Noncommutative self-dual gravity

H. García-Compeán*
Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, P.O. Box 14-740, 07000 México D.F., Mexico

O. Obregón†
Department of Applied Mathematics and Theoretical Physics, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

C. Ramírez‡
Instituto de Física de la Universidad de Guanajuato, P.O. Box E-143, 37150 León Gto., Mexico

M. Sabido§
Instituto de Física de la Universidad de Guanajuato, P.O. Box E-143, 37150 León Gto., Mexico

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Starting from a self-dual formulation of gravity, we obtain a noncommutative theory of pure Einstein theory in four dimensions. In order to do that, we use the Seiberg-Witten map. A procedure is outlined that allows one to find the solution of the noncommutative torsion constraint through the vanishing of the commutative one. Finally, the noncommutative corrections to the action are computed up to second order.

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I. INTRODUCTION

Nowadays, there are two main candidates for a quantum theory of the gravitational field: string theory (M theory) and loop quantum gravity. From the description of the low energy excitations of open strings, in the presence of a Neveu-Schwarz (NS) constant background B field, a noncommutative effective low energy gauge action [1,2] appears in a natural way. It is known, from M(atrix) theory that, at low energies, the coordinates of a gas of D0-branes are described by matrices, which cause virtual effects [3,4]. Such effects give rise to a supergravity interaction in 11 dimensions. Thus, gravity seems to arise from noncommutativity.

Along these lines, noncommutative gauge theory, as a continuous deformation of the usual theory, has attracted a lot of attention. Although gravitation does not arise in the low energy limit of open string theory as a gauge theory, some interesting effects of gravity processes (such as the graviton–graviton–D-brane scattering, in the presence of a constant B field) can be computed [5]. However, a deeper study of the deformations of pure gravitational theories is still needed. Thus, the study of models of noncommutative gravity, independently of how they could arise from string or M theory, might be important. Such models could be obtained starting from those formulations of gravitation that are based on a gauge principle. One of these formulations is self-dual gravity (for a review, see [6,7]), from which the Hamiltonian Ashtekar formulation [8] can be obtained [9,10]. The properties of this formulation have allowed the exploration of quantum gravity in the framework of loop quantum gravity and quantum geometry (for a review, see [7]). In this context also, it would be interesting to explore noncommutative quantum gravity.

On the other hand, there are proposals for a noncommutative formulation of gravitation [11], motivated by the understanding of the short distance behavior of the gravitational field [12]. Proposals based on the recent developments are given in [13–17]. In particular, in [15–17] a Seiberg-Witten map for the tetrad and the Lorentz connection is given, where these fields are taken as components of a SO(4,1) connection in the first work, and of a U(2,2) connection in the others. In these works a MacDowell-Mansouri (MM) type of action is considered, invariant under the subgroup $U(1,1)\times U(1,1)$, and the excess of degrees of freedom, additional to the ones of the commutative theory, is handled by means of constraints. For other recent proposals of noncommutative gravity, see [18]. In particular, in [17], from the chosen constraints, a consistent noncommutative SO(3,1) extension arises.

On the other hand, in [19] it was shown that noncommutative gauge theories, based on the Seiberg-Witten map, for any commutative theory invariant under a gauge group G can be constructed. The resulting noncommutative theory can be seen as an effective theory, invariant under the noncommutative enveloping algebra transformations, and also under the commutative transformations of G. This results from the fact that the Seiberg-Witten map may be seen as a sort of gauge fixing, in which the degrees of freedom added by noncommutativity to the fields and to the transformation parameters are mapped to expressions depending on the commutative fields, in such a way that in the commutative limit the original theory is obtained. In this way, a minimal version of the noncommutative standard model with the gauge group $SU(3)\times SU(2)\times U(1)$ has been proposed [20].

*Electronic address: compean@fis.cinvestav.mx
†Permanent address: Instituto de Física de la Universidad de Guanajuato, P.O. Box E-143, 37150 León Gto., Mexico. Electronic address: octavio@ifug3.ugto.mx
‡Permanent address: Facultad de Ciencias Físico Matemáticas, Universidad Autónoma de Puebla, P.O. Box 1364, 72000 Puebla, Mexico. Electronic address: cramirez@fcfm.buap.mx
§Electronic address: msabido@ifug3.ugto.mx

Following these ideas, starting from a SL(2,**C**) self-dual connection, in a previous work [21] we gave a formulation for quadratic noncommutative topological gravitation, which contains the SO(3,1) topological invariants, namely, the signature and Euler characteristic. In fact, the noncommutative signature can be straightforwardly obtained, but the Euler invariant cannot, as it involves the same difficulty as the MM action, which contains a contraction with the Levi-Cività tensor, instead of the SO(3,1) trace. However, both invariants can be combined into an expression given by the signature with a SO(3,1) self-dual connection, which amounts to the SL(2,**C**) signature.

In this paper, taking the same SL(2,**C**) connection as in [21], considering the Plebański formulation [22], we make a proposal for a noncommutative theory of gravity, which is fully invariant under the noncommutative gauge transformations [23]. The Plebański formulation is written as a SL(2,**C**) topological BF formulation, given by the trace of the two-form B times the field strength [24]. The contact with Einstein gravitation is done through constraints on the B field, which are solved using the square of the tetrad one-form [22]. This theory can be restated in terms of self-dual SO(3,1) fields, the connection, and the antisymmetric tensor B. After the identification of the B two-form with the tetrad one-form squared, a variation of this action with respect to the connection gives the vanishing of the torsion. The resulting action contains Einstein gravitation plus an imaginary term, which is identically zero due to the Bianchi identities. The noncommutative version is obtained at the level of the SL(2,**C**) theory, by the application of the Moyal product and the Seiberg-Witten map. In this way, a highly nonlinear theory is obtained, which depends on the commutative SL(2,**C**) fields. These fields are then written in terms of the SO(3,1) fields, the self-dual connection, and the B field, and then the connection is written in terms of the tetrads. The consistency of the last step is ensured by the fact that the variation of the action with respect to the noncommutative SL(2,**C**) connection gives an equation which is solved by the vanishing of the commutative torsion. Other terms in this equation seem to vanish on shell, a fact explicitly shown to first order in the noncommutativity parameter.

The paper is organized as follows. In Sec. II we briefly overview the Seiberg-Witten map and enveloping algebra and state some results concerning noncommutative covariant equations. In Sec. III we formulate the commutative self-dual gravity theory from which the noncommutative one is obtained. In Sec. IV the noncommutative theory is formulated and corrections are computed. Finally, Sec. V contains our conclusions.

II. NONCOMMUTATIVE GAUGE SYMMETRY AND THE SEIBERG-WITTEN MAP

In this section, a few conventions and properties of noncommutative spaces will be given for future reference. For recent reviews, see, e.g., [25].

Noncommutative spaces can be understood as generalizations of the usual quantum mechanical commutation relations, by the introduction of noncommutative coordinates x^μ satisfying

$$[x^\mu, x^\nu] = i\,\theta^{\mu\nu}, \tag{1}$$

where x^μ are linear operators acting on the Hilbert space $L^2(\mathbf{R}^n)$, and $\theta^{\mu\nu} = -\theta^{\nu\mu}$ are real numbers. Given this linear operator algebra $\mathcal{A}$, the Weyl-Wigner-Moyal correspondence establishes an isomorphic relation between it and the algebra of functions on $\mathbf{R}^n$, with an associative and noncommutative star product, the Moyal $\star$ product. Thus, the Moyal algebra $\mathcal{A}_\star \equiv \mathbf{R}^n_\star$ is, under certain conditions, equivalent to the Heisenberg algebra (1). The Moyal product is given by

$$f(x)\star g(x) \equiv \left[\exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\varepsilon^\mu}\frac{\partial}{\partial\eta^\nu}\right) \times f(x+\varepsilon)g(x+\eta)\right]_{\varepsilon=\eta=0}. \tag{2}$$

Under complex conjugation it satisfies $(\overline{f\star g}) = \overline{g}\star\overline{f}$.

Since we will be working with a non-Abelian group, we must include also matrix multiplication, so an * product will be used as the external product of matrix multiplication with the $\star$ product. In this case, Hermitian conjugation is given by $(f*g)^\dagger = g^\dagger * f^\dagger$. Inside integrals on closed manifolds, this product has the cyclicity property $\mathrm{Tr}\int f_1^* f_2^* f_3^* \cdots * f_n = \mathrm{Tr}\int f_n^* f_1^* f_2^* f_3^* \cdots * f_{n-1}$. In particular, $\mathrm{Tr}\int f_1^* f_2 = \mathrm{Tr}\int f_1 f_2$. From now on we will understand that the multiplication of noncommutative quantities is given by this * product.

Thus, with any expression containing space-time functions, a noncommutative expression can be associated by substitution of the usual product by this * product. However, this procedure has the well known ambiguity of the ordering of the resulting expression, which could be fixed by physical considerations. In particular, in the case of gauge theories, we wish to have a noncommutative theory, invariant under a suitable generalization of the gauge transformations. This generalization frequently is used to fix, to some extent, the ordering ambiguities.

Let us consider a theory, invariant under the action of the Lie group G, with gauge fields A_μ, and matter fields Φ which transform under the adjoint representation **ad**,

$$\delta_\lambda A_\mu = \partial_\mu\lambda + i[\lambda, A_\mu],$$
$$\delta_\lambda \Phi = i[\lambda, \Phi], \tag{3}$$

where $\lambda = \lambda^i T_i$, and $T_i (i = 1, \ldots, \dim G)$ are the generators of the Lie algebra $\mathcal{G}$ of G, in the adjoint representation. These transformations are generalized for the noncommutative connection [2] and for the adjoint representation as

$$\delta_{\hat\lambda}\hat A_\mu = \partial_\mu\hat\lambda + i[\hat\lambda \overset{*}{,} \hat A_\mu], \tag{4}$$

$$\delta_{\hat\lambda}\hat\Phi = i[\hat\lambda \overset{*}{,} \hat\Phi]. \tag{5}$$

The commutator $[A\overset{*}{,}B] \equiv A*B - B*A$, satisfies the Leibnitz rule when acting on products of noncommutative fields. Due to noncommutativity, commutators like $[\hat{\lambda}\overset{*}{,}\hat{A}_\mu]$ take values in the enveloping algebra $\mathcal{U}(\mathcal{G},\mathbf{ad})$ of the adjoint representation of $\mathcal{G}$. Therefore, $\hat{\lambda}$ and the gauge fields $\hat{A}_\mu$ will also take values in this algebra. In general, for some representation $\mathbf{R}$, we will denote by $\mathcal{U}(\mathcal{G},\mathbf{R})$ the section of the enveloping algebra $\mathcal{U}$ of $\mathcal{G}$ that corresponds to the representation $\mathbf{R}$.

Let us write, for instance, $\hat{\lambda} = \hat{\lambda}^I T_I$ and $\hat{A} = \hat{A}^I T_I$; then

$$[\hat{\lambda}\overset{*}{,}\hat{A}_\mu] = \frac{1}{2}\{\hat{\lambda}^I\overset{*}{,}\hat{A}^J_\mu\}[T_I,T_J] + \frac{1}{2}[\hat{\lambda}^I\overset{*}{,}\hat{A}^J_\mu]\{T_I,T_J\}. \quad (6)$$

Thus all the products of the generators T_I will be needed in order to close the algebra $\mathcal{U}(\mathcal{G},\mathbf{ad})$. Its structure can be obtained by successively computing the commutators and anticommutators starting from the generators of $\mathcal{G}$ in the corresponding representation, until it closes,

$$[T_I,T_J] = i f^K_{IJ} T_K, \quad \{T_I,T_J\} = d_{IJ}{}^K T_K.$$

The field strength is [2] $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu\overset{*}{,}\hat{A}_\nu]$; hence it also takes values in $\mathcal{U}(\mathcal{G},\mathbf{ad})$. From Eq. (5) it turns out that

$$\delta_{\hat{\lambda}} \hat{F}_{\mu\nu} = i(\hat{\lambda} * \hat{F}_{\mu\nu} - \hat{F}_{\mu\nu} * \hat{\lambda}). \quad (7)$$

We see that these noncommutative transformation rules can be obtained from the commutative ones, just by replacing the ordinary product of functions by the Moyal product, with a suitable product ordering. This allows us to construct invariant quantities in a simple way.

If we wish to have a continuous commutative limit, the noncommutative fields $\hat{\Phi}$ must be power series expansions of the noncommutativity parameter θ, starting from the commutative ones,

$$\hat{\Phi} = \Phi + \theta^{\mu\nu}\Phi^{(1)}_{\mu\nu} + \theta^{\mu\nu}\theta^{\rho\sigma}\Phi^{(2)}_{\mu\nu\rho\sigma} + \cdots. \quad (8)$$

Thus, in such an expansion, the noncommutative fields will have in general an infinity of independent components. Moreover, the noncommutative gauge fields will take values in the enveloping algebra and, unless the enveloping algebra coincides with the Lie algebra of the commutative theory, as is the case of $G = U(N)$, they will also have a bigger number of matrix components. As this is also the case for the transformation parameters, it will be possible to eliminate a lot of degrees of freedom by fixing the gauge.

In fact, the Seiberg-Witten map [2] establishes a one-to-one correspondence among the physical degrees of freedom of the noncommutative fields and the physical degrees of freedom of the commutative fields. This fact is used in Ref. [19] to construct noncommutative gauge theories, in principle, for any Lie group G.

The main point is that the Seiberg-Witten map allows for field dependent transformations. This means that if we combine two transformations the gauge parameters will be transformed as well. Thus, if for an infinitesimal transformation matrix we have the correspondence $\lambda \to \hat{\lambda}$ the Moyal commutator will not correspond simply to the commutator of two transformations but [19]

$$\widehat{[\lambda,\eta]} = [\hat{\lambda}\overset{*}{,}\hat{\eta}] + i(\delta_{\lambda}\hat{\eta} - \delta_{\eta}\hat{\lambda}). \quad (9)$$

If we write that there is an expansion like (8) for these matrices,

$$\hat{\lambda} = \lambda + \theta^{\mu\nu}\lambda^{(1)}_{\mu\nu} + \theta^{\mu\nu}\theta^{\rho\sigma}\lambda^{(2)}_{\mu\nu\rho\sigma} + \cdots, \quad (10)$$

then a solution for the coefficients can be obtained [2,19]:

$$\hat{\lambda}(\lambda,A) = \lambda + \frac{1}{4}\theta^{\mu\nu}\{\partial_\mu\lambda, A_\nu\} + \mathcal{O}(\theta^2). \quad (11)$$

Further, the Seiberg-Witten map determines the $\Phi^{(a)}$ terms in Eq. (8), from the fact that the noncommutative transformations are given by Eqs. (4), (5) and consequently, Eq. (7). These functions in Eq. (8) can be expressed in terms of the commutative fields and their derivatives. For the gauge fields, one solution is given by [2]

$$\hat{A}_\mu(A) = A_\mu - \frac{1}{4}\theta^{\nu\rho}\{A_\nu, \partial_\rho A_\mu + F_{\rho\mu}\} + \mathcal{O}(\theta^2), \quad (12)$$

from which, for the field strength, it turns out that

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{4}\theta^{\rho\sigma}(2\{F_{\mu\rho},F_{\nu\sigma}\} - \{A_\rho,(D_\sigma + \partial_\sigma)F_{\mu\nu}\}) + \mathcal{O}(\theta^2). \quad (13)$$

For fields in the adjoint representation we have the solution

$$\hat{\Phi}(\Phi,A) = \Phi - \frac{1}{4}\theta^{\mu\nu}\{A_\mu,(D_\nu + \partial_\nu)\Phi\} + \mathcal{O}(\theta^2). \quad (14)$$

It is well known that these solutions are not unique; other terms even depending on continuous parameters can be added to them. In [26] this freedom has been related to the renormalizability properties. However, it can also be used in order to simplify the structure of the theory [20]. In particular, it allows one to give simple forms of Eqs. (13) and (14), which have the interesting property that if the commutative fields vanish, the first order corrections will also vanish. In this case, there is a solution for which all higher order terms of the expansion (8) vanish as well. In fact, for a noncommutative field $\hat{\Phi}$, we can always add covariant terms, with the same tensor structure as Φ, and which depend on theta at least to first order.

These higher order terms can be obtained from the Seiberg-Witten maps for which $(\partial/\partial\theta^{\mu\nu})\hat{\lambda}(\theta)$ and $(\partial/\partial\theta^{\mu\nu})\hat{\Phi}(\theta)$ are solutions, i.e., from the solutions of the equations that result from the corresponding gauge transformations, given by the θ derivatives of Eqs. (10) and (5),

$$\theta^{\mu\nu}\left[\delta_\lambda \frac{\partial}{\partial\theta^{\mu\nu}}\hat{\eta} - \delta_\eta \frac{\partial}{\partial\theta^{\mu\nu}}\hat{\lambda}\right] = i\,\theta^{\mu\nu}\left(\left[\frac{\partial}{\partial\theta^{\mu\nu}}\hat{\lambda} \overset{*}{,} \hat{\eta}\right] + \left[\hat{\lambda} \overset{*}{,} \frac{\partial}{\partial\theta^{\mu\nu}}\hat{\eta}\right] - \frac{\partial}{\partial\theta^{\mu\nu}}\widehat{[\lambda,\eta]} + \frac{i}{2}\{\partial_\mu\hat{\lambda} \overset{*}{,} \partial_\nu\hat{\eta}\}\right) \tag{15}$$

and

$$\theta^{\mu\nu}\delta_\lambda \frac{\partial}{\partial\theta^{\mu\nu}}\hat{\Phi} = i\,\theta^{\mu\nu}\left(\left[\frac{\partial}{\partial\theta^{\mu\nu}}\hat{\lambda} \overset{*}{,} \hat{\Phi}\right] + \left[\hat{\lambda} \overset{*}{,} \frac{\partial}{\partial\theta^{\mu\nu}}\hat{\Phi}\right] + \frac{i}{2}\{\partial_\mu\hat{\lambda} \overset{*}{,} \partial_\nu\hat{\Phi}\}\right), \tag{16}$$

where the $\theta^{\mu\nu}$ factor is included in order to take into account the antisymmetry in μ and ν. A solution to this equation can be obtained [2] from the first order solution $\Phi^{(1)}_{\mu\nu}$. Indeed, from this first order term, by substitution of the commutative fields by the noncommutative ones, with multiplication given by the $*$ product, the full Seiberg-Witten mapped fields $\widehat{\lambda^{(1)}_{\mu\nu}}$ and $\widehat{\Phi^{(1)}_{\mu\nu}}$ can be constructed. Hence, from Eqs. (11) and (14), we have

$$\theta^{\mu\nu}\widehat{\lambda^{(1)}}_{\mu\nu} = \frac{1}{4}\,\theta^{\mu\nu}\{\partial_\mu\hat{\lambda} \overset{*}{,} \hat{A}_\nu\}, \tag{17}$$

$$\theta^{\mu\nu}\widehat{\Phi^{(1)}}_{\mu\nu} = -\frac{1}{4}\,\theta^{\mu\nu}\{\hat{A}_\mu \overset{*}{,} (\hat{D}_\nu + \partial_\nu)\hat{\Phi}\}. \tag{18}$$

Now, after some algebra, taking into account the noncommutative gauge transformations (4), (5), and (9), we get

$$\theta^{\mu\nu}[\delta_\lambda\widehat{\eta^{(1)}}_{\mu\nu} - \delta_\eta\widehat{\lambda^{(1)}}_{\mu\nu}] = i\,\theta^{\mu\nu}\left([\widehat{\lambda^{(1)}}_{\mu\nu} \overset{*}{,} \hat{\eta}] + [\hat{\lambda} \overset{*}{,} \widehat{\eta^{(1)}}_{\mu\nu}] - \widehat{[\lambda,\eta]^{(1)}}_{\mu\nu} + \frac{i}{2}\{\partial_\mu\hat{\lambda} \overset{*}{,} \partial_\nu\hat{\eta}\}\right),$$

$$\theta^{\mu\nu}\delta_\lambda\widehat{\Phi^{(1)}}_{\mu\nu} = i\,\theta^{\mu\nu}\left([\widehat{\lambda^{(1)}}_{\mu\nu} \overset{*}{,} \hat{\Phi}] + [\hat{\lambda} \overset{*}{,} \widehat{\Phi^{(1)}}_{\mu\nu}] + \frac{i}{2}\{\partial_\mu\hat{\lambda} \overset{*}{,} \partial_\nu\hat{\Phi}\}\right). \tag{19}$$

These equations give solutions for Eqs. (15) and (16), if the following identifications are made [2]:

$$\frac{\partial}{\partial\theta^{\mu\nu}}\hat{\lambda} = \widehat{\lambda^{(1)}}_{\mu\nu}, \tag{20}$$

$$\frac{\partial}{\partial\theta^{\mu\nu}}\hat{\Phi} = \widehat{\Phi^{(1)}}_{\mu\nu}, \tag{21}$$

which at $\theta=0$ are identically satisfied. In fact, Eq. (21) is more general; it is valid also for the connection [2], and as well for any field transforming under a linear representation. From it, together with Eq. (20), by successive derivations with respect to θ, a solution for all higher terms of the Seiberg-Witten map can be computed.

Now we see that, if the first order term $\Phi^{(1)}$ vanishes, by construction $\widehat{\Phi^{(1)}}$ will also vanish, and consequently in this case $\Phi=0$ is a consistent solution for $\hat{\Phi}=0$.

The fact that the components of the noncommutativity parameter θ are constant has the important consequence that Lorentz covariance and general covariance under diffeomorphisms of the underlying manifold are spoiled. The answer that is usually given to this question is that, at the scale where noncommutativity is relevant, it is possible that nature does not have the same symmetries as in the commutative limit.

III. DESCRIPTION OF SELF-DUAL GRAVITY

One of the main features of the tetrad formalism of the theory of gravitation [27] is that it introduces local Lorentz SO(3,1) transformations. In this case, the generalized Hilbert-Palatini formulation is written as $\int e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab}(\omega)d^4x$, where $e_a{}^\mu$ is the inverse tetrad, and $R_{\mu\nu}{}^{ab}(\omega)$ is the so(3,1) valued field strength. The decomposition of the Lorentz group as $SO(3,1)=SL(2,\mathbf{C})\otimes SL(2,\mathbf{C})$, and the geometrical structure of four-dimensional space-time, makes it possible to formulate gravitation as a complex theory, as in [8,22]. These formulations take advantage of the properties of the fundamental or spinorial representation of SL(2,**C**), which allows a simple separation of the action on the fields of both factors of SO(3,1), as shown in great detail in [22]. All the Lorentz Lie algebra valued quantities, in particular the connection and the field strength, decompose into self-dual and anti-self-dual parts, in the same way as the Lie algebra $so(3,1)=sl(2,\mathbf{C})\oplus sl(2,\mathbf{C})$. However, Lorentz vectors, like the tetrad, transform under mixed transformations of both factors and so this formulation cannot be written as a chiral SL(2,**C**) theory. Various proposals in this direction have been made (for a review, see [6]). In an early formulation, this problem was solved by Plebański [22], where by means of a constrained Lie algebra valued two-form Σ, the theory can be formulated as a chiral SL(2,**C**) invariant BF theory, $\mathrm{Tr}\int\Sigma\wedge R(\omega)$. In this formulation Σ has two SL(2,**C**) spinorial indices, and it is symmetric on them, $\Sigma^{AB}=\Sigma^{BA}$, like any such sl(2,**C**) valued quantity. The constraints are given by $\Sigma^{AB}\wedge\Sigma^{CD}=\frac{1}{3}\delta^C_{(A}\delta^D_{B)}\Sigma^{EF}\wedge\Sigma_{EF}$ and, as shown in [22], their solution implies the existence of a tetrad one-form, which squared gives the two-form Σ. In the language of SO(3,1), this two-form is a second rank antisymmetric self-dual two-form, $\Sigma^{+ab}=\Pi^{+ab}{}_{cd}\Sigma^{cd}$, where $\Pi^{+ab}{}_{cd}=\frac{1}{4}(\delta^{ab}_{cd}-i\varepsilon^{ab}{}_{cd})$. In this case, the constraints can be recast into the equivalent form $\Sigma^{+ab}\wedge\Sigma^{+cd}=-\frac{1}{3}\Pi^{+abcd}\Sigma^{+ef}\wedge\Sigma^{+}_{ef}$, with the solution $\Sigma^{ab}=2e^a\wedge e^b$.

For the purpose of the noncommutative formulation, we will consider self-dual gravity in a somewhat different way from the papers [8,22]. In this section we will fix our notation and conventions.

Let us take the self-dual SO(3,1) BF action, defined on a (3+1)-dimensional pseudo-Riemannian manifold $(X,g_{\mu\nu})$,

$$I=i\,\mathrm{Tr}\int_X \Sigma^+\wedge R^+=i\int_X \varepsilon^{\mu\nu\rho\sigma}\Sigma^{+ab}_{\mu\nu}R^+_{\rho\sigma ab}(\omega)d^4x, \tag{22}$$

where $R^+_{\rho\sigma ab}=\Pi^{+cd}_{ab}R_{\rho\sigma cd}$, is the self-dual SO(3,1) field strength tensor. This action can be rewritten as

$$I=\frac{1}{2}\int_X \varepsilon^{\mu\nu\rho\sigma}\left(i\Sigma_{\mu\nu}{}^{ab}R_{\rho\sigma ab}+\frac{1}{2}\varepsilon_{abcd}\Sigma_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{cd}\right)d^4x. \tag{23}$$

If now we take the solution of the constraints on Σ, which we now write as

$$\Sigma_{\mu\nu}{}^{ab}=e_\mu{}^a e_\nu{}^b-e_\mu{}^b e_\nu{}^a, \tag{24}$$

then

$$I=\int_X(\det eR+i\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})d^4x. \tag{25}$$

The real and imaginary parts of this action must be varied independently because the fields are real. The first part represents the Einstein action in the Palatini formalism, from which, after variation of the Lorentz connection, a vanishing torsion $T_{\mu\nu}{}^a=0$ results. As a consequence, the second term vanishes due to Bianchi identities.

The action (22) can be written as

$$I=i\int_X \varepsilon^{\mu\nu\rho\sigma}\Sigma^{+ab}_{\mu\nu}R_{\rho\sigma ab}(\omega^+)d^4x, \tag{26}$$

where $R_{\mu\nu}{}^{ab}(\omega^+)=\partial_\mu\omega^{+ab}_\nu-\partial_\nu\omega^{+ab}_\mu+\omega^{+ac}_\mu\omega^{+b}_{\nu c}-\omega^{+ac}_\nu\omega^{+b}_{\mu c}$. From the decomposition $SO(3,1)=SL(2,\mathbf{C})\times SL(2,\mathbf{C})$, it turns out that $\omega_\mu^{\ i}=\omega^{+0i}_\mu$ is a SL(2,**C**) connection. Further, if we take into account self-duality, $\varepsilon_{cd}{}^{ab}\omega^{+cd}_\mu=2i\omega^{+ab}_\mu$, we get $\omega^{+ij}_\mu=-i\varepsilon^{ij}_{\ \ k}\omega^k_\mu$. Therefore,

$$R_{\mu\nu}{}^{0i}(\omega^+)=\partial_\mu\omega^i_\nu-\partial_\nu\omega^i_\mu+2i\varepsilon^i_{\ jk}\omega^j_\mu\omega^k_\nu=R_{\mu\nu}{}^i(\omega), \tag{27}$$

$$\begin{aligned}R_{\mu\nu}{}^{ij}(\omega^+)&=\partial_\mu\omega^{+ij}_\nu-\partial_\nu\omega^{+ij}_\mu+2(\omega_\mu{}^i\omega_\nu{}^j-\omega_\nu{}^i\omega_\mu{}^j)\\&=-i\varepsilon^{ij}{}_kR_{\mu\nu}{}^k(\omega),\end{aligned} \tag{28}$$

where $R^i_{\mu\nu}$ is the SL(2,**C**) field strength.

Similarly, we define $\Sigma_{\mu\nu}{}^i=\Sigma^{+0i}_{\mu\nu}$, which transforms in the SL(2,**C**) adjoint representation. From it we get, $\Sigma^{+ij}_{\mu\nu}=-i\varepsilon^{ij}_{\ \ k}\Sigma^{\ k}_{\mu\nu}$. Thus the action (26) can be written as a SL(2,**C**) BF action

$$\begin{aligned}I&=i\int_X \varepsilon^{\mu\nu\rho\sigma}[\Sigma^{+0i}_{\mu\nu}R_{\rho\sigma 0i}(\omega^+)+\Sigma^{+ij}_{\mu\nu}R_{\rho\sigma ij}(\omega^+)]d^4x\\&=-4i\int_X\varepsilon^{\mu\nu\rho\sigma}\Sigma_{\mu\nu}{}^iR_{\rho\sigma i}(\omega)d^4x.\end{aligned} \tag{29}$$

Therefore, if we choose the algebra sl(2,**C**) to satisfy $[T_i,T_j]=-2\varepsilon_{ij}{}^kT_k$ and $\mathrm{Tr}(T_iT_j)=-2\delta_{ij}$, we have that Eq. (22) can be rewritten as the self-dual action [22]

$$I=2i\,\mathrm{Tr}\int_X\Sigma\wedge R, \tag{30}$$

which is invariant under the SL(2,**C**) transformations $\delta_\lambda\omega_\mu=\partial_\mu\lambda+i[\lambda,\omega_\mu]$ and $\delta_\lambda\Sigma_{\mu\nu}=i[\lambda,\Sigma_{\mu\nu}]$.

If the variation of this action with respect to the SL(2,**C**) connection ω is set to zero, we get the equations

$$\Psi^{\mu i}=\varepsilon^{\mu\nu\rho\sigma}D_\nu\Sigma_{\rho\sigma}{}^i=\varepsilon^{\mu\nu\rho\sigma}(\partial_\nu\Sigma_{\rho\sigma}{}^i+2i\varepsilon^i{}_{jk}\omega_\nu{}^j\Sigma_{\rho\sigma}{}^k)=0. \tag{31}$$

Taking into account separately both real and imaginary parts, we get, in terms of the SO(3,1) connection,

$$\begin{aligned}\varepsilon^{\mu\nu\rho\sigma}D_\nu\Sigma_{\rho\sigma}{}^{ab}&=\varepsilon^{\mu\nu\rho\sigma}(\partial_\nu\Sigma_{\rho\sigma}{}^{ab}+\omega_\nu{}^{ac}\Sigma_{\rho\sigma c}{}^b-\omega_\nu{}^{bc}\Sigma_{\rho\sigma c}{}^a)\\&=0,\end{aligned} \tag{32}$$

which after the identification (24) can be written as

$$\begin{aligned}&\varepsilon^{\mu\nu\rho\sigma}(\partial_\nu e_\rho{}^a e_\sigma{}^b-\partial_\nu e_\rho{}^b e_\sigma{}^a+\omega_\nu{}^{ac}e_{\rho c}e_\sigma{}^b-\omega_\nu{}^{bc}e_{\rho c}e_\sigma{}^a)\\&\quad=\varepsilon^{\mu\nu\rho\sigma}(T_{\nu\rho}^{\ \ a}e_\sigma{}^b-T_{\nu\rho}^{\ \ b}e_\sigma{}^a)=0.\end{aligned} \tag{33}$$

From this the vanishing torsion condition once more appears.

IV. THE NONCOMMUTATIVE ACTION

We start from the SL(2,**C**) invariant action (30). From it, the noncommutative action can be obtained straightforwardly as

$$\hat{I}=2i\,\mathrm{Tr}\int_X\hat{\Sigma}\wedge\hat{R}. \tag{34}$$

This action is invariant under the noncommutative SL(2,**C**) transformations

$$\delta_{\hat{\lambda}}\hat{\omega}_\mu=\partial_\mu\hat{\lambda}+i[\hat{\lambda}\overset{*}{,}\hat{\omega}_\mu], \tag{35}$$

$$\delta_{\hat{\lambda}}\hat{\Sigma}_{\mu\nu}=i[\hat{\lambda}\overset{*}{,}\hat{\Sigma}_{\mu\nu}]. \tag{36}$$

Actually, in order to obtain the noncommutative generalization of the Einstein equation, we could consider the real part of Eq. (34),

$$\widehat{I_E} = -i\,\mathrm{Tr}\int_X [\hat{\Sigma}\wedge\hat{R} - (\hat{\Sigma}\wedge\hat{R})^\dagger], \tag{37}$$

which is also invariant under Eqs. (35) and (36).

In order to obtain a result corresponding to the torsion condition, a ω variation of Eq. (34) must be done. Although we are considering the commutative fields as the fundamental ones, the action is written in terms of the noncommutative ones. Furthermore, the relation between the commutative and the noncommutative physical degrees of freedom is one to one [2]. So the equivalent to the variation of the action with respect to ω will be the variation with respect to $\hat{\omega}$. Thus we write

$$\delta_{\hat{\omega}}\hat{I} = 8i\,\mathrm{Tr}\int_X \varepsilon^{\mu\nu\rho\sigma}(\partial_\rho\hat{\Sigma}_{\mu\nu} - i[\hat{\omega}_{\rho,}{}^{*}\hat{\Sigma}_{\mu\nu}])*\delta\hat{\omega}_\sigma = 0, \tag{38}$$

from which we obtain the noncommutative version of Eq. (32):

$$\widehat{\Psi^\mu} = \varepsilon^{\mu\nu\rho\sigma}\hat{D}_\nu\hat{\Sigma}_{\rho\sigma} = 0. \tag{39}$$

These equations are covariant under the noncommutative transformations (35) and (36), which means that their Seiberg-Witten expansion should be similar to that of a matter field in the adjoint representation (14). In this case we would have that, if the commutative field vanishes, the first order term of the noncommutative one will also vanish. If this happens, as shown at the end of Sec. II, all the higher orders vanish as well. Thus, we could expect that a solution to Eq. (39) would be given by the solution of the commutative equation $\Psi^\mu = 0$. Making use of the ambiguity of the Seiberg-Witten map, we make the following choice for $\hat{\Sigma}$:

$$\hat{\Sigma}_{\mu\nu} = \Sigma_{\mu\nu} - \frac{1}{4}\theta^{\rho\sigma}[\{\omega_\rho,(D_\sigma+\partial_\sigma)\Sigma_{\mu\nu}\} - \{R_{\mu\nu},\Sigma_{\rho\sigma}\}] + \mathcal{O}(\theta^2), \tag{40}$$

from which it turns out that

$$\hat{\Psi}^\mu = \Psi^\mu - \frac{1}{4}\theta^{\nu\rho}[\{\omega_\nu,(D_\rho+\partial_\rho)\Psi^\mu\} - \{R_{\nu\rho},\Psi^\mu\} - 2\delta^\mu_\nu\varepsilon^{\sigma\tau\theta\zeta}D_\sigma\{R_{\tau\theta},\Sigma_{\rho\zeta}\}] + \mathcal{O}(\theta^2). \tag{41}$$

Hence, if the zeroth order terms vanish, $\Psi^\mu = 0$, then the first two terms in Eq. (41) will vanish. These equations $\Psi^\mu = 0$ are equivalent to setting the commutative torsion equal to zero, that is, after the substitution $\Sigma_{\mu\nu}{}^{ab} = e_\mu{}^a e_\nu{}^b - e_\nu{}^a e_\mu{}^b$, their solution is given by

$$\omega_\mu{}^{ab} = -\frac{1}{2}e^{a\nu}e^{b\rho}[e_{\mu c}(\partial_\nu e_\rho{}^c - \partial_\rho e_\nu{}^c) - e_{\nu c}(\partial_\rho e_\mu{}^c - \partial_\mu e_\rho{}^c) - e_{\rho c}(\partial_\mu e_\nu{}^c - \partial_\nu e_\mu{}^c)]. \tag{42}$$

Furthermore, at first order, a computation of the last term in Eq. (41) shows that it is proportional to $\theta^{\mu\nu}\partial_\rho(e^{-1}G_\nu{}^\rho)$, where $G_{\mu\nu}$ is the Einstein tensor. If we now substitute Eq. (42) back into the action (37), the equations of motion to zeroth order will give the vanishing of the Einstein tensor, and the last term in Eq. (41) will be automatically satisfied. In order to explore more general, theta dependent solutions, to first and higher θ orders, a more detailed and involved analysis is forthcoming [28].

With this in mind, the corrections to the noncommutative action (37) can be computed as follows. First we write the Seiberg-Witten expansion of the SL(2,**C**) fields $\hat{\Sigma}$ and $\hat{\omega}$. Furthermore, the commutative SL(2,**C**) fields are written by means of the self-dual SO(3,1) fields, $\omega_\mu{}^i = \omega_\mu^{+0i}$ and $\Sigma_{\mu\nu}{}^i = \Sigma_{\mu\nu}^{+0i}$. Then we decompose these self-dual fields into the real ones $\omega_\mu{}^{ab}$ and $\Sigma_{\mu\nu}{}^{ab}$ and then substitute $\Sigma_{\mu\nu}{}^{ab} = e_\mu{}^a e_\nu{}^b - e_\nu{}^a e_\mu{}^b$ and write the connection as in Eq. (42). In this case we have a noncommutative action that depends only on the tetrad.

If we consider the real part, as in Eq. (37), the first order correction vanishes, and, after a lengthy calculation, the second order one turns out to be, already written in terms of commutative SO(3,1) fields,

$$\begin{aligned}\hat{I}_{\theta^2} = \frac{1}{2^4}\theta^{\gamma\delta}\theta^{\tau\xi}\int d^4x[&4e\{4R_\delta{}^\rho(R_{\rho\tau}{}^{ab}R_{\gamma\xi ab} - \omega_\tau{}^{ab}\partial_\xi R_{\rho\gamma ab}) + \omega_\gamma{}^{\rho\sigma}\partial_\tau\omega_\delta{}^{ab}\partial_\xi R_{\rho\sigma ab} + R\partial_\delta[\omega_\tau{}^{ab}(\partial_\xi\omega_{\gamma ab} + R_{\gamma\xi ab})]\\ &+ 2\omega_\gamma{}^{\rho\sigma}\partial_\delta(R_{\rho\tau}{}^{ab}R_{\sigma\xi ab} - \omega_\tau{}^{ab}\partial_\xi R_{\rho\sigma ab})\} + \epsilon^{\mu\nu\rho\sigma}(4e[\epsilon_{\gamma\delta\alpha\beta}R_{\rho\sigma}{}^{\alpha\beta}(R_{\mu\tau}{}^{ab}R_{\nu\xi ab} - \omega_\tau{}^{ab}\partial_\xi R_{\mu\nu ab})\\ &+ 2\epsilon_{\tau\xi\alpha\beta}R_{\mu\nu}{}^{ab}R_{\rho\sigma ab}R_{\gamma\delta}{}^{\alpha\beta}] + \epsilon_{abcd}\{4R_{\rho\sigma\gamma\delta}(R_{\mu\tau}{}^{ab}R_{\nu\xi}{}^{cd} - \omega_\tau{}^{ab}\partial_\xi R_{\mu\nu}{}^{cd}) + 4R_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{cd}[2R_{\gamma\delta\tau\xi} - \omega_{\tau ef}\partial_\xi(e_\gamma{}^e e_\delta{}^f)]\\ &- 2\omega_{\gamma\mu\nu}\partial_\delta(R_{\rho\tau}{}^{ab}R_{\sigma\xi}{}^{cd} - \omega_\tau{}^{ab}\partial_\xi R_{\rho\sigma}{}^{cd}) - 2\omega_\gamma{}^{ef}R_{\rho\sigma ef}\partial_\delta[2R_{\mu\nu}{}^{ab}e_\tau{}^c e_\xi{}^d - \omega_\tau{}^{ab}\partial_\xi(e_\mu{}^c e_\nu{}^d)]\\ &- 2\omega_\gamma{}^{ab}R_{\rho\sigma}{}^{cd}\partial_\delta[2R_{\mu\nu\tau\xi} - \omega_{\tau ef}\partial_\xi(e_\mu{}^e e_\nu{}^f)] - \omega_{\gamma\mu\nu}\partial_\tau\omega_\delta{}^{ab}\partial_\xi R_{\rho\sigma}{}^{cd} - \omega_\gamma{}^{ef}R_{\rho\sigma ef}\partial_\tau\omega_\delta{}^{ab}\partial_\xi(e_\mu{}^c e_\nu{}^d)\\ &- \omega_\gamma{}^{ab}R_{\rho\sigma}{}^{cd}\partial_\tau\omega_{\delta ef}\partial_\xi(e_\mu{}^e e_\nu{}^f) - 4R_{\mu\nu}{}^{ef}R_{\rho\sigma ef}\omega_\tau{}^{ab}\partial_\xi(e_\gamma{}^c e_\delta{}^d)\})],\end{aligned} \tag{43}$$

where the connection $\omega_\mu{}^{ab}$ is given by Eq. (42). From these correction terms, the explicit computation of deformed known gravitational metrics could be done.

V. CONCLUSIONS

In this work we propose an ansatz to obtain a noncommutative formulation of standard four-dimensional Einstein gravitation. We start from a self-dual SO(3,1) BF action, which is equivalent to Einstein gravitation after substitution of the B field in terms of the tetrad. This action is reformulated as the chiral SL(2,**C**) invariant self-dual action (30), from which the noncommutative action (34) is straightforwardly obtained. This chiral action allows us to find an alternative noncommutative gravity action in four dimensions, which generalizes the usual general relativity (37). As mentioned, there are other proposals already introduced in the literature [14–17]. In our proposed action the noncommutative spin connection variation gives the noncommutative ‘‘torsion condition’’ (39), which seems to be solved, at any order, by Eq. (42), as explicitly shown to first order. This allows us to introduce the tetrad at the commutative level in a consistent way (24). For this solution, the second order corrections to the action are computed after laborious algebra [Eq. (43)]. More general, θ dependent solutions for the spin connection, as well as the consequences of the higher order terms corresponding to the last term in Eq. (41), will be studied elsewhere [28].

In the process we have used the results from [19,20], developed there to construct the noncommutative versions of the standard model and grand unified theories. In the present paper, the Seiberg-Witten map for matter fields in the adjoint representation of any gauge group has been constructed in order to get the Seiberg-Witten map for the Σ field.

The physical consequences of the noncommutative extension of standard general relativity remain to be studied. An interesting possibility seems to be the study of inflation in this model. It is well known, in a very different physical setting, that the trace anomaly that leads to higher derivative corrections in the corresponding effective action could produce inflation [29,30].

Finally, the results presented here can be regarded as a preliminary step for the construction of a noncommutative version of the Ashtekar Hamiltonian formulation through a noncommutative Legendre transformation. Moreover, it would be interesting to search for a quantization of the results obtained in the present paper and proceed to find the corresponding loop quantum gravity. Furthermore, the computation of noncommutative gravitational effects can be done, for instance, from the corrections to known metrics. Details of work in these directions will be reported elsewhere.

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[1] A. Connes, M.R. Douglas, and A. Schwarz, J. High Energy Phys. **02**, 003 (1998).
[2] N. Seiberg and E. Witten, J. High Energy Phys. **09**, 032 (1999).
[3] J. Polchinski, ‘‘M Theory: Uncertainty and Unification,’’ hep-th/0209105.
[4] T. Banks, W. Fischler, S.H. Shenker, and L. Susskind, Phys. Rev. D **55**, 5112 (1997).
[5] F. Ardalan, H. Arafaei, M.R. Garousi, and A. Ghodsi, Int. J. Mod. Phys. A **18**, 1051 (2003); S.I. Vacaru, ‘‘(Non)Commutative Finsler Geometry from String/M-Theory,’’ hep-th/0211068.
[6] P. Peldán, Class. Quantum Grav. **11**, 1087 (1994).
[7] S. Carlip, Rep. Prog. Phys. **64**, 885 (2001); G.T. Horowitz, ‘‘Quantum Gravity at the Turn of the Millennium,’’ gr-qc/0011089.
[8] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986); Phys. Rev. D **36**, 1587 (1987); *Lectures on Non-Perturbative Canonical Gravity* (World Scientific, Singapore, 1991).
[9] J. Samuel, Pramana, J. Phys. **28**, L429 (1987).
[10] T. Jacobson and L. Smolin, Phys. Lett. B **196**, 39 (1987); Class. Quantum Grav. **5**, 583 (1988).
[11] A.H. Chamseddine, G. Felder, and J. Fröhlich, Commun. Math. Phys. **155**, 205 (1993).
[12] A. Connes, *Noncommutative Geometry* (Academic, London, 1994).
[13] A.H. Chamseddine, Commun. Math. Phys. **218**, 283 (2001).
[14] J.W. Moffat, Phys. Lett. B **491**, 345 (2000); **493**, 142 (2000).
[15] A.H. Chamseddine, Phys. Lett. B **504**, 33 (2001).
[16] A.H. Chamseddine, ‘‘Invariant Actions for Noncommutative Gravity,’’ hep-th/0202137.
[17] M.A. Cardella and Daniela Zanon, Class. Quantum Grav. **20**, L95 (2003).
[18] M. Bañados, O. Chandia, N. Grandi, F.A. Schaposnik, and G.A. Silva, Phys. Rev. D **64**, 084012 (2001); H. Nishino and S. Rajpoot, Phys. Lett. B **532**, 334 (2002); V.P. Nair, Nucl. Phys. **B651**, 313 (2003); S. Cacciatori, D. Klemm, L. Martucci, and D. Zanon, Phys. Lett. B **536**, 101 (2002); S. Cacciatori, A.H. Chamseddine, D. Klemm, L. Martucci, W.A. Sabra, and D. Zanon, Class. Quantum Grav. **69**, 4029 (2002); Y. Abe and V.P. Nair, Phys. Rev. D **68**, 025002 (2003).
[19] B. Jurco, L. Moller, S. Schraml, P. Schupp, and J. Wess, Eur. Phys. J. C **21**, 383 (2001).
[20] X. Calmet, B. Jurco, P. Schupp, J. Wess, and M. Wohlgenannt, Eur. Phys. J. C **23**, 363 (2002).
[21] H. García-Compeán, O. Obregón, C. Ramírez, and M. Sabido, Phys. Rev. D (to be published), hep-th/0210203.
[22] J. Plebański, J. Math. Phys. **18**, 2511 (1977).
[23] H. García-Compeán, O. Obregón, C. Ramírez, and M. Sabido, ‘‘Noncommutative Plebański-Ashtekar Theory of Gravity,’’ talk presented at Topics in Mathematical Physics, General Relativity, and Cosmology, on the Occasion of the 75th Birth-

day of Jerzy F. Plebański, México City, 2002.
[24] G.T. Horowitz, Commun. Math. Phys. **125**, 417 (1989); G.T. Horowitz and M. Srednicki, *ibid.* **130**, 83 (1990).
[25] M.R. Douglas and N.A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001); R.J. Szabo, Phys. Rep. **378**, 207 (2003); C.K. Zachos, Int. J. Mod. Phys. A **17**, 297 (2002).
[26] A. Bichl, J. Grimstrup, H. Grosse, L. Popp, M. Schweda, and R. Wulkenhaar, J. High Energy Phys. **06**, 013 (2001).
[27] R. Utiyama, Phys. Rev. **101**, 1597 (1956); T.W.B. Kibble, J. Math. Phys. **2**, 212 (1961).
[28] H. García-Compeán, O. Obregón, C. Ramírez, and M. Sabido (in preparation).
[29] A.A. Starobinsky, Phys. Lett. **91B**, 99 (1980).
[30] S.W. Hawking and T. Hertog, Phys. Rev. D **66**, 123509 (2002).