Large-N collective fields and holography

Sumit R. Das*
Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506, USA

Antal Jevicki†
Department of Physics, Brown University, Providence, Rhode Island 02192, USA

(Received 14 May 2003; published 12 August 2003)

We propose that the Euclidean bilocal collective field theory of critical large-N vector models provides a complete definition of the proposed dual theory of higher spin fields in anti–de Sitter spaces. We show how this bilocal field can be decomposed into an infinite number of even spin fields in one more dimension. The collective field has a nontrivial classical solution which leads to an $O(N)$ thermodynamic entropy characteristic of the lower dimensional theory, as required by general considerations of holography. A subtle cancellation of the entropy coming from the bulk fields in one higher dimension with $O(1)$ contributions from the classical solution ensures that the subleading terms in thermodynamic quantities are of the expected form. While the spin components of the collective field transform properly under dilatational, translational, and rotational isometries of AdS, special conformal transformations mix fields of different spins indicating a need for a nonlocal map between the two sets of fields. We discuss the nature of the propagating degrees of freedom through a Hamiltonian form of collective field theory and argue that nonsinglet states which are present in an Euclidean version are related to nontrivial backgrounds.

DOI: 10.1103/PhysRevD.68.044011 PACS number(s): 04.70.Dy, 11.15.Pg

I. INTRODUCTION AND SUMMARY

It has been known for a long time that theories of $N\times N$ matrices, e.g., gauge theories, become string theories at large N, with $1/N$ playing the role of string coupling constant [1]. A new element of this correspondence was learned in the early 1990, viz., the string theory lives in a higher dimensional space. An early example is the quantum mechanics of a single Hermitian matrix $M_{ij}(t)$—the $c=1$ matrix model. In this case the string theory is in $1+1$ dimensions, whose only dynamical field is a massless scalar. This scalar is related to the density of eigenvalues $\phi(x,t)$ of the matrix, so that the space of eigenvalues x provides the extra coordinate [2]. The Hamiltonian for this collective field can be written down following [3]. The classical value of $\phi(x,t)$ corresponds to the linear dilaton background of string theory while the fluctuations are related to the massless scalar.[1] Thus the singlet sector of the model contains the propagating degrees of freedom of the two-dimensional string theory. This is, in a sense, holography [5]. However, the singlet sector thermodynamic entropy of the collective field theory is $O(1)$ rather than $O(N^2)$ and is that of a $(1+1)$-dimensional theory rather than that of a $(0+1)$-dimensional theory [2]. This is different from what one expects from the holographic principle [5]. In fact the contribution of nonsinglet states should be significant in the thermodynamic partition function at high temperatures which is precisely the regime where a $(0+1)$-dimensional behavior with entropy proportional to N^2 is expected. Such nonsinglet states are, however, not part of the perturbative string theory spectrum.

The most concrete example of holography is of course the AdS/conformal field theory (CFT) correspondence [6,7]. Here, the high temperature thermodynamics is dominated by AdS black holes [8]. For AdS$_{d+1}$ with length scale R, and a $(d+1)$-dimensional Newton constant G, the entropy of a black hole S_{bulk} is related to the temperature T by the relation

$$S_{bulk}G\sim(TR^2)^{d-1}. \tag{1.1}$$

On the other hand, the conformal field theory on the boundary at the same temperature should have an entropy $S_{boundary}$ given by

$$S_{boundary}\sim N_f(TR)^{d-1}, \tag{1.2}$$

where N_f is the number of degrees of freedom. These two expressions agree if

$$\frac{G}{R^{d-1}}\sim\frac{1}{N_f}, \tag{1.3}$$

a relation which is satisfied for all known examples of AdS/CFT.

There are black holes in $(1+1)$-dimensional string theory as well [9]. However, despite considerable effort [10], such black holes are not fully understood in the matrix model, though there has been significant recent progress [11]. The above discussion shows that these black holes are intimately related to nonsinglet states, along the lines of [11]. In the AdS/CFT examples all the physical states are singlets in any case, which may be the reason why the thermodynamics is reproduced faithfully.

*Email address: das@pa.uky.edu
†Email address: antal@het.brown.edu

[1] The detailed relationship between $\phi(x,t)$ and the scalar which couples simply to the worldsheet is, however, rather complicated [4].

In models with matrices, there are an exponentially growing number of single trace singlet operators, which is one reason why they represent string theories. In this paper we will consider models with fields in the vector representation of groups like $O(N)$ or $U(N)$. These models are known to be solvable in the large-N limit [12]. The singlet operators can involve products of pairs of fields and therefore there are no exponentially growing number of "single particle" states. Indeed, the Feynman diagrams are made of bubbles and resemble branched polymers rather than string worldsheets.

In a recent work, Klebanov and Polyakov [13] have proposed that critical vector models are dual to certain higher spin gauge theories [14,15] defined on AdS spaces. Such higher spin fields include gravity, but are not string theories. (Higher spin gauge theory with $N=8$ supersymmetry in AdS_5 is, however, related to *free* $N=4$ Yang-Mills theory and hence to $\alpha' \to \infty$ limit of IIB string theory [16,17].) The complete set of interaction in these theories is still not known.

Of particular interest is the critical three (Euclidean) dimensional $O(N)$ vector model

$$S=\int d^3x\left[(\partial\vec{\phi})\cdot(\partial\vec{\phi})+\frac{\lambda}{2N}(\vec{\phi}\cdot\vec{\phi})^2\right]. \tag{1.4}$$

This has two fixed points. In terms of a running coupling $\lambda(k)$ these are at $\lambda(k)=0$ and $\lambda(k)=4k$. According to the proposal of [13], the three dimensional conformal field theories at these fixed points are dual to a theory of higher spin fields with one field for each even spin defined on AdS_4 in two different senses. The correspondence of bulk and boundary quantities for the theory at the nontrivial fixed point is standard, with the generating functional for correlators of singlet operators being equal to the effective action in the bulk with the boundary values of the fields set equal to the currents. For the theory at the Gaussian fixed point the correspondence proceeds via a Legendre transform of the generating functional as in [18].

In this paper we show that standard methods of collective field theory can be used to start with a vector model in Euclidean space and *construct* a theory of even spins in one higher dimension. The main ingredient is the fact that all singlet correlations of the model may be expressed in terms of a bilocal field $\sigma(\vec{x},\vec{y})$ [19,20]

$$\sigma(\vec{x},\vec{y})=\frac{1}{N}\vec{\phi}(\vec{x})\cdot\vec{\phi}(\vec{y}). \tag{1.5}$$

The higher dimensional space is made out of the center of mass coordinate

$$\vec{u}=\frac{1}{2}(\vec{x}+\vec{y}) \tag{1.6}$$

and the magnitude r of the relative coordinate

$$\vec{v}=\frac{1}{2}(\vec{x}-\vec{y}), \quad r^2=\vec{v}\cdot\vec{v}. \tag{1.7}$$

A spherical harmonic decomposition in the angles of the coordinate $\vec{v}$ then yields a collection of higher spin fields in one higher dimension. The symmetry of the collective field $\sigma(\vec{x},\vec{y})$ under interchanges of $\vec{x}$ and $\vec{y}$ implies that these spins are zero or even integers. While this construction works for any number of dimensions and for generic values of the coupling, things become more interesting when the theory is conformally invariant. By examining the transformation properties of the field $\sigma(\vec{x},\vec{y})$ under conformal transformations we find that the various spin components of $\sigma(u,v)$ transform as tensors under translation, rotation, and dilatation isometries of (Euclidean) AdS space, where r is a coordinate in the following form of the AdS metric:

$$ds^2=\frac{1}{r^2}[dr^2+d\vec{u}\cdot d\vec{u}]. \tag{1.8}$$

However, these components *do not* transform properly under special conformal isometries. We suspect that this indicates that these modes are related to the standard fields in AdS by a field redefinition which is possibly nonlocal. This is similar to the fact that in the $c=1$ matrix model the collective field is not really the field which follows from a worldsheet formulation of the dual string theory. Our considerations may be easily extended to vector models with other symmetries, in which case one would get even spins as well.

From the point of view of the vector model, the bilocal collective field represents a collection of local composite operators. This may be seen by performing a Taylor expansion in the coordinate $\vec{v}$ so that one has

$$\sigma(\vec{x},\vec{y})=\vec{\phi}(\vec{x})\vec{\phi}(\vec{x})+[\vec{\phi}(\vec{x})\partial_i\partial_j\vec{\phi}(\vec{x}) -(\partial_i\vec{\phi}(\vec{x}))(\partial_j\vec{\phi}(\vec{x}))]v^iv^j+\cdots. \tag{1.9}$$

The coefficients of powers of v in this expansion are related, but *not identical*, to the infinite set of conserved currents of the free theory which are conjectured to be the operators dual to the higher spin fields. The nontrivial relationship between these currents and the coefficients in the expansion (1.9) is another indication of the fact that the relationship between σ and the higher spin fields is nonlocal. Nevertheless, since these currents are all contained in the bilocal field one can construct bulk fields in AdS by folding in with the appropriate bulk-boundary Green's function as in [17].

In this paper we construct the Euclidean collective field theory with special attention to subleading effects in $1/N$. As is well known, the collective field has a nontrivial classical solution. In our interpretation this provides the four dimensional "spacetime" on which the physical excitations propagate. By considering fluctuations around the classical solution we demonstrate the existence of a nontrivial IR stable fixed point in three dimensions and reproduce the well known results for conformal dimensions of composite operators at this fixed point.

One finite N effect which is not discussed in this paper in detail is the emergence of an *exclusion principle*. Consider the problem on a finite lattice with M sites in each of the d directions. Then the functional integral over ϕ^i is a multiple integral over NM^d variables. However, the functional integral over the collective field $\sigma(\vec{x},\vec{y})$ is a multiple integral over M^{2d} variables. Thus if $N<M^d$ there are too many degrees of freedom in the collective field. This would lead to rather nontrivial constraints on σ. In terms of Fourier modes of σ this means that all these modes are not independent. Roughly speaking, for each component of the momenta, one may regard the first $N^{1/d}$ values to be independent, while the remaining modes are related to these by nontrivial relations. For the bulk theory this is a kind of exclusion principle which has appeared in the context of AdS/CFT correspondence and which arises because of the same reason [21].

One of our main results relates to the thermodynamics of the model. From the point of view of the critical vector model defined in d Euclidean dimensions, the finite temperature properties are those appropriate to d dimensions. Furthermore, one would expect that the leading free energy and the entropy are both proportional to N. This, however, appears quite mysterious from the point of view of a $(d+1)$-dimensional theory, where $1/\sqrt{N}$ appears as a *coupling constant*, so that the natural expectation is that in a $1/N$ expansion the leading entropy comes from the free theory and hence of $O(1)$. We show that the leading thermodynamic behavior is a *classical* contribution in the collective field theory coming from the presence of a nontrivial classical solution. Perhaps more significantly, there is a partial cancellation between the $O(1)$ contribution obtained by integrating out fluctuations and a $O(1)$ term present in the classical action. In particular, for the Gaussian fixed point this cancellation is complete and ensures that the entire answer is $O(N)$ as expected. We speculate that this result is indicative of the presence of black holes in the bulk theory. The fact that the thermodynamics is reproduced correctly is an indication that these models, like string theory examples, have the right ingredients to provide a holographic description of theories containing gravity. Unlike string theory examples, however, we have an explicit construction of the higher dimensional theory in terms of the fields of the lower dimensional theory. The hope is that this will facilitate a better understanding of holography.

This explicit construction in fact shows a special feature of the bulk theory. We show that the interactions of the collective field theory have a coupling constant $1/\sqrt{N}$ *with no other free parameter*. On the other hand, the bulk theory has *a priori* two dimensional parameters, the Newton constant G and the AdS scale R. The collective field theory seems to indicate, however, that the bulk theory must be characterized by only the dimensionless combination G/R^2. This should be exactly $1/N$. Indeed, this is exactly what is required in $d=3$ from Eq. (1.3). This is related to the fact that the conformal field theory of the vector model lives at *fixed points* rather than on *fixed lines*. There is no free coupling constant which would be the analog of the gauge coupling constant for string theory on $\mathrm{AdS}_5\times S^5$. In this sense this is similar to M-theory examples of holography.

The *Euclidean* collective field, however, contains *more* than *propagating* degrees of freedom. This is related to the fact that the collective field is made out of currents of the vector model which are conserved. Thus all the normal modes of the current do not create independent and orthogonal states. This feature is in fact well known in the AdS/CFT correspondence. For example, the operator in the dual theory which represents the bulk graviton is the energy momentum tensor which seems to have more components than the graviton. However, the energy momentum tensor is conserved, which reduces the number of independent modes to the correct value.

To formulate the theory in terms of the physical propagating degrees of freedom it is more useful to consider a Hamiltonian formulation of collective field theory where the dynamical variables are $\psi(\tilde{x},\tilde{y},t)$, where $\tilde{x}$ and $\tilde{y}$ denote points in *space* and t is the time. These have canonical conjugate momenta $\Pi(\tilde{x},\tilde{y},t)$ and one can derive a Hamiltonian which reproduces the correlators of such singlet operators. However, like other examples of Hamiltonian collective fields (notably the $c=1$ matrix model), it is difficult to describe nontrivial *backgrounds* and to describe the finite temperature thermodynamics fully.

In Sec. II we derive the Euclidean collective field action for any dimension after a careful derivation of the Jacobian, and discuss the saddle point solution at large N, the action for quadratic fluctuations and the $O(1)$ partition function, the nature of interactions, and the appearance of the nontrivial fixed point in $d=3$. In Sec. III we discuss the implications of our results for the holographic correspondence: the finite temperature thermodynamics, the identification of dilatation, rotation, and translation isometries of the bulk, the nature of interactions in the bulk theory, and the question of physical propagating modes and its relationship to Hamiltonian collective field theory. Section IV contains conclusions and comments.

II. EUCLIDEAN COLLECTIVE FIELD THEORY: SADDLE POINT SOLUTION AND FLUCTUATIONS

We start with the following action in d Euclidean dimensions:

$$S[\vec{\phi}]=\int d^dx\left[(\partial\vec{\phi})\cdot(\partial\vec{\phi})+m^2\vec{\phi}\cdot\vec{\phi}+\frac{\lambda}{2N}(\vec{\phi}\cdot\vec{\phi})^2\right]. \tag{2.1}$$

The collective field is defined as in Eq. (1.5). The collective field action $\mathcal{S}[\sigma]$ is then defined via the relation

$$\int \mathcal{D}\vec{\phi}(x)e^{-S[\vec{\phi}]}=\int \mathcal{D}\sigma(\vec{x},\vec{y})e^{-\mathcal{S}[\sigma(\vec{x},\vec{y})]}. \tag{2.2}$$

A. Derivation of the action

The action $\mathcal{S}$ has a piece $\mathcal{S}_0$ which comes from rewriting Eq. (2.1) in terms of σ and a second piece coming from the Jacobian J in the change of variables in the path integral measure,

$$\mathcal{S}=\mathcal{S}_0-\log J. \tag{2.3}$$

$\mathcal{S}_0$ may be written down easily from Eq. (2.1)

$$\mathcal{S}_0 = N\int d^dx\Big\{-[\nabla_x^2\sigma(\vec{x},\vec{y})]_{\vec{x}=\vec{y}} + m^2\sigma(\vec{x},\vec{x}) + \frac{\lambda}{2}(\sigma(\vec{x},\vec{x}))^2\Big\}. \quad (2.4)$$

To treat singular terms which appear in J we will work on a square lattice with M points in each direction. The fields ϕ will be also rescaled appropriately to make them dimensionless, so that the lattice spacing disappears from the expressions. At the end of the calculation one may of course restore the lattice spacing and take the continuum and thermodynamic limit. A point on the lattice will be denoted by an integer-valued vector $\vec{m}$. The Jacobian can be then determined by comparing Dyson-Schwinger equations for invariant correlators obtained from the ensembles on the two sides of Eq. (2.2) [20,22,23]. First consider the identity

$$\int [\mathcal{D}\phi]\frac{\delta}{\delta\phi^i(\vec{m})}(\phi^i(\vec{m}')F[\sigma])e^{-S[\vec{\phi}]}=0 \quad (2.5)$$

for some arbitrary functional $F[\sigma]$ of the bilocal collective field. This leads to the equation

$$\langle N\delta_{\vec{m},\vec{m}'}F[\sigma]\rangle + \Big\langle \phi^i(\vec{m}')\frac{\delta F}{\delta\phi^i(\vec{m})}\Big\rangle - \Big\langle \phi^i(\vec{m}')\frac{\delta S}{\delta\phi^i(\vec{m})}F[\sigma]\Big\rangle = 0, \quad (2.6)$$

where the averages are evaluated with the action $S[\vec{\phi}]$. Using

$$\frac{\delta}{\delta\phi^i(\vec{m})} = \sum_{\vec{m}_1,\vec{m}_2}\frac{\delta\sigma(\vec{m}_1,\vec{m}_2)}{\delta\phi^i(\vec{m})}\frac{\delta}{\delta\sigma(\vec{m}_1,\vec{m}_2)} = \sum_{\vec{m}_1}\phi^i(\vec{m}_1)\Big[\frac{\delta}{\delta\sigma(\vec{m}_1,\vec{m})} + \frac{\delta}{\delta\sigma(\vec{m},\vec{m}_1)}\Big], \quad (2.7)$$

Eq. (2.6) becomes

$$N\delta_{\vec{m},\vec{m}'}\langle F\rangle + 2\sum_{\vec{m}_1}\sigma(\vec{m}_1,\vec{m}')\Big[\frac{\delta F}{\delta\sigma(\vec{m}_1,\vec{m})} - F\frac{\delta S}{\delta\sigma(\vec{m}_1,\vec{m})}\Big] = 0. \quad (2.8)$$

Next consider a change of variables to the collective fields $\sigma(\vec{m},\vec{m}')$ and consider the identity

$$\sum_{\vec{m}_1}\int [\mathcal{D}\sigma]\frac{\delta}{\delta\sigma(\vec{m}_1,\vec{m})}(\sigma(\vec{m}_1,\vec{m}')J[\sigma]F[\sigma]e^{-\mathcal{S}})=0, \quad (2.9)$$

where $J[\sigma]$ is the Jacobian that we want to determine. The averages $\langle\langle\cdot\rangle\rangle$ are defined in the ensemble with the action $\mathcal{S}$ and measure $[\mathcal{D}\sigma]J[\sigma]$. Note that for any observable A one has the identity

$$\langle A\rangle = \langle\langle JA\rangle\rangle. \quad (2.10)$$

Then Eq. (2.9) becomes

$$M^d\delta_{\vec{m},\vec{m}'}\langle\langle F\rangle\rangle + \sum_{\vec{m}_1}\Big\langle\Big\langle \sigma(\vec{m}_1,\vec{m}')\frac{\delta\log J}{\delta\sigma(\vec{m}_1,\vec{m})}F\Big\rangle\Big\rangle + \sum_{\vec{m}_1}\Big\langle\Big\langle \sigma(\vec{m}_1,\vec{m}')\frac{\delta F}{\delta\sigma(\vec{m}_1,\vec{m})}\Big\rangle\Big\rangle - \sum_{\vec{m}_1}\Big\langle\Big\langle \sigma(\vec{m}_1,\vec{m}')F\frac{\delta\mathcal{S}}{\delta\sigma(\vec{m}_1,\vec{m})}\Big\rangle\Big\rangle = 0. \quad (2.11)$$

Since F is arbitrary, comparing Eq. (2.11) with Eq. (2.8) one gets an equation for J

$$\sum_{\vec{m}_1}\sigma(\vec{m}_1,\vec{m}')\frac{\delta\log J}{\delta\sigma(\vec{m}_1,\vec{m})} = \frac{1}{2}(N-2M^d)\delta_{\vec{m},\vec{m}'}, \quad (2.12)$$

which may be solved by

$$\log J = \frac{1}{2}(N-2M^d)\mathrm{Tr}\log\sigma \quad (2.13)$$

up to a constant. Here the trace is taken over the indices $\vec{m}$. The final expression (2.13) can be, of course, written in continuum notation, in which the role of the factor M^d is given by $V\delta^d(0)$, where V is the volume of the d dimensional space.

B. The saddle point solution

Since both terms in Eq. (2.3) have pieces proportional to N, the functional integral may be performed by a saddle point method at $N\to\infty$. It is convenient to work in momentum space by defining Fourier components for any bilocal field $A(\vec{m}_1,\vec{m}_2)$ by

$$A(\vec{m}_1,\vec{m}_2) = \sum_{\vec{n}_1,\vec{n}_2}\tilde{A}(\vec{n}_1,\vec{n}_2)e^{(2\pi i/M)(\vec{n}_1\cdot\vec{m}_1+\vec{n}_2\cdot\vec{m}_2)}. \quad (2.14)$$

Then the action $\mathcal{S}$ becomes

$$\mathcal{S} = NM^d\Big[\sum_{\vec{n}}(p_n^2+m_0^2)\tilde{\sigma}(\vec{n},-\vec{n}) + \frac{\lambda_0}{2}\sum_{\vec{n}_1,\vec{n}_2,\vec{n}_3}\tilde{\sigma}(\vec{n}_1,\vec{n}_2)\tilde{\sigma}(\vec{n}_3,-(\vec{n}_1+\vec{n}_2+\vec{n}_3))\Big] - \frac{1}{2}(N-2M^d)\mathrm{Tr}\log\sigma, \quad (2.15)$$

where m_0 and λ_0 are the dimensionless mass and coupling constant, respectively (with appropriate powers of the cutoff multiplying the dimensional quantities m and λ) and

$$p_n^2 = 4\sin^2\frac{n}{2}, \quad n=|\vec{n}|. \quad (2.16)$$

The saddle point is translationally invariant so that

$$\tilde{\sigma}(\vec{n}_1,\vec{n}_2)=\xi(\vec{n}_1)\,\delta_{\vec{n}_1,-\vec{n}_2}. \tag{2.17}$$

With this ansatz, the term in the action becomes

$$\mathcal{S}=N\Bigg[M^d\sum_{\vec{n}}\ (p_n^2+m_0^2)\xi(\vec{n})+M^d\frac{\lambda_0}{2}\sum_{\vec{n},\vec{n}'}\ \xi(\vec{n})\xi(\vec{n}')-\frac{1}{2}\sum_{\vec{n}}\ \log\xi(\vec{n})\Bigg]+O(1/N) \tag{2.18}$$

so that the saddle point becomes

$$\xi(\vec{n})=\frac{1}{2M^d}\,\frac{1}{p_n^2+m_0^2+\lambda_0 s}, \tag{2.19}$$

where

$$s=\sum_{\vec{n}}\ \xi(\vec{n}). \tag{2.20}$$

Equation (2.20) is of course the lowest order (in $1/N$) propagator $\langle\vec{\phi}(\vec{x})\cdot\vec{\phi}(\vec{y})\rangle$ in momentum space. The gap s is then determined by a gap equation

$$s=\frac{1}{2M^d}\sum_{\vec{n}}\ \frac{1}{p_n^2+m_0^2+\lambda_0 s}. \tag{2.21}$$

In the continuum limit the equation (2.21) reads

$$s=\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2+m^2+\lambda s}. \tag{2.22}$$

The saddle point value of the action is then given by

$$\mathcal{S}_0=\frac{N}{2}\Bigg[\sum_{\vec{n}}\ \log(p_n^2+m_0^2+\lambda_0 s)-M^d\lambda_0 s^2\Bigg] \tag{2.23}$$

up to an unimportant constant.

The theory is on the critical surface when the renormalized mass vanishes. In this critical theory, one has

$$\xi_c(p)=\frac{1}{2|p|^2}. \tag{2.24}$$

For any dimension d the point $\lambda_0=0$ is of course a fixed point. For $d=3$ this Gaussian fixed point is unstable and there is an IR stable fixed point at a finite value of λ_0, as will be explained in a following section.

C. Leading 1/N correction and propagator

The "classical" action $\mathcal{S}$ evaluated at the saddle point already has an $O(1)$ piece which is given by

$$\mathcal{S}_1^{(1)}=-M^d\sum_{\vec{n}}\ \log(p_n^2+m_0^2+\lambda_0 s). \tag{2.25}$$

As we will see below, the extra power of the number of lattice points M^d in Eq. (2.25) is significant, as is its sign. Other contributions to this order are obtained by expanding the collective field as

$$\tilde{\sigma}(\vec{n}_1,\vec{n}_2)=\tilde{\sigma}_0(\vec{n}_1,\vec{n}_2)+\frac{1}{\sqrt{N}}\tilde{\eta}(\vec{n}_1,\vec{n}_2). \tag{2.26}$$

Then the quadratic action for $\tilde{\eta}$ is

$$\delta\mathcal{S}=\frac{\lambda_0 M^d}{2}\sum_{\vec{n}_1,\vec{n}_2,\vec{n}_3}\ \tilde{\eta}(\vec{n}_1,\vec{n}_2)\,\tilde{\eta}(\vec{n}_3,-(\vec{n}_1+\vec{n}_2+\vec{n}_3))+\frac{1}{4}\sum_{\vec{n}_1,\vec{n}_2}\ \xi^{-1}(-\vec{n}_1)\xi^{-1}(\vec{n}_2)\,\tilde{\eta}(\vec{n}_1,\vec{n}_2)\,\tilde{\eta}(-\vec{n}_2,-\vec{n}_1). \tag{2.27}$$

The continuum expression for the quadratic action is

$$\delta\mathcal{S}=\frac{1}{4}\int\frac{d^dp_1d^dp_2d^dp_3d^dp_4}{(2\pi)^{2d}}\Bigg\{\frac{2\lambda}{(2\pi)^d}\,\delta^{(d)}(\vec{p}_1+\vec{p}_2+\vec{p}_3+\vec{p}_4)+\delta^{(d)}(\vec{p}_3+\vec{p}_2)\,\delta^{(d)}(\vec{p}_4+\vec{p}_1)\,\sigma_0^{-1}(\vec{p}_1)\sigma_0^{-1}(\vec{p}_2)\Bigg\}\,\tilde{\eta}(p_1,p_2)\,\tilde{\eta}(p_3,p_4), \tag{2.28}$$

where

$$\sigma_0(\vec{p})=M^d\xi(\vec{n}),\qquad \vec{p}=\frac{2\pi\vec{n}}{Ma}. \tag{2.29}$$

From Eq. (2.28) one can calculate the two-point function of the fluctuations [23]

$$\langle\tilde{\eta}(p_1,p_2)\,\tilde{\eta}(p_3,p_4)\rangle=\delta^d(p_1+p_4)\,\delta^d(p_2+p_3)\,\sigma_0(p_1)\times\sigma_0(p_2)-G(p_1,p_2,p_3,p_4), \tag{2.30}$$

where

$$G(p_1,p_2,p_3,p_4)=\frac{2\lambda\,\sigma_0(p_1)\sigma_0(p_2)\sigma_0(p_3)\sigma_0(p_4)}{1+2\lambda\int\frac{d^dk}{(2\pi)^d}\,\sigma_0(-k)\,\sigma_0(k-p_1-p_2)}\times\delta^d(p_1+p_2+p_3+p_4). \tag{2.31}$$

D. $O(1)$ partition function

As advertised in the Introduction, there is an interesting cancellation between contributions coming from the $O(1)$ terms in the "classical" action and those coming from integrating out the fluctuations. Consider, for example, the free theory at $\lambda_0=0$. Then from the formulation in terms of the fields $\vec{\phi}$ the partition function may be exactly evaluated

$$\log Z=\frac{N}{2}\Bigg[\sum_{\vec{n}}\ \log(p_n^2+m_0^2)\Bigg]. \tag{2.32}$$

Clearly this exact answer is reproduced by the $O(N)$ classical value of the collective field action in Eq. (2.23). Therefore the $O(1)$ contributions from Eq. (2.25) should cancel whatever one gets by integrating out the fluctuations $\tilde{\eta}$. This is straightforward to check. The $O(1)$ contribution to the effective action coming from the fluctuations is given by (for $\lambda_0 = 0$)

$$S_1^{(2)} = \frac{1}{2}\sum_{\vec{n}_1,\vec{n}_2} \log(p_{\vec{n}_1}^2 p_{\vec{n}_2}^2) = M^d \sum_{\vec{n}} \log p_{\vec{n}}^2. \quad (2.33)$$

Adding the contribution $S_1^{(1)}$ from Eq. (2.25) (with $m_0 = \lambda_0 = 0$) we see that the total $O(1)$ contribution to the partition function is

$$(\log Z)_1 = -(S_1^{(1)} + S_1^{(2)}) = 0 \quad (2.34)$$

as expected. From the collective field theory point of view, at any stage of the $(1/N)$ expansion, there are two contributions, one from the ''classical'' action and one from the fluctuations. For the free theory these should cancel precisely.

For $\lambda_0 \neq 0$ the situation is more complicated. Here there are nonzero subleading terms in the partition function. Now contributions from the classical action *partially* cancel those coming from the fluctuations. Significantly, in the continuum limit the ultraviolet divergent terms cancel at the level of leading $1/N$ correction—this is evident from the fact that at this level the effect of a nonzero λ_0 is to simply change the mass gap, and this does not affect the ultraviolet behavior. These cancellations have important consequences for a holographic interpretation.

E. Nature of interactions

The cubic and higher order interactions in the collective field theory come entirely from the Jacobian factor. On the finite lattice this has the structure

$$(N - 2M^d)\mathrm{Tr}\left[\log \sigma_0 - \sum_{k=2}^{\infty} \frac{(-1)^k}{kN^{k/2}} (\sigma_0^{-1}\eta)^k\right]. \quad (2.35)$$

In each order of $1/\sqrt{N}$ there are generically two terms which come from the two terms in the overall coefficient.

The interactions have an interesting scale invariant form in the critical theory. In this case the classical value of the collective field σ_0 is simply the massless propagator and it is straightforward to see that the term which contains k factors of the fluctuation η has the following form in the continuum and thermodynamic limits:

$$\int \prod_{i=1}^{k} (d^d x_i)[\partial_{x_1}\partial_{x_2}\eta(x_1,x_2)\partial_{x_2}\partial_{x_3} \times \eta(x_2,x_3)\cdots\partial_{x_k}\partial_{x_1}\eta(x_k,x_1)]. \quad (2.36)$$

It is interesting to note that the cubic and higher order interactions do not depend on λ. The coupling constant of the collective field theory is $1/\sqrt{N}$ as expected.

F. Fixed points for $d=3$

For the special case of $d=3$ there is a nontrivial fixed point away from $\lambda = 0$. This may be seen in the collective theory in the following way.

To arrive at the IR fixed point one has to first put the theory on the critical surface by tuning the renormalized mass to zero. In this case the saddle point value of the bilocal field is, in momentum space

$$\sigma_0(p) = \frac{1}{2p^2}. \quad (2.37)$$

The momentum space propagator for the bilocal field fluctuation now reads

$$\langle \tilde{\eta}(p_1,p_2)\tilde{\eta}(p_3,p_4)\rangle = \frac{1}{p_1^2 p_2^2}\Bigg[\delta^d(p_1+p_4)\delta^d(p_2+p_3) - \frac{2\lambda}{p_3^2 p_4^2}\frac{\delta^d(p_1+p_2+p_3+p_4)}{1+2\lambda\int\frac{d^dk}{(2\pi)^d}\frac{1}{(p_1+p_2-k)^2k^2}}\Bigg]. \quad (2.38)$$

It is clear from the form of the classical solution that there is no anomalous dimension for the fundamental field ϕ^i, whereas the expression (2.38) shows that there would be anomalous dimensions for composite operators in general. The first term in Eq. (2.38) is the contribution of free field theory. Thus the second term may be used to define a dimensional *running* coupling constant

$$\alpha(p) = \frac{\lambda}{1+2\lambda\int\frac{d^dk}{(2\pi)^d}\frac{1}{(p-k)^2k^2}}. \quad (2.39)$$

The basic integral is given by

$$I(p) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(p-k)^2k^2} = \frac{1}{8|p|} \quad (2.40)$$

so that

$$\alpha(p) = \frac{\lambda}{1+\frac{\lambda}{4|p|}}. \quad (2.41)$$

Clearly in the infrared

$$\lim_{p\to 0}\alpha(p) = 4|p|, \quad (2.42)$$

the running coupling becomes independent of the original bare coupling of the theory. The *dimensionless* running coupling $\alpha(p)/p$ approaches a constant nemerical value, 4. Alternatively one may define a dimensionless renormalized coupling at some scale μ, $g(\mu)$ by the relation

$$\mu g(\mu)=\frac{\lambda_0\Lambda}{1+\frac{\lambda_0\Lambda}{4\mu}},\tag{2.43}$$

where Λ denotes the momentum space cutoff of the bare theory and λ_0 the bare dimensionless coupling. Then as one approaches the continuum limit $\Lambda\to\infty$ the renormalized coupling g tends to a fixed value 4 independently of λ_0. Thus $g=4$ is an infrared fixed point. The full beta function, at leading order of large-N expansion can be read off from Eq. (2.43)

$$\beta(g)=\mu\frac{\partial g}{\partial\mu}=-g\left(1-\frac{1}{4}g\right).\tag{2.44}$$

The scaling behavior of the correlators (2.38) at this IR fixed point may be read off by considering the continuum limit approached by starting with any arbitrary bare coupling λ_0. Instead of considering the bilocal field by itself it is instructive to consider various moments which define local composite operators as in Eq. (1.9). Consider the simplest such operator which is the scalar composite

$$\sigma(\vec{x},\vec{x})=\vec{\phi}(\vec{x})\cdot\vec{\phi}(\vec{x}).\tag{2.45}$$

The Fourier components of this composite may be expressed in terms of the Fourier components $\tilde{\sigma}(p,q)$ as

$$\tilde{\sigma}(k)=\int[d^3x]e^{i\vec{k}\cdot\vec{x}}\sigma(\vec{x},\vec{x})=\int\frac{d^3q}{(2\pi)^3}\tilde{\sigma}(k-q,q).\tag{2.46}$$

Thus the connected two-point function of the composite operator is given by

$$\langle\tilde{\zeta}(k_1)\tilde{\zeta}(k_2)\rangle=\int\frac{d^3q}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3}\times\langle\tilde{\eta}(k_1-q,q)\tilde{\eta}(k_2-q',q')\rangle,\tag{2.47}$$

where $\tilde{\zeta}(k)$ denotes the fluctuation in $\tilde{\sigma}(k)$. Using Eqs. (2.38) and (2.41) this may be easily evaluated to yield

$$\langle\tilde{\zeta}(k_1)\tilde{\zeta}(k_2)\rangle=\frac{\delta(k_1+k_2)}{8|k_1|}\left[1-\frac{1}{8|k_1|}\frac{2\lambda}{1+\frac{2\lambda}{8|k_1|}}\right]=\frac{\delta(k_1+k_2)}{\lambda+8|k_1|}.\tag{2.48}$$

At the trivial fixed point one gets

$$\langle\tilde{\zeta}(k_1)\tilde{\zeta}(k_2)\rangle=\frac{\delta(k_1+k_2)}{8|k_1|},\tag{2.49}$$

which implies that the dimension of the operator $\sigma(\vec{x},\vec{x})$ is 1. To see the behavior at the nontrivial fixed point one has to rewrite this expression in terms of the bare dimensionless coupling λ_0 and the momentum cutoff Λ and perform the continuum limit with nonzero λ_0. This yields

$$\langle\tilde{\zeta}(k_1)\tilde{\zeta}(k_2)\rangle=\delta(k_1+k_2)\left[\frac{1}{\lambda_0\Lambda}-\frac{8|k_1|}{\lambda_0\Lambda^2}+O\left(\frac{|k_1|^2}{\Lambda^3}\right)\right].\tag{2.50}$$

In the limit $\Lambda\to\infty$ the first term gives rise to a short distance contact term which has to be subtracted in the renormalized theory. The nontrivial part scales as $|k_1|$ which means that the dimension of the operator is 2 at the nontrivial fixed point.

III. HOLOGRAPHIC CORRESPONDENCE

Our proposal is that the collective field theory described above provides a description of the singlet sector of the vector model in terms of a $(d+1)$-dimensional theory of higher spins. The basic idea is to write the collective field as a function of the center of mass coordinates $\vec{u}$ and the relative coordinate $\vec{v}$ as in Eqs. (1.6) and (1.7). One can then expand the field σ as

$$\sigma(\vec{u},\vec{v})=\sum_{l,m}\sigma_{l\vec{m}}(\vec{u},r)Y_{l\vec{m}}(\theta_i),\tag{3.1}$$

where we have written the d relative coordinates $\vec{v}$ in terms of its magnitude r and $(d-1)$ angles $\theta_1\cdots\theta_{d-1}$. $Y_{l\vec{m}}(\theta_i)$ denote the spherical harmonics on S^{d-1}. Since the original field $\sigma(\vec{x},\vec{y})$ is symmetric under interchange of $\vec{x}$ and $\vec{y}$, it should be symmetric under $\vec{v}\to-\vec{v}$. This means that only even (or zero) values of l appear in the expansion (3.1). Thus the collective field is equivalent to a collection of higher spin fields living in $d+1$ dimensions spanned by $(\vec{u},r)$ and there is exactly one field for each even spin. Note that if we had a $U(N)$ rather than a $O(N)$ symmetry one would have odd spins as well.

For $d=3$ we thus have a four dimensional theory. When the vector model is at one of its fixed points, the theory is conformally invariant and has a symmetry group $SO(4,1)$. It is then natural to expect that the four dimensional theory is defined on AdS_4 which has the same isometry. We will see later in what sense this is true.

In the remaining part of this section we will discuss several issues which point towards an interpretation of the collective field theory for the fixed point models as a *holographic* theory defined on the boundary of AdS_4.

A. Finite temperature thermodynamics

One of the crucial aspects of holography is that the high temperature thermodynamics of the bulk theory in $(d+1)$ dimensions is appropriate to a theory in d dimensions. Furthermore, the result involves N which is the coupling constant of the bulk theory. This leading result cannot come from counting of the states of the bulk theory since N appears in the latter only through the coupling constant and in a $1/N$ expansion one would expect an $O(1)$ answer which reflects that the propagating modes live in $(d+1)$ dimensions. In known examples of holography, however, an

N-dependent answer characteristic of a d dimensional theory comes from the fact that the bulk theory is typically a theory of gravity and its high temperature properties are dominated by black holes whose entropies are proportional to their areas and whose thermodynamics is appropriate to that of theory in d dimensions. Furthermore, the black hole entropy is a ''classical'' effect and goes as the inverse of the square of the coupling constant and therefore contains the right power of N.

In the previous section we have calculated the leading order and the $(1/N)$ corrections to the partition function of the collective field theory defined on a periodic lattice with M sites in each direction. We now use these results to discuss the finite temperature behavior. To do this all we have to do is take M large but consider different lattice spacings in the ''space'' and the ''Euclidean time'' directions. Finally we have to consider a continuum limit and a thermodynamic limit in which the physical extent of the Euclidean time direction is a finite quantity $\beta = 1/T$ while those in the space directions are L with $L \gg \beta$.

First consider the Gaussian fixed point at $\lambda = 0$ in any number of dimensions. The finite temperature free energy may be read off from Eq. (2.23) in a standard fashion. The expression (2.23) has a leading divergent term which is extensive, proportional to $L^{d-1}\beta$—the coefficient being the ground state energy density. The next subleading term, which we denote by $\mathcal{S}'$, is proportional to L^{d-1} and has the form

$$\mathcal{S}' = -NL^{d-1}\int \frac{d^{d-1}p}{(2\pi)^{d-1}} \log(1-e^{-\beta|p|}). \tag{3.2}$$

This is then related to the thermodynamic free energy F by

$$F = \frac{1}{\beta}\mathcal{S}'. \tag{3.3}$$

It is clear that

$$F \sim NL^{d-1}T^{d}, \tag{3.4}$$

which is nothing but the free energy of N species of massless particles in $d-1$ space dimensions. In particular the entropy scales as

$$S \sim NL^{d-1}T^{d-1}. \tag{3.5}$$

From the point of view of the vector model this result is of course obvious. However, from the point of view of the collective field theory this is a rather nontrivial result. As we saw, we can interpret this theory as a theory of higher spin fields in $d+1$ dimensions. Naively one would expect that the thermodynamics would be the one appropriate to d, rather than $(d-1)$, space dimensions and should be of $O(1)$ as explained above.

The point is that all these expectations are based on the usual situation where the leading thermodynamic free energy comes from the ''one loop'' contribution. In the present context this is the leading $1/N$ correction, which actually gives an $O(1)$ contribution to the free energy. What we saw above, however, is that there is a *classical* contribution to the free energy which is proportional to N *which leads not only to a nontrivial internal energy, but a nontrivial entropy proportional to N*. In fact there are no $1/N$ corrections to this result for the $\lambda = 0$ theory due to the complete cancellation discussed above. For $\lambda \neq 0$ the cancellation is not complete and there are finite $O(1)$ corrections. However, the divergent terms which contribute to the vacuum energy cancels. The finite temperature behavior of the interacting $O(N)$ model was discussed a long time ago in [24] and more recently in [25].

The fact that the leading thermodynamics comes from the classical contribution to the action is reminiscent of the Gibbons-Hawking calculation of the entropy of a black hole. As will be seen below, the space-time interpretation of the four dimensional collective field theory does not appear to be straightforward and it is difficult to identify what kind of space-time configurations give rise to this classical contribution. Nevertheless our result strongly suggests that the four dimensional bulk theory of higher spin fields have black holes.

B. Conformal transformations and AdS

While the fields $\sigma_{lm}(\vec{u},r)$ do represent higher spin fields in $d+1$ dimensions, they are not the standard higher spin fields as discussed in [14], but related by some field redefinition. This may be seen from the fact that the quadratic action is not diagonal in the spins. A spin l field mixes with spins $l \pm 2$. We do not know what is the exact field redefinition which relates these components to the standard fields of higher spin theories.

An important indication of this fact comes from an examination of the transformation properties of the collective field under conformal transformations on the boundary. Consider for example the theory at the Gaussian fixed point. From the known conformal transformations it follows that the transformations of the bilocal field $\sigma(\vec{x},\vec{y})$ are given by

$$\begin{aligned}
\delta_D\sigma &= -\alpha\left(x^i\frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial y^i} + \Delta\right)\sigma(\vec{x},\vec{y}),\\
\delta_T\sigma &= t^i\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)\sigma,\\
\delta_R\sigma &= \theta^{ij}\left(x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial y^j} - y^j\frac{\partial}{\partial y^i}\right)\sigma,\\
\delta_S\sigma &= \Bigg\{[2(\epsilon\cdot x)x^i - |x|^2\epsilon^i]\frac{\partial}{\partial x^i} + [2(\epsilon\cdot y)y^i\\
&\quad - |y|^2\epsilon^i]\frac{\partial}{\partial y^i} + \Delta\epsilon\cdot(x+y)\Bigg\}\sigma.
\end{aligned} \tag{3.6}$$

Here Δ is the scaling dimension of σ and α, t^i, θ^{ij} and ϵ^i are the parameters of dilatations, translations, rotations, and special conformal transformations, respectively. Rewriting these expressions in terms of $\vec{u}$ and $\vec{v}$ we get

$$\delta_D\sigma = -\left(u^i\frac{\partial}{\partial u^i} + r\frac{\partial}{\partial r} + \Delta\right)\sigma,$$

$$\delta_T\sigma = t^i\frac{\partial}{\partial u^i}\sigma, \tag{3.7}$$

$$\delta_R\sigma = \theta^{ij}\left(u^i\frac{\partial}{\partial u^j} - u^j\frac{\partial}{\partial u^i} + L_{ij}\right)\sigma,$$

$$\delta_S\sigma = \left\{[2(\epsilon\cdot u)u^i - |u|^2\epsilon^i - r^2\epsilon^i]\frac{\partial}{\partial u^i} + 2(\epsilon\cdot u)r\frac{\partial}{\partial r} + 2\Delta(\epsilon\cdot u) + (\epsilon^j u^i - u^j\epsilon^i)L_{ji}\right\}\sigma + 2(\epsilon\cdot v)v^i\frac{\partial}{\partial u^i}\sigma, \tag{3.8}$$

where

$$L_{ij} = v^i\frac{\partial}{\partial v^j} - v^j\frac{\partial}{\partial v^i}. \tag{3.9}$$

To see the action on the individual components $\sigma_{lm}(\vec{u},r)$ one needs to substitute these expressions in the expansion (3.1). It is clear from the above expressions that the dilatations, translations, and rotations act on the components $\sigma_{lm}(\vec{u},r)$ diagonally, i.e., the action does not mix up various spins. The factor of L_{ij} in the transformation $\delta_R\sigma$ mixes fields of different m for the *same* l exactly as rotation generators should. On the other hand, the last term in the special conformal transformation on σ shows that this mixes up fields with *different* spin.

In fact, if we define new component fields

$$\chi_{lm}(\vec{u},r) = r^{l+\Delta}\sigma_{lm}(u,r) \tag{3.10}$$

the generators for dilatations, translations, and rotations on χ_{lm} are *exactly* the generators of the corresponding isometries on tensor fields of rank l defined on an AdS space, with the metric given in Eq. (1.8),

$$ds^2 = \frac{1}{r^2}[dr^2 + d\vec{u}\cdot d\vec{u}]. \tag{3.11}$$

In particular this means that the magnitude of the relative coordinate behaves as a scale, as it should.

For special conformal transformations, the story is different. Here all the terms in Eq. (3.8), *except the last term*, are the correct expressions for generators of the corresponding isometries of the metric (3.11). The last term, however, clearly mixes different spins.

The correct higher spin fields [14–17], however, transform homogeneously under the Killing isometries of AdS and do not mix up fields with different spin. This shows that the fields $\sigma_{lm}(\vec{u},r)$, while containing the complete physics of higher spin theories, are not themselves the correct higher spin fields.

In fact there are indications that the correct higher spin fields are related to the components $\sigma_{lm}(\vec{u},r)$ by nonlocal transformations. This may be seen from various points of view. The exercise we have done above is in fact an attempt to rewrite conformal transformations on a pair of vectors $(\vec{x},\vec{y})$ as isometries in an AdS space by identifying the correct coordinates in the latter. It is straightforward to see that this works for dilatations, translations, and rotations with the identification of $\vec{u}$ and r as the coordinates in AdS as in the metric (3.11). However, this cannot work for the special conformal transformation. To see this consider the case of $d=1$. In this case the collective field should contain only one field in AdS since there is no spin in AdS_2. The special conformal transformations are then

$$\delta x = \epsilon x^2, \quad \delta y = \epsilon y^2, \tag{3.12}$$

which leads to

$$\delta u = \epsilon(u^2 + v^2), \quad \delta v = 2\epsilon uv. \tag{3.13}$$

This is to be compared with the corresponding Killing isometry of the metric (3.11) for $d=1$, viz.,

$$\delta' u = \epsilon(u^2 - v^2), \quad \delta v = 2\epsilon uv. \tag{3.14}$$

Another indication comes from the relationship of the components σ_{lm} with the infinite set of conserved currents in the vector model (at $\lambda=0$) [26,17]. These currents are symmetric and traceless and given by (in $d=3$)

$$J_{i_1\cdots i_s} = \sum_{k=0}^{s}\frac{(-1)^k(\partial_{i_1}\cdots\partial_{i_k}\phi)(\partial_{i_{k+1}}\cdots\partial_{i_s}\phi)}{\Gamma(k+1)\Gamma\left(k+\frac{1}{2}\right)\Gamma(s-k+1)\Gamma\left(s-k+\frac{1}{2}\right)} - \text{traces}. \tag{3.15}$$

These currents are conserved

$$\partial^{i_1}J_{i_1\cdots i_s} = 0. \tag{3.16}$$

These currents can be expressed in terms of the collective field. Consider for example the first few currents. These may be expressed as follows:

$$J_0 = \sigma(u,v)_{v=0},$$

$$J_{ij} = \left[\frac{\partial^2}{\partial v^i\partial v^j} - \frac{1}{3}\delta^{ij}\frac{\partial^2}{\partial v^k\partial v_k}\right]\sigma(\vec{u},\vec{v})|_{v=0} - \frac{1}{2}\left[\frac{\partial^2}{\partial u^i\partial u^j} - \frac{1}{3}\delta^{ij}\frac{\partial^2}{\partial u^k\partial u_k}\right]\sigma(u,0). \tag{3.17}$$

These currents transform homogeneously under conformal transformations.

The collective field expansion in Eq. (3.1) can be also reorganized in terms of derivatives of the form which appear in Eqs. (3.17), since spherical harmonics are in one-to-one correspondence with traceless symmetric tensors made out of products of v^i. Thus we have expansions of the form

$$\sigma(\vec{u},\vec{v}) = \left[\sigma(\vec{u},0) + \frac{1}{6}r^2\left(\frac{\partial^2\sigma}{\partial v^2}\right)_{v=0} + O(r^4)\right] + \frac{1}{2}r^2\left(\hat{v}^i\hat{v}^j - \frac{1}{3}\delta^{ij}\right)\left[\left(\frac{\partial^2\sigma}{\partial v^i \partial v^j}\right)_{v=0} - \frac{1}{3}\delta^{ij}\left(\frac{\partial^2\sigma}{\partial v^2}\right)_{v=0} + O(r^2)\right], \quad (3.18)$$

where we have performed a Taylor expansion in v^i and reorganized it in terms of traceless symmetric products of the unit vectors $\hat{v}^i$. Thus the components $\sigma_{l,m}$ are given by

$$\sigma_{00} \sim \left[\sigma(\vec{u},0) + \frac{1}{6}r^2\left(\frac{\partial^2\sigma}{\partial v^2}\right)_{v=0} + O(r^4)\right],$$

$$\sigma_{1m} \sim \left[\left(\frac{\partial^2\sigma}{\partial v^i \partial v^j}\right)_{v=0} - \frac{1}{3}\delta^{ij}\left(\frac{\partial^2\sigma}{\partial v^2}\right)_{v=0} + O(r^2)\right]. \quad (3.19)$$

Comparing Eqs. (3.19) and (3.17) it is clear that the fields $\sigma_{lm}(u,0)$ *do not* reduce to the currents $J_{i_1 \cdots i_l}$. This is the basic reason why these components do not transform properly under special conformal transformations.

It must be emphasized that the collective field theory contains all the information contained in the vector model singlet correlators and hence serves as a *complete definition* of the higher spin theory, including all interactions. However, the relationship between the components of the collective field and the higher spin fields which propagate independently at $N=\infty$ appears to be rather nontrivial. The key to uncovering the precise relationship is conformal invariance. We hope to report results about this connection soon [27].

C. Interactions and the bulk theory

As shown in the previous section, for the critical theory the interactions of the collective field theory are characterized by a coupling constant which is $1/\sqrt{N}$ and independent of the bare coupling λ_0 of the underlying vector model. We now make several comments about how this may come about in a bulk theory defined on AdS space.

Since the bulk theory contains gravity, it is characterized by a Newton's gravitational constant G which has dimensions of $(\text{length})^2$ in four dimensions. In flat space, the interaction terms in the theory have coefficients which depend on the coupling constant $\sqrt{G}$ and the terms have a number of derivatives which make the action dimensionless. Typically, again for four dimensions, each $\sqrt{G}$ is accompanied by a single derivative. In AdS space, however, there is another length scale R, where $1/R^2$ is the constant curvature. Consequently, instead of derivatives there could be inverse powers of R which account for the correct dimensions. This is familiar in supergravity in, e.g., $\text{AdS}_5 \times S^5$. Here, there is a class of couplings which do not depend on G and R individually, but only on the dimensionless combination G/R^3. By virtue of the AdS/CFT correspondence one has $G/R^3 \sim 1/N^2$, where the four dimensional dual super-Yang-Mills theory has a gauge group $SU(N)$. Since the gauge theory coupling g_{YM} is related to the bulk parameters by the relation $g_{YM} = G^{1/4}R^{5/4}l_s^{-2}$, such bulk couplings are computed in terms of gauge theory three-point functions which are completely independent of g_{YM} and only depends on N. This is possible since one is computing three-point functions of composite operators which have nonzero values in free field theory. Indeed in this particular case, the underlying supersymmetry ensures that the three-point functions of a class of operators are given exactly by their free field values [28].

For the vector model the conformal field theory is at a fixed point rather than on a line of fixed points, so that there is no analog of a gauge theory coupling constant. This is the reason why the couplings in the collective field theory are characterized only by N and by no other parameter. The value of the bare coupling drops out since in the continuum limits one approaches the infrared fixed point.

This fact therefore implies that if the dual theory is a higher spin theory in AdS, then *all* the couplings of that theory are characterized by the dimensionless combination G/R^2. This is a rather nontrivial prediction for the higher spin theory.

D. Propagating modes and Hamiltonian collective theory

We have so far considered the Euclidean version of the collective field theory as derived in [20]. For the three dimensional vector model this is a collection of higher spin fields in four dimensions—one field for each even spin. A component field $\sigma_{lm}(\vec{u},r)$ has $2l+1$ components. However, if all the four dimensional fields of the dual theory are massless—as conjectured—there are precisely two propagating polarizations for each spin. σ_{lm} clearly contains too many independent propagating modes.

The key reason behind this overcounting is the fact that the Euclidean collective field is a way to organize an infinite set of higher spin currents in the boundary theory, as indicated above. These are symmetric and traceless in the three dimensional indices, with the spin-l current containing l indices, leading to $(2l+1)$ components. However, these currents are *conserved* in the $\lambda=0$ theory, and conserved to leading order in $1/N$ in the interacting theory, so that there are $2(l-1)+1$ conditions relating the components. Thus the number of independent components is $(2l+1)-[2(l-1)+1]=2$, which is the correct value for the number of propagating modes for each spin. This counting can be easily seen to work in any number of dimensions.

The meaning of all this is that the Euclidean collective field theory must have a gauge invariance which follows from the current conservation conditions in the vector model. We do not know how to display this symmetry, but we know it is there because of the one-to-one correspondence between the spin components of the collective field and the currents.

This situation is not new and has been encountered before in other examples of the AdS/CFT correspondence. Consider the cases where the bulk theory on AdS_{d+1} contains a massless graviton. This has $(d+1)(d-2)/2$ propagating components. The operator which is dual to the graviton is the energy momentum tensor of the boundary theory which has

$(d^2+d-2)/2$ components because of the tracelessness. However, the energy-momentum tensor is conserved so that the number of independent polarizations is $(d^2+d-2)/2 - d=(d+1)(d-2)/2$. In the coordinates of Eq. (3.11), the CFT is defined on the boundary at $r=0$. The energy momentum tensor then computes correlators of the graviton field $h_{\mu\nu}$ in a gauge where $h_{r\mu}=0$, but this gauge still retains some gauge symmetries.

Our situation is rather similar. In fact, given the conserved currents in the vector model one may construct bulk fields using a bulk-to-boundary propagator. As shown in [17], the conservation of currents then lead to gauge conditions on the bulk fields.

It is not surprising to find that the Euclidean collective field contains redundant degrees of freedom. The collective field theory we have considered reproduces all singlet correlators of the theory. However, among these correlators are those which receive contributions from *nonsinglet* intermediate states. The simplest example is $\langle\sigma(x,y)\rangle$ itself, which is the propagator of the elementary field $\vec{\phi}$. On the other physical propagating states in the bulk must be singlet states.

To look at the propagating modes it is instructive to consider the Hamiltonian version of collective field theory [3,29]. In this formulation the collective fields are Schrödinger picture operators $\psi(\tilde{x},\tilde{y})$ defined by

$$\psi(\tilde{x},\tilde{y})=\frac{1}{N}\vec{\phi}(\tilde{x})\cdot\vec{\phi}(\tilde{y}) \tag{3.20}$$

and their canonically conjugate momenta $\Pi(\tilde{x},\tilde{y})$. Here $\tilde{x},\tilde{y}$ denote the *spatial* [i.e., $(d-1)$ dimensional] components of the space-time locations. These operators create all the singlet states of the theory, whose dynamics is governed by the collective field Hamiltonian,

$$H=2\,\mathrm{Tr}(\Pi\psi\Pi)+V_{coll}, \tag{3.21}$$

with

$$V_{coll}=\frac{1}{2}\int dx\left[-\nabla_x^2\psi(\tilde{x},\tilde{y})|_{\tilde{x}=\tilde{y}}+m^2\psi(\tilde{x},\tilde{x})+\frac{\lambda}{2}(\psi(\tilde{x},\tilde{x}))^2\right] + \frac{1}{8}\mathrm{Tr}\psi^{-1} \tag{3.22}$$

and

$$\Pi(\tilde{x},\tilde{y})=\frac{\delta}{\delta\psi(\tilde{x},\tilde{y})} \tag{3.23}$$

being the canonically conjugate variable. In Eqs. (3.21), (3.23) ψ should be regarded as a matrix in $\tilde{x},\tilde{y}$ and the trace refers to the trace of this matrix.

One may expand the corresponding Heisenberg picture operator in a manner similar to the spherical harmonic expansion of the Euclidean collective field

$$\psi(\tilde{x},\tilde{y};t)=\sum_{l,m}\psi_{lm}(\tilde{u},r,t)Y_{lm}, \tag{3.24}$$

where as usual

$$\tilde{u}=\frac{1}{2}(\tilde{x}+\tilde{y}),\quad \tilde{v}=\frac{1}{2}(\tilde{x}-\tilde{y}),\quad r^2=\tilde{v}\cdot\tilde{v}. \tag{3.25}$$

In Eq. (3.24) Y_{lm} are spherical harmonics on a S^{d-2} rather than on S^{d-1}. Consequently the number of components of ψ_{lm} for a given l is exactly the same as the number of propagating polarizations of a massless spin-l field in $d+1$ dimensions. For example, for $d=3$ for a given l we have precisely two values of m, i.e., $m=\pm l$ which count the two polarizations of massless four dimensional fields with any spin. This Hamiltonian collective field theory therefore correctly counts the propagating modes of higher spin fields.

The manner in which the action formulation reduces to the canonical. Hamiltonian representation is interesting and rather nontrivial. The Langragian formulation was characterized by being bilocal in time as well as in space while the canonical, Hamiltonian formulation is local in time. The reduction from one to another

$$\sigma(\tilde{x},t;\tilde{x}',t')\to\psi(\tilde{x},\tilde{x}';t) \tag{3.26}$$

involves a formal reduction in the number of degrees of freedom as we have seen. One has indications that this reduction can be understood in terms of a gauge principle in analogy with a connection between a covariant and canonical gauge description of gravity. We should also emphasize implications on thermodynamics contained in the two descriptions. By its nature the canonical description naturally leads to a thermodynamics with entropy of order one rather than of order N and would seemingly miss one of the main ingredients of holography. [There is a vacuum energy of $O(N)$ but not an entropy.] The reason behind this may be gleaned from an understanding of the free theory. Here the exact entropy is of order N and clearly counts the number of states created by the elementary fields $\vec{\phi}$. In other words this leading classical contribution to the thermodynamics comes from the *nonsinglet* states of the theory. Analogous states have an interpretation of winding modes in matrix theories and they are not contained in the Hamiltonian collective theory. On the other hand, as we have seen the Euclidean collective field theory does capture the contribution from these nonsinglet states. The importance of nonsinglet states for thermodynamics also makes its appearance in the $d=1$ matrix model [11].

We therefore see that the nonsinglet states of the vector model correspond to nonpropagating modes in the bulk. The thermodynamics in the bulk description comes from a classical contribution in the Euclidean collective field theory and hence from these nonpropagating modes. This is consistent with the conjecture that in this model, like in other string theory examples, the thermodynamics is dominated by black holes, which are, of course, examples of condensates of nonpropagating modes. The relationship between nonsinglet states, nonpropagating modes, black holes, etc., has been a matter of considerable discussion in matrix models. For vector models, the availability of a tractable Euclidean collective theory provides an opportunity to understand this important issue.

IV. CONCLUSIONS

We have argued that the *Euclidean* bilocal collective field is capable of describing a higher spin theory of a single field for each even spin in one higher dimension. However, while the spherical harmonic decomposition of the bilocal field gives the correct count of the higher spins, these are *not* the standard higher spin fields. We suspect that there is possibly a nonlocal field redefinition between σ_{lm} and the standard fields.

We have also argued that the Euclidean collective field contains more degrees of freedom than the *propagating* modes. On the other hand, the Hamiltonian collective theory precisely counts the propagating modes. The distinction is important, since as we have found, the leading thermodynamics in fact receives contributions from nonsinglet states. In the bulk description this means that nonpropagating backgrounds dominate the thermodynamics (which is why the result is *classical*). One of course knows examples of this in string theory realizations of the AdS/CFT duality: here black holes provide the right thermodynamics.

One of the most interesting aspects of the duality between vector models and higher spin theories is that one has a fully consistent quantum theory in four dimensions which contain gravity—and this is not a string theory. In string theory, however, higher spin fields are massive with the masses determined by the string length. Thus at distances much larger than the string length, string theory is essentially (super) gravity. In this case, there is no analog of a string length. This means that there is no separation of the spin-2 field with all the other higher spin fields. In other words, considered as a theory of gravity there is no reason to expect that this theory has nice locality properties.

Nevertheless, it is conceivable that due to unknown reasons there is a sense in which this theory may be considered as a theory of gravity.[2] This may happen as in the $AdS_5 \times S^5$–Yang-Mills duality. Here at the AdS scale, the bulk theory cannot be *a priori* described in terms of a five dimensional theory of gravity, since the Kaluza-Klein modes from the S^5 have the same scale. However, it turns out that one does much better than this naive expectation. For many purposes, the theory can indeed be regarded as five dimensional gravity even at the AdS scale. This is evidenced by the fact that the thermodynamics of the Yang-Mills theory is correctly reproduced by *five dimensional* AdS-Schwarzschild black holes. We have no idea whether a similar decoupling holds in this case.

ACKNOWLEDGMENTS

S.R.D. would like to thank S. Kachru, I. Klebanov, S. Mathur, S. Minwalla, G. Murthy, A. Shapere, E. Silverstein, S. Shenker, and L. Susskind for discussions. Some of the results were presented at ''Workshop on String Theory'' held at Harishchandra Research Institute, Allahabad. S.R.D. would like to thank the organizers and participants of the workshop for providing a stimulating atmosphere. The work of S.R.D. was supported in part by the U.S. Department of Energy contract DE-FG01-00ER45832. A.J. was partially supported by the U.S. Department of Energy under contract DE-FG02-91ER4068.

[2] This possibility was suggested by Shiraz Minwalla.

[1] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).

[2] S.R. Das and A. Jevicki, Mod. Phys. Lett. A **5**, 1060 (1990).

[3] A. Jevicki and B. Sakita, Nucl. Phys. **B165**, 511 (1980).

[4] See, e.g., J. Polchinski, hep-th/9411028; A. Jevicki, M. Li, and T. Yoneya, Nucl. Phys. **B448**, 277 (1995).

[5] G. 't Hooft, in ''Salamfest,'' gr-qc/9310026; L. Susskind, J. Math. Phys. **36**, 6377 (1995).

[6] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).

[7] S. Gubser, I. Klebanov, and A. Polyakov, Phys. Lett. B **428**, 105 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).

[8] L. Susskind and E. Witten, hep-th/9805114.

[9] G. Mandal, A. Sengupta, and S.R. Wadia, Mod. Phys. Lett. A **6**, 1685 (1991); E. Witten, Phys. Rev. D **44**, 314 (1991).

[10] S.R. Das, Mod. Phys. Lett. A **8**, 69 (1993); **8**, 1331 (1993); A. Dhar, G. Mandal, and S.R. Wadia, *ibid.* **7**, 3703 (1992); **8**, 1701 (1993); A. Jevicki and T. Yoneya, Nucl. Phys. **B411**, 64 (1994).

[11] V. Kazakov, D. Kutasov, and I. Kostov, Nucl. Phys. **B622**, 141 (2002); S. Alexandrov, V. Kazakov, and I. Kostov, hep-th/0302106.

[12] See, e.g., S. Coleman, in Erice lectures, 1979 and references to the original literature therein.

[13] I. Klebanov and A.M. Polyakov, Phys. Lett. B **550**, 213 (2002).

[14] C. Fronsdal, Phys. Rev. D **18**, 3624 (1978); E. Fradkin and M. Vasiliev, Phys. Lett. B **189**, 89 (1987); Nucl. Phys. **B291**, 141 (1987); M. Vasiliev, hep-th/9910096.

[15] E. Sezgin and P. Sundell, J. High Energy Phys. **07**, 055 (2002); J. Engquist, E. Sezgin, and P. Sundell, Class. Quantum Grav. **19**, 6175 (2002); M. Vasiliev, Int. J. Mod. Phys. D **5**, 763 (1996).

[16] P. Haggi-Mani and B. Sundborg, J. High Energy Phys. **04**, 031 (2000); B. Sundborg, Nucl. Phys. B (Proc. Suppl.) **102**, 113 (2001); E. Witten, http://theory.caltech.edu/jhs60/witten/1.html; E. Sezgin and P. Sundell, J. High Energy Phys. **09**, 036 (2001).

[17] A. Mikhailov, hep-th/0201019.

[18] I. Klebanov and E. Witten, Nucl. Phys. **B556**, 89 (1999).

[19] J. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974).

[20] A. Jevicki and B. Sakita, Nucl. Phys. **B185**, 89 (1981).

[21] J. Maldacena and A. Strominger, J. High Energy Phys. **12**, 005 (1998); A. Jevicki and S. Ramgoolam, *ibid.* **04**, 032 (1999).

[22] S.R. Wadia, Phys. Rev. D **24**, 970 (1981).

[23] R. de Mello Koch and J.P. Rodrigues, Phys. Rev. D **54**, 7794 (1996).

[24] L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).
[25] S. Sachdev, Phys. Lett. B **309**, 285 (1993).
[26] D. Anselmi, Class. Quantum Grav. **17**, 1383 (2000).
[27] S.R. Das and A. Jevicki (work in progress).
[28] D. Freedman, S.D. Mathur, A. Matusis, and L. Rastelli, Nucl. Phys. **B546**, 96 (1999); H. Liu and A. Tseytlin, *ibid.* **B533**, 88 (1998); G. Chalmers, H. Nastase, K. Schalm, and R. Siebelink, *ibid.* **B540**, 247 (1999); S. Lee, S. Minwalla, M. Rangamani, and N. Seiberg, Adv. Theor. Math. Phys. **2**, 697 (1998).
[29] A. Jevicki and J. Rodrigues, Nucl. Phys. **B230**, 317 (1984).