Asymptotic safety of gravity coupled to matter

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Nonperturbative treatments of the UV limit of pure gravity suggest that it admits a stable fixed point with a positive Newton constant and a cosmological constant. We prove that this result is stable under the addition of a scalar field with a generic potential and nonminimal coupling to the scalar curvature. There is a fixed point where the mass and all nonminimal scalar interactions vanish, while the gravitational couplings have values which are almost identical to those in the pure gravity case. We discuss the linearized flow around this fixed point and find that the critical surface is four dimensional. In the presence of other, arbitrary, massless minimally coupled matter fields, the existence of the fixed point, the sign of the cosmological constant, and the dimension of the critical surface depend on the type and number of fields. In particular, for some matter content, there exist polynomial asymptotically free scalar potentials, suggesting a gravitational solution to the well-known problem of triviality.

I. INTRODUCTION

The failure of perturbative approaches to quantum gravity does not necessarily imply that quantum gravity does not exist as a field theory. There is still in principle the possibility that the theory could be "nonperturbatively quantized." To understand what this means, one has to look at the renormalization group (RG), i.e., the flow of the coupling constants $\tilde{g}_i(k)$, as a certain external momentum parameter k is changed.[1] It is customary to take k as a unit of mass; if $\tilde{g}_i$ has dimension d_i in units of mass, we define dimensionless couplings $g_i = \tilde{g}_i k^{-d_i}$. The RG flow is then given by the integral curves of a vector field β in the space of all couplings, whose components $\beta_i(g) = \partial_t g_i$ (with $t = \ln k$) are the beta functions. A fixed point (FP) is a point g_* in the space of all couplings, where

$$\beta_i(g_*) = 0. \tag{1}$$

For example, in ordinary quantum field theories in Minkowski space, a Gaussian theory is always a FP (called the Gaussian FP), because it does not have quantum corrections.

Suppose that the theory admits a FP. We define the UV critical surface to be the locus of points that, under the RG evolution, are attracted toward the FP for $t \to \infty$.[2] Starting from any point on the critical surface, the UV limit can be taken safely, because the couplings, and as a consequence the physical reaction rates, will be drawn toward the FP and hence remain finite.[3] On the other hand, if one starts from a point not belonging to the critical surface, the RG evolution will generally lead to divergences. If the critical surface has finite dimension c, the theory will be predictive, because only $c-1$ parameters will be left undetermined and will have to be fixed by experiment at a given energy scale (the last remaining parameter being the scale itself). The special case when the FP with a finite dimensional critical surface is the Gaussian FP is equivalent to the usual perturbative notion of renormalizability and asymptotic freedom.

This scenario for nonperturbative renormalizability has been discussed specifically in a gravitational context in [1] where this good property was called "asymptotic safety." At the time, some encouraging results were obtained by studying gravity in $2+\epsilon$ dimensions [1,2], but the program soon came to a halt essentially for want of technical tools. It now appears that the right tool to tackle this problem is the exact RG equation (ERGE), in one of several guises that have appeared in the literature in the last decade [3–5].

The ERGE is a differential equation that determines the RG flow of the action. It can be viewed as a set of infinitely many first order differential equations for infinitely many variables (the coupling constants) and therefore cannot generally be solved in practice. A method that is commonly used to calculate nonperturbative beta functions is to make a

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[1]The precise physical meaning of the parameter k depends on the specific problem that one is addressing; it is usually the momentum of some particle entering into the process under study, or the inverse of some characteristic length of the system. In general, k has the meaning of an IR cutoff, because the effective theory describing the process must include the effect of all the fluctuations of the fields with momenta larger than k.

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[2]In statistical mechanics the term "critical surface" is used for the IR critical surface, i.e., the locus of points that are attracted toward the FP when $t \to -\infty$. Note, also, that the "RG time" t is often defined with the opposite sign with respect to ours.

[3]In general, in the action there will be couplings whose values do not affect cross sections and reaction rates. They are called inessential couplings. Our reasoning applies only to the essential couplings.

physically motivated ansatz for the running effective action, typically containing a finite number of parameters, and insert this ansatz into the ERGE.

The ERGE, in the specific form discussed in [5], has been applied to Einstein's theory in [6,7], where the beta functions for Newton's constant and for the cosmological constant were derived. It was later realized that these beta functions actually admit a nontrivial UV-attractive FP [8]. The properties of this FP were further discussed in greater detail in [9]. A particularly important issue is to prove that the FP is not an artifact of the ansatz but is a genuine property of gravitation. Several facts seem to indicate that the FP is quite robust. It has been shown to exist in four spacetime dimensions for many different shapes of the cutoff, whereas in other dimensions it exists only for certain cutoffs but not others. Its properties have been shown to be only weakly dependent on the shape of the cutoff, indicating that the truncation is self-consistent [10]. It has been found in certain dimensionally reduced versions of the theory [11]. It is also remarkable that Newton's constant always turns out to be positive, a fact that could not in any way be guaranteed by the general form of the equations.

The most important test, however, is the stability of the FP against the addition of new couplings. Each time we consider a new coupling, whether remaining in the context of pure gravity or if we introduce matter fields, a new beta function has to vanish, and therefore a new constraint has to be satisfied by the set of all couplings at the FP. It is therefore nontrivial that the FP still exists when we take into account additional couplings.

In the context of pure gravity, an important progress was made in [12], where it was shown that the addition of a term quadratic in curvature does not spoil the existence of the FP. In fact it turned out that the values of the cosmological and Newton's constants at the FP are almost unaffected by the new interaction, while the new coupling constant is quite small at the FP. This is far from being conclusive evidence, but it is nevertheless an important result, especially in view of the fact that another FP that was present in the truncation with only two couplings—the Gaussian FP—does not exist in the three-coupling truncation.

As far as matter is concerned, we have recently considered the effect of minimally coupled, massless quantum fields of arbitrary spin [13]. The only couplings taken into account were the cosmological and Newton's constant, since the coefficients of the matter kinetic terms can be normalized to their standard values by field rescalings. It was shown that the existence of the FP, the values of the cosmological constant and Newton's constant at the FP, and the dimension of the critical surface all depend on the type and number of fields present. Altogether, the existence and attractiveness of the FP puts some constraints on the number of matter fields that are present.

In this paper we continue the analysis of coupled gravity and matter systems and we begin to address the issue of matter couplings. There are many different couplings that are necessary to construct realistic theories of the world, and we cannot possibly take them all into account, so as a first step we shall consider the simplest example, that of a self-interacting scalar field. Aside from its role as a model for the Higgs field in unified theories, a scalar field (the dilaton) appears in many popular theories of gravity. It can therefore sometimes be regarded as part of the gravitational sector, rather than the matter sector. This makes its properties especially interesting in a gravitational context. In this paper we shall not make any assumption about the physical interpretation of the scalar field.

The class of actions that we consider is

$$\Gamma[g,\phi] = \int d^4x \sqrt{g}\left(V(\phi^2) - F(\phi^2)R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right),$$

(2)

where the potential V and the scalar-tensor coupling F are arbitrary real analytic functions. (The RGE's for this system were studied earlier in [14].) Although it is not necessary for some of the results of this paper, we shall assume that the potential has its minimum at the origin. Then we can identify $F(0) = \kappa \equiv 1/16\pi G$, G being Newton's constant, and $V(0) = 2\kappa\Lambda$, where Λ is the (dimension-2) cosmological constant. It will appear that the behavior of the couplings implicit in the functions V and F is sufficiently systematic that we are actually able to draw several conclusions involving an infinite number of couplings. For some purposes, however, we shall restrict our attention to a five-parameter ansatz, where V is at most quartic in ϕ and F is at most quadratic.

Aside from establishing the existence of a nontrivial FP, the main question addressed in this paper will be the dimension of the critical surface. In practice, this is done by linearizing the flow around the FP. We define $v_i = g_i - g_{i*}$ as the shift from the fixed point. The linearized flow around the fixed point is described by the equations

$$\partial_t v_i = M_{ij}v_j,$$

(3)

where $M_{ij} = \partial\beta_i/\partial g_j$. Let P be the (generally complex) linear transformation that diagonalizes M: $P^{-1}MP = \text{diag}(\alpha_1, \ldots, \alpha_N)$ (the columns of P are the eigenvectors of M). Defining $f = P^{-1}v$, one finds $\partial_t f_k = \alpha_k f_k$, so $f_k(t) = e^{\alpha_k t}$. Transforming back to the original variables, the solution can be written $g_i(t) = g_{i*} + \text{Re}[P_{ij}f_j(t)]$. It is easy to show that the eigenvalues α_i are invariant under redefinitions of the couplings.

The eigenvalues with negative real part (which for brevity we shall call the "negative eigenvalues") correspond to directions for which the RG flow approaches the FP in the UV. The corresponding parameter f_i is called a relevant parameter. Those with positive real part (the "positive eigenvalues") correspond to directions for which the RG flow moves away from the FP in the UV. The corresponding parameter f_i is called an irrelevant parameter. The parameters corresponding to purely imaginary eigenvalues are called marginal.[4] In the linearized theory, in order to approach the FP in the UV, it is therefore necessary to stay on the hyperplane spanned by

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[4]Thus, in the conventions used in statistical mechanics, the (IR) critical surface is spanned by the irrelevant parameters.

the eigenvectors with negative eigenvalues. This hyperplane is the tangent space to the critical surface at the FP. Therefore, the dimension of the critical surface is equal to the number of negative eigenvalues. On general grounds, one expects the critical surface to be finite dimensional. In this way all but a finite number of couplings would be fixed and the theory would be as predictive as a perturbatively renormalizable theory.

We now give a brief summary of the results of this paper. First of all, a nontrivial FP still exists with the ansatz (2). The purely gravitational couplings (the cosmological constant and Newton's constant, which appear as the ϕ^2-independent terms in the functions V and F) have the same values as in [13] and all the other couplings are equal to zero. In a sense this is therefore "the same" fixed point that was considered in [8] and in [9,15]. It can also be regarded as a generalization of the Gaussian FP of the pure scalar theory in flat space. We therefore call it the Gaussian-matter FP (GMFP). We have performed a systematic search for other FP's within a five-parameter truncation of the action, where V and F are polynomials containing at most terms of order ϕ^4 and ϕ^2, respectively. Thus, in addition to the cosmological and Newton's constants, we consider a scalar mass term, a quartic self-interaction, and a nonminimal coupling of the scalar field to the scalar curvature. Detailed numerical analyses have convinced us that there are no FP's with nonzero scalar mass and couplings, for values of the cosmological and Newton constant close to those of the pure-gravity FP.

Comparing to the results of the pure scalar theory, the main effect of the coupling to gravity is to change the exponents α_i. It turns out that of the two canonically marginal couplings, the ϕ^4 coupling becomes irrelevant while the $\phi^2 R$ coupling becomes relevant. The other couplings preserve the character that is implied by their canonical dimension; the critical dimension would thus be equal to 4.

We then look at the effect of other matter fields on the FP. The results of this investigation generalize those already reported in [13]. The behavior of the beta functions is determined by two parameters that depend on the number of fields, and the existence of the FP depends on the values of these parameters. In this way the existence of the FP yields constraints on the type and number of matter fields. These constraints appear to be satisfied by popular unified models. The existence region is subdivided into subregions with varying numbers of attractive directions. In the region that we have explored, comprising large numbers of matter fields, the dimension of the critical surface is always finite. In particular, there are regions in which the attractive directions correspond to nontrivial polynomial potentials of degree 4 or higher. This may yield a solution of a long-standing puzzle. In a pure scalar theory in flat space, the Gaussian FP is IR attractive (all couplings are irrelevant). As a consequence, when one takes the continuum limit at the Gaussian FP, the renormalized theory is free. This result is not an obstacle in the context of an effective field theory, but it has to be somehow circumvented if scalar fields have to appear in a fundamental field theory. The coupling to gravity is a natural context for a solution of this issue. Our results suggest that there exist theories of gravity coupled to matter such that the renormalized scalar potential has finitely many nonzero couplings in the continuum limit. This seems to indicate that the interaction with gravity (and, indirectly, with the other matter fields) solves the problem of the triviality of the scalar theory.

This paper is organized as follows. In Sec. II, by way of introduction, we derive the ERGE and we use it to prove some well-known results on the Gaussian FP of a pure scalar theory in flat space. In Sec. III we consider the modifications of the beta functions due to gravity, we prove the existence of the Gaussian-matter FP, and we discuss the (negative) results of the numerical search for other FP's. Section IV is devoted to the properties of the GMFP. We analyze the linearized flow around the GMFP and show that the coupling to gravity affects the dimensions of the couplings, shifting them relative to the canonical values. In Sec. V we consider the effect of other massless, minimally coupled matter fields on the GMFP. In Sec. VI we will consider in some detail the dependence of our results on the shape of the cutoff function and on the gauge-fixing parameter. Finally, in Sec. VII we make some concluding remarks.

All results are derived in the case of Euclidean signature, in four dimensions. Since the expressions of the beta functions are extremely lengthy, in deriving our results we have made extensive use of algebraic manipulation software.

II. THE GAUSSIAN FP IN PURE SCALAR THEORY

We begin by considering the case of a single scalar field without gravity and a generic even potential

$$V(\phi) = \sum_{n=0}^{\infty} \tilde{\lambda}_{2n} \phi^{2n}. \tag{4}$$

In this section we assume that $V(0) = \tilde{\lambda}_0 = 2\Lambda\kappa = 0$; the first nonzero term is the mass $\tilde{\lambda}_2 = \frac{1}{2}m^2$, while $\tilde{\lambda}_4$ is the usual quartic coupling. The couplings $\tilde{\lambda}_{2n}$ have dimension $(\text{mass})^{2(2-n)}$, so the usual power-counting arguments tell us that the terms in Eq. (4) with $n>2$ are perturbatively nonrenormalizable, while the term $n=2$ is marginal. We will now rederive this result within the formalism of the ERGE's. This will set the stage for further developments in later sections.

To derive the ERGE, one begins by modifying the classical propagator by adding to the action a term quadratic in the fields which in momentum space can be written $\Delta S_k(\phi) = \frac{1}{2}\int d^4q\, \phi(-q)R_k(z)\phi(q)$, where $z = q^2$. The effect of this term must be to suppress the propagation of field modes with momenta smaller than k, while leaving the modes with momenta larger than k unaffected. This is the case if the smooth cutoff function R_k is chosen to tend to zero for $z \gg k^2$ and to a constant for $z \to 0$. For numerical work, in this paper we will work with cutoffs of the form

$$R_k(z) = \frac{2aze^{-2az/k^2}}{1 - e^{-2az/k^2}}, \tag{5}$$

with a a free parameter. We then define a scale-dependent generating functional of connected Green's functions

$$W_k[J] = -\ln \int (D\phi) \exp\left[-\left(S + \Delta S_k + \int J\phi\right)\right]$$

such that $(\delta W_k/\delta J)|_{J=0} = \langle\phi\rangle$ and a scale-dependent effective action $\Gamma_k[\phi_{cl}] = \tilde{\Gamma}_k[\phi_{cl}] - \Delta S_k[\phi_{cl}]$, where $\tilde{\Gamma}_k[\phi_{cl}] = W_k[J] - \int J\phi_{cl}$ is obtained from $W_k[J]$ by the usual Legendre-transform procedure. The scale-dependent effective action tends to the bare action S when k tends to the UV cutoff, and to the ordinary effective action for $k \to 0$. We have

$$\partial_t W_k = \partial_t \langle \Delta S_k \rangle = \frac{1}{2} \text{Tr}\langle\phi\phi\rangle \partial_t R_k, \tag{6}$$

where the trace is over all Fourier modes (and internal indices, if there were any). Then,

$$\partial_t \Gamma_k[\phi_{cl}] = \partial_t W_k[J] - \partial_t \Delta S_k[\phi_{cl}]$$

$$= \frac{1}{2} \text{Tr}(\langle\phi\phi\rangle - \langle\phi\rangle\langle\phi\rangle)\partial_t R_k$$

$$= -\frac{1}{2} \text{Tr} \frac{\delta^2 W_k}{\delta J \delta J} \partial_t R_k. \tag{7}$$

Applying the standard identity

$$\frac{\delta^2 W_k}{\delta J \delta J} = -\left(\frac{\delta^2 \tilde{\Gamma}_k}{\delta\phi_{cl}\delta\phi_{cl}}\right)^{-1}, \tag{8}$$

one then obtains the ERGE [16]

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr}\left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1} \partial_t R_k. \tag{9}$$

In the previous formula and in the following we shall drop the subscript in ϕ_{cl}; this should not cause any confusion. Note that, although the definition of Γ_k would require an UV regulator, the trace in Eq. (9) is automatically finite due to the term $\partial_t R_k$, which effectively restricts the integration to a small range of momenta about k. The ERGE describes the flow of the functional Γ_k with the scale k. In order to extract beta functions, one has to resort to approximations. A common procedure is to make an ansatz about the form of Γ_k and to insert it into the ERGE. Of course the beta functions obtained in this way are no longer exact: one loses all information about the dependence of the beta functions on the parameters that have been left out of the ansatz. Nevertheless, the results do contain information that is not accessible in perturbation theory and they have been shown to yield numerically accurate values in many circumstances [4,17]. We now apply this procedure to the scalar theory.

Introducing in Eq. (9) the truncation $\Gamma_k(\phi) = \int d^4x[-\frac{1}{2}\phi\partial^2\phi + V(\phi^2)]$, where V is a k-dependent potential, gives

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr}\left(\frac{\partial_t P_k}{P_k + V' + 4\phi^2 V''}\right), \tag{10}$$

where a prime denotes the derivative with respect to ϕ^2, the trace can be understood as an integration over momenta. It can be reexpressed as

$$\partial_t \Gamma_k = \frac{\mathcal{V}}{32\pi^2} \tilde{Q}_2\left(\frac{\partial_t P_k}{P_k + \mathcal{V}' + 4\phi^2 \mathcal{V}'}\right) \tag{11}$$

where $\mathcal{V} = \int d^4x$ denotes the volume of spacetime and

$$\tilde{Q}_n[f] = \frac{1}{\Gamma(n)} \int_0^{+\infty} dz\, z^{n-1} f(z). \tag{12}$$

The coupling constants can be extracted from the potential by

$$\tilde{\lambda}_{2n} = \frac{1}{n!} \frac{\partial^n V}{\partial(\phi^2)^n}\bigg|_{\phi=0}. \tag{13}$$

In order to look for a fixed point one has to define dimensionless couplings $\lambda_{2n} = k^{2(n-2)}\tilde{\lambda}_{2n}$. The corresponding beta functions are given by

$$\partial_t \lambda_{2n} = 2(n-2)\lambda_{2n} + \frac{k^{2(n-2)}}{\mathcal{V}} \frac{1}{n!} \frac{\partial^n}{\partial(\phi^2)^n} \partial_t \Gamma_k\bigg|_{\phi=0}. \tag{14}$$

Explicitly, the first few beta functions are given by

$$\partial_t \lambda_2 = -2\lambda_2 - \frac{12\lambda_4}{32\pi^2} Q_2\left(\frac{\partial_t \mathcal{P}}{\mathcal{P} + 2\lambda_2}\right), \tag{15a}$$

$$\partial_t \lambda_4 = \frac{1}{32\pi^2}\left[-30\lambda_6 Q_2\left(\frac{\partial_t \mathcal{P}}{(\mathcal{P} + 2\lambda_2)^2}\right) + 144\lambda_4^2 Q_2\left(\frac{\partial_t \mathcal{P}}{(\mathcal{P} + 2\lambda_2)^3}\right)\right], \tag{15b}$$

$$\partial_t \lambda_6 = 2\lambda_6 + \frac{1}{32\pi^2}\left[-56\lambda_8 Q_2\left(\frac{\partial_t \mathcal{P}}{(\mathcal{P} + 2\lambda_2)^2}\right) + 720\lambda_4\lambda_6 Q_2\left(\frac{\partial_t \mathcal{P}}{(\mathcal{P} + 2\lambda_2)^3}\right) - 1728\lambda_4^3 Q_2\left(\frac{\partial_t \mathcal{P}}{(\mathcal{P} + 2\lambda_2)^4}\right)\right], \tag{15c}$$

$$\partial_t \lambda_8 = 4\lambda_8 + \frac{1}{32\pi^2}\left[-90\lambda_{10} Q_2\left(\frac{\partial_t \mathcal{P}}{(\mathcal{P} + 2\lambda_2)^2}\right) + 1344\lambda_4\lambda_8 Q_2\left(\frac{\partial_t \mathcal{P}}{(\mathcal{P} + 2\lambda_2)^3}\right) + 900\lambda_6^2 Q_2\left(\frac{\partial_t \mathcal{P}}{(\mathcal{P} + 2\lambda_2)^3}\right)\right.$$

$$-8640\lambda_6\lambda_4^2 Q_2\left(\frac{\partial_t \mathcal{P}}{(\mathcal{P}+2\lambda_2)^4}\right)$$

$$+20736\lambda_4^4 Q_2\left(\frac{\partial_t \mathcal{P}}{(\mathcal{P}+2\lambda_2)^5}\right)\Bigg], \tag{15d}$$

where $Q_n[f]=k^{-2n}\tilde{Q}_n[f]$ is a dimensionless integral, $\mathcal{R}=k^{-2}R_k$ is a dimensionless cutoff, and $\mathcal{P}=k^{-2}P_k$ is a dimensionless modified propagator.

This theory admits a well-known Gaussian FP: if we set $\lambda_2=0$, Eq. (15a) implies $\lambda_4=0$, Eq. (15b) then implies $\lambda_6=0$, and so on: recursively all couplings are found to be zero. This is not the only solution of the coupled system. One can fix an arbitrary value of λ_2 and the equations then recursively determine all the other couplings [18]. However, when $\lambda_2\neq0$ these potentials become singular at a finite value of ϕ and therefore are not considered to be physically acceptable [19]. In what follows we will restrict our attention to the Gaussian FP.

We now study the critical surface in the neighborhood of the FP using the linearized RG equation (3). Let $\beta_{2n}=\partial_t\lambda_{2n}$ and let $M_{ij}=\partial\beta_{2i}/\partial\lambda_{2j}$. It appears from Eqs. (15) (as well as from dimensional and diagrammatic considerations) that the $2n$th beta function is a polynomial in the couplings $\lambda_4,\ldots,\lambda_{2n+2}$, linear in λ_{2n+2}. Therefore the elements of the matrix M_{ij} with $j>i+1$ are zero. On the other hand, since all λ_{2n} are zero at the FP, when the derivatives are evaluated at the FP only the terms linear in the couplings remain. These are exactly the terms on the diagonal, which are equal to the canonical dimensions of the couplings, and the terms on the second diagonal ($i=j+1$), which are equal to

$$M_{ii+1}=\frac{\partial\beta_{2n}}{\partial\lambda_{2n+2}}=(2n+1)(n+1)c \tag{16}$$

where $c=-(1/16\pi^2)Q_2(\partial_t\mathcal{P}/\mathcal{P}^2)$. All other terms are zero. Numerically, the integral $Q_2(\partial_t\mathcal{P}/\mathcal{P}^2)$ is equal to 0.924 for $a=2$.

Therefore, the matrix M has the following form:

$$\begin{pmatrix} -2 & 6c & 0 & 0 & \ldots \\ 0 & 0 & 15c & 0 & \ldots \\ 0 & 0 & 2 & 28c & \ldots \\ 0 & 0 & 0 & 4 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}. \tag{17}$$

The eigenvalue problem for this infinite matrix yields the recursion relation

$$\lambda_{2n+2}=\frac{2(n-2)-\mu}{(2n+1)(n+1)c}\lambda_{2n}, \tag{18}$$

where μ is the eigenvalue. This relation can have two types of solution. If we assume that the potential is a finite polynomial of order K, Eq. (18) implies that $\mu=2(K-2)$. These eigenvalues are just the diagonal elements of the matrix (17). The corresponding eigenvectors are the columns of the following matrix P:

$$\begin{pmatrix} 1 & -0.0175512 & 3.84804\times10^{-3} & 1.04825\times10^{-5} & \ldots \\ 0 & 0.999846 & -0.0438425 & 1.79148\times10^{-3} & \ldots \\ 0 & 0 & 0.999038 & -0.0816446 & \ldots \\ 0 & 0 & 0 & 0.99666 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}. \tag{19}$$

The eigenvalues are equal to the canonical dimensions of the couplings, so that the relevant, irrelevant, and marginal couplings correspond exactly to the couplings that are superrenormalizable, nonrenormalizable, and renormalizable in the perturbative sense.

These polynomial potentials suffer from the well-known problem of triviality. Consider the scalar theory regularized with a UV cutoff Λ_{UV} and an IR cutoff k. Keeping k fixed and letting $\Lambda_{UV}\to\infty$ (the continuum limit) has the same effect as keeping Λ_{UV} fixed and letting $k\to0$. An irrelevant coupling tends to zero for $k\to0$, and therefore for any fixed k it will tend to zero in the continuum limit. This will be the case for all λ_{2i} with $i\geq2$, so the theory is noninteracting in the continuum limit. (Our analysis says only that the couplings from λ_6 upward have to be zero; the hard part is to prove that also the marginal coupling λ_4 tends to zero.

For this, one has to go beyond the linearized analysis [20].)

There is also another type of eigenvector, corresponding to nonpolynomial potentials, that avoids the problem of triviality. If we do not assume that $\lambda_{2K+2}=0$ for some K, the recursion relation (18) can be solved for the λ_{2n} in terms of the free parameters λ_2 and μ, yielding a potential that can be written as a Kummer function [18]. There are (negative) values of μ for which the potential has all the physically desirable properties (positivity at ∞, symmetry breaking). They are therefore nontrivial asymptotically free scalar theories. However, there are infinitely many attractive directions and therefore these theories do not satisfy the conditions for asymptotic safety.

This concludes our brief review of the ERGE for a scalar field theory.

III. THE COUPLED SYSTEM

We now consider the coupling of the scalar theory to gravity, using the ansatz (2) for the running effective action. This will obviously change the beta functions of the scalar potential; in addition, we will have to take into account also the beta functions of the gravitational couplings. These are given by the Taylor expansion coefficients of the function $F(\phi^2)$ of Eq. (2), which we write as follows:

$$F(\phi^2) = \sum_{n=0}^{\infty} \tilde{\xi}_{2n} \phi^{2n}. \tag{20}$$

The first term in the expansion can be identified with the (inverse) Newton constant: $\xi_0 = \kappa = 1/(16\pi G)$, while the second term is the well-known scalar tensor interaction term $\phi^2 R$ with dimensionless coefficient $\xi_2 = \xi$. The running couplings are given by

$$\tilde{\xi}_{2n} = \frac{1}{n!} \frac{\partial^n F}{\partial(\phi^2)^n}\bigg|_{\phi=0}. \tag{21}$$

As before, we define dimensionless couplings $\xi_{2n} = k^{2(n-1)}\tilde{\xi}_{2n}$. The corresponding beta functions are given by

$$\partial_t \xi_{2n} = 2(n-1)\xi_{2n} + \frac{k^{2(n-1)}}{\mathcal{V}} \frac{1}{n!} \frac{\partial^{n+1}}{\partial R \partial(\phi^2)^n} \partial_t \Gamma_k \bigg|_{\phi=0, R=0}. \tag{22}$$

We now have to insert this ansatz into the appropriate ERGE. The derivation of Eq. (9) in the previous section was quite general and therefore the ERGE for gravity coupled to a scalar field has again the same form, except for two generalizations: first, the field ϕ is to be reinterpreted as a matrix consisting of the components of the metric and a scalar field; second, since gravity is a gauge theory, one has to take into account the effect of gauge fixing and ghost terms.

Here we mention some points that are necessary to understand the results; we refer to [7] and [9] for details. In deriving the ERGE, one encounters the *quantum metric* (to be integrated out in the functional integral), say $\gamma_{\mu\nu}$, which can be decomposed into the sum of an arbitrary background metric $\bar{g}_{\mu\nu}$ and a quantum fluctuation $h_{\mu\nu}$. The background metric is used in the gauge fixing terms (23) below and also in the cutoff terms ΔS_k, which have to be quadratic in $h_{\mu\nu}$. In the Legendre transformation one encounters also the *classical metric* $g_{\mu\nu}$, which is the canonically conjugate variable of the source associated with the quantum metric. Thus, in general, the action Γ_k will depend on both $\bar{g}$ and g. On the other hand the ansatz (2) depends only on one metric. In order to derive the beta functions for the couplings in Eq. (2) we proceed as follows. In the right-hand side (RHS) of the ERGE, one first takes the functional derivatives with respect to the *classical field* g, and then one sets the

background metric equal to the classical one, i.e., $g_{\mu\nu} = \bar{g}_{\mu\nu}$, so that many contributions disappear. From these equations one can read off the beta functions of the couplings.

The gauge-fixing action is chosen as

$$S_{GF} = \frac{1}{2\alpha} \int d^4 x \sqrt{\bar{g}} \bar{g}^{\mu\nu} F_\mu F_\nu, \tag{23}$$

$$F_\mu = \bar{g}^{\nu\rho} \left(\bar{\nabla}_\nu g_{\rho\mu} - \frac{\beta+1}{4} \bar{\nabla}_\mu g_{\nu\rho} \right),$$

so that the corresponding ghost action will be

$$S_{gh} = \int d^4 x \sqrt{\bar{g}} \bar{C}_\mu \left(-\bar{\nabla}^2 g^{\mu\nu} + \frac{\beta-1}{2} \nabla^\mu \nabla^\nu - R^{\mu\nu} \right) C_\nu. \tag{24}$$

In principle, α and β are running parameters in the effective action, so one should take into account their beta functions, too. However, as will be discussed in Sec. VI, there are arguments to the effect that $\alpha = 0$ at the FP. Therefore, unless otherwise stated, we will always work in the gauge $\alpha = 0$ and $\beta = 1$.

The kinetic term of the gravitons is obtained by linearizing the action around a de Sitter metric with scalar curvature R and a constant scalar background ϕ. Using the method of [7], the RHS of Eq. (9) can be written as a sum of several terms, corresponding to the spin-2, -1, and -0-components of the fields, and has to be completed by adding the ghost contributions.

The spin-2 component of the metric has the inverse propagator

$$\frac{1}{2} F(\phi^2) \left(z + \frac{2}{3} R \right) - \frac{1}{2} V(\phi^2), \tag{25}$$

where now $z = -\nabla_\mu \nabla^\mu$. The spin-1 component of the metric has the inverse propagator

$$\frac{1}{\alpha} F(\phi^2) \left(z + \frac{2\alpha-1}{4} R \right) - V(\phi^2), \tag{26}$$

where α is the gauge-fixing parameter. The two spin-0 components of the metric mix with the scalar field; the resulting inverse propagator is given by the matrix

$$\begin{pmatrix} \dfrac{3}{16}F(\phi^2)\left(\dfrac{3-\alpha}{\alpha}z+\dfrac{\alpha-1}{\alpha}R\right)-\dfrac{3}{8}V(\phi^2) & \dfrac{3}{16}\dfrac{\beta-\alpha}{\alpha}F(\phi^2)\sqrt{z}\sqrt{z-\dfrac{R}{3}} & -\dfrac{3}{2}F'(\phi^2)\phi\sqrt{P_k}\sqrt{P_k-\dfrac{R}{3}} \\[2ex] \dfrac{3}{16}\dfrac{\beta-\alpha}{\alpha}F(\phi^2)\sqrt{z}\sqrt{z-\dfrac{R}{3}} & -\dfrac{1}{16}\dfrac{3\alpha-\beta^2}{\alpha}F(\phi^2)z+\dfrac{1}{8}V(\phi^2) & -\dfrac{3}{2}F'(\phi^2)\phi\left(z-\dfrac{R}{3}\right)+\phi V' \\[2ex] -\dfrac{3}{2}F'(\phi^2)\sqrt{z}\sqrt{z-\dfrac{R}{3}} & -\dfrac{3}{2}F'(\phi^2)\left(z-\dfrac{R}{3}\right)+\phi V' & z+2V'+4\phi^2V''-R(2F'+4\phi^2F'') \end{pmatrix}.$$

$$(27)$$

The two factors under the trace in the RHS of Eq. (9) are obtained from these expressions as follows. The modified (cutoff) propagators are given by the inverses of the expressions in Eqs. (25),(26),(27), with z replaced by $P_k(z)$. The function R_k for each spin component is given by the difference of the cutoff propagator and the original propagator. In the case of spin 2 and spin 1, this is just the function $R_k(z)$ defined in Eq. (5), whereas for the spin-0 components it is a 3×3 matrix [the difference of Eq. (27) with z replaced by $P_k(z)$ and Eq. (27)].

Since in Eqs. (25), (26), (27) the momentum variable z always appears multiplied by the function $F(\phi^2)$, the matrices R_k appearing in Eq. (9) also contain $F(\phi^2)$. When inserted in the RHS of Eq. (22), in addition to the explicit dependence of $P_k(z)$ on k, one has to take into account the

dependence on k of all coupling constants that are present in $F(\phi^2)$ and its derivatives (this is related to the "renormalization group improvement" that turns the one-loop RG into an exact equation). This generates terms proportional to the beta functions in the RHS of the equations, so that the ERGE does not immediately yield expressions for the beta functions but rather linear equations for the beta functions.[5]

The beta functions themselves are then obtained by inverting the matrix of coefficients, and this introduces further nonlinearities into the system. We will not write the expressions for the beta functions themselves but only the linear equations that determine the beta functions. We will order the couplings in order of decreasing mass dimension (before dividing by powers of k): $\lambda_0,\xi_0,\lambda_2,\xi_2,\lambda_4,\xi_4,\ldots$. These are the first five equations:

$$\partial_t\lambda_0=\frac{1}{32\pi^2}\left\{Q_2\left[\frac{\partial_t\mathcal{P}(\lambda_0(3\mathcal{P}+8\lambda_2)+\mathcal{P}(3\mathcal{P}+4\lambda_2)\xi_0)}{\mathcal{P}(\mathcal{P}+2\lambda_2)(\xi_0\mathcal{P}-\lambda_0)}\right]+2\frac{\xi_0'}{\xi_0}Q_2\left[\frac{\mathcal{R}(2\lambda_0-5\xi_0\mathcal{P})}{\mathcal{P}(\lambda_0-\xi_0\mathcal{P})}\right]\right\},\tag{28}$$

$$\begin{aligned}
\partial_t\xi_0=\frac{1}{384\pi^2}\Bigg\{&Q_1\left[\frac{\partial_t\mathcal{P}[-\lambda_0(3\mathcal{P}+10\lambda_2)+\mathcal{P}(11\mathcal{P}+26\lambda_2)\xi_0]}{\mathcal{P}(\mathcal{P}+2\lambda_2)(\xi_0\mathcal{P}-\lambda_0)}\right] \\
&-Q_2\left[\frac{\partial_t\mathcal{P}\{6\lambda_0\xi_0\mathcal{P}[20\lambda_2\mathcal{P}+20\lambda_2^2+\mathcal{P}^2(5-8\xi_2)]+3\lambda_0^2[-20\lambda_2\mathcal{P}-20\lambda_2^2+\mathcal{P}^2(8\xi_2-5)]\}}{\mathcal{P}^2(\mathcal{P}+2\lambda_2)^2(\xi_0\mathcal{P}-\lambda_0)^2}\right] \\
&+Q_2\left[\frac{\partial_t\mathcal{P}\{\xi_0^2\mathcal{P}^2[-220\lambda_2\mathcal{P}-220\lambda_2^2+\mathcal{P}^2(24\xi_2-55)]\}}{\mathcal{P}^2(\mathcal{P}+2\lambda_2)^2(\xi_0\mathcal{P}-\lambda_0)^2}\right]\Bigg\}+\frac{1}{384\pi^2}\frac{\partial_t\xi_0}{\xi_0}\Bigg\{Q_1\left[\frac{\mathcal{R}(5\lambda_0+3\xi_0\mathcal{P})}{\mathcal{P}(\xi_0\mathcal{P}-\lambda_0)}\right] \\
&+5Q_2\left[\frac{\mathcal{R}(-3\lambda_0^2+6\lambda_0\xi_0\mathcal{P}+5\xi_0^2\mathcal{P}^2)}{\mathcal{P}^2(\xi_0\mathcal{P}-\lambda_0)^2}\right]\Bigg\},
\end{aligned}\tag{29}$$

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[5]This did not happen in the pure scalar case because there the propagator was fixed to be equal to 1. It would have happened if we had written a more general action containing a term $(1/2)Z(\phi^2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Then the expressions for all beta functions would contain on the RHS the beta functions of the couplings that appear in the function $Z(\phi^2)$.

$$\partial_t \lambda_2 = -\frac{3}{16\pi^2} Q_2 \left[\frac{\partial_t \mathcal{P}\{2\lambda_0^2\lambda_4 + \xi_0\mathcal{P}^2[2\lambda_4\xi_0 - \lambda_2(1+2\xi_2)^2] + \lambda_0[(1+2\xi_2)(-2\lambda_2^2 + 4\lambda_2\xi_2\mathcal{P} + \xi_2\mathcal{P}^2) - 4\lambda_4\xi_0\mathcal{P}]\}}{(\mathcal{P}+2\lambda_2)^2(\xi_0\mathcal{P}-\lambda_0)^2} \right]$$

$$-\frac{\partial_t\xi_0}{16\pi^2\xi_0^2} Q_2 \left[\frac{\mathcal{R}\{-\lambda_2[-4\lambda_0^2\xi_2 + 8\lambda_0\xi_0\xi_2\mathcal{P} + \xi_0^2\mathcal{P}^2(3+2\xi_2)] + \xi_2\mathcal{P}[2\lambda_0^2 - 4\lambda_0\xi_0\mathcal{P} + (5+6\xi_2)\xi_0^2\mathcal{P}^2]\}}{\mathcal{P}(\mathcal{P}+2\lambda_2)(\xi_0\mathcal{P}-\lambda_0)^2} \right]$$

$$+\frac{\partial_t\xi_2}{16\pi^2\xi_0} Q_2 \left[\frac{\mathcal{R}[-2\lambda_0(\mathcal{P}+2\lambda_2) + \xi_0\mathcal{P}(5\mathcal{P}-2\lambda_2+12\xi_2\mathcal{P})]}{\mathcal{P}(\mathcal{P}+2\lambda_2)(\xi_0\mathcal{P}-\lambda_0)} \right], \tag{30}$$

$$\partial_t\xi_2 = \frac{1}{48\pi^2} Q_1 \left[\frac{\partial_t\mathcal{P}}{(\mathcal{P}+2\lambda_2)^2(\xi_0\mathcal{P}-\lambda_0)^2} \cdot (3\lambda_0^2\lambda_4 - \lambda_0\{4\lambda_2\mathcal{P}(1-3\xi_2)\xi_2 + \lambda_2^2(3+16\xi_2) + \mathcal{P}[6\lambda_4\lambda_0 + \mathcal{P}(1-3\xi_2)\xi_2]\} \right.$$

$$\left. + \xi_0[10\lambda_2^2\mathcal{P} + 10\lambda_2^3 + 3\lambda_4\xi_0\mathcal{P}^2 + \lambda_2\mathcal{P}^2(1-6\xi_2-6\xi_2^2)]) \right] - \frac{1}{48\pi^2}$$

$$\times Q_2 \left[\frac{\partial_t\mathcal{P}}{\mathcal{P}^2(\mathcal{P}+2\lambda_2)^3(\xi_0\mathcal{P}-\lambda_0)^3} \cdot (-18\lambda_0^3\mathcal{P}^2[-4\lambda_4\xi_2 + (\mathcal{P}+2\lambda_2)\xi_4] \right.$$

$$-6\lambda_0^2[3\lambda_2^3\mathcal{P} + 2\lambda_2^4 + \lambda_2^2\mathcal{P}^2(1+4\xi_2+10\xi_2^2)] + 6\lambda_0^2[9\xi_0\mathcal{P}^3(-4\lambda_4\xi_2+\xi_4\mathcal{P}) + \lambda_2\mathcal{P}^3(4\xi_2+19\xi_2^2+24\xi_2^3+18\xi_0\xi_4)]$$

$$+\lambda_0\xi_0\mathcal{P}[36\lambda_2^4 + 8\lambda_2^3\mathcal{P}(6+7\xi_2) - 12\lambda_2\mathcal{P}^3(\xi_2+18\xi_2^2+18\xi_2^3+9\xi_0\xi_4)] + \lambda_0\xi_0\mathcal{P}^3\{3\lambda_2^2(5+36\xi_2+20\xi_2^2)$$

$$+\mathcal{P}[216\xi_0\lambda_4\xi_2 + \mathcal{P}(10\xi_2+21\xi_2^2+36\xi_2^3-54\xi_0\xi_4)]\} - \xi_0^2\mathcal{P}^2[104\lambda_2^4 + 3\lambda_2^2\mathcal{P}^2(23-12\xi_2) - 6\lambda_2^3\mathcal{P}(-25+4\xi_2)]$$

$$\left. + 2\xi_0^2\lambda_2\mathcal{P}^5(-5+24\xi_2+51\xi_2^2+36\xi_2^3+18\xi_0\xi_4) + 3\xi_0^2\mathcal{P}^5(-24\lambda_4\xi_0\xi_2 - 7\xi_2^2\mathcal{P} - 12\xi_2^3\mathcal{P} + 6\xi_0\xi_4\mathcal{P})) \right]$$

$$+\frac{\partial_t\xi_0}{384\pi^2\xi_0^2} Q_1 \left[\frac{\mathcal{R}}{\mathcal{P}(\mathcal{P}+2\lambda_2)(\xi_0\mathcal{P}-\lambda_0)^2} \cdot \{40\lambda_2^2\xi_0^2\mathcal{P} + \xi_2\mathcal{P}[5\lambda_0^2 - 10\lambda_0\xi_0\mathcal{P} + 3\xi_0^2\mathcal{P}^2(-1+8\xi_2)] \right.$$

$$\left. - 2\lambda_2[-5\lambda_0^2\xi_2 + 10\lambda_0\xi_0\xi_2\mathcal{P} + \xi_0^2\mathcal{P}^2(-4+27\xi_2)]\} \right] - \frac{\partial_t\xi_0}{384\pi^2\xi_0^2}$$

$$\times Q_2 \left[\frac{\mathcal{R}}{\mathcal{P}^2(\mathcal{P}+2\lambda_2)^2(\xi_0\mathcal{P}-\lambda_0)^3} \cdot \{15\lambda_0^3(\mathcal{P}+2\lambda_2)^2\xi_2 - 3\lambda_0^2\xi_0(\mathcal{P}+2\lambda_2)(8\lambda_2^2+15\xi_2\mathcal{P}^2+22\lambda_2\xi_2\mathcal{P}) \right.$$

$$+\xi_0^3\mathcal{P}^2[-416\lambda_2^3 + 4\lambda_2^2\mathcal{P}(-92+61\xi_2)] + \xi_0^3\mathcal{P}^4[\xi_2\mathcal{P}(25-168\xi_2-288\xi_2^2) + 4\lambda_2(-20+79\xi_2+60\xi_2^2)]$$

$$\left. + \lambda_0\xi_0^2\mathcal{P}[144\lambda_2^3 - 20\lambda_2\xi_2\mathcal{P}^2(-5+12\xi_2)] + \lambda_0\xi_0^2\mathcal{P}^2[4\lambda_2^2(18+37\xi_2) + \xi_2\mathcal{P}^2(85+168\xi_2+288\xi_2^2)]\} \right]$$

$$-\frac{\partial_t\xi_2}{384\pi^2\xi_0} Q_1 \left[\frac{\mathcal{R}}{\mathcal{P}(\mathcal{P}+2\lambda_2)(\xi_0\mathcal{P}-\lambda_0)} \cdot \{-5\lambda_0(\mathcal{P}+2\lambda_2) + 3\xi_0\mathcal{P}(-\mathcal{P}-18\lambda_2+16\xi_2\mathcal{P})\} \right]$$

$$-\frac{\partial_t\xi_2}{384\pi^2\xi_0} Q_2 \left[\frac{\mathcal{R}}{\mathcal{P}^2(\mathcal{P}+2\lambda_2)^2(\xi_0\mathcal{P}-\lambda_0)^2} \cdot \{15\lambda_0^2(\mathcal{P}+2\lambda_2)^2 - 6\lambda_0\xi_0\mathcal{P}[12\lambda_2^2 + 8\lambda_2\mathcal{P}(2-5\xi_2) \right.$$

$$\left. + \mathcal{P}^2(5+28\xi_2+96\xi_2^2)] + \xi_0^2\mathcal{P}^2[-148\lambda_2^2 - 4\lambda_2\mathcal{P}(31+60\xi_2) + \mathcal{P}^2(-25+168\xi_2+576\xi_2^2)]\} \right], \tag{31}$$

$$\partial_t \lambda_4 = \frac{3}{16\pi^2} Q_2 \left[\frac{\partial_t \mathcal{P}}{(\mathcal{P}+2\lambda_2)^3(\xi_0\mathcal{P}-\lambda_0)^3} \cdot (\lambda_0^3[-24\lambda_4^2+5(\mathcal{P}+2\lambda_2)\lambda_6] \right.$$

$$+\lambda_0^2\{-24\lambda_2^3\xi_4+4\lambda_2^2[8\lambda_4(1+2\xi_2)+\mathcal{P}(-1+16\xi_2)\xi_4]\}+\lambda_0^2\mathcal{P}\{72\lambda_4^2\xi_0+\mathcal{P}(-15\lambda_6\xi_0+\xi_4\mathcal{P}[1+8\xi_2])\}\}$$

$$+2\lambda_0^2\lambda_2\mathcal{P}\{-4\lambda_4(1+8\xi_2+12\xi_2^2)+3[-5\lambda_6\xi_0+\mathcal{P}(1+8\xi_2)\xi_4]\}+\xi_0[40\lambda_2^5-40\mathcal{P}\lambda_2^4(-1+2\xi_2)$$

$$-24\mathcal{P}\lambda_2^3(-\lambda_4\xi_0+5\xi_2\mathcal{P})]+\xi_0^2\mathcal{P}^3[24\lambda_4^2\xi_0-5\lambda_6\xi_0\mathcal{P}+\lambda_4\mathcal{P}(1+8\xi_2+24\xi_2^2)]$$

$$+\xi_0\lambda_2\mathcal{P}^3\{-10\lambda_6\xi_0^2-2\lambda_4\xi_0(5+40\xi_2+24\xi_2^2)-\mathcal{P}[6\xi_2^2+52\xi_2^3+48\xi_2^4-8\xi_0\xi_4+\xi_2(1-16\xi_0\xi_4)]\}$$

$$+\lambda_0\{20\lambda_2^4(1+4\xi_2)-2\lambda_2^3[-12\lambda_4\xi_0+\mathcal{P}(1+24\xi_2+124\xi_2^2+12\xi_0\xi_4)]\}+2\lambda_0\lambda_2^2\mathcal{P}[2\lambda_4\xi_0(17+16\xi_2)$$

$$+3\mathcal{P}(\xi_2+4\xi_2^2+44\xi_2^3+2\xi_0\xi_4+16\xi_0\xi_2\xi_4)]+\lambda_0\lambda_2\mathcal{P}^2\{-30\lambda_6\xi_0^2-18\lambda_4\xi_0(1+8\xi_2+8\xi_2^2)+\mathcal{P}[28\xi_2^3-96\xi_2^4+14\xi_0\xi_4$$

$$+\xi_2(1+64\xi_0\xi_4)]\}+\lambda_0\mathcal{P}^2\{72\lambda_4^2\xi_0^2+\mathcal{P}\lambda_4\xi_0(1+8\xi_2+24\xi_2^2)+\mathcal{P}[-15\lambda_6\xi_0^2-\mathcal{P}(\xi_2+4\xi_2^3+24\xi_2^4-\xi_0\xi_4$$

$$\left.-8\xi_0\xi_2\xi_4)]\}) \right] + \frac{\partial_t\xi_0}{16\pi^2\xi_0^3}Q_2\left[\frac{\mathcal{R}}{\mathcal{P}(\mathcal{P}+2\lambda_2)^2(\xi_0\mathcal{P}-\lambda_0)^3} \cdot (\{60\lambda_2^4\xi_0^3\mathcal{P}-240\lambda_2^3\xi_0^3\xi_2\mathcal{P}^2 \right.$$

$$+\lambda_2^2[-8\lambda_0^3\xi_2^2+4\xi_0^4\mathcal{P}^2(9\lambda_4+7\xi_4\mathcal{P})-24\lambda_0\xi_0^2\mathcal{P}(\xi_2^2\mathcal{P}+\lambda_0\xi_4)]\}+\lambda_2^2\{8\lambda_0^2\xi_0(3\xi_2^2\mathcal{P}+\lambda_0\xi_4)$$

$$+\xi_0^3\mathcal{P}[\mathcal{P}^2(3+12\xi_2+380\xi_2^2)-12\lambda_0(3\lambda_4+\xi_4\mathcal{P})]\}+\mathcal{P}^2(-2\lambda_0^3+6\lambda_0^2\xi_0\mathcal{P})(\xi_2^2-\xi_0\xi_4)$$

$$+\mathcal{P}^3\{\xi_0^3\mathcal{P}[3\lambda_4\xi_0(1+8\xi_2+24\xi_2^2)+5\xi_2^2+12\xi_2^3+72\xi_2^4-5\xi_0\xi_4-24\xi_0\xi_2\xi_4]\}+3\lambda_0\xi_0^2\mathcal{P}^3[-\lambda_4\xi_0(1+8\xi_2+24\xi_2^2)$$

$$+\mathcal{P}(-2\xi_2^2+3\xi_0\xi_4+8\xi_0\xi_2\xi_4)]-2\lambda_2\mathcal{P}\{(4\lambda_0^3-12\lambda_0^2\xi_0\mathcal{P})(\xi_2^2-\xi_0\xi_4)-6\lambda_0\xi_0^2\mathcal{P}[\lambda_4\xi_0(1+8\xi_2)$$

$$+\mathcal{P}(-2\xi_2^2+\xi_0\xi_4+4\xi_0\xi_2\xi_4)]\}-2\lambda_2\xi_0^3\mathcal{P}^2\{6\lambda_4\xi_0(1+8\xi_2)+\mathcal{P}[8\xi_2^2+132\xi_2^3-2\xi_0\xi_4+\xi_2(3+24\xi_0\xi_4)]\}) \right]$$

$$+\frac{\partial_t\xi_2}{16\pi^2\xi_0^2}Q_2\left[\frac{\mathcal{R}}{\mathcal{P}(\mathcal{P}+2\lambda_2)^2(\xi_0\mathcal{P}-\lambda_0)^2} \cdot (120\lambda_2^3\xi_0^2\mathcal{P}-2\lambda_2^2[4\lambda_0^2\xi_2-8\lambda_0\xi_0\xi_2\mathcal{P}+\xi_0^2\mathcal{P}^2(3+190\xi_2)] \right.$$

$$+\lambda_2\mathcal{P}[-8\lambda_0^2\xi_2+16\lambda_0\xi_0\xi_2\mathcal{P}+48\xi_0^3\mathcal{P}(2\lambda_4+\xi_4\mathcal{P})]+\xi_0^2[\mathcal{P}^2(3+16\xi_2+396\xi_2^2)-48\lambda_0(2\lambda_4+\xi_4\mathcal{P})]$$

$$+\mathcal{P}^2\{-2\lambda_0^2\xi_2-\xi_0^2\mathcal{P}[24\lambda_4\xi_0(1+6\xi_2)+\mathcal{P}(\xi_2+18\xi_2^2+144\xi_2^3-24\xi_0\xi_4)]\}+4\lambda_0\xi_0\mathcal{P}^2[6\lambda_4\xi_0(1+6\xi_2)$$

$$\left. +\mathcal{P}(\xi_2-6\xi_0\xi_4)])]+\frac{\partial_t\xi_4}{16\pi^2\xi_0}Q_2\left[\frac{\mathcal{R}}{(\mathcal{P}+2\lambda_2)^2(\xi_0\mathcal{P}-\lambda_0)^2} \cdot \{-2\lambda_0(\mathcal{P}+2\lambda_2)+\xi_0\mathcal{P}(5\mathcal{P}-14\lambda_2+24\xi_2\mathcal{P})\} \right]. \tag{32}$$

This system of linear equations can be solved for $\beta_{2i}^\lambda = \partial_t\lambda_{2i}$ and $\beta_{2i}^\xi = \partial_t\xi_{2i}$. Note that we have not made any truncation of the functions V and F: no couplings have been assumed to be zero.

We do not exhibit the equations for the higher couplings. By means of algebraic manipulation software we have calculated the beta functions up to λ_8 and ξ_6. The general pattern however, is, already clear from the equations shown here. The functions Q_1 and Q_2 always contain denominators involving only the couplings λ_0, λ_2, and ξ_0. Aside from these couplings appearing inside Q_1 and Q_2, the equation for β_{2i}^λ is a polynomial in the couplings $\lambda_0,\lambda_2,\ldots,\lambda_{2i+2}$ and $\xi_0,\xi_2,\ldots,\xi_{2i}$, with the functions Q_1 and Q_2 as coefficients, while the equation for β_{2i}^ξ is a polynomial in the couplings $\lambda_0,\lambda_2,\ldots,\lambda_{2i+2}$ and $\xi_0,\xi_2,\ldots,\xi_{2i+2}$, with the functions Q_1 and Q_2 as coefficients. When these equations

are solved, the beta function β_{2i}^λ is written as a rational function of $\lambda_0,\lambda_2,\ldots,\lambda_{2i+2}$ and $\xi_0,\xi_2,\ldots,\xi_{2i}$, while the beta function β_{2i}^ξ is written as a rational function of $\lambda_0,\lambda_2,\ldots,\lambda_{2i+2}$ and $\xi_0,\xi_2,\ldots,\xi_{2i+2}$.

It is clear that the system $\beta_{2i}^\lambda=0$, $\beta_{2i}^\xi=0$ admits a FP for which all couplings λ_{2i} and ξ_{2i} vanish for $i>0$, while for $i=0$ $\lambda_0=2\kappa\Lambda$, and $\xi_0=\kappa$ have the same values that they would have in the presence of a single free scalar field, as discussed in [13] (these values are numerically very close to those of pure gravity, discussed in [9]), namely, for $a=2$,

$$\lambda_{0*}=0.0080022,$$

$$\xi_{0*}=0.023500. \tag{33}$$

To compare with the results of [9], we define the dimensionless variables $\lambda = \Lambda/k^2 = \lambda_0/2\xi_0$ and $g = Gk^2 = 1/16\pi\xi_0$. At the GMFP

$$\lambda_* = \frac{\lambda_{0*}}{2\xi_{0*}} = 0.1703,$$

$$g_* = \frac{1}{16\pi\xi_{0*}} = 0.8466. \tag{34}$$

These values differ from those in Eq. (5.25) of [9] on two accounts: they are calculated for different values of the cutoff parameter a, and here the FP is shifted due to the presence of the scalar field. When these factors are taken into account there is perfect agreement. [See Fig. 5(b) of [9] for the dependence of results on $s = 2a$ and compare with Fig. 8 below.]

This FP can be viewed alternatively as the FP of pure gravity, slightly shifted due to the presence of a free, massless, minimally coupled scalar, or as the Gaussian FP of the pure scalar theory, generalized to include gravitational interactions. It is remarkable that matter remains ''non-self-interacting'' at this FP, and that the only nonzero couplings are those that affect only the gravitational degrees of freedom. (This goes some way toward justifying the assumption in [13] that matter fields are non-self-interacting.) For want of a better terminology, we shall refer to this FP as the Gaussian-matter FP.

The issue arises whether the coupled system of equations admits other nontrivial FP's. The complexity of the equations has prevented us from deriving definite results on this issue. We have looked for other FP's using numerical methods in a five-parameter truncation of the theory containing the couplings λ_{2n} for $n = 0,1,2$ and the couplings ξ_{2n} for $n = 0,1$. Our method consists in considering a grid in the space of all parameters and evaluating the beta functions by numerical integration at a point and then at all neighboring points. If all beta functions change sign simultaneously when going from a point to a neighbor, then generically there will be a FP somewhere near the link between the two points. The region is then examined with a finer grid until the position of the FP is located with sufficient accuracy. We started off with the 2×2 grid given by λ_0 and ξ_0, confirming the results of [9]; we then added one by one the other variables involved in the five-parameter truncation, getting increasingly complicated systems of equations.

Because of the complexity of the beta functions, the numerical evaluation takes considerable time. The largest range we have explored is a 5×5 grid with the dimensionless cosmological constant λ ranging from 0.010 to 0.045 in steps of 0.005; the dimensionless Newton constant g ranging from 0.01 to 0.06 in steps of 0.01; the dimensionless scalar mass $2\lambda_2$ ranging from -1 to 1 in steps of 0.2; the dimensionless quartic scalar coupling λ_4 ranging from -15 to 5 in steps of 1; the dimensionless scalar-tensor coupling $\xi = \xi_2$ ranging from -5 to 5 in steps of 1. This makes a lattice with more than 120 000 points. Many other attempts have been tried with finer lattices and/or fewer parameters. We did find some nontrivial solutions when considering fewer than five param-

eters, but none of them survived the addition of an extra parameter, so that we had to conclude that they were all spurious FP's due to the truncation. As a further check we also resorted to series expansions around the GMFP or one of the spurious FP's mentioned above. All the results obtained in this way are perfectly consistent with the other calculations. The outcome of all these efforts is that no FP other than the GMFP was found.

This is of course not a proof that a FP does not exist in this range of couplings. For example, a beta function may change sign twice on a link, once from positive to negative and once from negative to positive, and if the distance between the zeros is smaller than the size of the step, it may well escape detection by our methods. Nevertheless, after this numerical work, we consider it quite unlikely that another nontrivial FP exists in the range of values for the couplings that we have considered.

This result is corroborated by the following observation. If one does not truncate the functions V and F to polynomials, as in the pure scalar case the structure of the beta functions seems to allow for a recursive solution depending on two free parameters. If we fix arbitrary values for λ_0 and ξ_0, from Eqs. (28),(29) one derives λ_2 and ξ_2; substituting them into Eqs. (30),(31), one can solve for λ_4 and ξ_4, and so on. This will determine the functions $V = V_*$ and $F = F_*$ up to two arbitrary parameters. It will be interesting to analyze this in detail and to see whether the resulting functions V_* and F_* are regular or still present the problems discussed in [19]. In any case, it seems highly unlikely that the solutions will be polynomial. This point of view also sheds a different light on the FP found in [8,9]. The values of λ_{0*} and ξ_{0*} at the GMFP are the only ones for which $\lambda_{2*} = 0$ and $\xi_{2*} = 0$, and as a consequence all the higher couplings turn out to be zero, in accordance with the truncation made there. We shall not pursue this issue any more here. In the rest of this paper we shall restrict our attention to the GMFP, which is a special member of this family of solutions, and is definitely a physically acceptable solution.

IV. LINEARIZED FLOW AROUND THE GMFP

Having established the existence of the GMFP in the truncation defined by the action (2), we have to study its properties, in particular, to find the dimension of the critical surface.

We begin by calculating the matrix M_{ij}. Again, we order the couplings in order of decreasing mass dimension: $\lambda_0, \xi_0, \lambda_2, \xi_2, \lambda_4, \xi_4, \lambda_6, \ldots$. As in the pure scalar theory, due to the functional dependences of β_{2i}^λ and β_{2i}^ξ on the couplings, an infinite triangle above the diagonal is zero. Furthermore, due to the fact that only the ''purely gravitational'' couplings λ_0 and ξ_0 are nonzero at the GMFP, an infinite triangle below the diagonal is zero. The structure of the matrix M is therefore remarkably simple:

$$\begin{pmatrix} M_{00} & M_{02} & 0 & 0 & \cdots \\ 0 & M_{22} & M_{24} & 0 & \cdots \\ 0 & 0 & M_{44} & M_{46} & \cdots \\ 0 & 0 & 0 & M_{66} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \tag{35}$$

where each one of the nonzero entries is a 2×2 matrix of the form

$$M_{ij}=\begin{pmatrix} \dfrac{\partial\beta^{\lambda}_{(2i)}}{\partial\lambda_{(2j)}} & \dfrac{\partial\beta^{\lambda}_{(2i)}}{\partial\xi_{(2j)}} \\[2ex] \dfrac{\partial\beta^{\xi}_{(2i)}}{\partial\lambda_{(2j)}} & \dfrac{\partial\beta^{\xi}_{(2i)}}{\partial\xi_{(2j)}} \end{pmatrix}. \tag{36}$$

For the calculation of the dimension of the critical surface we need to count the number of negative eigenvalues of the matrix M. The eigenvalue problem for the matrix M could be turned into recursion relations for λ_{2i} and ξ_{2i}, as for the pure

scalar theory. However, if we restrict ourselves to solutions where V and F are polynomials, given the almost-block-diagonal structure of M, the eigenvalues of M are just the eigenvalues of the diagonal blocks M_{ii}. Explicitly, the diagonal blocks have the following form:

$$M_{ii}=\begin{pmatrix} 2(i-2) & 0 \\ 0 & 2(i-1) \end{pmatrix}+\begin{pmatrix} \delta M_{\lambda\lambda} & \delta M_{\lambda\xi} \\ \delta M_{\xi\lambda} & \delta M_{\xi\xi} \end{pmatrix}, \tag{37}$$

where the first term contains the canonical dimensions of the couplings and the second term, which contains the quantum corrections, has the following form:

$$\delta M_{\lambda\lambda}=\frac{\xi_0}{16\pi^2}\left(3Q_2\left[\frac{\partial_t\mathcal{P}}{(\lambda_0-\mathcal{P}\xi_0)^2}\right]-\frac{1}{\Delta^2}\cdot\left\{-3Q_2\left[\frac{\mathcal{R}}{(\lambda_0-\xi_0\mathcal{P})^2}\right]\cdot\left(Q_1\left[\frac{\partial_t\mathcal{P}(-3\lambda_0+11\xi_0\mathcal{P})}{\mathcal{P}(-\lambda_0+\xi_0\mathcal{P})}\right]\right.\right.\right.$$

$$+5Q_2\left[\frac{\partial_t\mathcal{P}(3\lambda_0^2-6\lambda_0\xi_0\mathcal{P}+11\xi_0^2\mathcal{P}^2)}{\mathcal{P}^2(\lambda_0-\xi_0\mathcal{P})^2}\right]\right)\cdot\Delta-8Q_2\left[\frac{\mathcal{R}(-2\lambda_0+5\xi_0\mathcal{P})}{\mathcal{P}(-\lambda_0+\xi_0\mathcal{P})}\right]\cdot\left(Q_1\left[\frac{\partial_t\mathcal{P}}{(\lambda_0-\xi_0\mathcal{P})^2}\right]\right.$$

$$+10\xi_0Q_2\left[\frac{\partial_t\mathcal{P}}{(\xi_0\mathcal{P}-\lambda_0)^3}\right]\right)\cdot\Delta-8Q_2\left[\frac{\mathcal{R}(-2\lambda_0+5\xi_0\mathcal{P})}{\mathcal{P}(-\lambda_0+\xi_0\mathcal{P})}\right]\cdot\left(Q_1\left[\frac{\mathcal{R}}{(\lambda_0-\xi_0\mathcal{P})^2}\right]\right.$$

$$+10\xi_0Q_2\left[\frac{\mathcal{R}}{(\xi_0\mathcal{P}-\lambda_0)^3}\right]\right)\cdot\left(Q_1\left[\frac{\partial_t\mathcal{P}(-3\lambda_0+11\xi_0\mathcal{P})}{\mathcal{P}(-\lambda_0+\xi_0\mathcal{P})}\right]+5Q_2\left[\frac{\partial_t\mathcal{P}(3\lambda_0^2-6\lambda_0\xi_0\mathcal{P}+11\xi_0^2\mathcal{P}^2)}{\mathcal{P}^2(\lambda_0-\xi_0\mathcal{P})^2}\right]\right)\right\}\right), \tag{38}$$

$$\delta M_{\lambda\xi}=-\frac{3\lambda_0}{16\pi^2}Q_2\left[\frac{\partial_t\mathcal{P}}{(\lambda_0-\xi_0\mathcal{P})^2}\right]-\frac{1}{8\pi^2\Delta}Q_2\left[\mathcal{R}\left(\frac{2}{\xi_0\mathcal{P}}+\frac{3}{\xi_0\mathcal{P}-\lambda_0}\right)\right]\left\{Q_1\left[\frac{\partial_t\mathcal{P}(-3\lambda_0+11\xi_0\mathcal{P})}{\mathcal{P}(-\lambda_0+\xi_0\mathcal{P})}\right]\right.$$

$$+5Q_2\left[\frac{\partial_t\mathcal{P}(3\lambda_0^2-6\lambda_0\xi_0\mathcal{P}+11\xi_0^2\mathcal{P}^2)}{\mathcal{P}^2(\lambda_0-\xi_0\mathcal{P})^2}\right]\right\}$$

$$-\frac{1}{32\pi^2\Delta^2}\left\{6\lambda_0\xi_0\cdot\Delta\cdot Q_2\left[\frac{\mathcal{R}}{(\lambda_0-\xi_0\mathcal{P})^2}\right]\cdot\left(Q_1\left[\frac{\partial_t\mathcal{P}(-3\lambda_0+11\xi_0\mathcal{P})}{\mathcal{P}(-\lambda_0+\xi_0\mathcal{P})}\right]\right.\right.$$

$$+5Q_2\left[\frac{\partial_t\mathcal{P}(3\lambda_0^2-6\lambda_0\xi_0\mathcal{P}+11\xi_0^2\mathcal{P}^2)}{\mathcal{P}^2(\lambda_0-\xi_0\mathcal{P})^2}\right]\right)-2Q_2\left[\frac{\mathcal{R}(-2\lambda_0+5\xi_0\mathcal{P})}{\mathcal{P}(-\lambda_0+\xi_0\mathcal{P})}\right]\cdot\left(Q_1\left[\frac{\partial_t\mathcal{P}(-3\lambda_0+11\xi_0\mathcal{P})}{\mathcal{P}(-\lambda_0+\xi_0\mathcal{P})}\right]\right.$$

$$+5Q_2\left[\frac{\partial_t\mathcal{P}(3\lambda_0^2-6\lambda_0\xi_0\mathcal{P}+11\xi_0^2\mathcal{P}^2)}{\mathcal{P}^2(\lambda_0-\xi_0\mathcal{P})^2}\right]\right)\cdot\left(Q_1\left[\frac{\partial_t\mathcal{P}(5\lambda_0^2-10\lambda_0\xi_0\mathcal{P}-3\xi_0\mathcal{P}^2)}{\mathcal{P}(\lambda_0-\xi_0\mathcal{P})^2}\right]\right.$$

$$\left.-5Q_2\left[\frac{\mathcal{R}(3\lambda_0^3-9\lambda_0^2\xi_0\mathcal{P}+17\lambda_0\xi_0^2\mathcal{P}+5\xi_0^3\mathcal{P}^3)}{\mathcal{P}^2(-\lambda_0+\xi_0\mathcal{P})^3}\right]\right)$$

$$-2\cdot\Delta\cdot Q_2\left[\frac{\mathcal{R}(-2\lambda_0+5\xi_0\mathcal{P})}{\mathcal{P}(-\lambda_0+\xi_0\mathcal{P})}\right]\cdot\left(Q_1\left[\frac{\partial_t\mathcal{P}(3\lambda_0^2-22\xi_0\lambda_0\mathcal{P}+11\xi_0^2\mathcal{P}^2)}{\mathcal{P}(\lambda_0-\xi_0\mathcal{P})^2}\right]\right.$$

$$\left.\left.+5Q_2\left[\frac{\partial_t\mathcal{P}(-3\lambda_0^3+9\lambda_0^2\xi_0\mathcal{P}-33\lambda_0\xi_0^2\mathcal{P}^2+11\xi_0^3\mathcal{P}^3)}{\mathcal{P}^2(-\lambda_0+\xi_0\mathcal{P})^3}\right]\right)\right\}, \tag{39}$$

$$\delta M_{\xi\lambda} = -2\frac{\xi_0}{\Delta^2} \cdot \left\{ -4\xi_0\left(Q_1\left[\frac{\partial_t \mathcal{P}}{(\lambda_0 - \xi_0\mathcal{P})^2}\right] + 10\xi_0 Q_2\left[\frac{\partial_t \mathcal{P}}{(-\lambda_0 + \xi_0\mathcal{P})^3}\right]\right)\cdot\Delta - 4\xi_0\left(Q_1\left[\frac{\partial_t\mathcal{P}(-3\lambda_0 + 11\xi_0\mathcal{P})}{\mathcal{P}(-\lambda_0 + \xi_0\mathcal{P})}\right]\right.\right.$$
$$\left.\left. + 5Q_2\left[\frac{\partial_t\mathcal{P}(3\lambda_0^2 - 6\lambda_0\xi_0\mathcal{P} + 11\xi_0^2\mathcal{P}^2)}{\mathcal{P}^2(\lambda_0 - \xi_0\mathcal{P})^2}\right]\right)\cdot\left(Q_1\left[\frac{\mathcal{R}}{(\lambda_0 - \xi_0\mathcal{P})^2}\right] + 10Q_2\left[\frac{\mathcal{R}}{(-\lambda_0 + \xi_0\mathcal{P})^3}\right]\right)\right\}, \tag{40}$$

$$\delta M_{\xi\xi} = -\frac{1}{\Delta^2}\cdot\left\{ -\left(Q_1\left[\frac{\partial_t\mathcal{P}(-3\lambda_0 + 11\xi_0\mathcal{P})}{\mathcal{P}(-\lambda_0 + \xi_0\mathcal{P})}\right]\right.\right.$$
$$\left. + 5Q_2\left[\frac{\partial_t\mathcal{P}(3\lambda_0^2 - 6\lambda_0\xi_0\mathcal{P} + 11\xi_0^2\mathcal{P}^2)}{\mathcal{P}^2(\lambda_0 - \xi_0\mathcal{P})^2}\right]\right)\cdot\left(Q_1\left[\frac{\mathcal{R}(5\lambda_0 - 10\lambda_0\xi_0\mathcal{P} - 3\xi_0^2\mathcal{P}^2)}{\mathcal{P}(\lambda_0 - \xi_0\mathcal{P})^2}\right]\right.$$
$$\left. - 5Q_2\left[\frac{\mathcal{R}(3\lambda_0^3 - 9\lambda_0^2\xi_0\mathcal{P} + 17\lambda_0\xi_0^2\mathcal{P}^2 + 5\xi_0^3\mathcal{P}^3)}{\mathcal{P}^2(-\lambda_0 + \xi_0\mathcal{P})^3}\right]\right) + \left(Q_1\left[\frac{\partial_t\mathcal{P}(-3\lambda_0 + 11\xi_0\mathcal{P})}{\mathcal{P}(-\lambda_0 + \xi_0\mathcal{P})}\right]\right.$$
$$\left. + 5Q_2\left[\frac{\partial_t\mathcal{P}(3\lambda_0^2 - 6\lambda_0\xi_0\mathcal{P} + 11\xi_0^2\mathcal{P}^2)}{\mathcal{P}^2(\lambda_0 - \xi_0\mathcal{P})^2}\right]\right)\cdot\Delta - \left(Q_1\left[\frac{\partial_t\mathcal{P}(3\lambda_0^2 - 22\lambda_0\xi_0\mathcal{P} + 11\xi_0^2\mathcal{P}^2)}{\mathcal{P}(\lambda_0 - \xi_0\mathcal{P})^2}\right]\right.$$
$$\left.\left. - 5Q_2\left[\frac{\partial_t\mathcal{P}(-3\lambda_0^3 + 9\lambda_0^2\xi_0\mathcal{P} - 33\lambda_0\xi_0^2\mathcal{P}^2 + 11\xi_0^3\mathcal{P}^3)}{\mathcal{P}^2(-\lambda_0 + \xi_0\mathcal{P})^3}\right]\right)\cdot\Delta\right\}, \tag{41}$$

where

$$\Delta = Q_1\left[\frac{\mathcal{R}(5\lambda_0 + 3\xi_0\mathcal{P})}{\mathcal{P}(\lambda_0 - \xi_0\mathcal{P})}\right] + 5Q_2\left[\frac{\mathcal{R}(3\lambda_0^2 - 6\lambda_0\xi_0\mathcal{P} - 5\xi_0^2\mathcal{P}^2)}{\mathcal{P}^2(\lambda_0 - \xi_0\mathcal{P})^2}\right] + 384\pi^2\xi_0. \tag{42}$$

The most remarkable property of these quantum corrections is that they are independent of i, so that the eigenvalues of M_{2i2i} simply grow by 2 whenever i is increased by 1. For example, choosing the cutoff with $a = 2$, we have the following numerical results:

$$M_{00} = \begin{pmatrix} 1.1257 & -2.5192 \\ 8.1295 & -5.3604 \end{pmatrix}, \tag{43}$$

which has eigenvalues $-2.1173 \pm 3.1563i$;

$$M_{22} = \begin{pmatrix} 3.1257 & -2.5192 \\ 8.1295 & -3.3604 \end{pmatrix}, \tag{44}$$

with eigenvalues $-0.1173 \pm 3.1563i$;

$$M_{44} = \begin{pmatrix} 5.1257 & -2.5192 \\ 8.1295 & -1.3604 \end{pmatrix}, \tag{45}$$

with eigenvalues $1.8826 \pm 3.1563i$;

$$M_{66} = \begin{pmatrix} 7.1257 & -2.5192 \\ 8.1295 & 0.6396 \end{pmatrix}, \tag{46}$$

with eigenvalues $3.8826 \pm 3.1563i$; and so on.

The off-diagonal blocks M_{ii+1} in Eq. (35) do not affect the eigenvalues but determine the mixing between the couplings. Numerically, we have

$$M_{02} = \begin{pmatrix} -0.005036 & -0.002264 \\ 0.002736 & -0.007585 \end{pmatrix}, \tag{47}$$

$$M_{24} = \begin{pmatrix} -0.03021 & -0.01359 \\ 0.01642 & -0.04551 \end{pmatrix}, \tag{48}$$

$$M_{46} = \begin{pmatrix} -0.07554 & -0.03340 \\ 0.04104 & -0.1138 \end{pmatrix}, \tag{49}$$

and so on.

The first two (complex conjugate) eigenvectors have components

$$
\begin{pmatrix}
-0.3486 \pm 0.3392i \\
0.8737 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\cdots
\end{pmatrix}. \qquad (50)
$$

They are a mixing of λ_0 and ξ_0; the corresponding (complex conjugate) eigenvalues have negative real part -2.1173 and therefore these are relevant couplings.

The second and third eigenvectors have components

$$
\begin{pmatrix}
(-16.59 \mp 5.343i) \times 10^{-4} \\
(-2.970 \pm 1.136i) \times 10^{-3} \\
-0.3485 \pm 0.3392i \\
0.8736 \\
0 \\
0 \\
0 \\
0 \\
\cdots
\end{pmatrix}. \qquad (51)
$$

They are essentially a mixing of λ_2 and ξ_2, with small contributions from λ_0 and ξ_0; the corresponding (complex conjugate) eigenvalues have negative real part -0.1173 and therefore these couplings are also relevant. Since they lie very close to the plane spanned by the relevant coupling λ_2 (canonical dimension 2) and the marginal coupling ξ_2 (canonical dimension 0), we can say by a slight abuse of language that the quantum corrections change the dimension of λ_2 and ξ_2 making them both relevant.

The fifth and sixth eigenvectors have components

$$
\begin{pmatrix}
(19.34 \mp 2.834i) \times 10^{-6} \\
(2.653 \mp 2.310i) \times 10^{-5} \\
(-9.951 \mp 3.203i) \times 10^{-3} \\
(-17.81 \pm 6.820i) \times 10^{-3} \\
-0.3485 \pm 0.3392i \\
0.8727 \\
0 \\
0 \\
\cdots
\end{pmatrix}. \qquad (52)
$$

They are essentially a mixing of λ_4 and ξ_4, with small contributions from λ_0, ξ_0, λ_2, and ξ_2; the corresponding (com-plex conjugate) eigenvalues have positive real part 1.8826 and therefore these couplings are irrelevant. Since they lie very close to the plane spanned by the marginal coupling λ_4 (canonical dimension 0) and the irrelevant coupling ξ_4 (canonical dimension -2), we can say by a slight abuse of language that the quantum corrections change the dimension of λ_4 and ξ_4, making them both irrelevant.

The pattern continues. The eigenvalues come in complex conjugate pairs, and are formed by mixing the couplings λ_{2i} and ξ_{2i}, with small contributions from the lower couplings. The eigenvalues also occur in complex conjugate pairs, and are equal to (minus) the canonical dimensions of the couplings [$2(i-2)$ and $2(i-1)$, respectively] plus a quantum correction. The correction is positive for the couplings of the series λ_{2i} and negative for those of the series ξ_{2i}, and the resulting dimension is always contained between those of the two main couplings that enter into the mix.

All eigenvalues differ from the first two by multiples of 2. In particular, all the eigenvalues from the fifth onward have positive real parts, so that the dimension of the critical surface is 4. The naive expectation based on canonical dimensions would have been 5 (or 3, if we do not count the two marginal couplings). The quantum corrections modify the dimension of the two marginal couplings λ_4 and ξ_2 so that ξ_2 (after mixing with λ_2) becomes relevant while λ_4 (after mixing with ξ_4) becomes irrelevant.

V. EFFECT OF OTHER MATTER FIELDS

In this section, we assume that in addition to the graviton and the scalar field discussed in the previous sections there are $n_S - 1$ new real scalar fields, n_W Weyl fields, n_M Maxwell fields, and n_{RS} (Majorana) Rarita-Schwinger fields, all minimally coupled. We neglect all masses and interactions of these additional matter fields. The only interactions are the ones discussed in the previous sections. This generalizes the results of [13] where only the couplings λ_0 and ξ_0 were taken into account. We also give some more details of the calculations.

In the presence of these new fields, Eqs. (28),(29) for the beta functions are modified by the addition of the following terms:

$$
\delta \partial_t \lambda_0 = \frac{1}{32\pi^2} (n_S - 2n_W + 2n_M - 4n_{RS}) Q_2\left[\frac{\partial_t \mathcal{P}}{\mathcal{P}}\right], \quad (53)
$$

$$
\delta \partial_t \xi_0 = \frac{1}{384\pi^2}\Bigg\{ (-2n_S + 4n_W - 4n_M) Q_1\left[\frac{\partial_t \mathcal{P}}{\mathcal{P}}\right]
$$

$$
+ (-6n_W + 9n_M - 16n_{RS}) Q_2\left[\frac{\partial_t \mathcal{P}}{\mathcal{P}^2}\right] \Bigg\}. \quad (54)
$$

Since the contribution of the new fields to the effective action is independent of ϕ, they do not affect at all the beta functions of all couplings λ_{2i} and ξ_{2i} for $i \geq 1$.

The equations for the couplings λ_{2i} for $i \geq 1$ and ξ_{2i} for $i \geq 1$ are automatically satisfied at the GMFP. Therefore the only equations that remain to solve are the ones for λ_0 and

ξ_0. For the sake of comparison with [13] we will use the couplings λ and g in place of λ_0 and ξ_0. The system of these two equations is the same as the one discussed in [13], and therefore the values of λ_0 and ξ_0 at the GMFP coincide with the ones calculated therein.

We recall some of the calculations in [13]. For the purpose of finding the fixed points, one can use the following trick. We observe that at a fixed point $\partial_t\xi_0/\xi_0 = -\partial_t g/g = 2$. Therefore, the equations for the fixed points are equivalent to another, simpler, set of equations which is obtained by replacing $\partial_t\xi_0/\xi_0$ with 2 in the RHS's of Eqs. (28),(29). Then the equation $\beta^\lambda = 0$ can be replaced by

$$g \cdot c(\lambda) - 2\lambda = 0, \tag{55}$$

where $c(\lambda)$ is obtained by formally replacing G with 1 and $\partial_t G$ with -2 in the expression for $\partial_t\Lambda/k^4$. When $c(\lambda) \neq 0$, we can solve Eq. (55) for g and substitute the result into $\beta^g = 0$. We shall denote

$$h(\lambda) = \beta^g\left(\lambda, \frac{2\lambda}{c(\lambda)}\right), \tag{56}$$

so that the zeros of h correspond to the FP's.

The general behavior of the function h is controlled by the values of the two parameters Δ' and τ, which in turn depend on the type and number of matter fields. The parameter Δ' is equal to $\Delta + \sigma$, where $\Delta = n_b - n_f$ is the difference of the total numbers of bosonic and fermionic degrees of freedom ($n_f = 2n_W + 4n_{RS}$ and $n_b = n_S + 2n_M + 2$) and $\sigma \equiv 20Q_2[\mathcal{R}/\mathcal{P}]/Q_2[\partial_t\mathcal{P}/\mathcal{P}]$ is approximately equal to 3.64 (for $a=2$).

The value of the cosmological constant at the FP, λ_*, is zero on the hyperplane $\Delta' = 0$. To see this, note that when $c(\lambda) = 0$, Eq. (55) implies $\lambda = 0$. Therefore, if $c(0) \neq 0$ the only solution with $\lambda_* = 0$ is the Gaussian FP, but if $c(0) = 0$ we can have a GMFP with $\lambda_* = 0$. Explicitly,

$$c(0) = \frac{1}{4\pi k^4}\left((n_b - n_f)Q_2\left[\frac{\partial_t\mathcal{P}}{\mathcal{P}}\right] + 20Q_2\left[\frac{\mathcal{R}}{\mathcal{P}}\right]\right), \tag{57}$$

so that the condition for the existence of a non-Gaussian FP with zero cosmological constant is precisely

$$\Delta' = 0. \tag{58}$$

Due to the irrationality of σ, there is in general no combination of matter fields that satisfies this condition; however, the hyperplane defined by Eq. (58) has an important physical significance: it separates the regions with positive and negative λ_*, as will become clear below.

The function h tends to zero when $\lambda \to \min_{z \in [0,\infty]}(\mathcal{P})/2$. However, this point does not correspond to a FP: For this value of λ the denominators in the functions Q_1 and Q_2 appearing in the beta functions vanish and the beta functions themselves blow up. Moreover, it was shown in [6] that the Ward identities break down near this point. Consequently, only values of λ strictly less than $\min_{z \in [0,\infty]}(\mathcal{P})/2$ will be considered.

The function $h(\lambda)$ always has a zero at the origin, corresponding to the Gaussian FP. The derivative of $h(\lambda)$ at the origin is given by

$$h'(0) = \frac{4}{c(0)} = \frac{16\pi}{Q_2[\partial_t\mathcal{P}/\mathcal{P}]} \cdot \frac{1}{\Delta'} \tag{59}$$

and therefore has the same sign as Δ'. When $\Delta' > 0$, the function $h(\lambda)$ tends to $-\infty$ for λ somewhere between 0 and $\min_{z \in [0,\infty]}(\mathcal{P})/2$ [namely, where $c(\lambda) = 0$]. Conseqeuntly, there exists a non-Gaussian FP (NGFP) with positive λ_*. On the other hand, when $\Delta' < 0$, h has no positive zeros and the existence of the NGFP for negative λ_* hinges on the asymptotic behavior of h for $\lambda \to -\infty$: it exists only if h tends to a negative asymptote. The asymptotic behavior of h is given by $\lim_{\lambda \to -\infty} h(\lambda) = 192\pi/\tau$, where

$$\tau = \tau_0 + n_S\tau_S + n_W\tau_W + n_M\tau_M + n_{RS}\tau_{RS}, \tag{60}$$

$$\tau_0 = -5Q_1\left[\frac{\partial_t\mathcal{P}}{\mathcal{P}}\right] - 15Q_2\left[\frac{\partial_t\mathcal{P}}{\mathcal{P}^2}\right] + 10Q_1\left[\frac{\mathcal{R}}{\mathcal{P}}\right]$$
$$+ 30Q_2\left[\frac{\mathcal{R}}{\mathcal{P}^2}\right] \approx -12.82,$$

$$\tau_S = 2Q_1\left[\frac{\partial_t\mathcal{P}}{\mathcal{P}}\right] \approx 3.58,$$

$$\tau_W = -4Q_1\left[\frac{\partial_t\mathcal{P}}{\mathcal{P}}\right] + 6Q_2\left[\frac{\partial_t\mathcal{P}}{\mathcal{P}^2}\right] \approx -1.62,$$

$$\tau_M = Q_1\left[\frac{\partial_t\mathcal{P}}{\mathcal{P}}\right] - 9Q_2\left[\frac{\partial_t\mathcal{P}}{\mathcal{P}^2}\right] \approx -6.52,$$

$$\tau_{RS} = 16Q_2\left[\frac{\partial_t\mathcal{P}}{\mathcal{P}^2}\right] \approx 14.79.$$

(The numerical values are given for $a=2$.)

Depending on the sign of the two parameters Δ' and τ, the space spanned by the variables n_S, n_W, n_M and n_{RS} can be divided into four regions that we shall label as follows:

	$\tau < 0$	$\tau > 0$
$\Delta' < 0$	III	IV
$\Delta' > 0$	I	II

The behavior of the function h is shown in Fig. 1 for pure gravity, which lies in region I. There are no zeros for negative λ, since h grows monotonically from the asymptote at $t \to -\infty$ to zero at the origin (Gaussian FP). It then has a positive zero and tends to $-\infty$. There is another apparent zero for $\lambda \approx 0.4$, but it is not an acceptable solution: it corresponds to the point where the denominators in the function Q_i vanish. Thus, in region I there is always a single FP with positive λ_*.
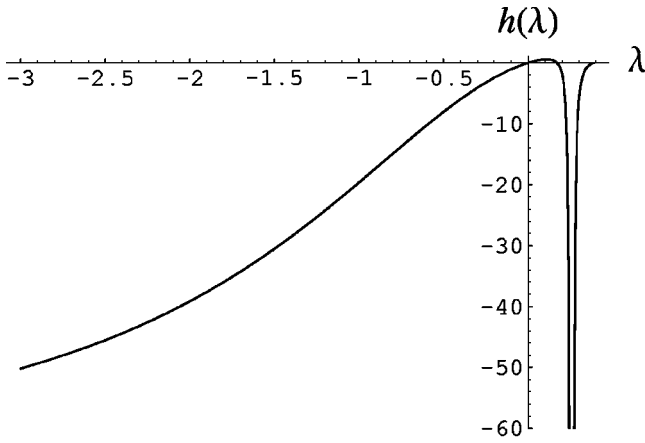
FIG. 1. Plot of the function h for $n_S=0, n_W=0, n_M=0$ (region I); the asymptotic value of h for $t \to -\infty$ is -65.3.

FIG. 2. Plot of the function h for $n_S=100, n_W=40, n_M=0$ (region II). The asymptotic value of h for $t \to -\infty$ is 0.939.
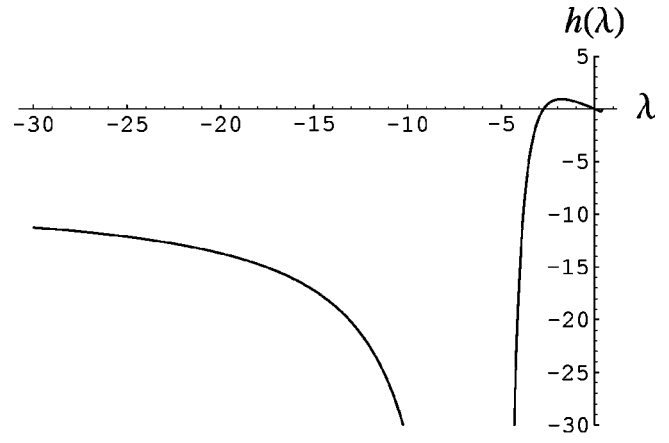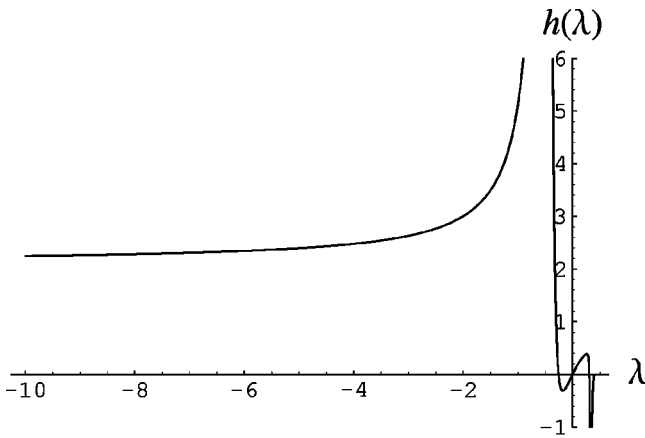

FIG. 3. Plot of the function h for $n_S=0, n_W=40, n_M=0$ (region III). The asymptotic value of h for $t \to -\infty$ is -8.13.
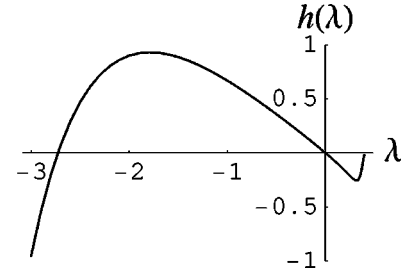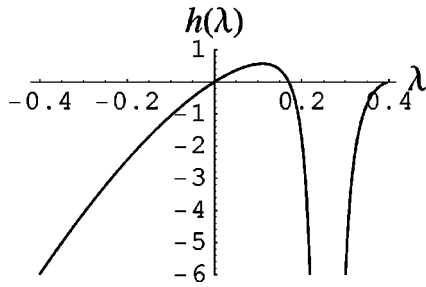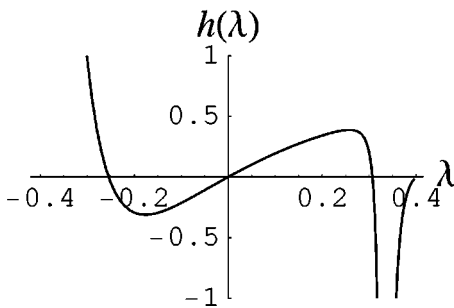
Figure 2 shows the function h for a theory in region II. The behavior for positive λ is very similar to that in region I, but the asymptote for $t \to -\infty$ is now positive, so that there exists a second FP for negative λ_*. This FP can be seen to yield negative g_* and is therefore physically uninteresting.

The behavior of the function h in region III is shown in Fig. 3. The positive zero is the unphysical one, so there is a single attractive GMFP with negative λ_*, which turns out to have positive g_*.

Finally, the behavior of the function h in region IV is shown in Fig. 4. It decreases monotonically from the positive asymptote $t \to -\infty$ to the Gaussian FP. For positive λ it behaves as in region III, having no zeros except for the unphysical one. Thus, in region IV there is no non-Gaussian FP. Region IV is the white wedge in Figs. 5, 6, and 7. One sees that it comes actually quite close to the origin; from this point of view the existence of the FP for pure gravity seems to be a lucky accident.

The value of λ_* in regions I and II is always less than $\min_{z \in [0,\infty]}(\mathcal{P})/2$, which is numerically equal to 0.402 (for $a=2$) and therefore reasonably within the bounds of the heat-kernel approximation. On the other hand, in region III λ_* becomes quickly rather large in absolute value; in this regime $R \gg k^2$ on shell and therefore the heat-kernel approximation ceases to be valid on shell. In this region the results are reliable only close to the surface $\Delta'=0$.

In order to determine the dimension of the critical surface we have calculated numerically the matrix M for many different combinations of fields. The results of such calculations are shown in Figs. 5, 6, and 7 for the case $n_{RS}=0$ and $n_M = 0$, $n_M=24$, and $n_M=45$, respectively (these numbers are
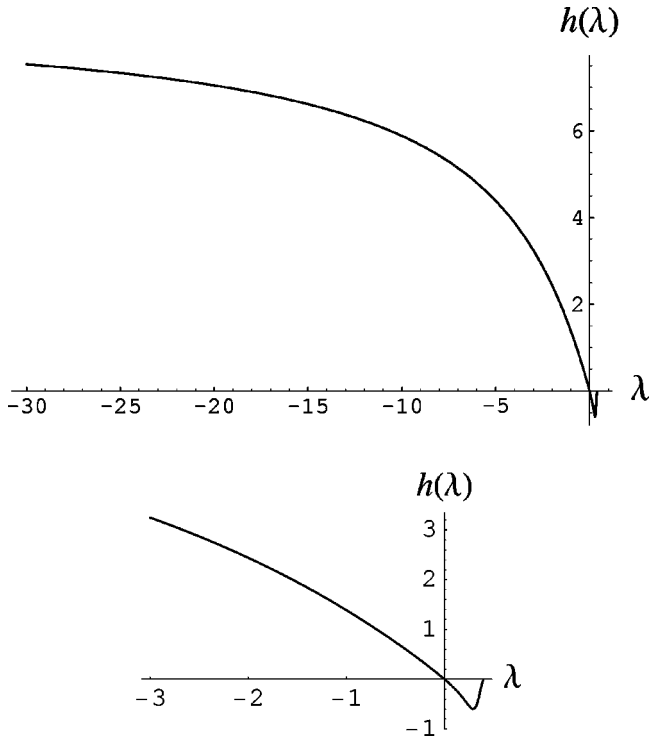
FIG. 6. Attractivity regions for $n_M = 24$. The gray scale corresponds to the number of attractive directions.

It is interesting to compare these results with the analysis of the pure scalar theory. Due to the fact that the coupling λ_4 is marginal in the pure scalar theory, the linearized analysis is not sufficient to determine its behavior. In the presence of gravity there is no zero eigenvalue and therefore the linearized analysis is sufficient to determine the dimension of the critical surface. In region III, the relevant directions correspond to potentials that are at most quadratic in ϕ. In region II, however, there can be a large number of negative eigenvalues, corresponding to nontrivial potentials V that are polynomial and asymptotically free. If, as at the end of Sec. II, we think of the theory as having a UV cutoff Λ_{UV}, then assuming that the limit $\Lambda_{UV} \to \infty$ is meaningful the problem of triviality is solved.

These theories are also predictive since they have a finite number of negative eigenvalues.[6] Therefore, they satisfy the conditions for asymptotic safety.

VI. CUTOFF AND GAUGE DEPENDENCE

The physical results are independent of the cutoff parameters in the exact theory, so the extent of parameter dependence that is observed in the truncated theory gives a quantitative measure of the errors. We have performed various tests on the parameter dependence of our results, and it is reassuring for the reliability of the truncation that this dependence turns out to be reasonably mild.

The dependence of λ_{0*} and ξ_{0*} on gauge and cutoff parameters was discussed in [9,21]. Figure 8 summarizes the cutoff dependence at the GMFP for gravity coupled to one scalar field in the gauge $\alpha = 0$. The results we obtain are very close to those of [9], since at the GMFP the only new contribution that we get for the values of λ_{0*} and ξ_{0*} is that of the kinetic term of the scalar field. It is apparent that, while λ_{0*} and ξ_{0*} are quite sensitive to the cutoff parameter a, the ratio λ_{0*}/ξ_{0*}^2 is not. As noted in [7], this quantity is, up to numerical factors, (the inverse of) the on-shell action, a physically observable quantity, so it must be independent of the cutoff scheme. It is seen in Fig. 8 that its a dependence is indeed pretty mild.

$h(\lambda)$

$h(\lambda)$

FIG. 4. Plot of the function h for $n_S = 40, n_W = 40, n_M = 0$ (region IV). The asymptotic value of h for $t \to -\infty$ is 8.72.

chosen to correspond to the gauge field content of popular grand unified theory models).

First of all, these numerical calculations exactly confirm the shape of the existence region of the FP that was derived analytically above. The structure of the eigenvalues is the same as in the pure gravity+scalar case, which was discussed in Sec. IV. The eigenvalues are given by the canonical dimensions plus a quantum correction which depends on the type and number of matter fields but otherwise is the same for every pair (λ_{2i}, ξ_{2i}). For any given number of matter fields, the GMFP has a finite-dimensional critical surface. In region III, the critical surface has mostly dimension 3 (three negative real eigenvalues), except for a narrow area close to the separatrix $\Delta' = 0$, where its dimension is 2 or 4. In regions I and II the critical dimension varies considerably, being roughly linear in the number of fields (it grows with n_S and decreases with n_W). These calculations generalize the results of [13], where the FP could have at most two attractive directions.
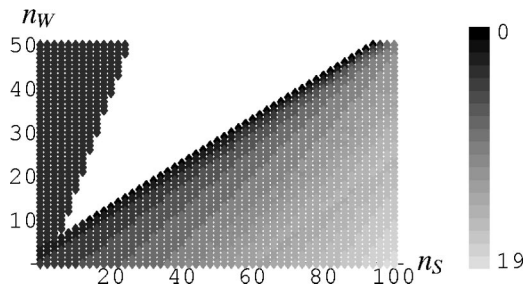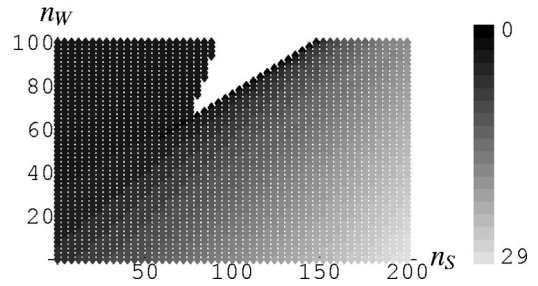

FIG. 5. Attractivity regions for $n_M = 0$. The gray scale corresponds to the number of attractive directions.

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[6] This should be contrasted with the asymptotically free nonpolynomial potentials of [18]. Those potentials are parametrized by a continuous parameter and therefore have an infinity of relevant couplings.
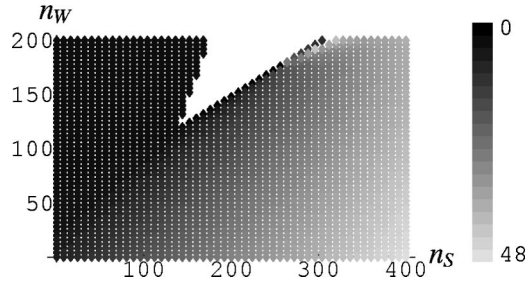
FIG. 7. Attractivity regions for $n_M = 45$. The gray scale corresponds to the number of attractive directions.

The dependence of $\theta'_{2i} \pm i\theta''_{2i}$, the eigenvalues of the stability matrix M, on the cutoff parameter a is shown in Figs. 9–11, for several values of α. We have calculated them in the range $1/5 \lesssim a \lesssim 50$, but they are reported only for $1/2 \lesssim a \lesssim 20$. Figure 9, giving the real parts of the eigenvalues of the submatrix M_{00}, agrees with Fig. 9 of [9], up to the small corrections due to the presence of a scalar field. The figures relative to the real parts of the remaining eigenvalues are simply shifted by the canonical dimension $2i$.

The first thing we can see in Fig. 9 is the presence a clear plateau with very weak (apparently logarithmic) variation of the eigenvalues, for $1 \lesssim a \lesssim 20$. Actually, the results for $1/5 \lesssim a \lesssim 1/2$ seem to indicate that there is a divergence as $a \to 0$. This is due to the fact that in this limit the cutoff function tends to become a constant, so it affects also the propagation of modes with momenta larger than k, and it does not work well as an IR cutoff. Clearly, larger values of a, of order unity, are preferred. This is in accordance with the generic features of the cutoff functions described in [5].

As far as the α dependence is concerned, we can see that it is quite weak. For all possible values of α all curves are contained between the curves $\alpha = 0$ and $\alpha \approx 3$, which differ by ~ 0.4. In order to better understand the dependence on α for different values of a, it is useful to plot the same results as a function of α (Fig. 10), with a being a parameter that labels the different curves (we shall restrict ourselves to only one plot of the real parts; the others can obviously be derived by shifting the graph by the canonical dimension of the operator involved).

Since the real parts of the eigenvalues of the matrix M_{22} are close to zero, this modest shift of the eigenvalues due to the change of gauge parameter is enough to change their
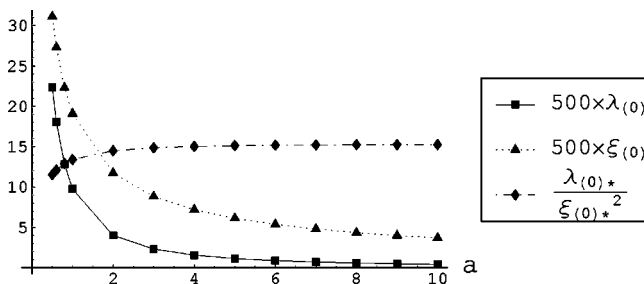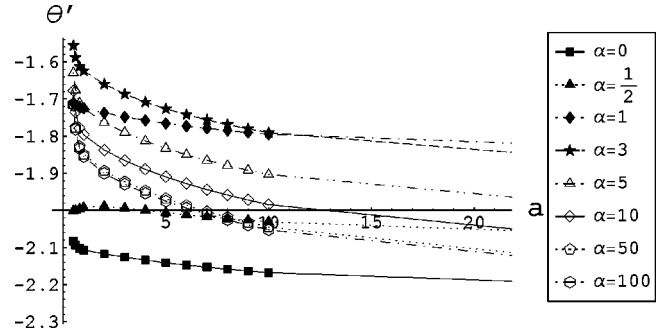

FIG. 9. a dependence of the real part of the eigenvalues of the stability matrix (first 2×2 submatrix).

sign. For instance, in the gauge $\alpha = 1$ one would compute the dimension of the critical surface to be 2. Looking at Fig. 10 one can most easily understand what the situation is like: for $\alpha = 0$ all cutoffs give a negative value of θ', then as a increases they change sign, but for large values of α they become negative again. Physical results such as the dimension of the critical surface cannot depend on the shape of the cutoff function, so this fact is certainly a shortcoming of our truncation. More work is needed to assess with greater confidence the dimension of the critical surface, but the considerations developed in [9], i.e., that α itself runs to 0 in the UV regime, suggest that the $\alpha = 0$ value is the physically correct result.

The same conclusions for the cutoff and gauge independence can be drawn for the imaginary parts, as can be seen from Fig. 11; they turn out to take the same values (up to a sign) for all the eigenvalues. The effect of nonvanishing imaginary parts is that the RG spirals around the FP, but they are not important in the discussion of the attractivity of the FP.

This discussion applies also to the higher couplings; their a and α dependence is given by curves that differ from those in Figs. 9–11 by a constant shift by a multiple of 2. The only important point that remains to some extent open, then, is the exact dimension of the UV critical surface, but nevertheless we can safely say that it is finite dimensional.

When other matter fields are present, the nature of the GMFP as a function of the number of matter fields is also a function of gauge and cutoff parameters. As already noted in [13], the constant σ is independent of the gauge parameter
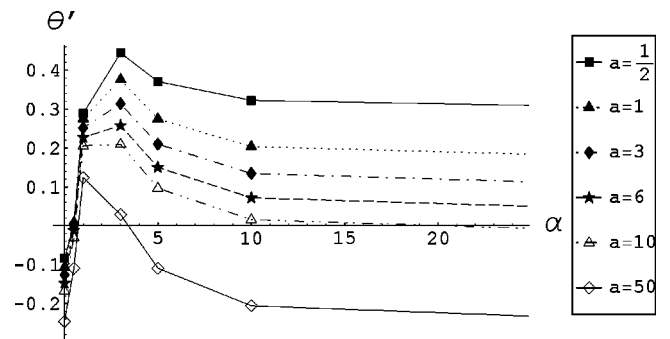

FIG. 8. a dependence of λ_{0*}, ξ_{0*}, and λ_{0*}/ξ_{0*}^2 in the gauge $\alpha = 0$. The values of λ_{0*} and ξ_{0*} have been magnified by a factor of 500 to display the three curves in the same range.

FIG. 10. α dependence of the real part of the eigenvalues of the stability matrix (second 2×2 submatrix).
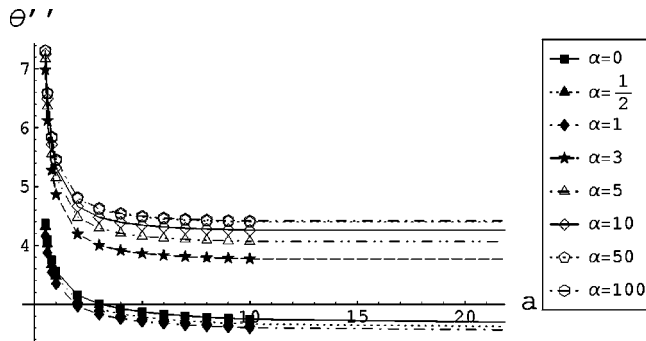
FIG. 11. *a* dependence of the imaginary part of the eigenvalues of the stability matrix.

and varies from 4.745 for $a = 0.05$ to 2.765 for $a = 20$. This corresponds to a vertical shift of the separatrix $\Delta' = 0$ by at most 2 in Figs. 5–7. The parameter τ is gauge independent for $\alpha \neq 0$ but shows discontinuity at $\alpha = 0$. The plane $\tau = 0$ is shifted and also slightly rotated to the right as a grows. Thus, region III becomes larger as a grows. Recall, however, that only the part of this region close to the separatrix $\Delta' = 0$ is trustworthy.

VII. CONCLUSIONS

In this paper we have considered the application of the ERGE's to a coupled system of gravity and matter fields. The main aim of this work was to verify that the conditions for asymptotic safety continue to hold in the presence of interacting matter fields. To make the problem manageable, we first dealt with a single scalar field ϕ with an arbitrary potential depending on ϕ^2, to see how this inclusion could change the picture of pure gravity; then we considered the effect of minimally coupled fields with different spins. Our results can be considered as a first step toward constructing a realistic theory of gravity and matter, but are also relevant to gravitational theories containing a dilaton.

In the context of the ansatz (2) we found that there exists a FP where only the cosmological constant and Newton's constant are nonzero. We called it the Gaussian matter FP. A detailed numerical search within a five-parameter truncation of the effective action has failed to yield any other FP. This is actually what one would expect from our understanding of the scalar theory [18].

The GMFP may be viewed in two ways. On one hand, the scalar field can be regarded as a "perturbation" of the pure theory of gravity considered in [9] and the GMFP as an extension of the FP found in [8]. The addition of the scalar field has the effect of shifting slightly the values of Λ and κ,

as already noticed in [13], while the attractivity property is preserved. On the other hand, we can consider the effect of adding gravity to a scalar theory with a generic potential and regard the GMFP as an extension of the Gaussian FP. The main effect is that the couplings that are present in the scalar potential V mix with those appearing in the function F, and the dimensions of the resulting couplings (which dictate the speed of the approach to the FP) is changed by a finite quantum correction. While at the Gaussian FP the quantum corrections vanish, so that the relevant couplings are, as usual, those with dimension less than 4, the gravitational contributions bring about modifications even if the matter sector allows for a perturbative treatment. At the GMFP, the stability matrix has a block-diagonal form so that there is strong mixing between the parameters λ_{2n} and ξ_{2m} for $n = m$, whereas for $n \neq m$ they are almost or completely decoupled. The eigenvalues, whose real part determines whether an operator is relevant or irrelevant, come in complex conjugate pairs, and grow systematically by a constant 2. For instance, the marginal operator of the pure scalar ϕ^4 theory becomes now an irrelevant operator, and the dimension of the UV critical surface is calculated to be 4 for a generic analytic potential. These results hold in the gauge $\alpha = 0$; they differ slightly for other values of the gauge-fixing parameter, but $\alpha = 0$ seems to be the physically correct value at the FP. The striking fact is that gravity gives calculable, finite contributions that change significantly the pure scalar theory. This is one of the most important results of our paper.

We have then considered the effect of adding other minimally coupled massless matter fields. For the existence of the GMFP, we obtain the same bounds presented in [13]. As to the attractivity of the FP, we have found that, when it exists, there are always finitely many attractive directions. Therefore gravity seems to remain asymptotically safe also in the presence of generic matter fields. We expect that this result will still hold if we add other interactions between matter fields that are asymptotically free. From this point of view, the scalar field posed a greater challenge, since the pure scalar theory is not asymptotically free. It is remarkable that the coupling to gravity fixes this problem and at the same time also opens a new path toward the solution of the issue of triviality.

All these results add on to the other proofs that have been collected in the literature about the physical reliability of this approach.

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