

First law of black hole mechanics in Einstein-Maxwell and Einstein-Yang-Mills theories

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The first law of black hole mechanics is derived from the Einstein-Maxwell Lagrangian by comparing two infinitesimally nearby stationary black holes. With similar arguments, the first law of black hole mechanics in Einstein-Yang-Mills theory is also derived.

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I. INTRODUCTION

According to the “no hair” theorem, a general stationary black hole is a charged and rotating black hole. The first law of black hole mechanics shows that the first order variations of the area A , mass M , angular momentum J , and charge Q are related by

$$\frac{1}{8\pi} \kappa \delta A = \delta M - \Omega_H \delta J - \Phi_{bh} \delta Q, \quad (1)$$

where κ denotes the surface gravity of the black hole, Ω_H denotes the angular velocity of the horizon, and Φ_{bh} denotes the electrostatic potential of the horizon. There are two versions of this law, referred to by Wald [1] as the “physical process version” and the “equilibrium state version.” The “physical process version” of the first law is obtained by changing a stationary black hole by some (infinitesimal) physical process. The black hole is assumed to have settled down to a new stationary final state. Then Eq. (1) is derived by comparing the final state of the black hole with the initial one [2]. The “equilibrium state” version of the first law simply compares the areas of two infinitesimally nearby stationary black hole solutions. The original derivation was given by Bardeen *et al.* [3]. However, since only a perfect fluid in circular orbit around a black hole was considered, the first law in [3] has a different form Eq. (1). A simple derivation in a general manner was given by Iyer and Wald [4] from the Lagrangian formulation of general relativity. The derivation makes essential use of the bifurcation two-sphere where the horizon Killing vector field vanishes. This treatment requires that all fields be smooth on the bifurcation surface, and consequently the “potential-charge” term does not appear explicitly in the first law. The first task of this paper is to extend the work of [4] to a general charged and rotating black hole where fields are not necessarily smooth through the horizon. The major modification is that, instead of choosing the bifurcation surface as the boundary of a hypersurface extending to spatial infinity, we replace it with any cross section of the event horizon to the future of the bifurcation surface (if one exists). We require that only the pullback [8] of the vector potential A_a to the horizon in the future of the bifurcation surface be smooth. Now we present such an example. The vector potential in the Reissner-Nordström spacetime is given by [8]

$$A_a = -\frac{Q}{r} (dt)_a. \quad (2)$$

To see the behavior of A_a on the horizon, we introduce the Kruskal coordinates (U, V) :

$$U = -e^{-\kappa u}, \quad (3)$$

$$V = e^{\kappa v}, \quad (4)$$

where

$$u = t - r_*, \quad (5)$$

$$v = t + r_*. \quad (6)$$

In terms of (U, V) , A_a can be written as

$$A_a = -\frac{Q}{2\kappa r} \left[-\frac{1}{U} (dU)_a + \frac{1}{V} (dV)_a \right]. \quad (7)$$

We see immediately that A_a is divergent at the bifurcation $U=V=0$. Although A_a is divergent on the future horizon $U=0, V>0$, the pullback of A_a to vectors tangent to the horizon (the restriction of A_a to vectors tangent to the horizon) is smooth. Since A_a falls off as $1/r$ at infinity, it will have no contribution to the canonical energy \mathcal{E} . As we shall see, the charge term in Eq. (1) emerges as an integration on the horizon. This modification also enables us to apply the result to black holes without a bifurcation surface, such as extremal black holes. A vector potential which is smooth through the horizon can easily be constructed by the gauge transformation

$$\tilde{A}_a = -\frac{Q}{r} (dt)_a + \frac{Q}{r_+} (dt)_a, \quad (8)$$

where r_+ is the radial coordinate of the event horizon. Since A_a is smooth through the horizon (identically zero), the potential-charge term will not appear in the integral over the horizon. However, \tilde{A}_a in Eq. (8) does not drop to zero at infinity; the potential-charge term will arise from infinity as part of the canonical energy.

The second task of this paper is to generalize the method above to Einstein-Yang-Mills (EYM) black holes. The discovery of “colored black holes,” such as black hole solutions in the Einstein-Yang-Mills theory, has been a great challenge to the traditional “no hair” conjecture. The first

law of black hole mechanics in the EYM case was discussed by Sudarsky and Wald [5] and the following result was obtained:

$$\frac{1}{8\pi} \kappa \delta A = \delta M + V \delta Q^\infty - \Omega_H \delta J, \quad (9)$$

where V and Q^∞ are the Yang-Mills potential and the charge evaluated at infinity. The presence of this term is due to the non-Abelian nature of the Yang-Mills field. The calculation also makes use of the bifurcation two-sphere and all fields are required to be smooth there. Again, we make no reference to the bifurcation surface, and an additional surface term evaluated on any cross section of the horizon is found [see Eq. (69)].

II. FIRST ORDER VARIATION OF STATIONARY SPACETIMES

In this section, we briefly introduce a general variation theory for stationary spacetimes in the framework of [4]. We start with the general issue of calculating the first order variation of conserved quantities. Consider a diffeomorphism covariant theory in four dimensions derived from a Lagrangian \mathbf{L} , where the dynamical fields consist of a Lorentz signature metric g_{ab} and other fields ψ . We follow the notational conventions of [4], and, in particular, we collectively refer to (g_{ab}, ψ) as ϕ and use boldface letters to denote differential forms. According to [4], the first order variation of the Lagrangian can always be expressed as

$$\delta \mathbf{L} = \mathbf{E}(\phi) \delta \phi + d\Theta(\phi, \delta \phi) \quad (10)$$

where $\mathbf{E}(\phi)$ is locally constructed out of ϕ and its derivatives and Θ is locally constructed out of $\phi, \delta \phi$ and their derivatives. The equations of motion can then be read off as

$$\mathbf{E}(\phi) = 0. \quad (11)$$

The symplectic current three-form ω is defined by

$$\omega(\phi, \delta_1 \phi, \delta_2 \phi) = \delta_1 \Theta(\phi, \delta_2 \phi) - \delta_2 \Theta(\phi, \delta_1 \phi). \quad (12)$$

The Noether current three-form associated with a smooth vector field ξ is defined by

$$\mathcal{J} = \Theta(\phi, \mathcal{L}_\xi \phi) - \xi \cdot \mathbf{L}, \quad (13)$$

where “ \cdot ” denotes contraction of the vector field ξ into the first index of \mathbf{L} . A simple calculation yields

$$d\mathcal{J} = -\mathbf{E}_\phi \mathcal{L}_\xi \phi. \quad (14)$$

It was proved in the Appendix of [6] that there exists a Noether charge two-form \mathbf{Q} , which is locally constructed from ϕ, ξ^a and their derivatives, such that

$$\mathcal{J}[\xi] = d\mathbf{Q}[\xi] + \xi^a \mathbf{C}_a \quad (15)$$

where \mathbf{C}_a is a three-form and $\mathbf{C}_a = 0$ when the equations of motion are satisfied. Now suppose that the spacetime satisfies asymptotic conditions at infinity corresponding to “case

I” of [7] and that ξ^a is an asymptotic symmetry. Then there exists a conserved quantity H_ξ , associated with ξ^a . Let $\delta\phi$ satisfy the linearized equations of motion in the neighborhood of infinity. Then δH_ξ is given by [7]

$$\delta H_\xi = \int_\infty (\bar{\delta} \mathbf{Q}[\xi] - \xi \cdot \Theta). \quad (16)$$

Since ξ^a is treated as a fixed background, it should not be varied in the expression above. So we used “ $\bar{\delta}$ ” to denote the variation that has no effect on ξ^a , in distinction to the total variation “ δ .” Let Σ be a hypersurface that extends to infinity and has an inner boundary $\partial \Sigma$. Now we consider the case where ξ^a is a symmetry of all the dynamical fields, i.e., $\mathcal{L}_\xi \phi = 0$, and $\delta\phi$ satisfies the linearized equations of motion. Then Eq. (76) in [4] shows that the integral in Eq. (16) over infinity can be turned into one on the inner boundary, i.e.,

$$\delta H_\xi = \int_{\partial \Sigma} (\bar{\delta} \mathbf{Q}[\xi] - \xi \cdot \Theta). \quad (17)$$

When ξ^a is taken to be an asymptotic time translation t^a and rotation ϕ^a , respectively, we obtain the variations of canonical energy \mathcal{E} and canonical angular momentum J [4]:

$$\delta \mathcal{E} = \int_\infty (\bar{\delta} \mathbf{Q}[t] - t \cdot \Theta), \quad (18)$$

$$\delta J = - \int_\infty (\bar{\delta} \mathbf{Q}[\varphi] - \varphi \cdot \Theta). \quad (19)$$

III. THE FIRST LAW OF BLACK HOLE MECHANICS IN EM THEORY

We now specialize to Einstein-Maxwell theory. The dynamical fields are (g_{ab}, A_a) and the Einstein-Maxwell Lagrangian is

$$\mathbf{L} = \frac{1}{16\pi} (\epsilon R - \epsilon g^{ac} g^{bd} F_{ab} F_{cd}). \quad (20)$$

The Noether charge two-form \mathbf{Q} and Θ have been calculated in [2] as

$$Q_{ab} = -\frac{1}{16\pi} \epsilon_{abcd} \nabla^c \xi^d - \frac{1}{8\pi} \epsilon_{abcd} F^{cd} A_e \xi^e \quad (21)$$

and

$$\Theta_{abc}(\phi, \delta\phi) = \frac{1}{16\pi} \epsilon_{dabc} v^d, \quad (22)$$

where

$$v_d = \nabla^e \delta g_{de} - g^{fe} \nabla_d \delta g_{fe} - 4F_d^b \delta A_b. \quad (23)$$

Let (g_{ab}, A_a) be a stationary solution to the Einstein-Maxwell equations derived from the Lagrangian (20). If the black hole possesses a bifurcation surface, we require that

the pullback of A_a to the future of the bifurcation surface be smooth, but not necessarily smooth on the bifurcation surface. Let

$$\xi^a = t^a + \Omega_H \varphi^a \quad (24)$$

denote the horizon Killing field of this black hole [1]. Let Σ be an asymptotic hypersurface which terminates on the portion of the event horizon \mathcal{H} to the future of the bifurcation surface. Denote the cross section on the horizon by $S_{\mathcal{H}}$, which is the inner boundary of Σ . Now consider a stationary perturbation $\delta\phi$ that generates a slightly different stationary axisymmetric black hole. When comparing two spacetimes, there is a certain freedom in which points are chosen to correspond. We shall adopt the gauge choice in [3], i.e., we make the hypersurface Σ , the event horizons, and the Killing vectors t^a and φ^a the same in the two solutions. Thus,

$$\delta t^a = \delta\varphi^a = 0, \quad (25)$$

$$\delta\xi^a = \delta\Omega_H \varphi^a. \quad (26)$$

Although the conditions above cannot be imposed on the bifurcation surface where ξ^a vanishes, our derivation will not be affected since we shall make no use of the bifurcation surface. If we assume that both A_a and δA_a fall off as fast as $1/r$ at infinity, as in the case in the Introduction, then the EM field contributes to neither $\delta\mathcal{E}$ nor δJ in Eqs. (18) and (19). Thus the variation of the canonical energy is the same as that of the Arnowitt-Deser-Misner (ADM) mass M and we shall rewrite $\delta\mathcal{E}$ as δM . Combining Eqs. (18), (19), (24), and (17), we have

$$\delta M - \Omega_H \delta J = \int_{S_{\mathcal{H}}} (\bar{\delta}\mathbf{Q}[\xi] - \xi \cdot \Theta). \quad (27)$$

Now we concentrate on the right-hand side of Eq. (27). We shall consider the contributions from the gravitational field and the EM field separately. From Eq. (21), we split \mathbf{Q} as

$$Q_{ab} = Q_{ab}^{GR} + Q_{ab}^{EM}, \quad (28)$$

where

$$Q_{ab}^{GR} = -\frac{1}{16\pi} \epsilon_{abcd} \nabla^c \xi^d, \quad (29)$$

$$Q_{ab}^{EM} = -\frac{1}{8\pi} \epsilon_{abcd} F^{cd} A_e \xi^e. \quad (30)$$

Similarly, we rewrite Θ as

$$\Theta_{abc} = \Theta_{abc}^{GR} + \Theta_{abc}^{EM}, \quad (31)$$

where

$$\Theta_{abc}^{GR} = \frac{1}{16\pi} \epsilon_{dabc} g^{dh} (\nabla^e \delta g_{he} - g^{fe} \nabla_h \delta g_{fe}), \quad (32)$$

$$\Theta_{abc}^{EM} = \frac{1}{16\pi} \epsilon_{dabc} (-4F^{db}) \delta A_b. \quad (33)$$

We first consider the term involving Q_{ab}^{GR} . On the horizon, we have [4]

$$\nabla_c \xi_d = \kappa \epsilon_{cd}, \quad (34)$$

where κ is the surface gravity and ϵ_{cd} is the binormal to $S_{\mathcal{H}}$ (see [4] for further details). Then

$$\int_{S_{\mathcal{H}}} Q_{ab}^{GR}[\xi] = \frac{1}{8\pi} \kappa A \quad (35)$$

where A is the area of the black hole. Remember that ξ^a is a fixed background quantity relative to the variation “ $\bar{\delta}$.” Using the identity

$$\bar{\delta} \int_{S_{\mathcal{H}}} Q_{ab}^{GR}[\xi] = \delta \int_{S_{\mathcal{H}}} Q_{ab}^{GR}[\xi] - \int_{S_{\mathcal{H}}} Q_{ab}^{GR}[\delta\xi], \quad (36)$$

we have

$$\begin{aligned} \bar{\delta} \int_{S_{\mathcal{H}}} Q_{ab}^{GR}[\xi] &= \frac{1}{8\pi} \delta(\kappa A) + \frac{1}{16\pi} \int_{S_{\mathcal{H}}} \epsilon_{abcd} \nabla^c \delta\xi^d \\ &= \frac{1}{8\pi} \delta(\kappa A) + \frac{\delta\Omega_H}{16\pi} \int_{S_{\mathcal{H}}} \epsilon_{abcd} \nabla^c \varphi^d \\ &= \frac{1}{8\pi} \delta(\kappa A) + \delta\Omega_H J_H, \end{aligned} \quad (37)$$

where Eqs. (35) and (26) were used and $J_H \equiv (1/16\pi) \int_{S_{\mathcal{H}}} \epsilon_{abcd} \nabla^c \varphi^d$ can be interpreted as the angular momentum of the black hole [8]. The computation in [3] reveals

$$\begin{aligned} \int_{S_{\mathcal{H}}} \xi \cdot \Theta^{GR} &= \frac{1}{16\pi} \int_{S_{\mathcal{H}}} \xi^a \epsilon_{dabc} g^{dh} (\nabla^e \delta g_{he} - g^{fe} \nabla_h \delta g_{fe}) \\ &= \frac{1}{8\pi} A \delta\kappa + \delta\Omega_H J_H. \end{aligned} \quad (38)$$

Thus, combining Eqs. (37) and (38), we have

$$\int_{S_{\mathcal{H}}} \bar{\delta}\mathbf{Q}^{GR} - \xi \cdot \Theta^{GR} = \frac{1}{8\pi} \kappa \delta A. \quad (39)$$

This result can be viewed as the net contribution from the gravitational field. We now consider the EM field. By using the smoothness of the pullback of A_a and the stationary condition, one can show that $\Phi^{EM} \equiv -\xi^a A_a|_{\mathcal{H}}$ is a constant in the portion of the horizon to the future of the bifurcation surface [2]. If A_a is smooth over the entire horizon, Φ^{EM} will be identically zero on the horizon since ξ^a vanishes on the bifurcation surface (in this case, the result in [4] is recovered). Together with Eq. (30), we have

$$\int_{S_{\mathcal{H}}} Q_{ab}^{EM} = \frac{\Phi^{EM}}{8\pi} \int_{S_{\mathcal{H}}} \epsilon_{abcd} F^{cd}. \quad (40)$$

In the asymptotic region, the total electric charge can be expressed as [8]

$$\frac{1}{8\pi} \int_{\infty} \epsilon_{abcd} F^{cd} = Q. \quad (41)$$

Since the Einstein-Maxwell Lagrangian we considered corresponds to the sourceless electromagnetic field, the same result must hold if the integral is performed on the horizon. Therefore

$$\int_{S_{\mathcal{H}}} Q_{ab}^{EM} = \Phi^{EM} Q. \quad (42)$$

Similar to the identity in Eq. (36), we have

$$\begin{aligned} \bar{\delta} \int_{S_{\mathcal{H}}} Q_{ab}^{EM}[\xi] &= \delta \int_{S_{\mathcal{H}}} Q_{ab}^{EM}[\xi] - \int_{S_{\mathcal{H}}} Q_{ab}^{EM}[\delta\xi] \\ &= \delta(\Phi^{EM} Q) \\ &\quad + \frac{1}{8\pi} \delta\Omega_H \int_{S_{\mathcal{H}}} \epsilon_{abcd} F^{cd} A_e \varphi^e. \end{aligned} \quad (43)$$

Now we compute

$$\int_{S_{\mathcal{H}}} \xi \cdot \Theta^{EM} = -\frac{1}{4\pi} \int_{S_{\mathcal{H}}} \epsilon_{cdab} F^{ce} \xi^d \delta A_e. \quad (44)$$

We first express the volume element in the form

$$\epsilon_{cdab} = \xi_c \wedge N_d \wedge \epsilon_{ab}, \quad (45)$$

where ϵ_{ab} is the volume element on $S_{\mathcal{H}}$ and N^a is the ‘‘in-going’’ future directed null normal to $S_{\mathcal{H}}$, normalized so that $N^a \xi_a = -1$ [8]. Thus, we have

$$\int_{S_{\mathcal{H}}} \xi \cdot \Theta^{EM} = \frac{1}{4\pi} \int_{S_{\mathcal{H}}} \epsilon_{ab} F^{ce} \xi_c \delta A_e. \quad (46)$$

By using the fact that on the horizon $F^{ce} \xi_c \propto \xi^e$ [2], together with $N^a \xi_a = -1$, we get immediately

$$F^{ce} \xi_c = F^{cf} N_c \xi_f \xi^e, \quad (47)$$

and hence

$$\int_{S_{\mathcal{H}}} \xi \cdot \Theta^{EM} = \frac{1}{4\pi} \int_{S_{\mathcal{H}}} \epsilon_{ab} F^{cf} N_c \xi_f \xi^e \delta A_e. \quad (48)$$

On the other hand,

$$\begin{aligned} Q \delta\Phi^{EM} &= -\frac{1}{8\pi} \int_{S_{\mathcal{H}}} \epsilon_{abcd} F^{cd} \delta(A_e \xi^e) \\ &= -\frac{1}{8\pi} \int_{S_{\mathcal{H}}} \epsilon_{abcd} F^{cd} (\delta A_e) \xi^e - \frac{\delta\Omega_H}{8\pi} \\ &\quad \times \int_{S_{\mathcal{H}}} \epsilon_{abcd} F^{cd} A_e \varphi^e \\ &= \frac{1}{8\pi} \int_{S_{\mathcal{H}}} \epsilon_{ab} \wedge N_c \wedge \xi_d F^{cd} \xi^e \delta A_e - \frac{\delta\Omega_H}{8\pi} \\ &\quad \times \int_{S_{\mathcal{H}}} \epsilon_{abcd} F^{cd} A_e \varphi^e \\ &= \frac{2}{8\pi} \int_{S_{\mathcal{H}}} \epsilon_{ab} F^{cd} N_c \xi_d \xi^e \delta A_e \\ &\quad - \frac{\delta\Omega_H}{8\pi} \int_{S_{\mathcal{H}}} \epsilon_{abcd} F^{cd} A_e \varphi^e \\ &= \int_{S_{\mathcal{H}}} \xi \cdot \Theta^{EM} - \frac{\delta\Omega_H}{8\pi} \int_{S_{\mathcal{H}}} \epsilon_{abcd} F^{cd} A_e \varphi^e. \end{aligned} \quad (49)$$

Using Eq. (43), we have

$$\int_{S_{\mathcal{H}}} \bar{\delta} Q_{ab}^{EM} - \xi \cdot \Theta^{EM} = \Phi^{EM} \delta Q. \quad (50)$$

Substitution of Eqs. (50) and (39) into the right-hand side of Eq. (27) yields Eq. (1), the desired first law of black hole mechanics in Einstein-Maxwell theory. As pointed out in the Introduction section, the potential-charge term (50) would have vanished if the EM field were smooth on the horizon and the integral were performed on the bifurcation surface.

IV. THE FIRST LAW IN EYM THEORY

In this section, we shall extend our derivation in the previous section to the EYM case. The assumptions and arguments will be similar to those in the previous section. The EYM Lagrangian takes the form

$$\mathbf{L} = \frac{1}{16\pi} \epsilon R - \frac{1}{16\pi} \epsilon g^{ac} g^{bd} F_{ab}^{\Lambda} F_{cd\Lambda}, \quad (51)$$

where F_{ab}^{Λ} is the Yang-Mills field strength:

$$F_{ab}^{\Lambda} = 2\nabla_{[a} A_{b]}^{\Lambda} + c_{\Gamma\Delta}^{\Lambda} A_a^{\Gamma} A_b^{\Delta}, \quad (52)$$

where $c_{\Gamma\Delta}^{\Lambda}$ denotes the structure tensor for the SU(2) Lie algebra and the Lie algebra indices are raised and lowered with the Killing metric $g_{\Gamma\Sigma} = -\frac{1}{2} c_{\Gamma\Sigma}^{\Lambda} c_{\Lambda}^{\Sigma}$.

Similarly to the EM case, the Lagrangian can be split into ‘‘GR’’ and ‘‘YM’’ parts. The contribution from the YM field gives

$$\theta_{bcd}^{YM} = -\frac{1}{4\pi} \epsilon_{abcd} F_{\Delta}^{ae} \delta A_e^{\Delta}, \quad (53)$$

$$Q_{ab}^{YM} = -\frac{1}{8\pi} \epsilon_{abcd} F_{\Lambda}^{cd} A_e^{\Lambda} \xi^e. \quad (54)$$

Then

$$\int_{\infty} Q[t] = -\frac{1}{8\pi} \int_{\infty} \epsilon_{abcd} F_{\Lambda}^{cd} A_0^{\Lambda}. \quad (55)$$

We choose a stationary solution of the EYM equations and then A_0^{Λ} is asymptotically constant [5]. The constant V is defined by

$$V = \lim_{r \rightarrow \infty} (A_0^{\Lambda} A_{0\Lambda})^{1/2}. \quad (56)$$

The electric field, viewed as a tensor density of weight, is

$$E_{\Lambda}^a = \sqrt{h} F_{\mu\Lambda}^a n^{\mu}, \quad (57)$$

where n^{μ} is the unit normal to the spacelike hypersurface ∞ . Reference [5] shows that, asymptotically, A_0^{Λ} and E_a^{Λ} point in the same Lie algebra direction and therefore

$$\int_{\infty} Q[t] = V Q^{\infty}, \quad (58)$$

where the Yang-Mills charge measured at infinity is defined by

$$Q^{\infty} = \frac{1}{4\pi} \int_{\infty} |E_{\Lambda}^a r_a|, \quad (59)$$

where r^a denotes the unit radial vector and vertical bars denote the Lie algebra norm. On the other hand,

$$\begin{aligned} \int_{\infty} t \cdot \theta^{YM} &= -\frac{1}{4\pi} \int_{\infty} \epsilon_{abcd} t^b F_{\Delta}^{ae} \delta A_e^{\Delta} \\ &= \frac{1}{4\pi} \int_{\infty} E_{\Delta}^a r_a \delta A_0^{\Delta} \\ &= Q^{\infty} \delta V. \end{aligned} \quad (60)$$

Therefore, the ‘‘YM’’ contribution to $\delta\mathcal{E}$ is

$$\delta\mathcal{E}_{YM} = V \delta Q^{\infty}. \quad (61)$$

Since the ‘‘GR’’ contribution gives the ADM mass M , we have the total variation of the canonical energy

$$\delta\mathcal{E} = \delta M + V \delta Q^{\infty}, \quad (62)$$

which agrees with the result in [5].

By using the arguments parallel to that in Sec. III, we obtain an expression similar to Eq. (27):

$$\delta\mathcal{E} - \Omega_H \delta J = \int_{S_{\mathcal{H}}} (\bar{\delta}\mathbf{Q}[\xi] - \xi \cdot \Theta). \quad (63)$$

Note that the ADM mass on the left-hand side of Eq. (27) has been replaced by \mathcal{E} . The canonical angular momentum J is defined by [4]

$$J = - \int_{\infty} \mathbf{Q}[\varphi]. \quad (64)$$

Combining Eqs. (29) and (54), we have

$$J = \frac{1}{16\pi} \int_{\infty} \epsilon_{abcd} \nabla^c \xi^d + \frac{1}{8\pi} \int_{\infty} \epsilon_{abcd} F_{\Lambda}^{cd} A_e^{\Lambda} \xi^e. \quad (65)$$

This formula agrees with that in [5]. This first term is just the expression for angular momentum in the vacuum case.

Since Eq. (39) also holds for the EYM case, we use it to rewrite the right-hand side of Eq. (63):

$$\delta\mathcal{E} - \Omega_H \delta J = \frac{1}{8\pi} \kappa \delta A + \int_{S_{\mathcal{H}}} (\bar{\delta}\mathbf{Q}^{YM}[\xi] - \xi \cdot \Theta^{YM}). \quad (66)$$

The same treatment used for the EM field gives

$$\begin{aligned} \bar{\delta} \int_{S_{\mathcal{H}}} Q_{ab}^{YM}[\xi] &= \delta \int_{S_{\mathcal{H}}} Q_{ab}^{YM}[\xi] - \int_{S_{\mathcal{H}}} Q_{ab}^{YM}[\delta\xi] \\ &= -\frac{1}{8\pi} \delta \int_{S_{\mathcal{H}}} \epsilon_{abcd} F_{\Lambda}^{cd} A_e^{\Lambda} \xi^e \\ &\quad + \frac{1}{8\pi} \delta \Omega_H \int_{S_{\mathcal{H}}} \epsilon_{abcd} F_{\Lambda}^{cd} A_e^{\Lambda} \varphi^e \\ &= -\frac{1}{8\pi} \int_{S_{\mathcal{H}}} A_e^{\Lambda} \xi^e \delta(\epsilon_{abcd} F_{\Lambda}^{cd}) \\ &\quad - \frac{1}{8\pi} \int_{S_{\mathcal{H}}} \epsilon_{abcd} F_{\Lambda}^{cd} \delta(A_e^{\Lambda} \xi^e) \\ &\quad + \frac{1}{8\pi} \delta \Omega_H \int_{S_{\mathcal{H}}} \epsilon_{abcd} F_{\Lambda}^{cd} A_e^{\Lambda} \varphi^e. \end{aligned} \quad (67)$$

Replacing the second term of Eq. (67) by an expression analogous to Eq. (49), we get

$$\bar{\delta} \int_{S_{\mathcal{H}}} Q_{ab}^{YM}[\xi] = -\frac{1}{8\pi} \int_{S_{\mathcal{H}}} A_e^{\Lambda} \xi^e \delta(\epsilon_{abcd} F_{\Lambda}^{cd}) + \int_{S_{\mathcal{H}}} \xi \cdot \Theta^{YM}. \quad (68)$$

Then, from Eqs. (66) and (68), we obtain the first law for a stationary EYM black hole:

$$\frac{1}{8\pi} \kappa \delta A = \delta\mathcal{E} - \Omega_H \delta J - \frac{1}{8\pi} \int_{S_{\mathcal{H}}} A_e^{\Lambda} \xi^e \delta(\epsilon_{abcd} F_{\Lambda}^{cd}). \quad (69)$$

This expression agrees with that in [11]. We cannot further evaluate the integral in the form of ‘‘ $\Phi \delta Q$ ’’ as in the EM

case because of the complicity of $SU(2)$ Lie algebra. Ash-
tekar, *et al.* [9] chose the following gauge conditions (see
also Corichi *et al.* [10]).

(i) The Yang-Mills potential

$$\Phi^{YM} = -|\xi \cdot \mathbf{A}| \quad (70)$$

is constant on the horizon.

(ii) The dual of the field strength ($*\mathbf{F}$) and $(\xi \cdot \mathbf{A})$ point in
the same Lie algebra direction

$$(\xi \cdot \mathbf{A})^\Sigma \propto ({}^2\epsilon \cdot *\mathbf{F})^\Sigma, \quad (71)$$

where ${}^2\epsilon$ is the pullback to the horizon of ϵ_{abcd} . Under
these two conditions, the integral in Eq. (69) can be evalu-
ated as

$$-\frac{1}{8\pi} \int_{S_{\mathcal{H}}} A_e^\Lambda \xi^e \delta(\epsilon_{abcd} F_\Lambda^{cd}) = \Phi^{YM} \delta Q_{\mathcal{H}}^{YM} \quad (72)$$

where $Q_{\mathcal{H}}^{YM} = -(1/4\pi) \int_{S_{\mathcal{H}}} |\mathbf{F}|$ is the electric Yang-Mills
charge evaluated on the horizon. However, there is no evi-

dence that our stationary gauge choice is consistent with con-
ditions (i) and (ii) above. Therefore, Eq. (69) is our final
form of the first law in EYM theory.

V. CONCLUSIONS

The first law of black hole mechanics for the EM and
EYM cases is derived in the framework of [4]. In contrast to
[4], we make no reference to the bifurcation surface. In the
EM case, when the pullback of A_a to the future horizon is
smooth, the desired charge-potential term is obtained. In the
EYM case, a corresponding surface integral on the horizon is
found. Since we avoid using the bifurcation surface, the deri-
vation and conclusions in this paper apply to extremal black
holes simply by taking $\kappa=0$.

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