Multi-black-hole solutions in five dimensions

H. S. Tan and Edward Teo

Department of Physics, National University of Singapore, Singapore 119260
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Using a recently developed generalized Weyl formalism, we construct an asymptotically flat, static vacuum Einstein solution that describes a superposition of multiple five-dimensional Schwarzschild black holes. The spacetime exhibits a $U(1) \times U(1)$ rotational symmetry. It is argued that for certain choices of parameters the black holes are collinear and so may be regarded as a five-dimensional generalization of the Israel-Khan solution. The black holes are kept in equilibrium by membranelike conical singularities along the two rotational axes; however, they still distort one another by their mutual gravitational attraction. We also generalize this solution to one describing multiple charged black holes, with fixed mass-to-charge ratio, in Einstein-Maxwell-dilaton theory.

I. INTRODUCTION

The Israel-Khan solution describing multiple collinear Schwarzschild black holes in four dimensions has been known for some time [1]. It belongs to a class of static, axisymmetric solutions first obtained by Weyl [2], who showed that the corresponding vacuum Einstein equations can be reduced to solving the Laplace equation in three-dimensional flat space. Although the Israel-Khan solution contains stringlike conical singularities, it nonetheless has a well-defined gravitational action [3], and this enables one to study their interactions using standard techniques of gravitational thermodynamics [4].

Recently, Emparan and Reall [5] generalized the Weyl formalism to arbitrary dimensions $D \geq 4$. They showed that the general solution of the D-dimensional vacuum Einstein equations which has $D - 2$ orthogonal commuting isometries is specified by $D - 3$ axisymmetric solutions of the Laplace equation in three-dimensional flat space. A way to classify these solutions was also presented in [5]. In particular, the five-dimensional (5D) Schwarzschild black hole, like its four-dimensional counterpart, belongs to the generalized Weyl class. Another noteworthy member of this class is the 5D black ring solution [5], which is the first example of an asymptotically flat vacuum spacetime with an event horizon of nonspherical topology.

The generalized Weyl formalism opens up, for the first time, the possibility of obtaining multi-Schwarzschild-black-hole solutions in five dimensions. Three possible configurations were briefly discussed in [5]. First, a two-black-hole solution was constructed, with the black holes located at the north and south poles of a Kaluza-Klein bubble.[1] This solution is not asymptotically flat, since one coordinate is asymptotically a Kaluza-Klein circle. The second solution considered in [5] was a three-black-hole solution that is asymptotically flat. However, it is not a collinear system, since the central black hole is only collinear with each of the other two black holes along *different* axes. Finally, a solution describing an infinite periodic array of black holes was also

considered, although it was argued that this solution cannot be interpreted as a black hole localized on a Kaluza-Klein circle because it does not have the correct asymptotic structure.

It would be very interesting to find a multi-black-hole solution in five dimensions that may be considered a generalization of the four-dimensional Israel-Khan solution. Such a solution should satisfy two conditions. First, it should be asymptotically flat, rather than asymptotically Kaluza-Klein as in the first solution mentioned above. Second, it should describe a ''collinear'' array of black holes, which rules out the second solution. Such a notion of collinearity would have to be compatible with the spatial symmetries imposed on 5D Weyl solutions, i.e., $U(1) \times U(1)$ corresponding to the two orthogonal commuting rotational Killing vectors.

In this paper, we construct a multi-Schwarzschild-black-hole solution (which is actually the finite version of the infinite-black-hole solution considered in [5]) which, we argue, satisfies the above two conditions and thus qualifies as a 5D analogue of the Israel-Khan solution. We begin in Sec. II with the explicit construction of our solution and a characterization of the conditions under which it may be considered a collinear array of black holes. In Sec. III, we examine some basic properties of our solution for the case of a two-black-hole system. In particular, we perform various limiting procedures and study the near-horizon geometries of the black holes, which turn out to be distorted by their mutual gravitational attraction. In Sec. IV, the analysis is briefly repeated for the three-black-hole system. In Sec. V, the charged version of our solution in the framework of Einstein-Maxwell-dilaton theory is obtained and studied. The extremal limit is then examined in Sec. VI. The paper ends with a discussion of some possible extensions of this work.

II. THE MULTI-SCHWARZSCHILD-BLACK-HOLE SOLUTION

In this paper, we are specifically interested in 5D static spacetimes belonging to the generalized Weyl class, i.e., possessing three orthogonal commuting Killing vectors. Such a spacetime metric can be written as

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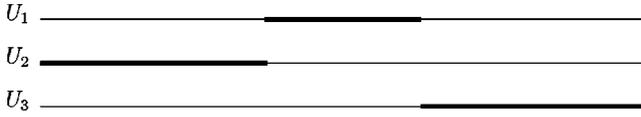[1] A detailed study of this solution recently appeared in [6].

FIG. 1. Rod structure of the 5D Schwarzschild-black-hole solution.

$$ds^2 = -e^{2U_1}dt^2 + e^{2U_2}d\varphi^2 + e^{2U_3}d\psi^2 + e^{2\nu}(dr^2 + dz^2),$$
(2.1)

where $\nu = \nu(r,z)$ and $U_\alpha = U_\alpha(r,z)$ for $\alpha = 1,2,3$. The vacuum Einstein equations can be shown to reduce to the Laplace equation:

$$\frac{\partial^2 U_\alpha}{\partial r^2} + \frac{1}{r}\frac{\partial U_\alpha}{\partial r} + \frac{\partial^2 U_\alpha}{\partial z^2} = 0,$$
(2.2)

with ν determined by quadratures [5]. The three harmonic functions U_α can be thought of as the Newtonian potentials produced by rods of zero thickness and density 1/2 along the z axis. They should add up to the potential of an infinite rod. An important example of a spacetime in this class is the 5D Schwarzschild solution, which has the potentials of the rod structure shown in Fig. 1.

As described in [5], certain important properties of a generalized Weyl spacetime can be read off from its corresponding rod structure, even without the explicit form of the metric. For example, rod sources for the two angular coordinates φ and ψ correspond to fixed points of these rotations, i.e., the symmetry axes. If the rod sources for the φ and ψ coordinates extend to infinity in either direction, then the spacetime is asymptotically flat. Another important fact is that a finite rod source for the time coordinate corresponds to an event horizon in the spacetime. Moreover, if either end of this rod continues with rods of *different* angular coordinates, then the event horizon will have S^3 topology. This means that the rod structure in Fig. 1 describes a black hole in an asymptotically flat spacetime, in agreement with the physical interpretation of the 5D Schwarzschild solution.

Bearing these facts in mind, let us now attempt to draw a rod structure corresponding to a superposition of N Schwarzschild black holes that generalizes the Israel-Khan

FIG. 2. Rod structure of an N-Schwarzschild-black-hole solution.

solution. We first note that it must have N finite rods for the time coordinate, corresponding to N disconnected event horizons. Furthermore, to ensure that each horizon has S^3 topology, the ends of each rod must continue with rods of different angular coordinates. Finally, the rod structure should have the same asymptotic form as that in Fig. 1, to ensure asymptotic flatness. These conditions leave us with the rod structure in Fig. 2 as one of the simplest possibilities. Of course, other rod structures are also possible; however, we do not consider them to be as natural or compelling as the one chosen above.

It is straightforward to write down the U_α's corresponding to the rod structure in Fig. 2. If we label the locations of the rod ends in order of increasing z (left to right in Fig. 2) by $a_1, a_2, \ldots, a_{3N-1}$, then we have

$$U_1 = \frac{1}{2}\sum_{k=1}^{N} \log\frac{R_{3k-2} - \zeta_{3k-2}}{R_{3k-1} - \zeta_{3k-1}},$$
(2.3a)

$$U_2 = \frac{1}{2}\sum_{k=2}^{N} \log\frac{R_{3k-3} - \zeta_{3k-3}}{R_{3k-2} - \zeta_{3k-2}} + \frac{1}{2}\log(R_1 + \zeta_1),$$
(2.3b)

$$U_3 = \frac{1}{2}\sum_{k=2}^{N} \log\frac{R_{3k-4} - \zeta_{3k-4}}{R_{3k-3} - \zeta_{3k-3}} + \frac{1}{2}\log(R_{3N-1} - \zeta_{3N-1}),$$
(2.3c)

where $R_i \equiv \sqrt{r^2 + \zeta_i^2}$ and $\zeta_i \equiv z - a_i$, for $1 \le i \le 3N-1$. Using the method described in [5], we can then solve for ν. After some calculation, we obtain the line element for this rod structure as

$$ds^2 = -\prod_{k=1}^{N}\left(\frac{R_{3k-2} - \zeta_{3k-2}}{R_{3k-1} - \zeta_{3k-1}}\right)dt^2 + (R_1 + \zeta_1)\prod_{k=2}^{N}\left(\frac{R_{3k-3} - \zeta_{3k-3}}{R_{3k-2} - \zeta_{3k-2}}\right)d\varphi^2 + (R_{3N-1} - \zeta_{3N-1})\prod_{k=2}^{N}\left(\frac{R_{3k-4} - \zeta_{3k-4}}{R_{3k-3} - \zeta_{3k-3}}\right)d\psi^2$$

$$+ e^{2\gamma_0}\frac{\sqrt{Y_{1,3N-1}(R_{3N-1} - \zeta_{3N-1})}}{R_1 R_2 \cdots R_{3N-1}\sqrt{R_1 - \zeta_1}}\prod_{k=2}^{N}\frac{\sqrt{Y_{3k-2,3N-1}Y_{3k-3,3N-1}Y_{1,3k-3}Y_{1,3k-4}Y_{3k-3,3k-2}Y_{3k-4,3k-3}Y_{3k-4,3k-2}}}{Y_{3k-4,3N-1}Y_{1,3k-2}}$$

$$\times \prod_{1<k<j}\frac{\sqrt{Y_{3k-3,3j-2}Y_{3k-2,3j-3}Y_{3k-4,3j-3}Y_{3k-3,3j-4}Y_{3k-2,3j-4}Y_{3k-4,3j-2}}}{Y_{3k-2,3j-2}Y_{3k-3,3j-3}Y_{3k-4,3j-4}}\prod_{k=2}^{N}\frac{\sqrt{R_{3k-4} - \zeta_{3k-4}}}{\sqrt{R_{3k-2} - \zeta_{3k-2}}}(dr^2 + dz^2),$$
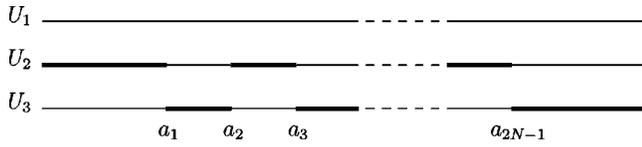(2.4)

FIG. 3. Rod structure of the background spacetime of Eq. (2.4).

where $Y_{ij} \equiv R_i R_j + \zeta_i \zeta_j + r^2$, and $e^{2\gamma_0}$ is a constant to be adjusted appropriately below. Note that this solution contains $3N-2$ free parameters, with N parameters related to the individual masses of the black holes and the rest determining their spatial arrangement.

Before embarking on a study of their spatial arrangement, let us consider the background spacetime limit of Eq. (2.4) in which all the black holes disappear. This corresponds to shrinking the rods for the time coordinate down to zero size, leaving just the rods for the φ and ψ coordinates as in Fig. 3. It can be seen that the resulting rod structure corresponds to the Euclidean version of the multiple C metric solution derived in [7] (in this case describing $N-1$ accelerating black holes), with a flat time direction added on. This spacetime is clearly nonflat when $N \geq 2$. In addition to the usual two semi-infinite rotational axes for the φ and ψ coordinates, it contains $N-1$ finite-length rotational axes for each coordinate. Observe that these axes are not one-dimensional lines, but rather two-dimensional membranes. While the semi-infinite axes have the topology of open disks D^2, the finite ones have the topology of spheres S^2. Furthermore, from the behavior of the multiple C metric, we know that there are in general conical singularities running along the axes. If we demand the two semi-infinite axes to be regular, then there are unavoidable conical singularities along the finite axes. Thus, this spacetime consists of $2(N-1)$ conical membranes with S^2 topology. They are orthogonal to one another, in the sense that any line of constant longitude of one S^2 is orthogonal to any line of constant longitude of an adjacent S^2 at their adjoining point.

At first, it may seem rather strange to have such a nontrivial spacetime as the background. However, this is basically forced upon us if we want to construct multiple-black-hole solutions within the generalized Weyl formalism, and can be seen as follows. A black hole with an event horizon of S^3 topology can be introduced only at points along the z axis where the rods for the φ and ψ coordinates meet, i.e., fixed points of the $U(1) \times U(1)$ rotational symmetry. There can only be one such point in 5D Minkowski space [5]. If we require two or more fixed points, then the only possible asymptotically flat background, with a flat time direction, is the Euclidean multiple C metric solution as described above. A solution corresponding to $N-1$ accelerating black holes has $2N-1$ such fixed points, although only N of them were used in the construction of the solution (2.4). These points are labeled $a_1, a_3, \ldots, a_{2N-1}$ in Fig. 3.

The reason for choosing these N points alternately is because any three adjacent fixed points cannot be collinear. For example, a_1 and a_2 are collinear along the ψ axis (with an S^2 conical membrane connecting them), while a_2 and a_3 are collinear along the φ axis (with another S^2 conical membrane connecting them). So the middle fixed point is collin-

ear with each of the other two points, but along different axes. This is precisely the reason why the three-black-hole system considered in [5] is not collinear. On the other hand, consider the three alternate fixed points a_1, a_3, and a_5. The points a_1 and a_3 are joined up along the z axis by a finite ψ axis and a finite φ axis (corresponding to two orthogonal S^2's); a similar situation occurs between a_3 and a_5. Suppose we measure the distances between these three points by coordinate displacements along the two angular axes, with $a_{ij} \equiv |a_i - a_j|$. Then we shall refer to the three points as being "collinear" if the ratios a_{12}/a_{23} and a_{34}/a_{45} are equal. In other words, one has to cover the same ratio of distances along the two orthogonal directions defined by the φ and ψ axes, in moving between collinear points.

This notion of collinearity is naturally compatible with the $U(1) \times U(1)$ generalized Weyl symmetry of the spacetime. It turns out there is another notion of collinearity in five dimensions that was alluded to in [5,6]: if a spacetime possesses an $SO(3)$ spatial isometry, then points on the symmetry axis can be regarded as collinear. [Both these notions actually coincide in four dimensions, since $SO(D-2) \cong U(1)^{D-3}$ when $D=4$.] However, since spacetimes in the generalized Weyl class will not possess $SO(3)$ symmetry in general, the latter notion of collinearity cannot be applied here. It should also be pointed out that our proposed notion of collinearity may not be the only possible one compatible with $U(1) \times U(1)$ symmetry, but it is certainly one of the simplest. Furthermore, as we shall see below, it passes a certain consistency check.[2]

It is now a straightforward matter to reintroduce the black holes into the background spacetime and extend our notion of collinearity to them. Stretching between any two adjacent black holes are now two orthogonal topological disks, with each D^2 terminating on a black hole event horizon. Otherwise, the picture is similar to that above. For definiteness, consider the first three black holes from the left in Fig. 2. Now, a_{23} is the coordinate distance from the horizon of the first black hole to the fixed point a_3 along the ψ axis, while a_{34} is the coordinate distance from a_3 to the horizon of the second black hole along the φ axis. If the ratio a_{23}/a_{34} is the same as the corresponding one between the second and third black holes, namely, a_{56}/a_{67}, then we shall refer to these three black holes as being collinear. Note that we are defining collinearity of the black holes with respect to the positions of their event horizons, rather than their centers, for simplicity. If collinearity is to be defined with respect to the centers, then one needs to take into account the masses (and hence radii) of the black holes.

In the interest of generality, we shall continue to keep the parameters of our solution a_i arbitrary for most of this paper. If a collinear array of black holes is desired, then the parameters can be specifically chosen to satisfy the conditions described above.

III. THE TWO-BLACK-HOLE SOLUTION

Having obtained a 5D analogue of the Israel-Khan solution, the next step is to study some of its properties in detail.

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[2]See the discussion surrounding Eq. (3.4).

We begin by focusing on the two-black-hole case, for simplicity and also because many of the characteristic properties of the general solution are already present in this case. Setting $N=2$ in Eq. (2.4) yields the metric

$$ds^2 = -\frac{(R_1-\zeta_1)(R_4-\zeta_4)}{(R_2-\zeta_2)(R_5-\zeta_5)}\,dt^2 + \frac{(R_1+\zeta_1)(R_3-\zeta_3)}{R_4-\zeta_4}\,d\varphi^2 + \frac{(R_2-\zeta_2)(R_5-\zeta_5)}{R_3-\zeta_3}\,d\psi^2$$
$$+ e^{2\gamma_0}\frac{\sqrt{(R_2-\zeta_2)(R_5-\zeta_5)Y_{15}Y_{45}Y_{13}Y_{34}Y_{23}Y_{35}Y_{24}Y_{12}}}{\sqrt{(R_1-\zeta_1)(R_4-\zeta_4)}R_1R_2R_3R_4R_5Y_{14}Y_{25}}(dr^2+dz^2), \tag{3.1}$$

which has rod structure and parameters as in Fig. 4.

Let us first check the regularity conditions for the spatial sections of Eq. (3.1). Consider the $ds^2_{\varphi r}$ part of the metric. It turns out that conical singularities cannot be avoided along the φ axis, and must be present at least along either the "inner" part $a_3 < z < a_4$ or the "outer" part $z < a_1$. By choosing $e^{2\gamma_0} = 1/8$ and the period of φ to be 2π, we have a regular outer axis and a conical singularity running along the inner one. A similar situation applies to the $ds^2_{\psi r}$ part of the metric. Explicitly, we find the conical excesses

$$\delta_\varphi = 2\pi\left(\frac{a_{14}a_{25}}{\sqrt{a_{15}a_{34}a_{35}a_{24}}} - 1\right) \quad \text{for } a_3 < z < a_4, \tag{3.2a}$$

$$\delta_\psi = 2\pi\left(\frac{a_{14}a_{25}}{\sqrt{a_{15}a_{13}a_{23}a_{24}}} - 1\right) \quad \text{for } a_2 < z < a_3, \tag{3.2b}$$

where $a_{ij} \equiv |a_i - a_j|$ denotes the coordinate distance between a_i and a_j along the z axis. That conical excesses, or struts, have appeared between the black holes agrees with our physical intuition: they provide the pressure necessary to counterbalance the gravitational attraction of the black holes and achieve a static configuration. This is analogous to the 4D case [4], but with one important difference: the struts are now extended in two spatial dimensions, and are therefore membranes. They have the topology of disks, as described in Sec. II, with their boundary circles wrapping around the black hole event horizons.

It was also pointed out in Sec. II that conical singularities remain in the background spacetime even when the black holes are removed with the choice $a_1 = a_2$ and $a_4 = a_5$. In this case, the result is just the Euclidean C metric solution with an added flat direction. The conical excesses along the inner axes are now

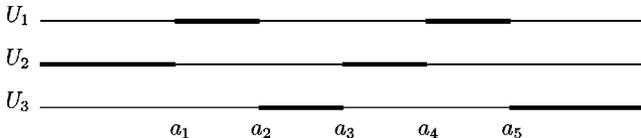$$\delta_\varphi = 2\pi\frac{a_{23}}{a_{34}} \quad \text{for } a_3 < z < a_4, \tag{3.3a}$$

$$\delta_\psi = 2\pi\frac{a_{34}}{a_{23}} \quad \text{for } a_2 < z < a_3. \tag{3.3b}$$

We proceed to show in detail that our solution really consists of a superposition of two Schwarzschild black holes. Suppose we center ourselves on one black hole, say the one on the left, and push the other infinitely far away. Note that there is an ambiguity in this procedure, since there are two possible directions in which this black hole can be pushed. We shall therefore demand that it be pushed to infinity in such a way that *it remains collinear with the original system*. In view of our notion of collinearity defined in Sec. II, this means we should take the limit $a_3 \to \infty$ while preserving the ratio

$$l \equiv \frac{a_{34}}{a_{23}}. \tag{3.4}$$

After taking this limit and performing the coordinate transformation

$$r = \frac{1}{2}\sqrt{1 - \frac{2a_{12}}{(1+l)R^2}}\,(1+l)R^2\sin 2\theta, \tag{3.5a}$$

$$z = -\frac{1}{2}\left(1 - \frac{a_{12}}{(1+l)R^2}\right)(1+l)R^2\cos 2\theta, \tag{3.5b}$$

we recover the metric

$$ds^2 = -\left(1 - \frac{2a_{12}}{(1+l)R^2}\right)dt^2 + \left(1 - \frac{2a_{12}}{(1+l)R^2}\right)^{-1}dR^2$$
$$+ R^2 d\theta^2 + R^2\sin^2\theta d\varphi^2 + (1+l)^2 R^2\cos^2\theta d\psi^2. \tag{3.6}$$

This is just the 5D Schwarzschild black hole, but there is a conical singularity, with excess angle $2\pi l$, attached to it and stretching to infinity along the ψ axis. It can be seen that this conical singularity is an artifact of the background spacetime, since the latter *has* a conical singularity with exactly the same excess angle given by Eq. (3.3b). This is a good consistency check, and it shows that we have taken the infinite-distance limit correctly. Now the presence of the conical singularity will affect the calculation of the Arnowitt-Deser-Misner (ADM) mass of this black hole [8], since

U_1

U_2

U_3

$\quad\quad a_1 \quad\quad a_2 \quad\quad a_3 \quad\quad a_4 \quad\quad a_5$

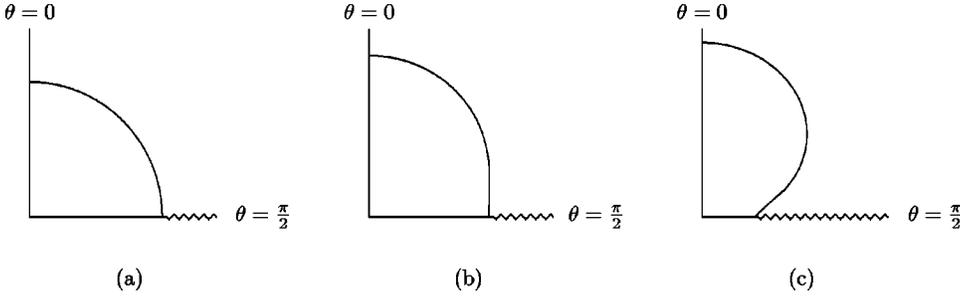FIG. 4. Rod structure of the two-black-hole solution.

FIG. 5. The horizon of the left black hole as represented by the quarter circle $0 \leq \theta \leq \pi/2$, with proper radius $\sqrt{g_{\theta\theta}}$, for (a) $w = 5$, (b) $w = 0.5$, and (c) $w = 0.05$. The jagged lines denote the conical singularities stretching between the two black holes.

spacetime is no longer asymptotically flat. Following the procedure of [8], we calculate its mass to be $(3/4)\pi a_{12}$.

In a similar fashion, we can center ourselves on the right black hole and push the left one to infinity. In doing so, we recover the limiting metric

$$ds^2 = -\left(1 - \frac{2la_{45}}{(1+l)R^2}\right)dt^2 + \left(1 - \frac{2la_{45}}{(1+l)R^2}\right)^{-1}dR^2$$

$$+ R^2 d\theta^2 + \left(\frac{1+l}{l}\right)^2 R^2\sin^2\theta\, d\varphi^2 + R^2\cos^2\theta\, d\psi^2.$$

(3.7)

Again we obtain a Schwarzschild black hole, with a conical singularity now stretching to infinity along the φ axis. It has excess angle $2\pi l^{-1}$, in agreement with Eq. (3.3a). The mass of this black hole can be calculated to be $(3/4)\pi a_{45}$. The sum of the masses of the two individual black holes is therefore

$$M = \frac{3}{4}\pi(a_{12} + a_{45}),$$

(3.8)

and it turns out to be equal to the calculated ADM mass of the full solution (3.1). This is to be expected since the interaction energy between the black holes (determined by the conical singularities [4]) vanishes in the infinite separation limit, and so the total energy of the system is just the sum of the masses of the separate black holes.

We shall now show that our solution describes two Schwarzschild black holes even if the distance between them is kept finite. This involves taking the near-horizon limit of each black hole. Let us focus on the left black hole. If we perform the coordinate transformation

$$r = \frac{1}{2}\sqrt{1 - \frac{2a_{12}}{R^2}}\, R^2\sin 2\theta,$$

(3.9a)

$$z = -\frac{1}{2}\left(1 - \frac{a_{12}}{R^2}\right)R^2\cos 2\theta,$$

(3.9b)

and then expand Eq. (3.1) near $R = \sqrt{2a_{12}}$, we obtain

$$ds^2 = f_1^2(\theta)\left\{-\left(1 - \frac{2a_{12}}{R^2}\right)dt^2 + \frac{a_{15}a_{13}}{a_{14}^2}\left[\left(1 - \frac{2a_{12}}{R^2}\right)^{-1}dR^2\right.\right.$$

$$\left.\left. + R^2 d\theta^2\right]\right\} + f_2^2(\theta)R^2\sin^2\theta\, d\varphi^2 + f_3^2(\theta)R^2\cos^2\theta d\psi^2,$$

(3.10)

where

$$f_1^2(\theta) \equiv \frac{a_{12}\cos^2\theta + a_{24}}{a_{12}\cos^2\theta + a_{25}},$$

(3.11a)

$$f_2^2(\theta) \equiv \frac{a_{12}\cos^2\theta + a_{23}}{a_{12}\cos^2\theta + a_{24}},$$

(3.11b)

$$f_3^2(\theta) \equiv \frac{a_{12}\cos^2\theta + a_{25}}{a_{12}\cos^2\theta + a_{23}}.$$

(3.11c)

The metric (3.10) describes the near-horizon geometry of a Schwarzschild black hole, albeit distorted away from spherical symmetry. This angular distortion is encoded by the three so-called distortion factors f_1, f_2, and f_3, and can be attributed to the gravitational pull of the other black hole. It is only when the latter is pushed to infinity that the distortion factors disappear.

One can similarly analyze the right black hole and find its near-horizon geometry to be given by the above expressions upon switching $\varphi \leftrightarrow \psi$, $\theta \to \pi/2 - \theta$, $a_{12} \to a_{45}$, $a_{23} \to a_{34}$, etc. For simplicity, let us assume here that $a_{12} = a_{45}$ and $a_{23} = a_{34}$, corresponding to a left-right symmetric system. We shall take a $t = R = \varphi = \psi = $ const slice of the near-horizon metric to see how the quarter circle $0 \leq \theta \leq \pi/2$, with proper radius $\sqrt{g_{\theta\theta}}$, of each black hole is affected by the other. For the left black hole, we have

$$g_{\theta\theta} \propto \frac{\cos^2\theta + w}{1 + \cos^2\theta + w},$$

(3.12)

where $w \equiv a_{24}/a_{12}$ is a parameter related to the coordinate separation-to-mass ratio of the two-black-hole system. It is readily seen (see Fig. 5) that when w is large we approach perfect quarter circles for both black holes. As w decreases, the two quarter circles start to deviate from circular symme-

try. In the limit $w \to 0$, the quarter circles pinch off along the axes joining them. This behavior is rather similar to that in the Israel-Khan solution [4].

Let us now turn briefly to some other properties of our two-black-hole solution. From the near-horizon metric of each black hole, one can calculate the three-areas of the event horizons to be

$$\mathcal{A}_{\text{left}} = \sqrt{2}(2\pi)^2 \frac{a_{12}\sqrt{a_{12}a_{13}a_{15}}}{a_{14}}, \qquad (3.13a)$$

$$\mathcal{A}_{\text{right}} = \sqrt{2}(2\pi)^2 \frac{a_{45}\sqrt{a_{45}a_{35}a_{15}}}{a_{25}}. \qquad (3.13b)$$

To calculate the Hawking temperature associated with each event horizon, it is convenient to Euclideanize our solution $t \to -i\tau$. The natural period of τ is then the inverse Hawking temperature. We obtain

$$T_{\text{left}} = \frac{1}{\sqrt{2}2\pi} \frac{a_{14}}{\sqrt{a_{15}a_{13}a_{12}}}, \qquad (3.14a)$$

$$T_{\text{right}} = \frac{1}{\sqrt{2}2\pi} \frac{a_{25}}{\sqrt{a_{15}a_{45}a_{35}}}. \qquad (3.14b)$$

A question then arises if the two black holes can be in thermodynamic equilibrium for some choice of parameters. In the 4D case, it was shown in [4] that the two black holes have to be of the same mass for the system to be in thermal equilibrium. To examine our solution likewise, we equate the expressions for the two temperatures to find

$$(a_{14}^2 - a_{13}a_{12})a_{45}^2 + (a_{14}^2 a_{34} - 2a_{13}a_{12}a_{24})a_{45} - a_{13}a_{12}a_{24}^2 = 0. \qquad (3.15)$$

It can be shown that Eq. (3.15) has positive solutions for a_{45} for any a_{12}, a_{23}, and a_{34}. In particular, we have the solution $a_{12} = a_{45}$ and $a_{23} = a_{34}$. Thus, we conclude that in our solution, the two black holes need not be of the same mass for them to have the same temperature. If they do, then the two finite rotational axes between them must be of the same length.

Finally we observe a Smarr relation for either black hole of our solution, consistent with that in [9]:

$$M_j = \frac{3}{2}T_j\left(\frac{\mathcal{A}_j}{4}\right), \qquad (3.16)$$

where M_j is the mass of the black hole. In its differential form, the Smarr relation may be identified with the first law of black hole thermodynamics, with $\mathcal{A}_j/4$ the entropy of the black hole.

IV. THE THREE-BLACK-HOLE SOLUTION

The techniques used in the preceding section to analyze the $N=2$ case of Eq. (2.4) can be straightforwardly extended to any other $N>2$. However, the various calculations will get much more tedious. In this section, we shall briefly study the $N=3$ solution, concentrating on the central black hole in this system as it exhibits some features not present in the $N=2$ case.

For a three-black-hole system, the metric (2.4) reduces to

$$ds^2 = -\frac{(R_1 - \zeta_1)(R_4 - \zeta_4)(R_7 - \zeta_7)}{(R_2 - \zeta_2)(R_5 - \zeta_5)(R_8 - \zeta_8)}dt^2 + (R_1 + \zeta_1)\frac{(R_3 - \zeta_3)(R_6 - \zeta_6)}{(R_4 - \zeta_4)(R_7 - \zeta_7)}d\varphi^2 + \frac{(R_2 - \zeta_2)(R_5 - \zeta_5)}{(R_3 - \zeta_3)(R_6 - \zeta_6)}(R_8 - \zeta_8)d\psi^2$$

$$+ \frac{1}{16\sqrt{2}} \frac{\sqrt{(R_2 - \zeta_2)(R_5 - \zeta_5)(R_8 - \zeta_8)}}{\sqrt{(R_1 - \zeta_1)(R_4 - \zeta_4)(R_7 - \zeta_7)}}$$

$$\times \frac{\sqrt{Y_{37}Y_{46}Y_{26}Y_{35}Y_{45}Y_{27}Y_{18}Y_{48}Y_{38}Y_{13}Y_{12}Y_{34}Y_{23}Y_{24}Y_{78}Y_{68}Y_{16}Y_{15}Y_{67}Y_{56}Y_{57}}}{R_1 R_2 \cdots R_8 Y_{28}Y_{14}Y_{58}Y_{17}Y_{47}Y_{36}Y_{25}} (dr^2 + dz^2), \qquad (4.1)$$

where the rod parameters are defined in Fig. 6, and the factor $1/(16\sqrt{2})$ has been chosen to make the outer rotational axes $z < a_1$ and $z > a_8$ regular. As usual, there are conical excesses resulting along the finite inner axes, but these cannot be removed by any choice of parameters. They are necessary to hold the system in static equilibrium.

We can show that Eq. (4.1) indeed describes a three-black-hole configuration by performing the same limiting procedures as in the two-black-hole case. In particular, to recover the central black hole, we center our coordinates on

it and push the other two black holes to infinity in a collinear fashion. This is done by taking the limit $a_6 \to \infty$ such that the three ratios

$$l_1 \equiv \frac{a_{56}}{a_{67}}, \quad l_2 \equiv \frac{a_{34}}{a_{67}}, \quad l_3 \equiv \frac{a_{24}}{a_{67}} \qquad (4.2)$$

remain fixed. After a coordinate transformation analogous to Eq. (3.5), the metric becomes
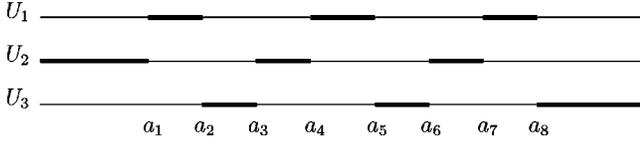
FIG. 6. Rod structure of the three-black-hole solution.

$$ds^2 = -\left(1 - \frac{2a_{45}}{\lambda_1\lambda_2 R^2}\right)dt^2 + \left(1 - \frac{2a_{45}}{\lambda_1\lambda_2 R^2}\right)^{-1}dR^2 + R^2 d\theta^2$$

$$+ \lambda_1^2 R^2 \sin^2\theta\, d\varphi^2 + \lambda_2^2 R^2 \cos^2\theta\, d\psi^2, \tag{4.3}$$

where

$$\lambda_1 \equiv \frac{l_3(l_1+l_2)(1+l_1+l_3)}{l_2(1+l_1+l_2)(l_1+l_3)},$$

$$\lambda_2 \equiv \frac{(1+l_1)(l_1+l_2)(1+l_1+l_3)}{l_1(1+l_1+l_2)(l_1+l_3)}. \tag{4.4}$$

Thus we recover the Schwarzschild black hole, but with conical singularities attached to it along two different directions. The calculated values of the conical excesses coincide with those of the corresponding Euclidean multiple C metric background.

To study the central black hole more carefully, let us consider its near-horizon geometry. As in the two-black-hole case, we center ourselves on it and perform a coordinate transformation analogous to (3.9). After expanding the metric near $R = \sqrt{2a_{45}}$, we get

$$ds^2 = g_1^2(\theta)\left\{-\left(1 - \frac{2a_{45}}{R^2}\right)dt^2 + B\left[\left(1 - \frac{2a_{45}}{R^2}\right)^{-1}dR^2\right.\right.$$

$$\left.\left. + R^2 d\theta^2\right]\right\} + g_2^2(\theta)R^2\sin^2\theta\, d\varphi^2$$

$$+ g_3^2(\theta)R^2\cos^2\theta\, d\psi^2, \tag{4.5}$$

where

$$B \equiv \frac{a_{18}a_{48}a_{38}a_{16}a_{15}a_{37}a_{46}a_{26}a_{35}a_{27}}{(a_{28}a_{17}a_{47}a_{36}a_{25})^2} \tag{4.6}$$

is a rather complicated constant term. More interestingly, the angular distortion factors which describe how the central black hole is affected by the other two black holes are

$$g_1^2(\theta) \equiv \frac{(a_{45}\sin^2\theta + a_{24})(a_{45}\cos^2\theta + a_{57})}{(a_{45}\sin^2\theta + a_{14})(a_{45}\cos^2\theta + a_{58})}, \tag{4.7a}$$

$$g_2^2(\theta) \equiv \frac{(a_{45}\sin^2\theta + a_{14})(a_{45}\cos^2\theta + a_{56})}{(a_{45}\sin^2\theta + a_{34})(a_{45}\cos^2\theta + a_{57})}, \tag{4.7b}$$

$$g_3^2(\theta) \equiv \frac{(a_{45}\sin^2\theta + a_{34})(a_{45}\cos^2\theta + a_{58})}{(a_{45}\sin^2\theta + a_{24})(a_{45}\cos^2\theta + a_{56})}. \tag{4.7c}$$

For simplicity, we now assume that $a_{12} = a_{78}$, $a_{23} = a_{67}$, and $a_{34} = a_{56}$, corresponding to a left-right symmetric system. Again, we shall take a $t = R = \varphi = \psi = $ const slice of the metric (4.5) and observe how the quarter circle $0 \le \theta \le \pi/2$, with proper radius $\sqrt{g_{\theta\theta}}$, is affected by the other two black holes. We have

$$g_{\theta\theta} \propto \frac{(\sin^2\theta + w)(\cos^2\theta + w)}{(\sin^2\theta + w + v)(\cos^2\theta + w + v)}, \tag{4.8}$$

where $w \equiv a_{24}/a_{45}$ and $v \equiv a_{12}/a_{45}$. This function encodes the distortion of the central black hole's horizon along the θ direction by the other two black holes. We can see from it, even without the aid of graphical plots, the characteristic effects of the various physical parameters as follows. First, note that w represents the ratio of the coordinate distance between the left and central black holes to the mass of the latter. As it increases, the quarter circle tends more toward a circular arc. Second, v represents the ratio of the mass of the left black hole to that of the central one. As it increases, the quarter circle deviates more from circular symmetry. This general behavior conforms to our Newtonian expectations.

Finally, we briefly study the temperatures of the black holes in this solution. They are

$$T_{\text{left}} = \frac{1}{\sqrt{2}2\pi} \frac{a_{14}a_{17}}{\sqrt{a_{18}a_{13}a_{12}a_{15}a_{16}}}, \tag{4.9a}$$

$$T_{\text{right}} = \frac{1}{\sqrt{2}2\pi} \frac{a_{28}a_{58}}{\sqrt{a_{18}a_{48}a_{38}a_{78}a_{68}}}, \tag{4.9b}$$

$$T_{\text{central}} = \frac{1}{\sqrt{2}2\pi} \frac{a_{28}a_{17}a_{47}a_{36}a_{25}}{\sqrt{a_{18}a_{48}a_{38}a_{16}a_{15}a_{37}a_{46}a_{26}a_{35}a_{45}a_{27}}}. \tag{4.9c}$$

The question then arises if the three black holes can be in thermodynamic equilibrium for some choice of parameters. Indeed, there exist infinitely many solutions for such a scenario, a particular solution being $a_{67} = a_{56} = a_{34} = a_{23} = a_{78}/2 = a_{12}/2$, and $a_{45} \simeq 1.5886a_{67}$. In this case, the central black hole has to have a smaller mass than the other two black holes in order to achieve thermodynamic equilibrium.

V. MULTIPLE CHARGED BLACK HOLES

We shall now generalize our solution (2.4) to a system of multiple charged black holes in a general 5D Einstein-Maxwell-dilaton theory. This would be a first step toward embedding the solution in a more complete framework such as string or M theory, which may then provide some insights into the microscopic description of such a system.

We begin by finding the corresponding multi-black-hole solution in the special case of 5D Kaluza-Klein theory. This can be done using the standard procedure [10] of embedding the spacetime (2.4) in six dimensions by adding a flat extra dimension:

$$ds^2_{(6)} = -e^{2U_1}dt^2 + e^{2U_2}d\varphi^2 + e^{2U_3}d\psi^2 + e^{2\nu}(dr^2 + dz^2) + dy^2. \quad (5.1)$$

Boosting along the y direction with rapidity σ, the metric becomes

$$ds^2_{(6)} = -\frac{e^{2U_1}}{\cosh^2\sigma - e^{2U_1}\sinh^2\sigma}dt^2$$
$$+ e^{2U_2}d\varphi^2 + e^{2U_3}d\psi^2 + e^{2\nu}(dr^2 + dz^2)$$
$$+ (\cosh^2\sigma - e^{2U_1}\sinh^2\sigma)$$
$$\times \left(dy - \frac{(1-e^{2U_1})\sinh\sigma\cosh\sigma}{\cosh^2\sigma - e^{2U_1}\sinh^2\sigma}dt\right)^2. \quad (5.2)$$

If we dimensionally reduce on y using the ansatz

$$ds^2_{(6)} = e^{-1/\sqrt{6}\phi}ds^2_{(5)} + e^{\sqrt{3/2}\phi}(dy - 2A_a dx^a)^2, \quad (5.3)$$

then the 5D metric $ds^2_{(5)}$, Abelian gauge field A_a, and dilaton ϕ can be read off from Eq. (5.2). They describe an electrically charged multi-black-hole solution in 5D Kaluza-Klein theory with the action

$$I = \frac{1}{16\pi}\int d^5x \sqrt{-g}\left(R - \frac{1}{2}\partial_a\phi\partial^a\phi - e^{\sqrt{8/3}\phi}F_{ab}F^{ab}\right), \quad (5.4)$$

where $F_{ab} \equiv \partial_a A_b - \partial_b A_a$.

Now Eq. (5.4) belongs to a general class of Einstein-Maxwell-dilaton theories with the action

$$I = \frac{1}{16\pi}\int d^5x \sqrt{-g}\left(R - \frac{1}{2}\partial_a\phi\partial^a\phi - e^{\alpha\phi}F_{ab}F^{ab}\right), \quad (5.5)$$

where α is a constant parametrizing the coupling of the dilaton to the gauge field. In particular, the Einstein-Maxwell case is recovered when $\alpha = 0$. It is fairly straightforward to generalize our static multi-black-hole solution in Kaluza-Klein theory to that of Eq. (5.5). The general-α solution turns out to be

$$ds^2 = -H^{-2\beta/3}e^{2U_1}dt^2 + H^{\beta/3}[e^{2U_2}d\varphi^2 + e^{2U_3}d\psi^2 + e^{2\nu}(dr^2 + dz^2)], \quad (5.6a)$$

$$A_t = \frac{\sqrt{\beta}}{2}H^{-1}(1-e^{2U_1})\sinh\sigma\cosh\sigma, \quad e^{\phi} = H^{\beta\alpha/2}, \quad (5.6b)$$

where

$$\beta \equiv \frac{12}{4+3\alpha^2}, \quad H \equiv 1 + \sinh^2\sigma\ (1-e^{2U_1}), \quad (5.7)$$

and U_α and ν can be read off from Eqs. (2.3) and (2.4), respectively. The corresponding magnetically charged solution can be obtained by the usual electromagnetic duality transformation.

The rod structure of this solution is still given by Fig. 2. Its ADM mass and electric and scalar charge [11] can be calculated in terms of the masses and electric and scalar charges of the individual black holes, as follows:

$$M_{\text{total}} = \sum_{j=1}^{N} M_j = \frac{3\pi}{8}\left(1 + \frac{2\beta}{3}\sinh^2\sigma\right)\sum_{j=1}^{N}\mu_j, \quad (5.8)$$

$$Q_{\text{total}} = \sum_{j=1}^{N} Q_j = \pi^2\sqrt{\beta}\sinh(2\sigma)\sum_{j=1}^{N}\mu_j, \quad (5.9)$$

$$\Sigma_{\text{total}} = \sum_{j=1}^{N}\Sigma_j = 2\pi^2\beta\alpha\sinh^2\sigma\sum_{j=1}^{N}\mu_j. \quad (5.10)$$

Here we have labeled the jth black hole from the left, and set $\mu_j \equiv 2\,|a_{3j-2} - a_{3j-1}|$. Note that the individual black holes all have the same mass-to-charge ratio. In accordance with the no-hair theorem [12], we observe as usual that the scalar charge is not an independent parameter, and it vanishes when the electric charge does so because

$$Q_j^2 = \Sigma_j\left(\frac{16\pi M_j}{3\alpha} + \frac{3-2\beta}{3\beta\alpha^2}\Sigma_j\right). \quad (5.11)$$

Furthermore, we have a relation between the rod length and the mass and charge of the black hole in question, given by

$$\mu_j^2 = \left(\frac{8M_j}{3\pi} + \frac{\alpha\Sigma_j}{4\pi^2}\right)^2 - \left(\frac{Q_j}{\pi^2\sqrt{\beta}}\right)^2. \quad (5.12)$$

The limit of vanishing rod lengths $\mu_j \to 0$ is the so-called extremal limit. In view of its relative importance, this case will be discussed separately in Sec. VI.

Let us now specialize to the two-black-hole case. If we center ourselves on the left black hole and push the right one to infinity as in Sec. III, we obtain

$$ds^2 = -\left(1 + \frac{\tilde{\mu}_1}{R^2}\sinh^2\sigma\right)^{-2\beta/3}\left(1 - \frac{\tilde{\mu}_1}{R^2}\right)dt^2$$
$$+ \left(1 + \frac{\tilde{\mu}_1}{R^2}\sinh^2\sigma\right)^{\beta/3}\left[\left(1 - \frac{\tilde{\mu}_1}{R^2}\right)^{-1}dR^2 + R^2d\theta^2\right.$$
$$\left. + R^2\sin^2\theta\ d\varphi^2 + (1+l)^2R^2\cos^2\theta\ d\psi^2\right], \quad (5.13a)$$

$$A_t = \frac{\sqrt{\beta}}{2} \frac{\tilde{\mu}_1 \sinh\sigma \cosh\sigma}{R^2 + \tilde{\mu}_1 \sinh^2\sigma},$$

$$e^\phi = \left(1 + \frac{\tilde{\mu}_1}{R^2}\sinh^2\sigma\right)^{\beta\alpha/2}, \tag{5.13b}$$

where $\tilde{\mu}_1 \equiv \mu_1/(1+l)$. Thus, we recover in this limit a single dilatonic black hole [11], except for a conical singularity attached to it along the ψ axis with excess angle $2\pi l$. In a procedure similar to that in Sec. III, we may also calculate its near-horizon geometry to obtain

$$ds^2 = f_1^2(\theta)\left\{ -\cosh^{-4\beta/3}\sigma\left(1 - \frac{\mu_1}{R^2}\right)dt^2 \right.$$
$$+ \cosh^{2\beta/3}\sigma\frac{a_{15}a_{13}}{a_{14}^2}\left[\left(1 - \frac{\mu_1}{R^2}\right)^{-1}dR^2 + R^2 d\theta^2\right]\right\}$$
$$+ \cosh^{2\beta/3}\sigma[f_2^2(\theta)R^2\sin^2\theta\, d\varphi^2$$
$$+ f_3^2(\theta)R^2\cos^2\theta\, d\psi^2], \tag{5.14}$$

where the distortion factors f_1, f_2, f_3 are the same as those in the vacuum case (3.11). The three-area and temperature of the event horizon are, respectively,

$$\mathcal{A} = \sqrt{2}(2\pi)^2\cosh^\beta\sigma\frac{a_{12}\sqrt{a_{12}a_{13}a_{15}}}{a_{14}}, \tag{5.15}$$

$$T = \frac{\cosh^{-\beta}\sigma}{\sqrt{2}2\pi}\frac{a_{14}}{\sqrt{a_{15}a_{13}a_{12}}}, \tag{5.16}$$

while the electrostatic potential at the horizon is

$$\Phi_{\text{horizon}} = \frac{\sqrt{\beta}}{2}\tanh\sigma. \tag{5.17}$$

Note that Φ_{horizon} is independent of which black hole we are considering.

A similar analysis can be performed on the right black hole, but we will not reproduce the results here. We end by remarking that a generalized Smarr relation holds for the individual black holes:

$$M_j = \frac{3}{2}T_j\left(\frac{\mathcal{A}_j}{4}\right) + \frac{\Phi_{\text{horizon}}Q_j}{4\pi}. \tag{5.18}$$

This relation can be explicitly checked for the left black hole using the above results, and is consistent with the Smarr formula for the electrically charged black holes found in [11].[3]

VI. EXTREMAL BLACK HOLES

The extremal limit of the charged multi-black-hole solution derived in the preceding section, is taken by sending $\mu_j \to 0$ and $\sigma \to \infty$ to infinity such that the charges Q_j remain fixed. The solution (5.6) becomes in this limit

$$ds^2 = -\left(1 + \sum_{j=1}^N \frac{\tilde{Q}_j}{2R_{2j-1}}\right)^{-2\beta/3}dt^2 + \left(1 + \sum_{j=1}^N \frac{\tilde{Q}_j}{2R_{2j-1}}\right)^{\beta/3}\left[(R_1 + \zeta_1)\prod_{j=1}^{N-1}\left(\frac{R_{2j} - \zeta_{2j}}{R_{2j+1} - \zeta_{2j+1}}\right)d\varphi^2 \right.$$

$$+ (R_{2N-1} - \zeta_{2N-1})\prod_{j=1}^{N-1}\left(\frac{R_{2j-1} - \zeta_{2j-1}}{R_{2j} - \zeta_{2j}}\right)d\psi^2 + e^{2\gamma_0}\prod_{k=1}^{N-1}\left(\frac{Y_{1,2k}}{Y_{1,2k+1}}\right)\frac{\displaystyle\prod_{l,m}^{N-1}Y_{2l+1,2m}}{\displaystyle\prod_{i=1}^{2N-1}R_i\prod_{j<s}^{N-1}Y_{2j,2s}Y_{2j+1,2s+1}}(dr^2 + dz^2)\right], \tag{6.1a}$$

$$A_t = -\frac{\sqrt{\beta}}{2}\left(1 + \sum_{j=1}^N \frac{\tilde{Q}_j}{2R_{2j-1}}\right)^{-1}, \qquad e^\phi = \left(1 + \sum_{j=1}^N \frac{\tilde{Q}_j}{2R_{2j-1}}\right)^{\beta\alpha/2}, \tag{6.1b}$$

where $\tilde{Q}_j \equiv Q_j/(2\pi^2\sqrt{\beta})$. Its corresponding rod structure is given by Fig. 3, but with the addition of point sources for the time coordinate at $z = a_{2j-1}$, where the N black holes are located.

Note that the part of the metric in square brackets is the Euclidean 4D multiple C metric solution. This is in contrast to the 5D multiextremal-black-hole solution previously considered in [13], in which the metric in the square brackets is the flat one. Our solution is more complicated due to the fact that we are adding black holes to the nontrivial background

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[3]There is a 4π in the denominator of the second term because the Maxwell term in Eq. (5.5) is $1/(4\pi)$ times that in [11].

of Fig. 3, instead of flat space. However, we believe it is still worth studying the solution (6.1), since the nonextremal generalization of the multi-black-hole solution of [13] is not known.

The ADM mass and scalar charge of the jth extremal black hole are expressed in terms of Q_j by

$$M_j = \frac{\sqrt{\beta}Q_j}{8\pi}, \tag{6.2}$$

$$\Sigma_j = \sqrt{\beta}\alpha Q_j, \tag{6.3}$$

with the total mass and scalar charge of the solution given by their respective sums.

Let us again consider the two-black-hole case for simplicity. The metric (6.1a) reduces to

$$ds^2 = -\left(1 + \frac{\tilde{Q}_1}{2R_1} + \frac{\tilde{Q}_2}{2R_3}\right)^{-2\beta/3} dt^2$$
$$+ \left(1 + \frac{\tilde{Q}_1}{2R_1} + \frac{\tilde{Q}_2}{2R_3}\right)^{\beta/3} \left[\frac{(R_1 + \zeta_1)(R_2 - \zeta_2)}{R_3 - \zeta_3} d\varphi^2\right.$$
$$+ \frac{(R_1 - \zeta_1)(R_3 - \zeta_3)}{R_2 - \zeta_2} d\psi^2$$
$$\left. + \frac{Y_{12}Y_{23}}{4R_1R_2R_3Y_{13}}(dr^2 + dz^2)\right], \tag{6.4}$$

where the part of the metric in square brackets is just the usual Euclidean C metric solution [14]. The black holes are located at $z = a_1$ and a_3. Centering on the left black hole and pushing the other to infinity as was done above, we obtain

$$ds^2 = -\left(1 + \frac{\tilde{Q}_1}{(1+l)R^2}\right)^{-2\beta/3} dt^2 + \left(1 + \frac{\tilde{Q}_1}{(1+l)R^2}\right)^{\beta/3} [dR^2$$
$$+ R^2 d\theta^2 + R^2\sin^2\theta \, d\varphi^2 + (1+l)^2 R^2 \cos^2\theta \, d\psi^2], \tag{6.5}$$

where now $l \equiv a_{23}/a_{12}$. This is the extreme dilatonic black hole metric of [11,13], but with a conical singularity attached to the ψ axis. Its near-horizon limit is simply

$$ds^2 = -\left(\frac{\tilde{Q}_1}{(1+l)R^2}\right)^{-2\beta/3} dt^2 + \left(\frac{\tilde{Q}_1}{(1+l)R^2}\right)^{\beta/3} [dR^2 + R^2 d\theta^2$$
$$+ R^2\sin^2\theta \, d\varphi^2 + (1+l)^2 R^2 \cos^2\theta \, d\psi^2]. \tag{6.6}$$

The corresponding limits for the right black hole are similar, with the conical singularity attached to the φ axis instead. Note that there is an absence of angular distortion in Eq. (6.6). This is due to the well-known fact that the electrostatic repulsion exactly balances the gravitational attraction between extremally charged black holes. There are, however, conical singularities still stretching between the black holes, but these are intrinsic to the background spacetime and cannot be avoided.

The areas $\mathcal{A}_i$ of the event horizons are zero except for the $\beta = 3$ (Einstein-Maxwell) case where we find

$$\mathcal{A}_{\text{left}} = \sqrt{\frac{1}{2\pi^2}\frac{a_{12}}{a_{13}}}\left(\frac{Q_1}{\sqrt{3}}\right)^{3/2}, \tag{6.7a}$$

$$\mathcal{A}_{\text{right}} = \sqrt{\frac{1}{2\pi^2}\frac{a_{23}}{a_{13}}}\left(\frac{Q_2}{\sqrt{3}}\right)^{3/2}. \tag{6.7b}$$

Furthermore, it is interesting to observe, as in [11], the dependence of the Hawking temperatures T of the two extremal black holes on the strength of the coupling constant:

$$T_{\text{both}} \to \infty \quad \text{for} \quad 0 < \beta < 1, \tag{6.8a}$$

$$T_{\text{both}} = 0 \quad \text{for} \quad 1 < \beta \leq 3, \tag{6.8b}$$

$$\left.\begin{array}{l} T_{\text{left}} = \sqrt{\dfrac{1}{2Q_1}\dfrac{a_{13}}{a_{12}}} \\[4mm] T_{\text{right}} = \sqrt{\dfrac{1}{2Q_2}\dfrac{a_{13}}{a_{23}}} \end{array}\right\} \quad \text{for} \quad \beta = 1. \tag{6.8c}$$

For $0 < \beta < 1$, the extremal limit brings the temperature to formal infinity, similar to the behavior of 4D Kaluza-Klein extremal black holes [15]. It was shown that these infinitely hot extremal black holes are protected by mass gaps or potential barriers which insulate them externally, and thus they can be treated as elementary particles [16]. For $1 < \beta \leq 3$, the temperature tends to zero smoothly, characteristic of extremal Einstein-Maxwell black holes, which are stable end points of black hole evaporation [17]. The $\beta = 1$ case has a finite temperature. This enigmatic case emerges from low-energy effective string theory, when compactified to five dimensions. The finite temperature might lead one to think that the extremal end point of black hole evaporation will result in the formation of a naked singularity, but there exist various arguments to avoid this conclusion [18,19].

VII. DISCUSSION

In this paper, we have constructed a static solution describing a superposition of N Schwarzschild black holes, which may be considered a 5D generalization of the Israel-Khan solution. For certain choices of parameters, the black holes may be regarded as collinear. The main properties of these solutions were then studied. While they share many properties with the Israel-Khan solution, there are also crucial differences, particularly in the structure of the conical singularities. The charged generalization of this solution was also considered.

There are a number of avenues for further research. For example, the interaction between two 4D near-extremal black holes was analyzed in [4] by embedding them in M theory as bound states of branes. Using an effective string description of these bound states, the semiclassical result for the entropy, and its correction due to the interaction between

the black holes, was reproduced for large separation. It would be very interesting to see if an effective string description can also be found for our 5D charged two-black-hole solution.

In four dimensions, there exists a class of solutions known as black diholes [20], which consist of pairs of black holes with equal mass, and charges of the same magnitude but opposite sign. This is in contrast to the multicharged-black-hole solutions of [4] and in this paper, whose black holes all carry charges of the same sign. An effective string model for near-extremal black diholes was found in [21], in terms of an interacting system of strings and antistrings. A natural question is whether these results would generalize to five dimensions. A first step in this direction was recently made in [22], in which a 5D extremal black dihole solution was found using the generalized Weyl formalism. Like the two-black-hole solutions considered in this paper, the black holes exist in the background of the Euclidean C metric.

We note that by removing all the finite rod sources for the φ coordinate and the left-most rod source for the time coordinate in Fig. 2, we obtain a limiting metric describing multiple concentric black rings. This solution can be analyzed almost in parallel with the multi-black-hole solution of this paper. Another possible black ring configuration that one could consider is obtained from the two-black-hole rod structure (Fig. 4) by moving the finite rod source for the φ coordinate to the ψ coordinate, and vice versa. The resulting solution describes a pair of orthogonal black rings. Superpositions of black rings and black holes are also possible.

Finally, there remains the open question of whether it is possible to construct a multi-black-hole solution in five dimensions with $SO(3)$ instead of $U(1)\times U(1)$ symmetry. As mentioned in Sec. II, such a solution would possess one symmetry axis rather than two, and so would in some sense resemble the Israel-Khan solution more closely. However, to construct such a solution requires one to move beyond the generalized Weyl formalism. Unfortunately, there has been little headway in this direction so far, mainly because the Einstein equations are no longer reducible to a linear equation, as in Eq. (2.2).

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