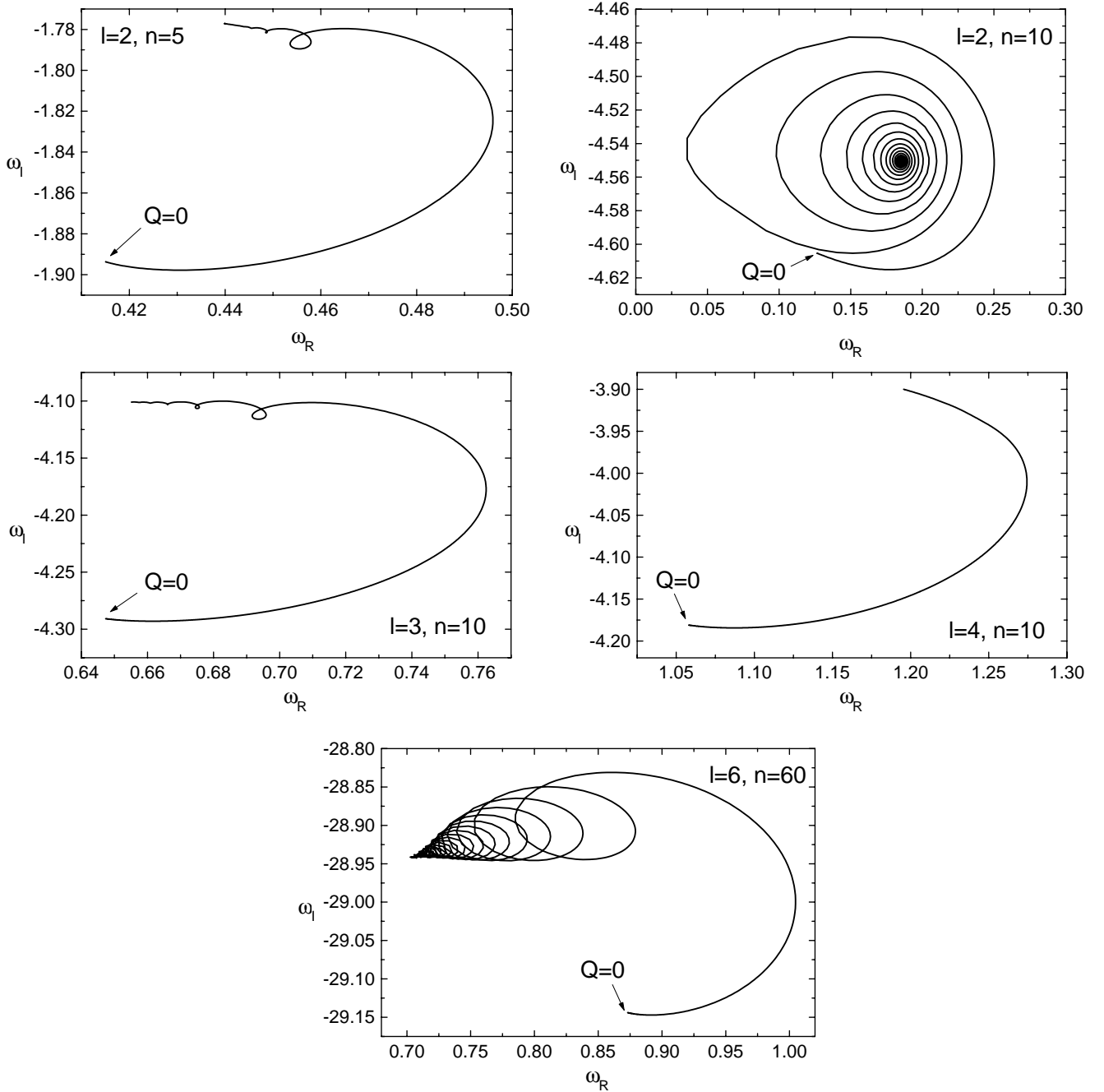
FIG. 1. The top two panels show the behavior of the $n=5$ and $n=10$ QNM frequencies in the complex ω plane. The $n=10$ mode "spirals in" towards its value in the extremal charge limit; the number of spirals described by each mode increases roughly as the mode order n. The panels in the second row show how the $n=10$ spiral "unwinds" as the angular index l is increased (in other words, the asymptotic behavior sets in later for larger l's). Finally, the bottom panel shows a high-l mode trajectory "pointing" to its limit as the charge becomes extremal. In all cases, we have marked by an arrow the frequency corresponding to the Schwarzschild limit ($Q=0$).

cated in Ref. [32]. These problems may be connected with our recent finding that extremally charged black holes in a (nonasymptotically flat) anti–de Sitter spacetime could be marginally unstable [33].

Our numerics seem to indicate that we can trust formula (6) in the large-charge regime. Then a very interesting conclusion follows [17]: the real part of the frequency for extremal RN black holes coincides with the Schwarzschild value, i.e.

$$\omega_R^{RN} \rightarrow \frac{\ln 3}{4\pi} \quad \text{as} \quad Q \rightarrow 1/2. \tag{20}$$

However, it is quite difficult to check this prediction numerically. In the extremal RN case, due to the coalescence of the inner and outer horizons, the singularity structure of the radial perturbation equations changes. Therefore one has to apply a different (and slightly more involved) procedure,
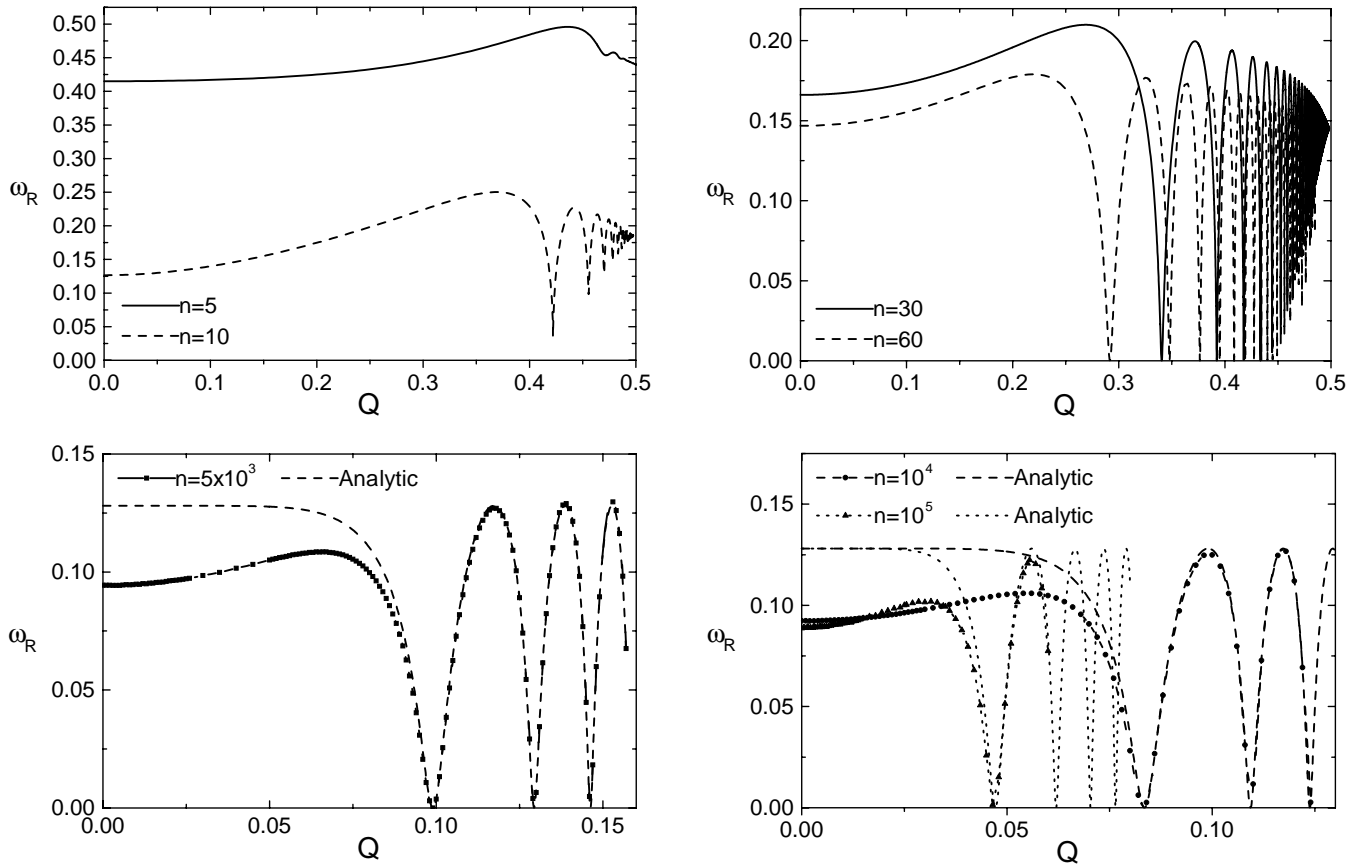
FIG. 2. Real part of the RN QNM frequencies as a function of charge for $n = 5$, 10, 30, 60, 5000, 10000, 100000. As the mode order increases, the computation becomes more and more time consuming, the oscillations become faster, and a good numerical sampling is rather difficult to achieve; therefore in the last plot we use different symbols (small squares, circles and triangles) to display the actually computed points. For $n = 5000$, 10000, 100000 we also compare to the prediction of the analytic formula (6) derived by Motl and Neitzke [16]. The oscillatory behavior is reproduced extremely well by their formula, but the disagreement increases for small charge: formula (6) does not yield the correct Schwarzschild limit.

which has been described in Ref. [27]. We have tried to apply that procedure to get highly damped QNM's. Apparently, the extremal RN QNM's show a behavior which is rather similar to that of Schwarzschild QNM's (see, e.g., Fig. 1 in Ref. [4]): they have finite real part for small values of ω_I, approach the pure-imaginary axis as ω_I is increased, and then the real part increases again. This would support the predictions of the asymptotic formula (6). Unfortunately, we have not yet managed to get stable numerical results for large values of ω_I. We plan to improve our codes and obtain more numerically stable results in the future.

If, supported by the agreement between our numerics and the analytical prediction, we assume that formula (20) holds, an interesting result emerges: Hod's conjecture (2) is *incompatible* with the "truncated" version of the first law of black hole mechanics (4) obtained *dropping the ΔQ term* [8]. Indeed, formula (5) predicts that the real part of the asymptotic frequencies in the extremal case should be zero. This does not imply that Hod's conjecture is wrong, but only that *dropping the ΔQ term* to deduce formula (5) is not a valid assumption.

B. Kerr black holes

Quasinormal frequencies of Kerr black holes were first studied by Leaver using continued fraction techniques [4] and subsequently investigated by other authors [34].

A systematic exploration of the behavior of the first few overtones was carried out only some years ago, using Leaver's continued fraction method, by Onozawa [19], who found some rather odd features as the mode damping increases. For example, Detweiler [35] showed that the first few modes having $l = m$ have vanishing imaginary part and real part equal to m as the black hole becomes extremal (in our units, as $a \to 1/2$). He also showed analytically that there can be infinite solutions to the Teukolsky equation having $a = 1/2$ and $\omega = m$, which led to the suspicion that *all* modes having $l = m$ should "cluster" on the real axis at $\omega = m$ as the black hole becomes extremal. An interesting outcome of Onozawa's investigation was that, for given values of l and $m = l$, there is at least one mode frequency which does not tend to m in the extremal limit. Onozawa also found that, when $n \sim 10$, modes having negative m (which for the first few overtones show a tendency to decrease in frequency as the hole is
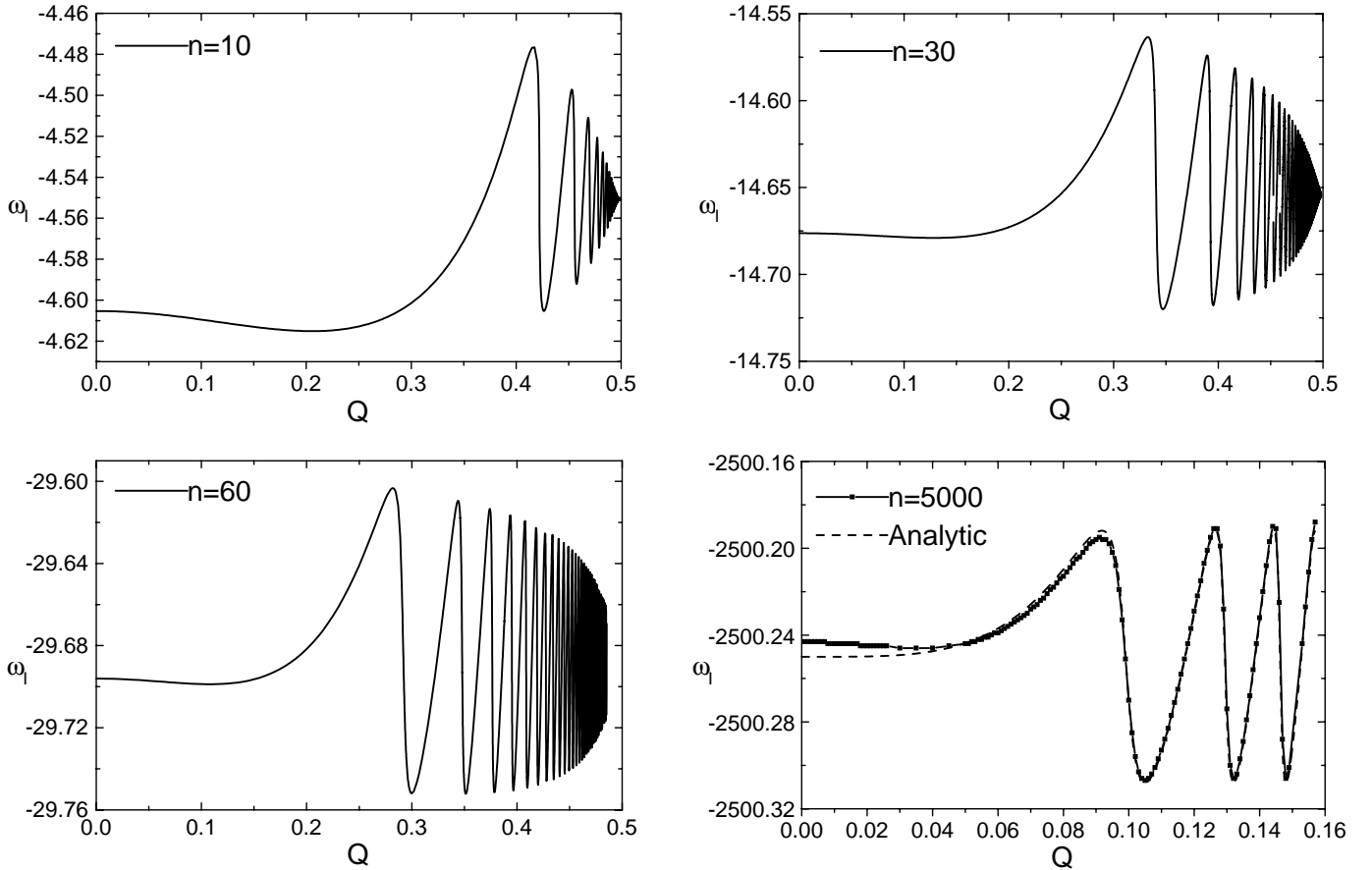
FIG. 3. Imaginary part of the RN QNM frequencies as a function of charge for $n = 10$, 30, 60, 5000. For $n = 5000$ we also display the actually computed points, and compare to the prediction of the analytic formula (6). As for the real part, the oscillations are reproduced extremely well, but the disagreement with our numerical data increases for small charge.

spun up) show instead a tendency to "turn around" and increase their frequency as a increases, sometimes showing strange "loops" in the complex-ω plane (see Figs. 3 and 4 in Ref. [19]).

We have confirmed Onozawa's results, and extended them to moderately high n. However, since one has to solve simultaneously the angular and the radial continued fraction, the numerical problem turned out to be much more tricky than the search of highly damped modes in the RN case. Comparing Nollert's technique and a standard summation of the continued fraction using Gautschi's algorithm [4], we found out that it is much harder to achieve a stable numerical computation of modes for $n \sim 50$ or higher [36]. Because of these convergence problems, even using Nollert's method, we did not manage to push the numerical calculation to very large values of n. However, even moderately high values of n shed some light on what should be the asymptotic behavior of Kerr QNM's.

Let us first consider Kerr perturbations having $m = 0$. We have been able to compute quite a few moderately highly damped modes for $l = 2$ and $l = 3$, and in Fig. 4 we show two of these modes. These plots should be compared to the RN modes we have shown in Fig. 1: there is a similar "looping" behavior, with the number of loops increasing as the damping of the mode increases. We notice that a similar looping

behavior has recently been found by Glampedakis and Andersson for *scalar* perturbations of Kerr black holes, using a different method [37]. Figures 5 and 6 show the real and imaginary parts (respectively) of some Kerr modes with $m = 0$. It is useful to compare these plots with Figs. 2 and 3: the behavior is extremely similar, and asymptotically it can probably be described by some formula reminiscent of Eq. (6). Of course, such a behavior is not even close to that predicted by formula (5).

Hod recently used Onozawa's data, which to our knowledge are the only available published data for highly damped Kerr modes, to show that the results predicted by formula (5) agree with the numerically computations in Ref. [19] within $\sim 5\%$, at least when $l = m$. We repeated Onozawa's calculations, finding excellent agreement with his results, and then extended it to higher-order modes. We found that, as n increases, the formula conjectured by Hod does not seem to provide a good fit to the asymptotic modes. As shown in Fig. 7, the proposed formula disagrees quite badly in the small rotation rate regime even for the low-lying modes which were used for comparison by Hod. In any event, this is to be expected and does not contradict Hod's claim, since it is known from previous calculations in the Schwarzschild limit [4] that modes with $n < 10$ have real parts of the QNM frequencies which are not close enough to their asymptotic
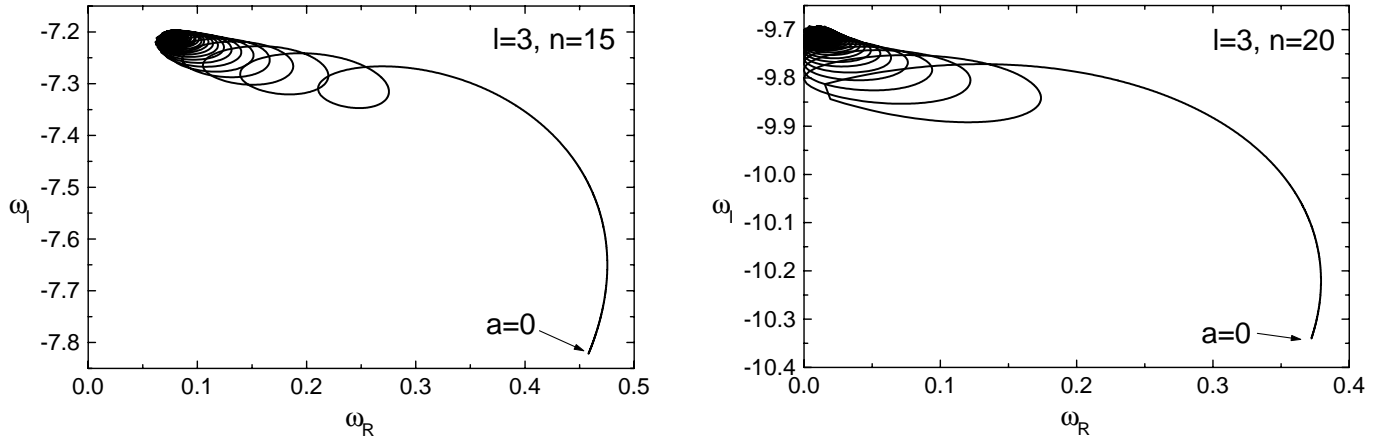
FIG. 4. Trajectories of Kerr modes having $m=0$ in the complex ω plane. The left panel corresponds to $l=3$, $n=15$ and the right panel corresponds to $l=3$, $n=20$. The number of spirals increases with the mode order, as in the RN case. We have marked by an arrow the point in the plane corresponding to the Schwarzschild limit.

value. However, if formula (5) really holds in the asymptotic limit, one would expect the agreement with numerical calculations to get better as the mode's imaginary part increases; on the contrary, Fig. 7 shows that, as n increases, the agreement gets worse, even for large rotation rates (which used to show rather good agreement for $n \sim 10$). So Hod's formula does not seem to provide an accurate fit to the asymptotic frequencies. We actually found that, as the mode order increases, modes having $l=m=2$ are fitted extremely well (relative errors being of order $\sim 0.1\%$ when $n \sim 50$) by the relation

$$\omega_{l=m=2}^{Kerr} = 2\Omega + i2\pi T_H n. \quad (21)$$

It is interesting to note that, although the real part tends to a different asymptotic value, the spacing in the imaginary parts *does* agree with the value conjectured by Hod in Ref. [18]. Indeed, Hod put forward his conjecture observing that in the Schwarzschild case the asymptotic QNM spacing is given by $2\pi T_H$. The convergence to the indicated asymptotic behav-
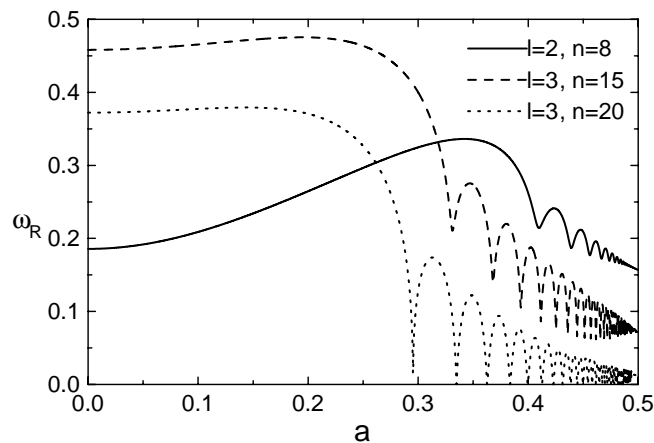
ior is faster for large values of a, and generally the agreement between formula (21) and our numerics is good for values of $a \gtrsim 0.1$. The index n appearing in formula (21) depends, of course, on the labelling convention we use to "count" QNM's.

Notice also that the modes' imaginary part does not show the typical "shift" $(-1/4)$ which is present in formula (3). In our opinion, this is another hint that the Schwarzschild limit of highly damped Kerr modes should be taken with special care, and that order of limits issues may be relevant in the asymptotic regime.

Preliminary calculations show that modes having $l=2$ and $m=1$, $m=-1$ and $m=-2$ show a more complicated behavior. We are currently trying to improve our understanding of highly damped Kerr QNM's using both analytical and numerical techniques. Some of our results will be shown in a separate paper [38].

IV. CONCLUSIONS

In this paper we have numerically investigated the asymptotic behavior of QNM's for charged (RN) and rotating (Kerr) black holes. We have first confirmed Nollert's results and extended them to scalar perturbations of Schwarzschild black holes. Our numerics are consistent with

$$\omega_R = \frac{\ln 3}{4\pi} + \frac{k_s[(s^2-1)-3l(l+1)]}{\sqrt{n}}, \quad (22)$$

where, within our accuracy, $k_0 = -0.0970 = 3k_{-2}$. Recently the constant k_{-2} has been determined analytically—see formula (19)—and our result may be useful to determine k_0 as well.

More importantly, our results for charged and rotating black holes do not agree with the simple behavior predicted by Hod's conjecture for the real part of the frequency, as given in formula (5). We have shown that both the real and the imaginary part of RN QNM's as functions of charge dis-
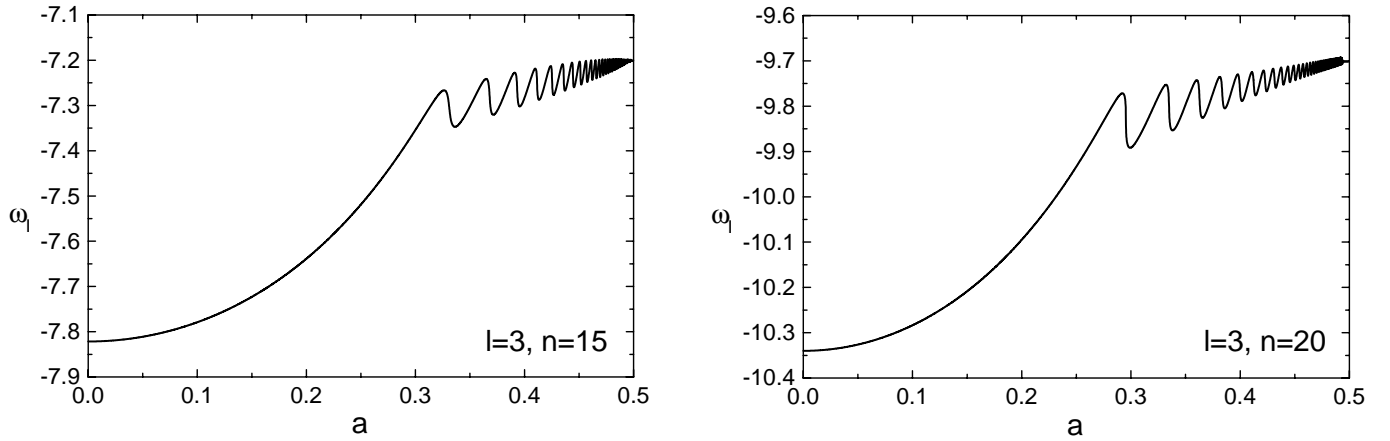

FIG. 5. Real part of Kerr modes having $m=0$ as a function of a. Labels indicate the corresponding values of l and of the mode order n.

FIG. 6. Imaginary part of Kerr modes having $m=0$ as a function of a. The left panel corresponds to $l=3$, $n=15$ and the right panel corresponds to $l=3$, $n=20$.

play an oscillating behavior. The oscillations start at smaller values of the charge and get faster as n increases. We have compared our numerical results to the predictions of the asymptotic formula (6) derived in Ref. [16] and found that, in general, they agree extremely well, especially for large values of the mode index n. The formula derived by Motl and Neitzke only fails to reproduce our numerical results in the limit of small charge. Quite likely, their expression has a wrong limit in the Schwarzschild case (the real part tending to $\ln 5/4\pi$ instead of $\ln 3/4\pi$) because finite-n corrections become relevant as Q tends to zero [17]. A computation of higher-order corrections to the asymptotic RN formula will probably give a final answer on the reasons for the small-Q discrepancy we observe. If we trust the predictions of formula (6), as the agreement between analytical and numerical results suggests to do, the asymptotic frequency for extremal RN black holes is the same as for Schwarzschild black holes.
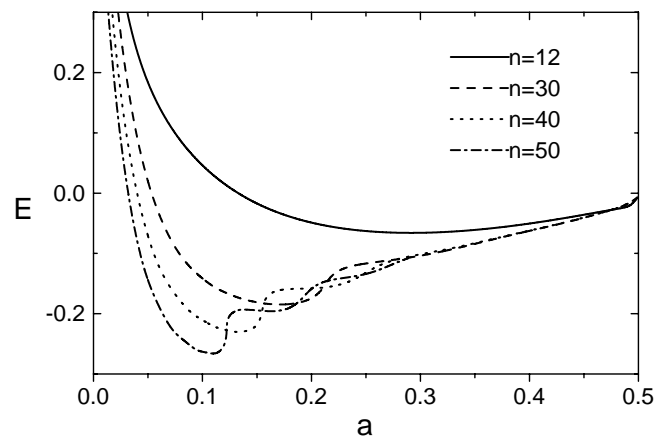
This implies that dropping the ΔQ term in the first law of black hole mechanics, as done by Hod to conjecture the validity of formula (5), is presumably *not* a justified assumption.

Highly damped gravitational quasinormal frequencies of rotating (Kerr) black holes proved more difficult to compute, even using Nollert's method, and show a more complicated behavior. For $m=0$, we observe spirals in the complex plane reminiscent of RN modes, confirming the behavior found in Ref. [37] for highly damped *scalar* perturbations. For $l=m=2$, the behavior is completely different. The asymptotic behavior of Kerr modes having $l=m=2$ is very well fitted by formula (21) that we rewrite here

$$\omega_{l=m=2}^{Kerr}=2\Omega+i2\pi T_H n,$$

where Ω is the angular velocity of the black hole horizon and T_H is its temperature. The convergence to the limiting value is faster when a is large, and the formula can be seen as an extension of formula (3) to Kerr modes having $l=m=2$. We think that a calculation of finite-n corrections to the asymptotics may help explain both the faster convergence rate at large a, and the apparent disagreement with formula (3) in the limit $a\rightarrow 0$. A more extensive investigation of asymptotic Kerr QNM's is ongoing [38].

FIG. 7. Relative error in the formula for the asymptotic frequency conjectured by Hod. The plots show $E=(\omega_R-\tilde{\omega}_R)/\tilde{\omega}_R$, where $\tilde{\omega}_R$ is defined in formula (5), for increasing values of the mode index n, namely, $n=12$, 30, 40, 50 and $l=m=2$. As n grows, the relative error in the conjectured asymptotic formula tends to be larger and larger.

ACKNOWLEDGMENTS

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