

Quasinormal frequencies of Schwarzschild black holes in anti-de Sitter spacetimes: A complete study of the overtone asymptotic behavior

Vitor Cardoso*

*Centro Multidisciplinar de Astrofísica-CENTRA, Departamento de Física, Instituto Superior Técnico, Av. Rovisco Pais 1,
1049-001 Lisboa, Portugal*

Roman Konoplya†

Department of Physics, Dnipropetrovsk National University, St. Naukova 13, Dnipropetrovsk 49050, Ukraine

José P. S. Lemos‡

*Department of Physics, Columbia University, New York, New York 10027, USA
and Centro Multidisciplinar de Astrofísica-CENTRA, Departamento de Física, Instituto Superior Técnico, Av. Rovisco Pais 1,
1049-001 Lisboa, Portugal*

(Received 9 May 2003; published 28 August 2003)

We present a thorough analysis of the quasinormal (QN) behavior associated with the decay of scalar, electromagnetic, and gravitational perturbations of Schwarzschild black holes in anti-de Sitter (AdS) spacetimes. As is known, the AdS QN spectrum crucially depends on the relative size of the black hole to the AdS radius. There are three different types of behavior depending on whether the black hole is large, intermediate, or small. The results of previous works, concerning lower overtones for large black holes, are completed here by obtaining higher overtones for all three black hole regimes. There are two major conclusions that one can draw from this work: First, asymptotically for high overtones, all the modes are evenly spaced, and this holds for all three types of regime, large, intermediate, and small black holes, independently of l , where l is the quantum number characterizing the angular distribution; second, the spacing between modes is apparently universal in that it does not depend on the field; i.e., scalar, electromagnetic, and gravitational QN modes all have the same spacing for high overtones. We are also able to prove why scalar and gravitational perturbations are isospectral, asymptotically for high overtones, by introducing appropriate superpartner potentials.

DOI: 10.1103/PhysRevD.68.044024

PACS number(s): 04.70.-s, 04.30.-w, 04.50.+h, 11.25.Hf

I. INTRODUCTION

Any physical system has its modes of vibration. For non-dissipative systems these modes, forming a complete set, are called normal, each mode having a given real frequency of oscillation and being independent of any other. The system once disturbed continues to vibrate in one or several of the normal modes. On the other hand, a black hole, as any other gravitational system, is a dissipative system since it emits gravitational radiation. One has to consider, instead, quasinormal modes (QNMs) for which the frequencies are no longer purely real, showing that the system is losing energy. QNMs are in general not complete and though insufficient to fully describe the dynamics, contain a great amount of information. For instance, they dominate the signal during the intermediate stages of the perturbation. Indeed, calculations ranging from the formation of a black hole in gravitational collapse to the collision of two black holes provide clear evidence that no matter how one perturbs a black hole, its response will be dominated by the QNMs. Through the QNMs, one can also probe the black hole mass, electric charge, and angular momentum by inspection of their characteristic waveform, as well as test the stability of the event

horizon against small perturbations. Moreover, the interest in QNMs has now broadened, they might be related to fundamental physics. In the context of black holes in asymptotically flat spacetimes the importance of QNMs has long ago been recognized. Due to a crescent increase of interest for black holes in asymptotically de Sitter (dS) and asymptotically anti-de Sitter (AdS) spacetimes the study of QNMs has now spread into these.

In asymptotically flat spacetimes the idea of QNMs started with the work of Regge and Wheeler [1], where the stability of a black hole was tested, and were actually first numerically computed by Chandrasekhar and Detweiler several years later [2]. It continues to be a very active field due to the eminent possibility of detecting gravitational waves from astrophysical sources. There are two standard reviews in the field [3]. QNMs in asymptotically flat spacetimes have recently acquired a further importance since it has been proposed that the Barbero-Immirzi parameter, a factor introduced by hand in order that Loop Quantum Gravity reproduces correctly the black hole entropy, is equal to the real part of the quasinormal (QN) frequencies with a large imaginary part [4] (see [5] for a short review). For further developments in calculating QN frequencies in Kerr spacetimes and in asymptotically flat black holes spacetimes in d -dimensions see [6]. Some other recent calculations of QN frequencies in asymptotically flat black hole spacetimes can be found in [7].

*Email address: vcardoso@fisica.ist.utl.pt

†Email address: konoplya_roma@yahoo.com

‡Email address: lemos@physics.columbia.edu

In asymptotically dS spacetimes the calculation of QN frequencies was first done by Moss and collaborators [8] in which the stability of the Cauchy horizon of a charge black hole was analyzed. In Cardoso and Lemos [9] an analytical method was devised to study the case in which the black hole and the cosmological horizons are very close to each other. This analytical method has recently been extended to higher dimensional Schwarzschild dS black holes [10] and also to higher order in the difference between the cosmological and horizon radius [11]. A different analytical approach has also been developed in [12].

In asymptotically AdS spacetimes, which is the background spacetime for our paper, there has been a great amount of work because the AdS conformal field theory (CFT) correspondence conjecture [13] makes the investigation of QNMs important. According to it, the black hole corresponds to a thermal state in the conformal field theory (CFT), and the decay of the test field in the black hole spacetime corresponds to the decay of the perturbed state in the CFT. The dynamical time scale for the return to thermal equilibrium can be done in AdS spacetime and then translated onto the CFT, using the AdS/CFT correspondence. Many authors have now delved into these calculations in several different black hole settings in several different dimensions (see [14] for a sample). In this paper we are interested in the four-dimensional Schwarzschild-AdS spacetime. The lowest lying modes (i.e., the less damped ones parametrized by the overtone number $n=0$) for this spacetime were found by Horowitz and Hubeny [15], and completed by Konoplya [16] for the scalar case, and by Cardoso and Lemos [17] for the electromagnetic and gravitational case. Recently, Berti and Kokkotas [18] have confirmed all these results and extended them to Reissner-Nordström-AdS black holes. Here, we shall take a step further in carrying on this program by computing numerically, through an extensive search, the high overtone QN frequencies for scalar, electromagnetic, and gravitational perturbations in the Schwarzschild-AdS black hole. We shall do an extensive search for the high overtone QN frequencies, ($n \geq 1$). We find that the modes are evenly spaced for frequencies with a large imaginary part. Moreover, the scalar, electromagnetic, and gravitational perturbations all possess, asymptotically for high overtones n , QN frequencies with the same spacing, and this spacing is l -independent. While we can numerically prove this with great accuracy for large black holes, it remains just a conjecture for small and intermediate black holes. We shall also see that the QN frequencies of the toroidal black hole with nontrivial topology [19] are identical to the QN frequencies of a large Schwarzschild-AdS black hole [20].

II. EQUATIONS AND NUMERICAL METHOD

We shall deal with the free evolution of massless classical fields in the background of a Schwarzschild-AdS spacetime with the metric given by

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where $f(r) = r^2/R^2 + 1 - 2M/r$, R is the AdS radius, and M the black hole mass (Newton's constant G_N and the velocity of light are set to one). The evolution of scalar, electromagnetic, and gravitational fields can be followed through the Klein-Gordon, Maxwell, and Einstein equations, respectively. If one assumes that the fields are a small perturbation in the background given by Eq. (1), then all the covariant derivatives can be taken with respect to the metric (1). It is then possible to show that the evolution equations are all of the same type, i.e., a second order radial differential equation (for more details we refer the reader to [15] for the scalar case and to [17] for the electromagnetic and gravitational case). The wave equation is

$$\frac{\partial^2 \Psi(r)}{\partial r_*^2} + [\omega^2 - V(r)]\Psi(r) = 0, \quad (2)$$

where the tortoise coordinate r_* is defined as

$$\frac{\partial r}{\partial r_*} = f(r), \quad (3)$$

and the potential V appearing in Eq. (2) depends on the specific field under consideration. Explicitly, for scalar perturbations,

$$V_s = f(r) \left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + \frac{2}{R^2} \right], \quad (4)$$

while for electromagnetic perturbations,

$$V_{em} = f(r) \left[\frac{l(l+1)}{r^2} \right]. \quad (5)$$

The gravitational perturbations decompose into two sets [17], the odd and the even parity one. For odd perturbations the potential $V(r)$ in Eq. (2) is

$$V_{odd} = f(r) \left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3} \right], \quad (6)$$

while for even perturbations, we have

$$V_{even} = \frac{2f(r)}{r^3} \frac{9M^3 + 3\alpha^2 M r^2 + \alpha^2(1+\alpha)r^3 + 3M^2 \left(3\alpha r + 3\frac{r^3}{R^2} \right)}{(3M + \alpha r)^2}, \quad (7)$$

where $\alpha = \frac{1}{2}[l(l+1) - 2]$. In all cases, we denote by l the angular quantum number, that gives the multipolarity of the field. We can of course rescale r , $r \rightarrow r/R$. If we do this, the wave equation takes again the form (2) with rescaled constants, i.e., $r_+ \rightarrow r_+/R$, $\omega \rightarrow \omega R$, where r_+ is the horizon radius. So, we can take $R=1$ and measure everything in terms of R , the AdS radius. Equation (2) should be solved under appropriate boundary conditions, i.e., incoming waves near the horizon,

$$\Psi \sim e^{-i\omega r_*}, \quad r \rightarrow r_+, \quad (8)$$

and no waves at infinity,

$$\Psi = 0, \quad r \rightarrow \infty. \quad (9)$$

We note that there are other reasonable boundary conditions at infinity, in particular for the gravitational perturbations. For instance, one can define Robin boundary conditions in such a way as to preserve certain dualities between the odd and the even gravitational perturbations. However, it was verified numerically by Moss and Norman [8] that Dirichlet or Robin boundary conditions yield approximately the same result, so we shall keep $\Psi = 0, r \rightarrow \infty$. Moreover, Cardoso and Lemos [17] proved that for high overtone QN frequencies the duality is preserved, so in this regime the distinction is irrelevant. Thus to compute the QN frequencies ω such that the boundary conditions (8) and (9) are preserved, we follow the Horowitz-Hubeny approach [15]. Within this approach we need to expand the solution to the wave equation around $x_+ = 1/r_+$ ($x = 1/r$),

$$\Psi(x) = \sum_{k=0}^{\infty} a_k(\omega)(x - x_+)^k, \quad (10)$$

and to find the roots of the equation $\Psi(x=0) = 0$. First, one should substitute Eq. (10) into the wave equation (2) in order to obtain a recursion relation for a_k [15]. Then, one has to truncate the sum (10) at some large $k = N$ and check that for greater k the roots converge to some true root which is the sought QN frequency. The higher the overtone number, and the smaller the black hole size, the larger the number N at which the roots of the equation $\Psi(x=0) = 0$ converge. Yet, since in the series (10) each next term depends on all the preceding terms through the recursion relations, when N is too large, the tiny numerical errors in the first terms start growing as $N \sim 10^2$ to 10^3 or greater. As a result the roots suffer a sharp change for a small change on any of the input parameters, displaying a “noisy” dependence. To avoid this we have to increase the precision of all the input data and the recursion relation we are dealing with from the standard 20-digit precision up to a precision such that further increasing of it will not influence the result for the QN frequency. For small black holes the roots start converging at very large N only, for instance, when $r_+ = 1/20$ we can truncate the series at $N \sim 3 \times 10^4$, but not before. Since for finding roots of Eq. (9) we have to resort to the trial and error method, the above procedure consumes much time, but nevertheless allows us to compute QNMs of small AdS black holes [16], and to find the higher overtones we are seeking.

III. NUMERICAL RESULTS

In this section we will present the numerical results obtained using the numerical procedure just outlined in the previous section. The results will be organized into three sections: scalar, electromagnetic, and gravitational perturbations. For each field we shall also divide the results into three different regimes: large, intermediate, and small black holes, since the results depend crucially on the regime one is dealing with. Here a large black hole stands for a black hole with $r_+ \gg 1$, an intermediate black hole is one with $r_+ \sim 1$, and a small black hole has a horizon radius $r_+ \ll 1$. We shall then try to unify these results. For each horizon radius r_+ and angular quantum number l there is an infinity of QN frequencies (or overtones). We shall order them according to the standard procedure, by increasing the imaginary part. Accordingly, the fundamental QN frequency is defined as the one having the lowest imaginary part (in absolute value) and will be labeled with the integer $n=0$. The first overtone has the second lowest imaginary part and is labeled with $n=1$, and so on. The QN frequencies also have a real part, which in general display an increase along with the imaginary part. To the lowest value of the imaginary part corresponds the lowest value of the real part, to the second lowest value of the imaginary part corresponds the second lowest value of the real part, and so on. Thus n , the overtone number, is also a number that in general increases with the real part of the frequency (or energy) of the mode. This seems to be a characteristic of AdS space only, due to the special boundary conditions associated with this spacetime. This, in a sense, is to be expected since the wave equation to be studied is a Schrödinger type equation, where for quantum nondissipative bound systems, such as the hydrogen atom or a particle in an infinite well potential, the principal quantum number n (which here has been called the overtone number) appears due to the boundary conditions of the radial equation, a typical eigenvalue problem, and is related directly with the frequency of vibration of the orbital. The similarity is not full, though, since the boundary condition at the black hole is of a different kind. However, for pure AdS spacetimes, when there is no black hole and the boundary conditions are of infinite well type, the overtone number n is indeed a principal quantum number (see the Appendix).

A. Scalar quasinormal frequencies

The fundamental scalar QN frequencies were first computed by Horowitz and Hubeny [15] for intermediate and large black holes. Konoplya [16] extended these calculations to the case of small black holes. Recently Berti and Kokkotas [18] rederived all these results. Here we do for the first time an extensive search for higher overtones of scalar perturbations. Some of the lowest lying modes we find are shown in Tables I, II, and III for large, intermediate, and small black holes, respectively.

(i) *Large black holes.* As proven by Horowitz and Hubeny [15] in the large black hole regime the frequencies must scale

TABLE I. QN frequencies corresponding to $l=0$ scalar perturbations of a large Schwarzschild-AdS black hole (BH) ($r_+ = 100$). It can be seen that for large n the modes become evenly spaced. Although not shown here, our numerical data indicates that this happens for all values of l and also that the spacing is the same, regardless of the value of l . For $l=0$ and for high n the QN frequencies go like $\omega_s/r_+ = (1.299 - 2.25i)n + 1.856 - 2.673i$. The corresponding spacing between consecutive modes seems to be l -independent.

n	$\text{Re}[\omega_{\text{QN}}]:$	$\text{Im}[\omega_{\text{QN}}]:$	n	$\text{Re}[\omega_{\text{QN}}]:$	$\text{Im}[\omega_{\text{QN}}]:$
0	184.95344	-266.38560	7	1096.44876	-1841.88813
1	316.14466	-491.64354	8	1226.38317	-2066.89596
2	446.46153	-716.75722	9	1356.31222	-2291.90222
3	576.55983	-941.81253	10	1486.23753	-2516.90740
4	706.57518	-1166.8440	50	6682.78814	-11516.9823
5	836.55136	-1391.8641	299	39030.810	-67542.308
6	966.50635	-1616.8779	300	39160.7272	-67767.3091

as the horizon radius [this can also be proven easily and directly from the differential equation (2)]. We show in Table I the results for a spherically symmetric mode ($l=0$) for a black hole with $r_+ = 100$ which is therefore sufficient to infer the behavior of all large black holes. The fundamental frequency agrees with previous results [15]. Perhaps the most interesting result in this large black hole regime is that asymptotically for high overtone number n the frequencies become evenly spaced and behave like, for $l=0$,

$$\frac{\omega_s}{r_+} = (1.299 - 2.25i)n + 1.856 - 2.673i, \quad (n, r_+) \rightarrow \infty. \quad (11)$$

TABLE II. QN frequencies corresponding to $l=0$ scalar perturbations of an intermediate Schwarzschild-AdS BH ($r_+ = 1$). Asymptotically for large n one finds approximately $\omega_s \sim (1.97 - 2.35i)n + 2.76 - 2.7i$.

n	$\text{Re}[\omega_{\text{QN}}]:$	$\text{Im}[\omega_{\text{QN}}]:$	n	$\text{Re}[\omega_{\text{QN}}]:$	$\text{Im}[\omega_{\text{QN}}]:$
0	2.7982	-2.6712	10	22.44671	-26.20913
1	4.75849	-5.03757	11	24.41443	-28.55989
2	6.71927	-7.39449	12	26.38230	-30.91059
3	8.66153	-9.74852	13	28.35029	-33.26123
4	10.6467	-12.1012	14	30.31839	-35.61183
5	12.6121	-14.4533	15	32.28658	-37.96238
6	14.5782	-16.8049	16	34.25485	-40.31290
7	16.5449	-19.1562	17	36.22318	-42.66340
8	18.5119	-21.5073	18	38.19157	-45.01387
9	20.4792	-23.8583	19	40.16002	-47.36431

TABLE III. QN frequencies corresponding to $l=0$ scalar perturbations of a small Schwarzschild-AdS BH ($r_+ = 0.2$). Asymptotically for large n one finds approximately $\omega_s \sim (1.69 - 0.57i)n + 2.29 - 0.46i$.

n	$\text{Re}[\omega_{\text{QN}}]:$	$\text{Im}[\omega_{\text{QN}}]:$	n	$\text{Re}[\omega_{\text{QN}}]:$	$\text{Im}[\omega_{\text{QN}}]:$
0	2.47511	-0.38990	6	12.45222	-3.89179
1	4.07086	-0.98966	7	14.14065	-4.46714
2	5.72783	-1.57600	8	15.83026	-5.04186
3	7.40091	-2.15869	9	17.52070	-5.61610
4	9.08118	-2.73809	10	19.21191	-6.18997
5	10.7655	-3.31557	11	20.90359	-6.76355

Thus the spacing between frequencies is

$$\frac{\omega_{s_{n+1}} - \omega_{s_n}}{r_+} = (1.299 - 2.25i), \quad (n, r_+) \rightarrow \infty. \quad (12)$$

Moreover, although the offset $1.856 - 2.673i$ in Eq. (11) is l -dependent (this number is different for $l=1$ scalar perturbations, for example), this asymptotic behavior for the spacing (12) holds for any value of l . In fact our search of the QN frequencies for higher values of l reveal that the results are very similar to those in Table I. We have gone up to $l=4$ for scalar perturbations and the results were quite insensitive to l . The asymptotic behavior sets in very quickly as one increases the mode number n . Typically for $n=10$ Eq. (11) already gives a very good approximation. Indeed, for $n=10$ we find numerically (see Table I) $\omega_s = 1486.23753 - 2516.90740i$ for a $r_+ = 100$ black hole, while the asymptotic expression gives $\omega_s = 1484.6 - 2517.3i$.

(ii) *Intermediate black holes.* In Table II we show some of the lowest lying scalar QN frequencies for an intermediate black hole with $r_+ = 1$. For a black hole with this size, one finds again that the spacing does not depend on the angular number l for very high overtone number n . With an error of about 2% the limiting value for the frequency is, for $l=0$,

$$\omega_s \sim (1.97 - 2.35i)n + 2.76 - 2.7i, \quad n \rightarrow \infty. \quad (13)$$

For QN frequencies belonging to different l 's the offset in Eq. (13) is different, but as far as we can tell numerically, not the asymptotic spacing implied by Eq. (13). Expression (13) for the asymptotic behavior works well again for $n > 10$.

(iii) *Small black holes.* Our search for the QN frequencies of small black holes, i.e., black holes with $r_+ \ll 1$, revealed what was expected on physical grounds, and was uncovered numerically for the first time in [16] for the fundamental mode: for small black holes, the QN frequencies approach the frequencies of pure AdS spacetime [21] (see also the Appendix). In fact we find

$$\omega_s = 2n + l + 3, \quad r_+ \rightarrow 0. \quad (14)$$

TABLE IV. QNMs corresponding to $l=1$ electromagnetic perturbations of a large Schwarzschild-AdS BH ($r_+=1000$). Notice that now there are eight pure imaginary modes, still well described by Liu's formula.

n	Re[ω_{QN}]:	Im[ω_{QN}]:	n	Re[ω_{QN}]:	Im[ω_{QN}]:
0	0	-1500.004789	5	0	-8985.232
1	0	-2999.982599	6	0	-10596.03
2	0	-4500.093600	7	0	-11644.76
3	0	-5999.513176	8	1219.7	-13566.42
4	0	-7502.69385	9	2494.6	-15847.06

In Table III we show some results for a small black hole with $r_+=0.2$. We stress that the values presented in Table III for the asymptotic spacing between modes may have an error of about 2%. In fact it is extremely difficult to find very high overtones of small black holes, and so it is hard to give a precise estimate of the value they asymptote to.

In summary, we can say that the QN frequencies tend to be evenly spaced asymptotically as n gets very large, no matter if the black hole is large, intermediate, or small. Moreover, the spacing between consecutive modes is, as far as we can tell, independent of the angular quantum number l .

B. Electromagnetic quasinormal frequencies

The fundamental electromagnetic QN frequencies were computed for the first time by Cardoso and Lemos [17]. Recently Berti and Kokkotas [18] have redone the calculation showing excellent agreement. Here we extend the results to higher overtones. Some of the lowest lying electromagnetic frequencies are shown in Tables IV–VIII.

(i) *Large black holes.* As found in [17] large black holes show a somewhat peculiar behavior: some of the lowest lying modes have pure imaginary frequencies, and these are well described by an analytical formula [17]. A surprising aspect unveiled for the first time by the present search is that the number of such modes decreases as the horizon radius becomes smaller, as can be seen from Tables IV and V. In other words, for very large black holes the number of imaginary modes grows. For example, for $r_+=1000$ (Table IV) there are eight pure imaginary modes, for $r_+=100$ there are four such modes (see Table V), and for $r_+=10$ there are only two. If one wants to go for r_+ larger than 1000, the computation is very time consuming since we use a trial and error method for finding new modes. However, not only is this a completely new piece of data, it also makes us think that infinitely large black holes may have pure imaginary electromagnetic QN frequencies for any overtone number. Perhaps, an infinitely large black hole cannot vibrate at all.

Again, we find that for large black holes and $l=1$, the frequencies are evenly spaced with

$$\frac{\omega_{\text{em}}}{r_+} = (1.299 - 2.25i)n - 11.501 + 12i, \quad (n, r_+) \rightarrow \infty, \tag{15}$$

TABLE V. QNMs corresponding to $l=1$ electromagnetic perturbations of a large Schwarzschild-AdS BH ($r_+=100$). The first four modes are pure imaginary and are well described by Liu's approximation [17]. For high n the QN frequencies obey, for $l=1$, $\omega_{\text{em}}/r_+ = (1.299 - 2.25i)n - 11.501 + 12i$. The corresponding spacing between consecutive modes seems to be l -independent.

n	Re[ω_{QN}]:	Im[ω_{QN}]:	n	Re[ω_{QN}]:	Im[ω_{QN}]:
0	0	-150.0479	10	799.6171	-2171.826
1	0	-299.8263	11	927.812	-2398.208
2	0	-450.9458	12	1056.153	-2624.438
3	0	-595.3691	13	1184.620	-2850.543
4	22.504	-799.194	14	1313.192	-3076.546
5	162.256	-1035.098	15	1441.856	-3302.464
6	289.028	-1263.537	16	1570.601	-3528.310
7	416.247	-1491.223	17	1699.416	-3754.094
8	543.792	-1718.409	18	1828.295	-3979.824
9	671.598	-1945.246	19	1957.229	-4205.508

and a spacing given by

$$\frac{\omega_{\text{em}_{n+1}} - \omega_{\text{em}_n}}{r_+} = (1.299 - 2.25i), \quad (n, r_+) \rightarrow \infty. \tag{16}$$

For different values of the angular quantum number l , we find the same spacing (16) between consecutive modes, although the offset in Eq. (15) depends on l . So, asymptotically for large n and large horizon radius the spacing is the same as for the scalar case. This is surprising, especially since the

TABLE VI. QNMs corresponding to $l=1$ electromagnetic perturbations of an intermediate Schwarzschild-AdS BH ($r_+=1$). Asymptotically for large n the modes become evenly spaced in mode number and behave as $\omega_{\text{em}} \sim (1.96 - 2.36i)n + 1.45 - 2.1i$.

n	Re[ω_{QN}]:	Im[ω_{QN}]:	n	Re[ω_{QN}]:	Im[ω_{QN}]:
0	2.163023	-1.699093	10	21.067466	-25.61714
1	3.843819	-4.151936	11	23.015470	-27.98278
2	5.673473	-6.576456	12	24.965381	-30.34713
3	7.553724	-8.980538	13	26.916889	-32.71037
4	9.458385	-11.37238	14	28.869756	-35.07265
5	11.37722	-13.75633	15	30.823790	-37.43413
6	13.30526	-16.13482	16	32.778838	-39.79488
7	15.23974	-18.50933	17	34.734776	-42.15499
8	17.17894	-20.88081	18	36.691500	-44.51455
9	19.12177	-23.24993	19	38.648922	-46.87360

TABLE VII. QNMs corresponding to $l=1$ electromagnetic perturbations of a small Schwarzschild-AdS BH ($r_+=0.2$). Asymptotically for large n one finds approximately $\omega_{\text{em}} \sim (1.68 - 0.59i)n + 1.87 - 0.04i$.

n	Re[ω_{QN}]:	Im[ω_{QN}]:	n	Re[ω_{QN}]:	Im[ω_{QN}]:
0	2.63842	-0.05795	6	12.00066	-3.53148
1	3.99070	-0.47770	7	13.66436	-4.12974
2	5.49193	-1.08951	8	15.33370	-4.72479
3	7.07835	-1.70859	9	17.00715	-5.31725
4	8.70165	-2.32191	10	18.68370	-5.90758
5	10.3450	-2.92920	11	20.36268	-6.49615

behavior of the scalar and electromagnetic potentials are radically different. It is even more surprising the fact that this asymptotic behavior does not depend on l , as the electromagnetic potential is strongly l -dependent. Furthermore, from the first electromagnetic overtones one could surely not anticipate this behavior.

(ii) *Intermediate black holes.* In Table VI we show some of the lowest lying electromagnetic QN frequencies for an intermediate black hole with $r_+=1$. For a black hole with this size, one finds again that the spacing does not depend on the angular number l for very high overtone number n . With an error of about 2% the limiting value for the frequency is, for $l=1$,

$$\omega_{\text{em}} \sim (1.96 - 2.36i)n + 1.45 - 2.1i, \quad n \rightarrow \infty. \quad (17)$$

We note that here too the offset in Eq. (17) does depend on l , but not the asymptotic spacing.

(iii) *Small black holes.* For small black holes, see Tables VII and VIII, the spacing seems also to be equal as for the scalar case, but since it is very difficult to go very high in mode number n in this regime, the error associated in estimating the asymptotic behavior is higher, and one cannot be completely sure. Again, the electromagnetic QN frequencies of very small black holes asymptote to the pure AdS electromagnetic modes (see the Appendix, where we sketch their computation). Indeed we find that

TABLE VIII. The fundamental ($n=0$) QNMs corresponding to $l=1$ electromagnetic perturbations of a small Schwarzschild-AdS BH for several values of r_+ .

r_+	Re[ω_{QN}]:	Im[ω_{QN}]:	r_+	Re[ω_{QN}]:	Im[ω_{QN}]:
1/2	2.25913	-0.65731	1/10	2.85188	-0.00064
1/3	2.40171	-0.29814	1/12	2.88058	-0.00030
1/4	2.53362	-0.13364	1/16	2.91363	-0.00016
1/5	2.63842	-0.05795	1/18	2.92406	-0.00009
1/8	2.80442	-0.00565	1/20	2.93200	-0.00002

TABLE IX. QNMs corresponding to $l=2$ odd gravitational perturbations of a large Schwarzschild-AdS BH ($r_+=100$). The fundamental QN frequency is purely imaginary and seems to be well described by the formula $\omega_{n=0} = -(l-1)(l+2)/3r_+ i$ valid only in the large black hole regime. In the large n limit one finds $\omega_{\text{odd}}/r_+ = (1.299 - 2.25i)n + 0.58 - 0.42i$. The corresponding spacing between consecutive modes seems to be l -independent.

n	Re[ω_{QN}]:	Im[ω_{QN}]:	n	Re[ω_{QN}]:	Im[ω_{QN}]:
0	0	-0.013255	6	836.55392	-1391.86345
1	184.95898	-266.38403	7	966.50872	-1616.87735
2	316.14887	-491.64242	8	1096.45098	-1841.88755
3	446.46505	-716.75629	9	1226.38527	-2066.89540
4	576.56293	-941.81172	10	1356.31422	-2291.90170
5	706.57797	-1166.8433	50	6552.87704	-11291.9807

$$\omega_{\text{em,AdS}} = 2n + l + 2, \quad r_+ \rightarrow 0. \quad (18)$$

This can be clearly seen from Table VIII, where we show the fundamental mode for small black holes of decreasing radius. As the horizon radius gets smaller and smaller, the fundamental frequency approaches the value of $3 + 0i$, which is indeed the correct pure AdS mode for $l=1, n=0$, electromagnetic perturbations. It was conjectured by Horowitz and Hubeny [15] that for very small black holes in AdS space, the imaginary part of the QN frequency for spherically symmetric perturbations should scale with the horizon area, i.e., with r_+^2 . Their argument was based on a previous result [22] for the absorption cross section for the $l=0$ component. This conjecture was later verified numerically to be correct by Konoplya [16] for the $l=0$ case. From Table VIII it is, however, apparent that this scaling is no longer valid for $l=1$ perturbation, and indeed we find it is not valid for $l \neq 0$ perturbations, be it scalar, electromagnetic, or gravitational perturbations. The reason why the imaginary part no longer scales with the horizon area for $l \neq 0$ perturbations is due to the fact that the partial absorption cross section only scales with the horizon area for $l=0$ perturbations. For other l 's the behavior is more complex, and it could be that there is no simple scaling, or even that the behavior is oscillatory with the mass M of the black hole. We refer the reader to [23] for details on the absorption cross section of black holes.

C. Gravitational quasinormal frequencies

The fundamental gravitational QN frequencies were computed for the first time by Cardoso and Lemos [17]. We remind that there are two sets of gravitational wave equations, the odd and even ones. Although it was found [17] that there is a family of the odd modes which is very slowly damped and purely imaginary, it was possible to prove that for high frequencies both odd and even perturbations must yield the same QN frequencies. We present the results for higher overtones of odd perturbations in Tables IX–XII, and even perturbations in Tables XIII–XVI.

TABLE X. QNMs corresponding to $l=2$ odd gravitational perturbations of an intermediate Schwarzschild-AdS BH ($r_+=1$). Asymptotically for large n one finds approximately $\omega_{\text{odd}} \sim (1.97 - 2.35i)n + 0.93 - 0.32i$.

n	$\text{Re}[\omega_{\text{QN}}]:$	$\text{Im}[\omega_{\text{QN}}]:$	n	$\text{Re}[\omega_{\text{QN}}]:$	$\text{Im}[\omega_{\text{QN}}]:$
0	0	-2	10	20.604949	-23.803860
1	3.033114	-2.404234	11	22.567854	-26.157246
2	4.960729	-4.898194	12	24.531429	-28.510214
3	6.905358	-7.289727	13	26.495564	-30.862849
4	8.854700	-9.660424	14	28.460169	-33.215214
5	10.80784	-12.02344	15	30.425175	-35.567355
6	12.76384	-14.38266	16	32.390524	-37.919308
7	14.72199	-16.73969	17	34.356173	-40.271103
8	16.68179	-19.09530	18	36.322082	-42.622761
9	18.64286	-21.44994	19	38.288221	-44.974301

1. Odd perturbations

(i) *Large black holes.* As discussed for the first time in [17] these exhibit a pure imaginary fundamental mode (see Table IX). For large black holes, this mode is slowly damped and scales as the inverse of the horizon radius. Our analysis for higher l 's indicates that in the large black hole regime an excellent fit to this fundamental pure imaginary mode is

$$\omega_{\text{odd}_{n=0}} = -\frac{(l-1)(l+2)}{3r_+}i, \quad r_+ \rightarrow \infty. \quad (19)$$

This generalizes a previous result by Berti and Kokkotas [18] for the $l=2$ case. The simplicity of this formula (which is just a fit to our numerical data) leads us to believe it is possible to find an analytical explanation for it, but such explanation is still lacking. In the large black hole regime, asymptotically for high overtones one finds, for $l=2$, for example,

TABLE XI. QNMs corresponding to $l=2$ odd gravitational perturbations of a small Schwarzschild-AdS BH ($r_+=0.2$). Asymptotically for large n one finds approximately $\omega_{\text{odd}} \sim (1.69 - 0.59i)n + 2.49 + 0.06i$.

n	$\text{Re}[\omega_{\text{QN}}]:$	$\text{Im}[\omega_{\text{QN}}]:$	n	$\text{Re}[\omega_{\text{QN}}]:$	$\text{Im}[\omega_{\text{QN}}]:$
0	2.404	-3.033	6	12.67161	-3.43609
1	4.91594	-0.30408	7	14.33020	-4.05366
2	6.30329	-0.89773	8	15.99881	-4.66448
3	7.82330	-1.53726	9	17.67433	-5.26955
4	9.40720	-2.17744	10	19.35465	-5.86978
5	11.0279	-2.81083	11	21.03839	-6.46596

$$\frac{\omega_{\text{odd}}}{r_+} = (1.299 - 2.25i)n + 0.58 - 0.42i, \quad (n, r_+) \rightarrow \infty. \quad (20)$$

This leads to the spacing

$$\frac{\omega_{\text{odd}_{n+1}} - \omega_{\text{odd}_n}}{r_+} = (1.299 - 2.25i), \quad (n, r_+) \rightarrow \infty, \quad (21)$$

which, as our results indicate is again l -independent. Again, the offset in Eq. (20) depends on l .

(ii) *Intermediate black holes.* Results for the odd QN frequencies of an intermediate ($r_+=1$) black hole are shown in Table X. With an error of about 5% the limiting value for the frequency is, for $l=2$,

$$\omega_{\text{odd}} \sim (1.97 - 2.35i)n + 0.93 - 0.32i, \quad n \rightarrow \infty. \quad (22)$$

We note that here too the offset in Eq. (22) does depend on l , but not the asymptotic spacing, with a numerical error of about 5%.

(iii) *Small black holes.* The behavior for small black holes is shown in Tables XI and XII. As the black hole gets smaller, the pure imaginary mode gets more damped: the imaginary part increases, as can be seen from Table XII,

TABLE XII. The fundamental ($n=0$) QNMs corresponding to $l=2$ odd gravitational perturbations of a small Schwarzschild-AdS BH for several values of r_+ .

r_+	$\text{Re}[\omega_{\text{QN}}]:$	$\text{Im}[\omega_{\text{QN}}]:$	r_+	$\text{Re}[\omega_{\text{QN}}]:$	$\text{Im}[\omega_{\text{QN}}]:$
0.8 ($n=0$)	0	-3.045373	0.5 ($n=1$)	3.03759	-0.71818
0.8 ($n=1$)	2.89739	-1.69556	0.4	3.16209	-0.43092
0.7 ($n=0$)	0	-3.83538	0.3	3.35487	-0.17320
0.7 ($n=1$)	2.90665	-1.34656	0.2	3.62697	-0.01792
0.6 ($n=0$)	0	-4.901973	0.1	3.84839	-0.00005
0.6 ($n=1$)	2.95550	-1.02196	1/15	3.90328	-0.00001
0.5 ($n=0$)	0	-6.40000	1/20	3.92882	-0.000002

TABLE XIII. QNMs corresponding to $l=2$ even gravitational perturbations of a large Schwarzschild-AdS BH ($r_+=100$). For large n , one finds $\omega_{\text{even}}/r_+ = (1.299 - 2.25i)n + 0.58 - 0.42i$. The corresponding spacing between consecutive modes seems to be l -independent.

n	Re[ω_{QN}]:	Im[ω_{QN}]:	n	Re[ω_{QN}]:	Im[ω_{QN}]:
0	184.97400	-266.351393	6	966.609780	-1616.695872
1	316.17838	-491.584999	7	1096.56635	-1841.681256
2	446.50884	-716.674054	8	1226.51495	-2066.664293
3	576.62103	-941.70468	9	1356.45821	-2291.645761
4	706.65039	-1166.71147	10	1486.39776	-2516.626168
5	836.64066	-1391.70679	50	6683.51993	-11515.70869

where we show the two lowest QN frequencies for small black holes with decreasing radius. As mentioned by Berti and Kokkotas [18] the ordering of the modes here should be different. However, since one can clearly distinguish this pure imaginary mode as belonging to a special family, we shall continue to label it with $n=0$. We have not been able to follow this mode for black holes with $r_+ < 0.5$, and so Table XII does not show any pure imaginary modes for a horizon radius smaller than 0.5. We note that, as for the scalar and electromagnetic cases, here too the modes are evenly spaced, with a spacing which seems to be independent of l no matter if the black hole is large or small. For very small black holes, the frequencies reduce to their pure AdS values, computed in the Appendix, to wit

$$\omega_{\text{odd}} = 2n + l + 2, \quad r_+ \rightarrow 0. \quad (23)$$

TABLE XIV. QNMs corresponding to $l=2$ even gravitational perturbations of an intermediate Schwarzschild-AdS BH ($r_+=1$). Asymptotically for large n one finds approximately $\omega_{\text{even}} \sim (1.96 - 2.35i)n + 2.01 - 1.5i$.

n	Re[ω_{QN}]:	Im[ω_{QN}]:	n	Re[ω_{QN}]:	Im[ω_{QN}]:
0	3.017795	-1.583879	10	21.68949	-24.98271
1	4.559333	-3.810220	11	23.64402	-27.33549
2	6.318337	-6.146587	12	25.60052	-29.68799
3	8.168524	-8.500194	13	27.55860	-32.04026
4	10.061220	-10.85631	14	29.51796	-34.39234
5	11.976813	-13.21224	15	31.47838	-36.74424
6	13.906140	-15.56749	16	33.43969	-39.09600
7	15.844371	-17.92208	17	35.40174	-41.44762
8	17.788721	-20.27609	18	37.36444	-43.79914
9	19.737469	-22.62960	19	39.32769	-46.15057

TABLE XV. QNMs corresponding to $l=2$ even gravitational perturbations of a small Schwarzschild-AdS BH ($r_+=0.2$). Asymptotically for large n one finds approximately $\omega_{\text{even}} \sim (1.61 - 0.6i)n + 2.7 + 0.37i$.

n	Re[ω_{QN}]:	Im[ω_{QN}]:	n	Re[ω_{QN}]:	Im[ω_{QN}]:
0	3.56571	-0.01432	3	7.65872	-1.42994
1	4.83170	-0.26470	4	9.20424	-2.04345
2	6.17832	-0.82063	5	10.78800	-2.65360

One can see this more clearly from Table XII, where in fact for very small black holes the frequency rapidly approaches Eq. (23). Again, for small black holes, the imaginary part does not scale with the horizon area, by the reasons explained before, in Sec. III B.

In conclusion, the higher overtones of odd perturbations follow a pattern very similar to the scalar case. We note that the asymptotic behavior sets in very quickly, much like what happened for scalar and electromagnetic perturbations. Typically the formulas yielding the asymptotic behavior work quite well for $n > 10$. We are now able to prove that for sufficiently high frequencies the scalar and gravitational perturbations are isospectral, a mystery that remained in [17], This is done in Sec. IV A below.

2. Even perturbations

Let us now briefly discuss the even modes. As found previously [17] these modes behave very similar to the scalar ones. Yet, the even gravitational modes are stipulated by a more complicated potential, and we have to truncate the series in power of $x - x_+$ at larger N , which makes the whole procedure more time consuming. That is why when considering small black holes we were restricted only by the first seven modes in that case. It is, however, sufficient to see that even gravitational QNMs, similar to other kinds of perturbations, tend to arrange into a equidistant spectrum under the increasing of n . We show in Tables XIII–XVI the numerical results for the QN frequencies of even gravitational perturbations.

(i) *Large black holes.* Results for the QN frequencies of large black holes are shown in Table XIII. In this regime one finds for $l=2$ even perturbations

TABLE XVI. The fundamental ($n=0$) QNMs corresponding to $l=2$ even gravitational perturbations of a small Schwarzschild-AdS BH for several values of r_+ .

r_+	Re[ω_{QN}]:	Im[ω_{QN}]:	r_+	Re[ω_{QN}]:	Im[ω_{QN}]:
0.8	2.91541	-1.18894	0.3	3.29299	-0.14103
0.7	2.90591	-0.98953	0.2	3.56571	-0.01432
0.6	2.92854	-0.78438	0.1	3.80611	-0.00005
0.5	2.98985	-0.57089	1/15	3.8735	-0.00001
0.4	3.10317	-0.35043	1/20	3.90852	-0.000002

$$\frac{\omega_{\text{even}}}{r_+} = (1.299 - 2.25i)n + 1.88 - 2.66i, \quad (n, r_+) \rightarrow \infty, \quad (24)$$

leading to the spacing

$$\frac{\omega_{\text{even}_{n+1}} - \omega_{\text{even}_n}}{r_+} = (1.299 - 2.25i), \quad (n, r_+) \rightarrow \infty, \quad (25)$$

which once more turns out to be l -independent. All the results concerning the spacing of frequencies for large black holes have a very good precision, since in this regime it is possible to go very far out in overtone number (typically $n = 300$ is enough to achieve a 0.1% accuracy for the spacing).

(ii) *Intermediate black holes.* In Table XIV we show some of the lowest lying even gravitational QN frequencies for an intermediate black hole with $r_+ = 1$. For a black hole with this size, one finds again that the spacing does not seem to depend on the angular number l for very high overtone number n . With an error of about 5% the limiting value for the frequency is, for $l = 2$,

$$\omega_{\text{even}} \sim (1.96 - 2.35i)n + 2.01 - 1.5i, \quad n \rightarrow \infty. \quad (26)$$

We note that here too the offset in Eq. (17) does depend on l , but not the asymptotic spacing.

(iii) *Small black holes.* The behavior for small black holes is shown in Tables XV and XVI. Our search for the QN frequencies of small black holes, i.e., black holes with $r_+ \ll 1$ revealed again what was expected on physical grounds: for small black holes, the QN frequencies approach the frequencies of pure AdS spacetime (see the Appendix). In fact we find

$$\omega_{\text{even}_{\text{AdS}}} = 2n + l + 2, \quad r_+ \rightarrow 0. \quad (27)$$

In Table XV we show the lowest lying QN frequencies for a small black hole ($r_+ = 0.2$). We stress that the values presented in Table XV (as a matter of fact, all the tables containing data for small black holes) for the asymptotic spacing between modes may have an error of about 2%. In fact it is extremely difficult to find very high overtones of small black holes, and so it is hard to give a precise estimate of the value they asymptote to. In Table XVI we show some of the fundamental even QN frequencies for small black holes of decreasing radius, and one can clearly see how the fundamental frequency approaches the pure AdS value given in the Appendix.

IV. DISCUSSION OF THE RESULTS

A. Why are the scalar and gravitational perturbations isospectral in the large black hole regime?

In a previous paper (Sec. III C in [17]), we have shown why the odd and even gravitational perturbations yield the same QN frequencies for large frequencies. The whole approach was based on the fact that the odd and even gravitational potentials are superpartner potentials [24], i.e., they

are related to one another via $V_{\text{odd}} = W^2 + dW/dr_* + \beta$, $V_{\text{even}} = W^2 - dW/dr_* + \beta$, where $\beta = -\alpha^2 + 2\alpha^3 + \alpha^4/9M^2$. The function W is $W = 2M/r^2 + -3 - 2\alpha/3r + 3\alpha^2 + 2\alpha^2 + 27M^2/3\alpha(3M + \alpha r) - \frac{1}{3}(\alpha/M + \alpha^2/M + 9M/\alpha)$. For more details we refer the reader to [17]. We shall now see that a similar method can be applied to show that in the large black hole regime, scalar and gravitational perturbations are isospectral for large QN frequencies. To begin with, we note that the potentials $V1$ and $V2$ defined by

$$V1 = \tilde{f} \left(\frac{2}{R^2} + \frac{2M}{r^3} \right), \quad (28)$$

and

$$V2 = \tilde{f} \left(\frac{a}{r^2} - \frac{6M}{r^3} \right), \quad (29)$$

with $\tilde{f} = r^2/R^2 + a/2 - 2M/r$, and a any constant, are superpartner potentials. The superpotential \tilde{W} is in this case is given by

$$\tilde{W} = \frac{r}{R^2} + \frac{a}{2r} - \frac{2M}{r^2}. \quad (30)$$

Thus the two superpartner potentials $V1$ and $V2$ can be expressed in terms of \tilde{W} as

$$V1 = \tilde{W}^2 + \frac{d\tilde{W}}{dr_*}, \quad V2 = \tilde{W}^2 - \frac{d\tilde{W}}{dr_*}. \quad (31)$$

Why are these two potentials of any interest? Because in the large r_+ limit, which we shall take to be $r_+ \gg a$, we have $\tilde{f} \sim r^2 - 2M/r$. Notice now that in this large r_+ limit the scalar potential (4) is $V_s \sim f(2 + 2M/r^3)$, with $f \sim r^2 - 2M/r$, since in this limit and with $r_+ \gg l$, one has $l(l+1)/r^2 \ll 2$. Thus $V1$ reduces to the scalar potential and $V2$ to the gravitational odd potential, provided we take $a = l(l+1)$. It then follows from the analysis in [17] (Sec. III C) that for large black holes these two potentials should yield the same frequencies.

B. Future directions

The preceding sections have shown that the QNMs of Schwarzschild-AdS black holes have a universal behavior in the asymptotic regime of high overtones. This was verified explicitly and with great accuracy for the large black hole regime, where we showed numerically that the spacing does not depend on the perturbation in question and is equal to

$$\frac{\omega_{n+1} - \omega_n}{r_+} = (1.299 - 2.25i), \quad (n, r_+) \rightarrow \infty. \quad (32)$$

We conjecture that the asymptotic behavior is the same for all kinds of perturbations irrespectively of the black hole size, i.e., a fixed horizon radius r_+ Schwarzschild-AdS black hole will have an asymptotic spacing between consecutive

QN frequencies which is the same for scalar, electromagnetic, and gravitational perturbations. The difficulty in extracting very high overtones for small black holes, however, prevents us from having an irrefutable numerical proof of this. It would be extremely valuable to have some kind of analytical scheme for extracting the asymptotic behavior, much as has been done for the asymptotically flat space by Motl and Neitzke [4]. However, it looks quite difficult to make any analytical approximation in asymptotically AdS spaces, although there have been some attempts at this recently (see, for example, Musiri and Siopsis [14]). We also note that the spacing (32) was already found to be true by Berti and Kokkotas [18] for the scalar and gravitational cases for the lowest radiatable multipole, i.e., $l=0$ and $l=2$ scalar and gravitational perturbations, respectively. We have concluded that, surprisingly, the spacing (32) also works for the electromagnetic case and for any value of l . It was observed that, despite having such different potentials the scalar, the electromagnetic, and gravitational QN frequencies have the same asymptotic behavior. Can one formulate some very general conditions the potentials should obey in order to have the same asymptotic solutions? This is still an open question.

There has been recently an exciting development trying to relate the asymptotic QN frequencies with the Barbero-Immirzi parameter [4,5]. In fact it was observed, in the Schwarzschild case, that asymptotically for high overtones, the real part of the QN frequencies was a constant, l -independent, and using some (not very clear yet) correspondence between classical and quantum states, was just the right constant to make Loop Quantum Gravity give the correct result for the black hole entropy. Of course it is only natural to ask whether such kind of numerical coincidence holds for other spacetimes. We have seen that apparently we are facing, in AdS space, a universal behavior, i.e., the asymptotic QN frequencies do not depend on the kind of perturbations, and also do not depend on l . However, and in contrast with asymptotically flat space, the real part of the asymptotic QN frequency is not a constant, but rather increases linearly with the mode number n . This is no reason to throw off the initial motivation of seeking some kind of relation between Loop Quantum Gravity and QNMs, after all, there are no predictions for AdS space.

Finally we point out that the asymptotic behavior studied here for the Schwarzschild-AdS black hole will hold also for other black holes in asymptotically AdS. One example of these is the black hole with nontrivial topology [19]. The general line element for this spacetime is [19]

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - r^2(d\theta^2 + d\phi^2), \quad (33)$$

where

$$f(r) = \frac{r^2}{R^2} - \frac{4MR}{r}, \quad (34)$$

where M is the ADM mass of the black hole, and R is the AdS radius. There is a horizon at $r_+ = (4M)^{1/3}R$. The range of the coordinates θ and ϕ dictates the topology of the black

hole spacetime. For a black hole with toroidal topology, a toroidal black hole, the coordinate θ ranges from 0 to 2π , and ϕ ranges from 0 to 2π as well. For the cylindrical black hole, or black string, the coordinate θ has the range $-\infty < R\theta < \infty$, and $0 \leq \phi < 2\pi$. For the planar black hole, or black membrane, the coordinate ϕ is further decompactified $-\infty < R\phi < \infty$ [19]. The fundamental QN frequencies for these black holes were computed in [20], where it was verified that they follow the same pattern as for Schwarzschild-AdS black holes. Indeed one easily sees that in the large black hole regime they both should yield the same results as the potentials are equal in this regime (compare the potentials in [20] with the ones in the present work). In particular the asymptotic behavior will be the same.

V. CONCLUSION

We have done an extensive search for higher overtones n of the QNMs of Schwarzschild-AdS BH corresponding to scalar, electromagnetic, and gravitational perturbations. We have shown that: (i) No matter what size the black hole is, the QN frequencies are evenly spaced, both in the real and in the imaginary component, for high overtone number n ; (ii) The spacing between consecutive modes is independent of the perturbation. This means that scalar, electromagnetic, and gravitational perturbations all have, asymptotically, the same spacing between modes. This is one of the major findings in this work, together with the fact that this spacing seems to be also independent of the angular quantum number l ; (iii) We were able to prove that the scalar and gravitational QN frequencies must asymptotically be the same; (iv) The electromagnetic QN frequencies of large black holes have a number of first overtones with pure imaginary parts, and the higher the black hole radius r_+ , the higher the number of these first pure damped, nonoscillating modes; (v) Finally, we have computed analytically the electromagnetic and gravitational pure AdS modes, and we have shown numerically that the QN frequencies of very small black holes asymptote to these pure AdS modes.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge stimulating correspondence related to this problem with Veronica Hubeny. This work was partially funded by Fundação para a Ciência e Tecnologia (FCT)–Portugal through project PESO/PRO/2000/4014. V.C. also acknowledges financial support from FCT through PRAXIS XXI program. J.P.S.L. acknowledges financial support from ICCTI/FCT and thanks Observatório Nacional do Rio de Janeiro for hospitality.

APPENDIX: PURE ADS NORMAL MODES FOR ELECTROMAGNETIC AND GRAVITATIONAL PERTURBATIONS

In this appendix we shall briefly outline how to compute the pure modes of AdS space (no black hole, $M=0$) for electromagnetic and gravitational perturbations. The scalar case was dealt with by Burgess and Lutken [21]. In pure AdS

space the electromagnetic and gravitational potentials (both odd and even) are

$$V = \left(\frac{r^2}{R^2} + 1 \right) \frac{l(l+1)}{r^2}, \quad (\text{A1})$$

as can be seen by substituting $M=0$ in Eqs. (5)–(7). Also in this case the relation $r(r_*)$ takes the simple form

$$r = R \tan \frac{r_*}{R}, \quad (\text{A2})$$

and therefore the potential (A1) takes a simple form in the r_* coordinate, namely

$$V = \frac{l(l+1)}{R^2 \sin^2 \left(\frac{r_*}{R} \right)}. \quad (\text{A3})$$

To proceed, we note that the change of variable $x = \sin^2(r_*/R)$ leads the wave equation to a hypergeometric equation,

$$\frac{\partial^2 \Psi(x)}{\partial x^2} + \frac{\tilde{\tau}}{\sigma} \frac{\partial \Psi(x)}{\partial x} + \frac{\tilde{\sigma}}{\sigma^2} \Psi(x) = 0, \quad (\text{A4})$$

with

$$\tilde{\sigma} = 4(\omega R)^2 x(1-x) - 4l(l+1)(1-x), \quad (\text{A5})$$

$$\sigma = 4x(1-x), \quad (\text{A6})$$

$$\tilde{\tau} = 2(1-2x). \quad (\text{A7})$$

To put this in a more standard form, one changes wave function by defining

$$\Psi(x) = \sqrt{x-1} x^{(l+1)/2} Z(x), \quad (\text{A8})$$

and one gets the following standard hypergeometric differential equation for Z :

$$\sigma \frac{\partial^2 Z(x)}{\partial x^2} + \tau \frac{\partial Z(x)}{\partial x} + \lambda Z(x) = 0, \quad (\text{A9})$$

with σ defined in Eq. (A6) and

$$\tau = 6 - 4l(x-1) - 12x, \quad (\text{A10})$$

$$\lambda = -4 - 4l - l^2 + \omega^2. \quad (\text{A11})$$

By requiring well behaved fields everywhere a simple analysis [25] then shows that the following constraint needs to be satisfied:

$$\omega R = 2n + l + 2. \quad (\text{A12})$$

These are the pure AdS frequencies for electromagnetic and gravitational perturbations, corresponding to pure AdS normal modes of the corresponding fields. One can compare the frequencies in Eq. (A12) with the scalar frequencies corresponding to pure AdS modes [21], $\omega_s R = 2n + l + 3$.

-
- [1] T. Regge and J.A. Wheeler, *Phys. Rev.* **108**, 1063 (1957).
 [2] S. Chandrasekhar and S. Detweiler, *Proc. R. Soc. London* **344**, 441 (1975).
 [3] K.D. Kokkotas and B.G. Schmidt, *Living Rev. Relativ.* **2**, 2 (1999); H.P. Nollert, *Class. Quantum Grav.* **16**, R159 (2000).
 [4] S. Hod, *Phys. Rev. Lett.* **81**, 4293 (1998); O. Dreyer, *ibid.* **90**, 081301 (2003); G. Kunstatter, *ibid.* **90**, 161301 (2003); A. Corichi, *Phys. Rev. D* **67**, 087502 (2003); L. Motl, gr-qc/0212096; L. Motl and A. Neitzke, hep-th/0301173; A. Maassen van den Brink, gr-qc/0303095.
 [5] J. Baez, gr-qc/0303027.
 [6] E. Berti and K.D. Kokkotas, hep-th/0303029; R.A. Konoplya, *Phys. Rev. D* **68**, 024018 (2003).
 [7] R.A. Konoplya, *Phys. Lett. B* **550**, 117 (2002); V. Ferrari, M. Pauri, and F. Piazza, *Phys. Rev. D* **63**, 064009 (2001); R.A. Konoplya, *Gen. Relativ. Gravit.* **34**, 329 (2002).
 [8] F. Mellor and I.G. Moss, *Phys. Rev. D* **41**, 403 (1990); I.G. Moss and J.P. Norman, *Class. Quantum Grav.* **19**, 2323 (2002).
 [9] V. Cardoso and J.P.S. Lemos, *Phys. Rev. D* **67**, 084020 (2003).
 [10] C. Molina, *Phys. Rev. D* (to be published), gr-qc/0304053.
 [11] A. Maassen van den Brink, gr-qc/0304092.
 [12] V. Suneeta, *Phys. Rev. D* **68**, 024020 (2003).
 [13] J. Maldacena, *Adv. Theor. Math. Phys.* **2**, 253 (1998).
 [14] J.S.F. Chan and R.B. Mann, *Phys. Rev. D* **55**, 7546 (1997); V. Cardoso and J.P.S. Lemos, *ibid.* **63**, 124015 (2001); D. Birmingham, I. Sachs, and S.N. Solodukhin, *Phys. Rev. Lett.* **88**, 151301 (2002); *Phys. Rev. D* **67**, 104026 (2003); B. Wang, C. Molina, and E. Abdalla, *ibid.* **63**, 084001 (2001); W.T. Kim and J.J. Oh, *Phys. Lett. B* **514**, 155 (2001); R.A. Konoplya, *Phys. Rev. D* **66**, 084007 (2002); A.O. Starinets, *ibid.* **66**, 124013 (2002); Y. Kurita and M. Sakagami, *ibid.* **67**, 024003 (2003); R. Aros, C. Martinez, R. Troncoso, and J. Zanelli, *ibid.* **67**, 044014 (2003); D.T. Son and A.O. Starinets, *J. High Energy Phys.* **09**, 042 (2002); V. Cardoso, O.J.C. Dias, and J.P.S. Lemos, *Phys. Rev. D* **67**, 064026 (2003); S. Musiri and G. Siopsis, *Phys. Lett. B* **563**, 102 (2003).
 [15] G.T. Horowitz and V.E. Hubeny, *Phys. Rev. D* **62**, 024027 (2000).
 [16] R.A. Konoplya, *Phys. Rev. D* **66**, 044009 (2002).
 [17] V. Cardoso and J.P.S. Lemos, *Phys. Rev. D* **64**, 084017 (2001).
 [18] E. Berti and K.D. Kokkotas, *Phys. Rev. D* **67**, 064020 (2003).
 [19] J.P.S. Lemos, *Class. Quantum Grav.* **12**, 1081 (1995); *Phys. Lett. B* **353**, 46 (1995); J.P.S. Lemos and V.T. Zanchin, *Phys. Rev. D* **54**, 3840 (1996); see also the review paper: J.P.S. Lemos, "Recent developments in astronomy and astrophysics," *Proceedings of the 10th Portuguese Meeting on Astronomy and Astrophysics*, edited by J.P.S. Lemos *et al.* (World Scientific, Singapore, 2001).
 [20] V. Cardoso and J.P.S. Lemos, *Class. Quantum Grav.* **18**, 5257 (2001).
 [21] C.P. Burgess and C.A. Lutken, *Phys. Lett.* **153B**, 137 (1985); see also I.I. Cotaescu, *Phys. Rev. D* **60**, 107504 (1999).

- [22] S.R. Das, G. Gibbons, and S.D. Mathur, Phys. Rev. Lett. **78**, 417 (1997).
- [23] J.A.H. Futterman, F.A. Handler, and R.A. Matzner, *Scattering from Black Holes* (Cambridge University Press, Cambridge, England, 1988); N. Sanchez, hep-th/9711068.
- [24] F. Cooper, A. Khare, and U. Sukhatme, Phys. Rep. **251**, 267 (1995).
- [25] A.F. Nikiforov and V.B. Uvarov, *Special Functions of Mathematical Physics* (Birkhäuser, Boston, 1988).