Level set method for the evolution of defect and brane networks

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A theory for studying the dynamic scaling properties of branes and relativistic topological defect networks is presented. The theory, based on a relativistic version of the level set method, well known in other contexts, possesses self-similar ''scaling'' solutions, for which one can calculate many quantities of interest. Here, the length and area densities of cosmic strings and domain walls are calculated in Minkowski space, and radiation, matter, and curvature-dominated Friedmann-Robertson-Walker cosmologies with two and three space dimensions. The scaling exponents agree with the naive ones based on dimensional analysis, except for cosmic strings in three-dimensional Minkowski space, which are predicted to have a logarithmic correction to the naive scaling form. The scaling *amplitudes* of the length and area densities are a factor of approximately 2 lower than the results from numerical simulations of classical field theories. An expression for the length density of strings in the condensed matter literature is corrected.

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I. INTRODUCTION

The solutions to some of the most interesting problems in physics depend on a better understanding of the dynamics of fields far from thermal equilibrium, particularly in particle cosmology, where we seek mechanisms for generating baryon asymmetry [1], density fluctuations [2], and perhaps primordial magnetic fields [3]. Significant advances have been made recently in studying nonequilibrium dynamics of phase transitions, both theoretically (see, e.g., [4] for a review) and numerically, where we can now perform real-time simulations of a quench with leading thermal corrections included [5,6]. One aspect is still not yet well understood: the approach to equilibrium after phase transitions of field theories with topological defects.

At the same time, the past few years have seen an explosion in theories involving various kinds of extended objects or branes, both solitonic (such as topological defects in field theory) and fundamental. Most of the interest has been in special configurations of branes of various dimensions, and the spectrum of states in those backgrounds. However, an interesting new scenario has emerged in which the Universe began with the branes in thermal equilibrium, the brane gas universe [7].

Both branes and topological defects in relativistic field theories obey the same equation of motion (at least for configurations with curvature small compared to the inverse width or fundamental scale), and so it clear that both may be discussed at the same time. Hence the theory presented in this paper can be applied to both brane gases and networks of topological defects. The general technique is independent of the space-time dimension and the codimension of the brane, but quantitative predictions must be taken case by case. The cases worked out in detail here concern defects of codimension 1 and 2 in Friedmann-Robertson-Walker (FRW) space-times of dimension $d=3$ and 4.

It is believed that when extended topological defects are formed, self-similar or scaling behavior emerges at large times, in which a characteristic length scale of the field configuration, ξ, increases with time as a power law:

$$\xi(t)\propto t^{z}.$$

Dynamic scaling can be seen in the order parameter of many condensed matter systems undergoing rapid quenches, and there are now quite sophisticated techniques for calculating correlation functions of the order parameter [8]. They fall into two classes. First, there are those based on a large-N expansion, where N is the number of components of the order parameter, which are applicable to Ginzburg-Landau theories. The second is applicable to systems with extended topological defects, in which the order parameter ϕ obeys an equation $\dot{\phi}\propto \delta F[\phi]/\delta\phi$, where F is the Ginzburg-Landau free energy. Allen and Cahn [9] proposed that the velocity of defects marking a phase boundary was proportional their local mean curvature. This proposal, now termed motion by mean curvature, was later rigorously proved [10].

Relativistic scalar field theories with spontaneously broken global symmetries (Goldstone models) also exhibit dynamic scaling. Significant progress has been made on the theory of $O(N)$ scalar field theories at large N, both classical [11–13] and quantum [14] (at large N the leading order in the quantum theory is the same as the classical theory). These works have established a theoretical basis for the scaling observed in numerical simulations [15–17]. The theory has also been used to calculate microwave background and density fluctuations. To date, however, analytic approaches to the dynamics of topological defects are few.

There are several numerical simulations which broadly support the dynamic scaling hypothesis for topological defects, including domain walls [18–20], gauge strings [21,22], and global strings [23,24]. All the simulations are consistent with the linear scaling law over the range of the simulations, although Press, Ryden, and Spergel suggested that the results for domain walls would be better fitted by $\xi\sim t/\ln(t)$; however, more recent simulations with a larger dynamic range [25] are not consistent with the logarithm.

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There are also string simulations based on direct integration of the equations of motion of one-dimensional objects, obeying the Nambu-Goto equations, which may be derived as the first approximation in an expansion in powers of string worldsheet curvature [26–28]. They do not include any way for energy to be lost from the network, but if one considers ''infinite'' strings only (strings longer than the causal horizon size), an approximately linear scaling law is found [29–31]. However, the simulations are plagued by kinkiness persisting at the resolution of the simulation, associated with the production of small loops of string, which does not appear to scale. It has been suggested that this is because the natural length scale for loop production is in fact the string width, where loops would become indistinguishable from large-amplitude oscillations in the field [32]. Indeed, numerical simulations of the fields [21,22] appear to support this hypothesis, although the latter authors suggest that the ''protoloops'' in their simulation are in fact a transient effect.

A program to understand analytically the results of the Nambu-Goto simulations has developed over the years [29,33–35]. In its simplest form, the model parametrizes the string with one length scale ξ, which is defined from the invariant length density of infinite string $\mathcal{L}_\infty$ through $\mathcal{L}_\infty = 1/\xi^2$. This length density can change in two ways: through stretching as the strings participate in the Hubble expansion, and through loop production. Loop production is parametrized by the so-called chopping efficiency c, the fraction of string lost to the network in the time scale ξ. The Hubble stretching depends on the mean-square string velocity v^2. The phenomenological equation is then

$$\dot{\xi} = H(1+v^2)\xi + c/2. \tag{1}$$

Further work [34] introduced two other length scales to describe the correlation length and the interkink distance. However, there are many unknown parameters in the model, which greatly restricts its predictive power, despite attempts to measure them [32]. A different approach was adopted by Martins and Shellard [35] who promoted the rms string velocity v to a time-dependent parameter to model the reduced rate of loop production of slower strings. The velocity-dependent one-scale model equations are (neglecting frictional terms)

$$\dot{\xi} = H(1+v^2)\xi + \tilde{c}v/2, \quad \dot{v} = -2Hv + k(1-v^2)\xi^{-1}, \tag{2}$$

where $\tilde{c}$ and k are, in the simplest version, constants. It is this velocity-dependent one-scale model which the authors of [22] use to make their claim that the production of loops on the scale of the string width seen in field theory simulations is a transient.

In this paper a potentially far more powerful analytic technique for describing the motion of strings is developed. The technique was outlined in [36,37] and applied to relativistic domain walls in two and three space dimensions. It is here further extended into a partial treatment of p-branes in D space dimensions, and fully applied to relativistic strings in three space dimensions. It is based on the u-theory of Ohta, Jasnow, and Kawasaki (OJK) [38], and its descendents [39], which describes the motion of defects obeying the Allen-Cahn equation. The relativistic generalization of the Allen-Cahn equation is the Nambu-Goto equation, in which, loosely speaking, the *acceleration* of the defect is proportional to its local curvature, with proportionality constant c^2, where c is the speed of light. More precisely, the Nambu-Goto equation is equivalent to the requirement that the world volume of the p-brane embedded in the d-dimensional spacetime has zero extrinsic curvature. How closely defects derived from a field theory obey this equation is a matter for debate [26–28,40]. The theoretical approach develops systematic expansions of the geometrical equations obeyed by the defect world volumes in powers of the width divided by the local curvature, which reduce to the Nambu-Goto equation in the limit of small curvature. The approach of Arodź [28] makes it particularly clear that the Nambu-Goto equation is really a consistency condition for a smooth defectlike solution to exist.

It is therefore plausible that we can forget about the details of the field theory and concentrate instead on the properties of extremal (zero extrinsic curvature) surfaces embedded in higher dimensions. If one finds such surfaces, then provided their curvature is small enough one can be confident that there is a solution of the field equations representing a smooth defect centered on that surface. A formalism for studying extremal, and more general, surfaces has been developed over the years by Carter [41], which makes clear the geometrical nature of the Nambu-Goto equations through close attention to the tensorial properties.

The present approach introduces scalar fields u^A with the intention that the loci of constant u^A should be extremal surfaces: these are the level sets of the title. The fields can also be interpreted as coordinates normal to the brane surface: in this sense the approach can be thought of as orthogonal to Carter's. We derive the equations that the u^A must satisfy, which are nonlinear, and so therefore do not seem to represent an improvement on the original field theory or the Nambu-Goto equations. However, one can derive equations for surfaces which are *on average* extremal, when we average the fields with a Gaussian probability distribution. With this Gaussian ansatz, one can also calculate analytically important quantities, such as the brane or defect density.

The results for $(D-1)$-branes (domain walls) are extremely encouraging when compared to the numerical simulations [18,42,43]. The theory predicts a scaling law for the area density in three dimensions, but not only does it predict the scaling exponent, it also predicts the scaling *amplitude* to within a factor of about 2, which is not bad given the approximations made. The prediction for $(D-2)$-branes in three dimensions (strings) is also challenging: the theory gives a logarithmic scaling violation in Minkowski space, with the length density doending on conformal time η as $\log(\eta)/\eta^2$. Looking for such scaling violations will be a good way to test the theory, although it is computationally very challenging.

The theory also describes the behavior of defects formed from initial conditions with a slight bias in the expectation value of the field favoring one vacuum over another [42–44]. It is found that the defects disappear exponentially fast at a critical conformal time η_c, which scales with the initial bias U as $\eta_c \sim U^{2/D}$. Indeed, part of the motivation for this work was to account for this kind of behavior observed in simulations by Coulson, Lalak, and Ovrut [42] and Larsson, Sarkar, and White [43].

Finally, in making comparisons with similar results in the condensed matter literature, an expression for the length density of strings in three space dimensions in the condensed matter is corrected (see Sec. V D).

In this paper we shall work a conformally flat d-dimensional Friedmann-Robertson-Walker space-time with coordinates $x^0, x^1, \ldots, x^D$, such that $d = D+1$. The metric is given by

$$g_{\mu\nu} = a^2(\eta)\mathrm{diag}(-1, \delta_{ij}), \tag{3}$$

where η is conformal time, giving an affine connection

$$\Gamma^{\rho}_{\mu\nu} = (\delta^{\rho}_{\mu}\delta^{0}_{\nu} + \delta^{\rho}_{\nu}\delta^{0}_{\mu} - g_{\mu\nu}g^{\rho 0})(\dot{a}/a). \tag{4}$$

II. FIELD EQUATIONS

In this section, we shall first study model field equations for topological defects of codimension $N=1$ and $N=2$, which correspond to walls and strings, respectively, in $D=3$. We shall see that we can find approximate solutions to the field equations near surfaces of codimension N which have zero extrinsic curvature, and whose other curvature radii are large compared with the width of the defect. These results are well known and have been shown in various ways in [26–28], but the approach here is slightly different and worth exhibiting in some detail for the later sections of the paper.

A. Domain walls

Let us first consider a theory with a single scalar field ϕ, with action

$$S = -\int d^d x \sqrt{-g}[\tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi + V(\phi)], \tag{5}$$

from which we derive the field equation

$$-\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\phi + \frac{dV}{d\phi} = 0. \tag{6}$$

We shall suppose that the potential V has the symmetry $\phi \to -\phi$, and moreover that its minima are at $\phi = \pm v$, with $V(\pm v) = 0$. If we impose the boundary conditions.

$$\phi(x^D \to -\infty) = -v, \quad \phi(x^D \to +\infty) = +v, \tag{7}$$

and make the ansatz

$$\partial_\mu \phi(x) = 0 \quad (\mu = 0, \ldots, D-1), \tag{8}$$

then the theory has a one-parameter family of domain-wall solutions, with $\phi = 0$ at $x^D = X^D$. If the potential is quartic,

$$V(\phi) = \tfrac{1}{4}\lambda(\phi^2 - v^2)^2, \tag{9}$$

then the solutions are

$$\overline{\phi}(x) = v \tanh[M(x^D - X^D)], \tag{10}$$

where $M = \sqrt{\lambda}v$. Thus the width of the defect is controlled by the parameter M^{-1}. The defect can be thought of as centered at X^D, where the field vanishes, with a width parameter M^{-1}.

B. Strings

The simplest theory to exhibit stringlike solutions is the Abelian-Higgs model, which has action

$$S = -\int d^d x \sqrt{-g}[\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu \phi^* D^\mu \phi + V(\phi)], \tag{11}$$

where ϕ is a complex scalar field with covariant derivative $D_\mu \phi = \partial_\mu \phi - ieA_\mu \phi$. The potential V is taken to respect a U(1) symmetry $\phi \to e^{i\alpha}\phi$, with a circle of minima at $|\phi| = v$. If we impose the boundary conditions in the $\{x^{D-1}, x^D\}$ plane

$$\phi(r \to \infty) = v e^{i\theta}, \tag{12}$$

where $r^2 = (x^{D-1})^2 + (x^D)^2$ and $\tan\theta = x^D/x^{D-1}$, then by continuity ϕ must vanish somewhere in the plane. If we furthermore assume translational invariance in the other $d-2$ directions in space-time, we find a two-parameter family of static string solutions, labeled by the coordinates of the center of the string, $\{X^{D-1}, X^D\}$. In the radial gauge $A_r = 0$ these solutions take the form

$$\overline{\phi}(x) = f(\rho)e^{i\varphi}, \quad \overline{A}_i = \frac{1}{e\rho}\hat{\varphi}_i a(m_v \rho), \quad \overline{A}_\alpha = 0, \tag{13}$$

where $\rho_1 = (x-X)^{D-1}$, $\rho_2 = (x-X)^D$, $\rho^2 = (\rho_1)^2 + (\rho_2)^2$, $\tan\varphi = \rho_2/\rho_1$, and $\hat{\varphi}^i$ is the unit azimuthal vector in the $\{x^{D-1}, x^D\}$ plane. These solutions cannot generally be found analytically, even when the potential has the renormalizable and gauge invariant form

$$V(\phi) = \tfrac{1}{2}\lambda(|\phi|^2 - v^2/2)^2. \tag{14}$$

However, they are easily found numerically, and exhibit similar properties to the domain wall in that away from the center of the defect the fields approach their vacuum values exponentially, at rates controlled by the masses of the fields $m_s = \sqrt{\lambda}v$ and $m_v = ev$. Defining a dimensionless coordinate $z = m_v \rho$, and $\beta = (m_s/m_v)^2 = \lambda/e^2$, one has [45]

$$f \sim 1 - f_1 z^{-1/2}\exp(-\sqrt{\beta}z),$$
$$a \sim 1 - a_1 z^{1/2}\exp(-z). \tag{15}$$

In the case $\beta > 4$, the asymptotic form of f is $1 - z^{-1}\exp(-2z)$.

Again, the string can be thought of as centered at $\{X^{D-1}, X^D\}$, with thickness m_v, although for light scalars

($\beta \ll 1$) there is a thicker scalar core where the scalar field asymptotes to its vacuum value.

C. Solutions in curvilinear coordinates

These are, however, rather special solutions with a high degree of symmetry. Let us instead look for (if necessary approximate) solutions, corresponding to defects centered on a more general surface $X^\mu(\sigma^\alpha)$, with $\alpha = 0,\ldots,p = D-N$. We choose a new set of coordinates $\xi^\mu = \{\sigma^\alpha, u^A\}$, where $A = 1,\ldots,N$, with the intention that the equations of the surfaces can be written

$$u^A(x) = 0. \tag{16}$$

We write the metric in these new coordinates

$$G_{\mu\nu} = \begin{pmatrix} \partial_\alpha x \cdot \partial_\beta x & \partial_\alpha x \cdot \partial_B x \\ \partial_A x \cdot \partial_\beta x & \partial_A x \cdot \partial_B x \end{pmatrix} = \begin{pmatrix} \gamma_{\alpha\beta} & N_{B\alpha} \\ N_{A\beta} & G_{AB} \end{pmatrix}, \tag{17}$$

where the dot indicates a contraction with respect to the original metric $g_{\mu\nu}$. We may choose the coordinates ξ^μ so that, at least at $u^A = 0$, the u^A and σ^α are locally orthogonal, or

$$N_{A\beta}|_{u^A=0} = 0. \tag{18}$$

In fact, with walls and strings in $D = 3$, these are only three or four conditions on the metric, respectively, so we know we can make a coordinate transformation so that this is true everywhere, and not just at $u^A = 0$.

Note that the upper left $(p+1)\times(p+1)$ block of $G_{\mu\nu}$, denoted $\gamma_{\alpha\beta}$ in Eq. (17), is the embedding metric on surfaces of constant u^A, which they acquire by virtue of being surfaces embedded in a space-time with metric $g_{\mu\nu}$.

We can also write the inverse metric

$$G^{\mu\nu} = \begin{pmatrix} \partial\sigma^\alpha \cdot \partial\sigma^\beta & \partial\sigma^\alpha \cdot \partial u^B \\ \partial u^A \cdot \partial\sigma^\beta & \partial u^A \cdot \partial u^B \end{pmatrix}. \tag{19}$$

We define

$$h^{AB} = \partial u^A \cdot \partial u^B, \tag{20}$$

and use the convention that the indices α, β, etc., are raised and lowered with $\gamma_{\alpha\beta}$ and $\gamma^{\alpha\beta}$ (defined as the matrix inverse), and that the indices A, B, etc., are raised and lowered with h^{AB} and its matrix inverse h_{AB}. Hence

$$G_{\mu\nu} = \begin{pmatrix} \gamma_{\alpha\beta} & N_{B\alpha} \\ N_{A\beta} & h_{AB} + N_{A\beta}N_B^\beta \end{pmatrix},$$

$$G^{\mu\nu} = \begin{pmatrix} \gamma^{\alpha\beta} + N_A^\alpha N^{A\beta} & -N^{B\alpha} \\ -N^{A\beta} & h^{AB} \end{pmatrix}. \tag{21}$$

One can show that

$$\det G^{\mu\nu} = \det \gamma^{\alpha\beta} \det h^{AB}, \tag{22}$$

and hence that $G = \gamma/h$, where $G = \det G_{\mu\nu}$, $\gamma = \det \gamma_{\alpha\beta}$, and $h = \det h^{AB}$.

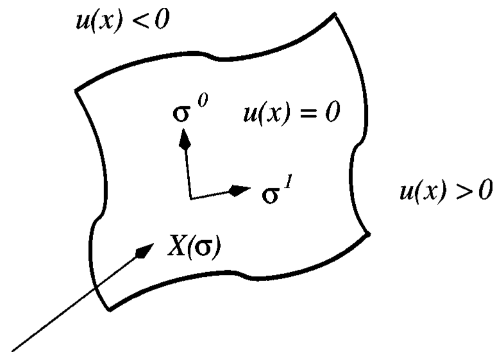

FIG. 1. The coordinates $\xi^\mu = \{\sigma^\alpha, u^A\}$, where $\alpha = 0,\ldots,p$ and $A = 1,\ldots,N$, which are chosen so that $u^A = 0$ will be the extremal surface on which the topological defect sits. Illustrated is a 1-brane in $2+1$ space-time dimensions, located at $X^\mu(\sigma) = x^\mu(\sigma,0)$.

We have two projectors associated with the constant u^A surfaces, one which projects onto the surface and the other which projects onto the subspace spanned by the vectors $\partial_\mu u^A$,

$$P^\mu_{\parallel\nu} = \gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\nu, \quad P^\mu_{\perp\nu} = h_{AB}\partial^\mu u^A \partial_\nu u^B, \tag{23}$$

with $P^\mu_{\parallel\nu} + P^\mu_{\perp\nu} = \delta^\mu_\nu$.

Let us study the field equation for the theory of an N-component scalar field Φ in these new coordinates:

$$-\frac{1}{\sqrt{-G}}\partial_\alpha(\sqrt{-G}\gamma^{\alpha\beta}\partial_\beta)\Phi$$

$$= \frac{1}{\sqrt{-G}}\hat{\partial}_A(\sqrt{-G}h^{AB}\hat{\partial}_B)\Phi - \frac{dV}{d\Phi}, \tag{24}$$

where $\hat{\partial}_A = \partial_A - N_A^\alpha \partial_\alpha$ (where we use notation mirroring that of Moss and Shiiki [46]). At the surface $u^A = 0$ it is possible to make a coordinate transformation among the u^A coordinates so that they are orthonormal, that is, $h^{AB} = \delta^{AB}$.

This choice of coordinates is different from the one used in other works on solving defect equations of motion in curvilinear coordinates [26–28], where coordinates $\{\sigma^\alpha, \rho^A\}$ are constructed away from the surface by setting

$$x^\mu(\sigma^\alpha, \rho^A) = X^\mu(\sigma^\alpha) + \rho^A n_A^\mu(\sigma^\alpha), \tag{25}$$

where $n_A^\mu = \partial_A x^\mu|_{u^A=0}$ and $n_A \cdot n_B = \delta_{AB}$. The coordinates coincide only when $N_{A\beta} = 0$. Carter [41] also uses orthonormal vectors in the surface, and is careful to express quantities as space-time tensors (see Fig. 1). Table I contains a summary which compares his notation and conventions with this work.

In contrast to previous work, here it is more convenient to use the unnormalized $\partial_\mu u^A$ as basis vectors, as we are interested in the surfaces generated by Gaussian random fields u^A, with unconstrained derivatives at $u^A = 0$.

We now try to find approximate solutions to Eq. (24). A promising avenue is to look for solutions which are independent of σ^α, in which case Eq. (24) becomes

TABLE I. Comparison of notation and conventions with that of Carter [41].

This work	Name	Carter	Relationship
$P^{\mu}_{\parallel\nu}$	First fundamental tensor	$\eta^{\mu}{}_{\nu}$	$P^{\mu}_{\parallel\nu}=\eta^{\mu}{}_{\nu}$
$P^{\mu}_{\perp\nu}$	Orthogonal projector	$\perp^{\mu}{}_{\nu}$	$P^{\mu}_{\perp\nu}=\perp^{\mu}{}_{\nu}$
$\partial_\alpha x^\mu$	Tangent vector	$\iota_A{}^{\mu}$	$\iota_A{}^{\mu}\iota_{B\mu}=\eta_{AB}$,
$(\alpha=0,\ldots,p)$		$(A=0,\ldots,p)$	$\partial_\alpha x^\mu\partial_\beta x_\mu=\gamma_{\alpha\beta}\neq\eta_{\alpha\beta}$
$\partial_\mu u^A$	Normal vector	$\lambda_X{}^{\mu}$	$\lambda_X{}^{\mu}\lambda_{Y\mu}=\delta_{XY}$,
$(A=1,\ldots,N)$		$(X=1,\ldots,N)$	$\partial_\mu u^A\partial^\mu u^B=h^{AB}\neq\delta^{AB}$
$K^A_{\mu\nu}$	Second fundamental tensor	$K_{\mu\nu}{}^{\rho}$	$K^A_{\mu\nu}=K_{\mu\nu}{}^{\rho}\partial_\rho u^A$

$$\sqrt{h}\partial_A\left(\frac{1}{\sqrt{h}}h^{AB}\partial_B\right)\Phi+K^A\partial_B\Phi-\frac{dV}{d\Phi}=0, \tag{26}$$

where K^A is the extrinsic curvature of the constant u^A hypersurfaces, given by

$$K^A=\frac{1}{\sqrt{-\gamma}}h^{AB}\partial_B(\sqrt{-\gamma}). \tag{27}$$

The ansatz $\Phi=\Phi(u^A)$ can only be self-consistent if both K^A and h^{AB} are independent of σ^α. This is still a difficult equation to solve, so the next step is to look near surfaces where the extrinsic curvature vanishes. Transforming to the orthonormal coordinates (25) near those surfaces, we have the approximate equations

$$-\frac{\partial}{\partial\rho^A}\frac{\partial}{\partial\rho^A}\Phi+\frac{dV}{d\Phi}\simeq 0. \tag{28}$$

By "near" we mean the region where $|K^A\partial_A\Phi|\ll|\partial_A\partial_A\Phi|$. Equation (28) is solved by $\bar{\Phi}(\rho^A)$, the original defect profile. Hence we are guaranteed approximate solutions to the field equations near smooth $K^A=0$ (extremal) surfaces. The argument in this section can be straightforwardly extended to gauge fields and so the task of solving the field equations has been replaced by the task of finding extremal surfaces.

The extrinsic curvature K^A will generically vanish only at $u^A=0$, and be nonzero elsewhere in space-time, and so the static solutions $\bar{\Phi}$ will not be exact. However, we should be able to find approximate solutions $\Phi=\bar{\Phi}+\varphi$, with the perturbation φ being sourced by the departures of K^A from zero,

$$\varphi(\xi)=\int d^4\xi'\sqrt{-G}\Delta_R(\xi,\xi')K^A(\xi')\partial_A\bar{\Phi}(u'), \tag{29}$$

where $\Delta_R(\xi,\xi')$ is the retarded Green's function for the scalar field fluctuation operator, given by

$$[-{}^{(d)}\Box+V''(\bar{\phi})]\Delta_R(\xi,\xi')=\delta^d(\xi-\xi'), \tag{30}$$

with $\Delta_R(\xi,\xi')=0$ for $\xi^0<\xi'^0$. If the extrinsic curvature decreases with time, the source for the perturbation φ dies away, and we should not have to worry that our initial assumption that $|\varphi|\ll|\bar{\Phi}|$ is rendered invalid. In fact, the dynamic scaling hypothesis holds that $K^A\sim\xi^{-1}$, where ξ is the average curvature radius of the defect network.

There are in fact special cases for which $K^A=0$ everywhere, and exact curved defect solutions exist. These represent traveling waves on walls and strings [47], although they do not obey a superposition principle because of the nonlinearities in the field theory.

This brings us close to the controversial subject of radiation from defect networks. We postpone this discussion until Sec. VI.

III. EXTREMAL SURFACES

We saw in the last section that if we could find a suitable surface of constant u^A (which without loss of generality we can choose to be $u^A=0$) satisfying $K^A=0$, an approximate solution of the field equations could be found. We shall now derive the equations that u^A must satisfy in order that $u^A(X)=0$ be an extremal surface.

Differentiating once with respect to the world-volume coordinates σ^α, we find

$$\partial_\beta X^\mu\partial_\mu u^A(X)=0. \tag{31}$$

(This equation is of course true independent of the choice of the coordinates ξ.) Using the embedding metric we can covariantly differentiate (31) by acting with $(-\gamma)^{-1/2}\partial_\alpha(-\gamma)^{1/2}\gamma^{\alpha\beta}$, where $\gamma=\det\gamma_{\alpha\beta}$, to obtain

$${}^{(p+1)}\Box X^\mu\partial_\mu u^A+\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\partial_\mu\partial_\nu u^A=0. \tag{32}$$

The operator

$${}^{(p+1)}\Box=(-\gamma)^{-1/2}\partial_\alpha(-\gamma)^{1/2}\gamma^{\alpha\beta}\partial_\beta \tag{33}$$

is the covariant d'Alembertian in the surface $u^A=0$.

The equations of motion are obtained by extremizing the invariant area of the surface [48,45],

$$A_{\rm inv}[X]=\int d^{p+1}\sigma\sqrt{-\gamma(X)} \tag{34}$$

with respect to the enbedding coordinates $X^\mu(\sigma)$. The result is

$${}^{(p+1)}\Box X^\mu+\Gamma^\mu_{\nu\rho}\gamma^{\alpha\beta}\partial_\alpha X^\nu\partial_\beta X^\rho=0, \tag{35}$$

where $\Gamma^{\mu}_{\nu\rho}$ is the affine connection derived from the metric $g_{\mu\nu}$.

The reader will notice the appearance of the tangential projector $P_{\parallel}^{\mu\nu}$ in Eqs. (32) and (35), which we replace by $g^{\mu\nu}-P_{\perp}^{\mu\nu}$ Combining Eqs. (35) and (32) to eliminate the d'Alembertian we find

$$[g^{\mu\nu}-h_{AB}\partial^{\mu}u^{A}\partial^{\nu}u^{B}](\partial_{\mu}\partial_{\nu}u^{C}-\Gamma^{\rho}_{\mu\nu}\partial_{\rho}u^{C})=0. \quad (36)$$

This is the fundamental equation of motion for the fields $u^{A}(x)$, which strictly only applies at $u^{A}=0$.

The equations also follow from a variational procedure. Using the fact that $G=\gamma/h$, and that $G=g$, one can show that the invariant area of a p-brane can be reexpressed in terms of the u^{A} as

$$\mathcal{A}_{\rm inv}[u^{A}]=\int d^{4}x\sqrt{-g}\sqrt{h}\,\delta^{N}(u). \quad (37)$$

Varying with respect to u^{A} and dividing by $\sqrt{-\gamma}$ gives us

$$\frac{\sqrt{h}}{\sqrt{-g}}\frac{\delta\mathcal{A}_{\rm inv}}{\delta u^{A}(x)}=-\delta^{N}(u)h^{1/2}\nabla_{\mu}(h^{-1/2}h_{AB}\partial^{\mu}u^{B})=0. \quad (38)$$

This can be shown to be equivalent to

$$\delta^{N}(u)P_{\parallel}^{\mu\nu}\nabla_{\mu}\partial_{\nu}u^{A}=0, \quad (39)$$

and hence Eq. (36) at $u^{A}=0$. In orthonormal coordinates, for which $h^{AB}=\delta_{AB}$, Eq. (38) becomes

$$\delta^{N}(u)\nabla_{\mu}n^{A\mu}\equiv\delta^{N}(u)K^{A}=0, \quad (40)$$

where K^{A} is the extrinsic curvature. Thus we can identify $K^{A}_{\mu\nu}=P^{\sigma}_{\parallel\mu}\nabla_{\sigma}\partial_{\nu}u^{A}$ as the extrinsic curvature tensor, or equivalently the second fundamental tensor (see Table I and [41]).

The restriction that the equations apply only at $u^{A}=0$ complicates the finding of solutions, and we assume that we can extend the equation $K^{A}=0$ to all u^{A}. It is not obvious that nontrivial solutions exist to the extended equations, because such a solution would be a foliation of space-time in which all leaves have zero extrinsic curvature. As mentioned above, some nontrivial solutions are known [47] but there is no general proof for the Allen-Cahn equation [10]. However, we could equally well look for solutions to $K^{A}=f(u^{A})$, with $f(u)$ any function which vanishes at $u=0$, so there should be a certain amount of freedom. Furthermore, we will be looking only for perturbative solutions to the extended equations.

IV. AVERAGE EXTREMAL SURFACES

The equations of motion (36) are not easy to solve, as they are nonlinear. However, they have distinct advantages over the alternatives. The equations of motion for the coordinates of the $u^{A}=0$ surfaces (35) are nonlocal: defects generically self-intersect. This nonlocality generally defeats analytic approaches, and also makes numerical simulations algorithmically difficult, as one must devise an efficient scheme for searching for self-intersections [29–31]. The equations of motion for the underlying field theory are also nonlinear, and in the gauge of the Abelian-Higgs model (and other gauge theories) they have a gauge covariance, which precludes the naive application of techniques such as large N. Numerical simulations of field theories are relatively straightforward, but require significant memory to allow the scale of the network to grow much larger than the width of the defect.

Instead of trying to find families of surfaces whose curvature is exactly zero, we shall find surfaces who curvature is zero *on average*. The average will be taken with respect to a Gaussian probability distribution for u^{A}. We assume that the distribution function remains Gaussian throughout the evolution, which is similar to the approximation underlying the large-N approximation in scalar field theory. Indeed, we should expect there to be a similar large-N limit in this theory.

A. Gaussian averaging

Our starting point is an ensemble of coordinate functions $u^{A}(x)$ with an assumed Gaussian distribution. Thus the average value of all observables of interest $\Omega(u^{A},\partial_{\mu}u^{A})$, which we take to be functions of u^{A} and its derivative $\partial_{\mu}u^{A}$, are evaluated with the probability distribution

$$dP[u^{A}]=\mathcal{D}u\,\exp\left(-\frac{1}{2}\int d^{d}x\,d^{d}y\,u^{A}(x)C^{-1}_{AB}(x,y)u^{B}(y)\right), \quad (41)$$

where $C^{AB}(x,y)$ is the two-point correlation function.

We are often interested in densities, which means that the observable Ω is evaluated at a particular point $\tilde{x}$. This means we can simplify the evaluation of the averages from a functional integral to an ordinary one, as we now demonstrate.

First, let us take the Fourier transform of the observable,

$$\Omega(u^{A}(\tilde{x}),\partial_{\mu}u^{A}(\tilde{x}))=\int\frac{d^{N}l}{(2\pi)^{N}}\frac{d^{Nd}k}{(2\pi)^{Nd}}\tilde{\Omega}(l,k)\times e^{il_{A}u^{A}(\tilde{x})+ik^{\mu}_{A}\partial_{\mu}u^{A}(\tilde{x})}. \quad (42)$$

We now introduce current densities $L_{A}(x)$ and $K^{\mu}_{A}(x)$, according to

$$L_{A}(x)=l_{A}\delta^{d}(x-\tilde{x}),\quad K^{\mu}_{A}(x)=k^{\mu}_{A}\delta^{d}(x-\tilde{x}), \quad (43)$$

so that the expectation value of $\Omega(\tilde{x})$ is given by

$$\langle\Omega(u^{A},\partial_{\mu}u^{A})\rangle=\int\frac{d^{N}l}{(2\pi)^{N}}\frac{d^{Nd}k}{(2\pi)^{Nd}}\int dP[u^{a}]\tilde{\Omega}(l,k)\times e^{i\int d^{d}x[L_{A}(x)-\partial\cdot K_{A}(x)]u^{A}(x)}. \quad (44)$$

Performing the integral of the random field u^{A}, we find

$$\langle\Omega(u^A,\partial_\mu u^A)\rangle=\int\frac{d^Nl}{(2\pi)^N}\frac{d^{Nd}k}{(2\pi)^{Nd}}\tilde{\Omega}(l,k)e^{-1/2\int_x\int_y(L_A-\partial\cdot K_A)C^{AB}(L_B-\partial\cdot K_B)}. \tag{45}$$

Substituting the form of the functions L_A and K_A^μ from Eq. (43), we find

$$\langle\Omega(u^A,\partial_\mu u^A)\rangle=\int\frac{d^Nl}{(2\pi)^N}\frac{d^{Nd}k}{(2\pi)^{Nd}}\tilde{\Omega}e^{(1/2)l_AC^{AB}(\eta)l_B+l_A[\partial_\mu C^{AB}(\eta)]k_B^\mu-(1/2)k_A^\mu[\partial_\mu\partial_\nu C^{AB}(\eta)]k_B^\nu}, \tag{46}$$

where

$$C^{AB}(\eta)=\lim_{x\to y}C^{AB}(x,y),$$

$$\partial_\mu C^{AB}(\eta)=\lim_{x\to y}\frac{\partial}{\partial x^\mu}C^{AB}(x,y),$$

$$\partial_\mu\partial_\nu C^{AB}(\eta)=\lim_{x\to y}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\mu}C^{AB}(x,y), \tag{47}$$

and $\eta=\tilde{x}^0$. We note that we expect correlation functions to be spatially homogeneous but to depend on conformal time nontrivially, reflecting the symmetries of the background space-time, hence the explicit conformal time dependence of the two-point correlators evaluated at the same two points. At this point we recall that the Fourier transform of the observable Ω may be written

$$\tilde{\Omega}(l,k)=\int d^Nud^{Nd}\pi\,\Omega(u^A,\pi_\mu^A)e^{-il_Au^A-ik_A^\mu\pi_\mu^A}. \tag{48}$$

We can economize slightly on the length of expressions by introducing some new notation. Defining $N(d+1)$-dimensional objects j and f by

$$j=\{l_A,k_A^\mu\},\quad f=\{u^A,\pi_\mu^A\}, \tag{49}$$

with a scalar product $(j,f)=l_Au^A+k_A^\mu\pi_\mu^A$, we can write

$$\langle\Omega(f)\rangle=\int d^{N(d+1)}f'\frac{d^{N(d+1)}j}{(2\pi)^{N(d+1)}}\Omega(f')e^{(1/2)(j,\mathcal{C}j)-i(j,f')}, \tag{50}$$

where $\mathcal{C}$ is the covariance matrix

$$\mathcal{C}=\begin{pmatrix}C^{AB}(\eta) & \partial_\mu C^{AB}(\eta)\\ \partial_\nu C^{AB}(\eta) & \partial_\mu\partial_\nu C^{AB}(\eta)\end{pmatrix}. \tag{51}$$

Finally, we may perform the integrations over the Fourier transform variables j to obtain

$$\langle\Omega(f)\rangle=[(2\pi)^{-N(d+1)/2}][\det\mathcal{C}]^{-1/2}\times\int d^{N(d+1)}f'\,\Omega(f')e^{-1/2(f',\mathcal{C}^{-1}f')}. \tag{52}$$

Hence the average of the observable can be found with an ordinary integral, as claimed.

B. The covariance matrix $\mathcal{C}$

In our case the covariance matrix $\mathcal{C}$ is restricted by the assumed FRW form of the background. It will be homogeneous and isotropic, but not time-independent. We will also assume an $O(N)$ symmetry between the N coordinate functions $u^A(x)$. Hence, the basic two-point correlation function at zero separation may be written

$$C^{AB}(\eta)\equiv\langle u^A(x)u^B(x)\rangle=\delta^{AB}C(\eta). \tag{53}$$

We shall also define a function $M_{\mu\nu}$ from the two-point correlator of $\partial_\mu u^A$:

$$\partial_\mu\partial_\nu C^{AB}(\eta)\equiv\langle\partial_\mu u^A(x)\partial_\nu u^B(x)\rangle=\delta^{AB}M_{\mu\nu}(\eta). \tag{54}$$

The assumed spatial isotropy of the distribution function dictates the form of $M_{\mu\nu}$:

$$M_{\mu\nu}=\begin{pmatrix}T(\eta) & 0\\ 0 & \delta_{mn}S(\eta)\end{pmatrix}. \tag{55}$$

With this definition it is not hard to show that $S(\eta)=-C''(\eta)$, where $C''(\eta)=\lim_{r\to 0}(\partial^2/\partial r^2)C(\eta,r)$.

Two-point correlators with odd numbers of derivatives also occur, as the ensemble is not time-translation-invariant. The correlator with one derivative is

$$\partial_\mu C^{AB}(\eta)\equiv\langle\partial_\mu u^A(x)u^B(x)\rangle=\frac{1}{2}\delta^{AB}\delta_\mu^0\dot{C}(\eta), \tag{56}$$

and with three,

$$\langle\partial_\mu u^A(x)\partial_\nu\partial_\rho u^B(x)\rangle=\gamma_{\mu\nu\rho}(\eta)\delta^{AB}. \tag{57}$$

Again, symmetry restricts the form of $\gamma_{\mu\nu\rho}$:

$$\gamma_{000}(\eta)=\tfrac{1}{2}\dot{T}(\eta),$$

$$\gamma_{0mn}(\eta)=-\tfrac{1}{2}\dot{S}(\eta)\delta_{mn},$$

$$\gamma_{m0n}(\eta)=\tfrac{1}{2}\dot{S}(\eta)\delta_{mn}. \tag{58}$$

It is interesting to note that $\gamma_{\mu\nu\rho}=\frac{1}{2}(M_{\mu\nu,\rho}+M_{\mu\rho,\nu}-M_{\nu\rho,\mu})$.

Thus the covariance matrix can be written

$$\mathcal{C}=\delta^{AB}\otimes\begin{pmatrix} C & \frac{1}{2}\dot{C} & 0 \\ \frac{1}{2}\dot{C} & T & 0 \\ 0 & 0 & \delta_{mn}S \end{pmatrix}. \tag{59}$$

Its inverse is easily found, and defining the determinant of the upper 2×2 block $\Delta=(TC-\frac{1}{4}\dot{C}^2)$, we can write

$$\mathcal{C}^{-1}=\delta_{AB}\otimes\begin{pmatrix} T & -\frac{1}{2}\dot{C} & \\ -\frac{1}{2}\dot{C} & C & 0 \\ 0 & & \delta_{mn}\frac{\Delta}{S} \end{pmatrix}\frac{1}{\Delta}. \tag{60}$$

The determinant factor in the probability distribution is also straightforward,

$$[\det\mathcal{C}]^{-1/2}=[S^D\Delta]^{-N/2}. \tag{61}$$

Often, we will want to find expectation values which are independent of $\partial_0 u^A$, mainly because the integrals are easier to evaluate. By integrating over π_0^A one can easily show that

$$\langle\Omega(u^A,\partial_i u^A)\rangle=[(2\pi)^d S^D C]^{-N/2}\int d^N u d^{ND}\pi\Omega(u^A,\pi_i^A)\times e^{-(1/2)u^A\delta_{AB}u^B/C-(1/2)\pi_i^A\delta_{AB}\delta^{ij}\pi_j^B/S}. \tag{62}$$

It is very convenient to rescale the integration variables in the probability distribution, $u^A\to u^A\sqrt{C}$ and $\pi_i^A\to\pi_i^A\sqrt{S}$, in which case

$$\langle\Omega(u^A,\partial_i u^A)\rangle=\frac{1}{(2\pi)^{dN/2}}\int d^N u d^{ND}\pi\Omega(u^A\sqrt{C},\pi_i^A\sqrt{S})\times e^{-(1/2)u^A\delta_{AB}u^B-(1/2)\pi_i^A\delta_{AB}\delta^{ij}\pi_j^B}. \tag{63}$$

C. Averaging the null extrinsic curvature condition

The averaging procedure is greatly aided by rewriting the equations of motion (36) in the following form:

$$\frac{1}{\sqrt{-g}}\left[\frac{\partial}{\partial g_{\mu\nu}}\sqrt{-g}\det h\right](\partial_\mu\partial_\nu u^C-\Gamma^\rho_{\mu\nu}\partial_\rho u^C)=0. \tag{64}$$

The procedure now is to linearize the equations of motion by taking the Gaussian average, and then to find a self-consistent solution for the fields $u^A(\mathbf{x},\eta)$. We will require the following identities, which are proved in Appendix B:

$$\langle\det h\,\partial_\mu\partial_\nu u^C\rangle=\langle\det h\rangle\partial_\mu\partial_\nu u^C+\frac{2}{N}\gamma_{\rho\mu\nu}\left(\frac{\partial}{\partial M_{\rho\sigma}}\langle\det h\rangle\right)\partial_\sigma u^C, \tag{65}$$

$$\langle\det h\,\partial_\rho u^C\rangle=\langle\det h\rangle\partial_\rho u^C+\frac{2}{N}\left(\frac{\partial}{\partial M_{\eta\sigma}}\langle\det h\rangle\right)M_{\eta\rho}\partial_\sigma u^C. \tag{66}$$

The expectation value of the determinants in Eqs. (65) and (66) can be expressed in terms of the two-point correlator $M_{\mu\nu}$ [defined in Eq. (54)],

$$\langle\det h\rangle=N!\,\Pi_{i=1}^N M_{\mu_i\nu_i}g^{\rho_i\sigma_i}\delta^{\mu_1\cdots\mu_N}_{\rho_1\cdots\rho_N}\delta^{\nu_1\cdots\nu_N}_{\sigma_1\cdots\sigma_N}, \tag{67}$$

where $\delta^{\mu_1\cdots\mu_N}_{\rho_1\cdots\rho_N}$ is the identity tensor in the space of rank N antisymmetric tensors, defined in Appendix A. The right-hand side of Eq. (67) resembles a determinant, and we introduce the notation $\overline{\det}\,M$ to refer to it. We can also define a kind of cofactor for $M_{\mu\nu}$:

$$\overline{M}^{\mu\nu}=N!\,\Pi_{i=1}^N g^{\rho_i\sigma_i}\delta^{\mu\mu_2\cdots\mu_N}_{\rho_1\cdots\rho_N}\delta^{\nu\nu_2\cdots\nu_N}_{\sigma_1\cdots\sigma_N}\times M_{\mu_2\nu_2}\cdots M_{\mu_N\nu_N}/\overline{\det}\,M. \tag{68}$$

Putting the pieces together we find that the linearized equations for surfaces which are on average extremal are

$$\left(g^{\mu\nu}-g^{\mu\rho}g^{\nu\sigma}\frac{\partial}{\partial g^{\rho\sigma}}\right)\overline{\det}\,M\left[\partial_\mu\partial_\nu u^C+\frac{2}{N}\overline{M}^{\kappa\eta}\gamma_{\kappa\mu\nu}\partial_\tau u^C-\Gamma^\tau_{\mu\nu}\left(\partial_\tau u^C+\frac{2}{N}\overline{M}^{\kappa\lambda}M_{\lambda\tau}\partial_\kappa u^C\right)\right]=0. \tag{69}$$

With the assumed symmetries for the correlation functions, these equations have the form

$$\ddot{u}^C+\frac{\mu(\eta)}{\eta}\dot{u}^C-v^2\nabla^2 u^C=0, \tag{70}$$

where $\mu(\eta)$ and v depend on T, S, and the background cosmology parametrized by α, and must be taken on a case-by-case basis for each N.

D. Linearized equations for walls and strings

In our three-dimensional universe, the cases of most interest are $N=1$ (domain walls) and $N=2$ (gauge strings). $N=3$ corresponds to gauge monopoles, which do not scale [48,49]. For $N=1$, $\langle\det h\rangle=M_{\mu\nu}g^{\mu\nu}$, while for $N=2$, $\langle\det h\rangle=\frac{1}{2}[(M_{\mu\nu}g^{\mu\nu})^2-M_{\mu\nu}\overline{M}^{\mu\nu}]$. We then find, for FRW backgrounds (see Appendixes D1 and E1),

$$\mu(\eta)=\begin{cases}-2\eta(\dot{S}/S)+\alpha(\eta)[D-3(T/S)] & (N=1)\\ -[2/(D-1)]\eta(\dot{S}/S)+\alpha(\eta)[(D-1)-4(T/S)] & (N=2),\end{cases} \tag{71}$$

where $\alpha(\eta)=\eta\dot{a}/a$, and

$$v^2=\begin{cases}[D-1-(T/S)]/D & (N=1)\\ [D-2-2(T/S)]/D & (N=2).\end{cases} \tag{72}$$

In scaling solutions, we expect S and T to have power-law behavior, and so as long as we are not near a transition in the equation of state of the Universe (such as that between the radiation- and matter-dominated eras), μ and v^2 are constant. Thus, imposing the boundary condition that u^C be regular as $\eta\to 0$, Eq. (70) has the simple solution

$$u^C_{\boldsymbol{k}}(\eta)=A^C_{\boldsymbol{k}}\left(\frac{\eta}{\eta_{\rm i}}\right)^{(1-\mu)/2+\nu}\frac{J_\nu(kv\eta)}{(kv\eta)^\nu}, \tag{73}$$

where $A^C_{\boldsymbol{k}}\to 2^\nu\Gamma(\nu+1)u^C_{\boldsymbol{k}}(\eta_{\rm i})$ as $\boldsymbol{k}\to 0$, and $(1-\mu)^2/4=\nu^2$. The form of the initial power spectrum is taken to be a power law, with index q, and an upper cutoff at $|\boldsymbol{k}|=\Lambda$.

We may now evaluate T/S and v^2, and self-consistently solve for the undetermined parameter μ. It turns out that one must take $\nu=-(1-\mu)/2$ if all the integrals are required not to diverge as $\Lambda\to\infty$. This also gives regular solutions as $\eta\to 0$, because as it turns out, $\mu>1$. With this choice, C scales as $\eta^{-(D+q)}$, S and T as $\eta^{-(D+q+2)}$.

In the following, we will take the power spectrum to be white noise, $q=0$, as is consistent with a causal origin for the defects in a phase transition. There the power spectrum of the scalar field from which the defects are made has a $q=0$ power spectrum at long wavelengths, and so we should take the fields u^A to have a similar power spectrum if we want to reproduce the statistics of the defects from the statistics of the zeros of u^A.

Using standard integrals of Bessel functions, and defining the parameter $\beta=2\nu-D-1=\mu-D-2$, we find (see Appendix G)

$$\frac{T}{S}=\begin{cases}\dfrac{(D+2)(D-1)}{2(D+2+\beta)} & (N=1)\\[2ex] \dfrac{(D+2)(D-2)}{3(D+2)+2\beta} & (N=2),\end{cases} \tag{74}$$

provided $\beta>0$, which ensures that the integrals for S and T are defined. Given the expressions for T/S, is easy to show that

$$v^2=\begin{cases}\dfrac{(D-1)(D+2+2\beta)}{2D(D+2+\beta)} & (N=1)\\[2ex] \dfrac{(D-2)(D+2+2\beta)}{D[3(D+2)+2\beta]} & (N=2).\end{cases} \tag{75}$$

To find β, we must solve the equations derived from Eq. (71):

$$\beta=\begin{cases}\alpha[D-3(T/S)]+(D+2) & (N=1)\\ \alpha[D-2-3(T/S)]+(D+2)(3-D)/(D-1) & (N=2),\end{cases} \tag{76}$$

which are quickly seen to be quadratic. One can obtain results in simple closed form in Minkowski space ($\alpha=0$) and curvature-dominated universes ($\alpha=\infty$) which are displayed in Table II. For other backgrounds the solutions may be written down in closed form, but are not particularly illuminating as they are fairly lengthy expressions.

Instead, numerical values of β, T/S, and v^2 for particular cases of interest are given: radiation-dominated ($\alpha=1$) and matter-dominated ($\alpha=2$) two and three-dimensional universes (Tables III and IV).

Note that for strings ($N=2$) in three dimensions in Minkowski space ($\alpha=0$), for which $\mu=D+2$, $\beta=0$, which does not satisfy the requirement $\beta>0$ for the integrals defining $S(\eta)$ and $T(\eta)$ to be convergent. One finds that a logarithmic scaling violation appears, and $S,T\propto\log(\Lambda\eta)\eta^{-(D+2+q)}$. We also have a solution with $\beta=0$, and

TABLE II. Values for parameters β, v^2, and T/S of the self-consistent solution to the linearized equations of motion (70) for the N fields u^A for $N=1$ (domain walls). In the special cases of Minkowksi space ($\alpha=0$), and curvature-dominated FRW cosmologies ($\alpha=\infty$), exact values can be found for all D.

N	α	β	(T/S)	v^2
1	0	$(D+2)$	$\frac{D-1}{4}$	$\frac{3}{4}\frac{D-1}{4}$
	∞	$\frac{(D+2)(D-3)}{2D}$	$\frac{D}{3}$	$\frac{2D-3}{3D}$
2	0	$\frac{(D+2)(3-D)}{D-1}$	$\frac{(D-2)(D-1)}{D+3}$	$\frac{(D-2)(D+5)}{D(D+3)}$
	∞	0	$\frac{D-2}{3}$	$\frac{D-2}{3D}$

TABLE III. Values for parameters β, v^2, and T/S of the self-consistent solution to the linearized equations of motion (70) for the N fields u^A for $N=1$ (domain walls) in $D=2,3$. Values listed are for Minkowksi space ($\alpha=0$), radiation-dominated ($\alpha=1$), matter-dominated ($\alpha=2$), and curvature-dominated FRW cosmologies ($\alpha=\infty$).

		$D=3$			$D=2$	
α	β	(T/S)	v^2	β	(T/S)	v^2
0	5	1/2	1/2	4	1/4	3/8
1	6.72	0.43	0.52	5.36	0.21	0.39
2	8.83	0.36	0.55	6.90	0.18	0.41
∞	0	1	1/3	-1	2/3	1/6

therefore logarithmically divergent S and T, for walls in three-dimensional curvature-dominated universes.

V. AREA DENSITIES FOR WALLS AND STRINGS

Armed with the mean-field solution for $u^A(x)$, we can now calculate anything that can be expressed in terms of local functions of the field and its derivatives, provided of course that we are able to perform the Gaussian integrals involved. Here we derive formulas for the area densities of defects, where by ''area'' we mean the world volume of the $(p+1)$-dimensional hypersurface $u^A=0$, which has dimensions of $(\text{length})^{-N}$. We must be careful to distinguish between various kinds of area: there is invariant or proper area which is a coordinate-independent quantity, and there is also the projected p-dimensional area. The latter quantity is what one would obtain by simply measuring the p-dimensional area of the defects at a particular time. This quantity is the most convenient to calculate for comparison with numerical simulations, which is a good thing as the proper area density is far harder to calculate. One must also bear in mind that area *densities* are coordinate-dependent quantities: in the cosmological setting we will need to convert between comoving area density and physical area density by multiplying by the appropriate power of the scale factor a, which is a^{-N}.

Here we give figures for the projected area densities of walls and strings in $D=3$. They can be compared with results from numerical simulations of the field theories and give surprisingly good agreement given the uncontrolled nature of the approximations made.

TABLE IV. Values for parameters β, v^2, and T/S of the self-consistent solution to the linearized equations of motion (70) for the N fields u^A for $N=2$ in $D=3$ (strings). Values listed are for Minkowksi space ($\alpha=0$), radiation-dominated ($\alpha=1$), matter-dominated ($\alpha=2$), and curvature-dominated FRW cosmologies ($\alpha=\infty$).

		$D=3$	
α	(T/S)	v^2	β
0	1/3	1/9	0
1	0.22	0.14	3.65
2	0.20	0.20	4.75
∞	1/3	1/9	0

A. Proper area density

The proper area density $\mathcal{A}$ of a p-dimensional defect in D space dimensions is

$$\mathcal{A}_D^p(x)=\int d^{p+1}\sigma'\sqrt{-\gamma}\,\delta^d(x-X(\sigma'))/\sqrt{-g}. \tag{77}$$

Making the coordinate transformation from x^μ to $\xi^\mu=\{\sigma^\alpha,u^A\}$ near the world volume of the defect, we have

$$\mathcal{A}_D^p(\xi)=\int d^{p+1}\sigma'\sqrt{-\gamma}\,\delta^{p+1}(\sigma-\sigma')\,\delta^N(u^A)/\sqrt{-G}. \tag{78}$$

Recalling the results of Sec. II C, we can perform the integration over σ' to obtain

$$\mathcal{A}_D^p=\delta^N(u^A)|\det h^{AB}|^{1/2}, \tag{79}$$

where the reader is reminded that

$$h^{AB}=\partial_\mu u^A\partial_\nu u^B g^{\mu\nu}. \tag{80}$$

Thus the problem of calculating the proper area density is reduced to finding the Gaussian average of $\mathcal{A}_D^p$ in Eq. (79). The conversion factor from comoving to physical area is given as

$$\mathcal{A}_{D,\mathrm{phys}}^p=a^{-N}\mathcal{A}_D^p, \tag{81}$$

with $N=D-p$.

B. Projected area density

Easier to measure and to calculate is the projected area density, which is defined as

$$A_D^p=\int d^p\sigma'\sqrt{\gamma_D}\,\delta^D(\boldsymbol{x}-\boldsymbol{X}(\sigma'))/\sqrt{g_D}, \tag{82}$$

where g_{Dij} is the spatial part of the metric. The induced D-dimensional metric on the p-dimensional surface $u^A=0$ is

$$\gamma_{Dab}=\partial_a X^i\partial_b X^j g_{Dij}, \tag{83}$$

where $a,b=1,\ldots,p$. As for the proper area density, one can show that

$$A_D^p=\delta^N(u^A)|\det h_D^{AB}|^{1/2}, \tag{84}$$

where

$$h^{AB}=\partial_i u^A\partial_j u^B g_D^{ij}. \tag{85}$$

Note that g_D^{ij} is defined as the matrix inverse of g_{Dij}, and is not the spatial part of $g^{\mu\nu}$. The conversion between physical and comoving area is again

$$A_{D,\mathrm{phys}}^p=a^{-N}A_D^p. \tag{86}$$

C. Average projected area density: Walls

We can now use the averaging formula (63) to find the mean value of the operator A, which when we specialize to domain walls $(N=1)$ gives

$$\langle A_D^{D-1}\rangle=\frac{1}{(2\pi)^{d/2}}\sqrt{\frac{S}{C}}\int du d^D\pi\,\delta(u)|\pi_i| \times e^{-(1/2)u^2-(1/2)\pi_i\delta^{ij}\pi_j}. \tag{87}$$

The integrals are easily performed to give

$$\langle A_D^{D-1}\rangle=\sqrt{\frac{S}{\pi C}}\frac{\Gamma[(D+1)/2]}{\Gamma(D/2)}, \tag{88}$$

a well-known result originally derived by Ohta, Jasnow, and Kawasaki [38]. This is the *comoving* projected area density: to obtain the physical projected area density, one multiplies by a^{-1}.

D. Average projected area density: Strings

For strings $(N=2)$, the average we need to calculate is

$$\langle A_D^{D-1}\rangle=\frac{1}{(2\pi)^d}\frac{S}{C}\int d^2u d^{2D}\pi\,\delta^2(u^A)|h^{AB}|^{1/2} \times e^{-(1/2)u^A\delta_{AB}u^B-(1/2)\pi_i^A\delta^{ij}\delta_{AB}\pi_j^B}, \tag{89}$$

where the rescaled quantity h^{AB} is given by

$$h^{AB}=\pi_i^A\pi_i^B. \tag{90}$$

Now,

$$\det h^{AB}=\tfrac{1}{2}\epsilon_{AC}\epsilon_{BD}h^{AB}h^{CD}, \tag{91}$$

$$=\tfrac{1}{2}\epsilon_{AC}\epsilon_{BD}\pi_i^A\pi_j^C\pi_i^B\pi_j^D, \tag{92}$$

which suggests that we construct the following antisymmetric matrix:

$$f_{ij}=\pi_i^A\pi_j^B\epsilon_{AB}, \tag{93}$$

such that

$$\det h^{AB}=\tfrac{1}{2}f_{ij}f_{ij}. \tag{94}$$

Thus in order to calculate the average area, we need the probability distribution for f_{ij}. At this point we specialize to $D=3$, as the calculations are considerably simplified by introducing the vector

$$\phi_k=\tfrac{1}{2}\epsilon_{ijk}f_{ij}, \tag{95}$$

whereupon

$$\det h^{AB}=|\phi_k\phi_k|^{1/2}. \tag{96}$$

The probability distribution for $\phi=|\phi_k\phi_k|^{1/2}$ is derived in Appendix F, and turns out to be remarkably simple, giving

$$\langle A_3^1\rangle=\frac{1}{2\pi}\frac{S}{C}\int d^2u^A d^3\phi\,\delta^2(u^A)\,\phi e^{-1/2u^Au^B\delta_{AB}}\frac{1}{4\pi\phi}e^{-\phi}. \tag{97}$$

A simple calculation now shows that the comoving projected length density for strings in $D=3$ is

$$\langle A_3^1\rangle=\frac{S}{\pi C}. \tag{98}$$

Note that this disagrees with the formula derived by Toyoki and Honda [50], but agrees with Scherrer and Vilenkin [51]. Toyoki and Honda write the 3D string length density as

$$A_3^1=\delta(u^1)\,\delta(u^2)|\nabla u^1\times\nabla u^2| = \delta(u^1)\,\delta(u^2)|\nabla u^1||\nabla u^2|\cos\theta_{12}, \tag{99}$$

where θ_{12} is the angle between the vectors ∇u^1 and ∇u^2. They then average θ_{12} over a uniform distribution, separately from u^1 and u^2, which is incorrect.

E. Projected area density: Higher *N*

Scherrer and Vilenkin [51] used an elegant argument to derive their value for the projected area densities of walls, strings, and monopoles in $D=3$, which can be generalized to any N and D. They noted that a string was located at the intersection of two surfaces $u^1=0$ and $u^2=0$, and therefore the length density string could be found by computing the length per unit area of the lines of $u^2=0$ in the surface $u^1=0$, and then multiplying by the area per unit volume of the surface $u^1=0$. That is,

$$A_3^1=A_2^1A_3^2, \tag{100}$$

which clearly has the correct dimensions. One can easily check that this gives the correct result $A_3^1=(S/\pi C)$. It is immediately obvious how to generalize the formula to any D and N,

$$A_D^p=\prod_{n=p}^{D-1}A_{n+1}^n. \tag{101}$$

Thus

$$A_D^p=\left(\frac{S}{\pi C}\right)^{N/2}\frac{\Gamma[(D+1)/2]}{\Gamma[(D-N+1)/2]}, \tag{102}$$

where $N=D-p$.

F. Quantitative results

It is shown in Appendix G that

$$\frac{S}{C}=\frac{1}{\eta^2}\frac{D+2+\beta}{4v^2}\frac{\beta+1}{\beta}. \tag{103}$$

In the special cases of $N=1,2$, one can substitute for v^2 from Eq. (75) to obtain

TABLE V. Comparison between theoretical and numerical simulation values of the domain-wall defect scaling density in Minkowski space, FRW radiation, and FRW matter-dominated universes ($\alpha=0,1,2$, respectively) in two and three dimensions. The numerical values are taken from [25].

	$D=3$ $A_3^2=\frac{2}{\pi}\sqrt{\frac{S}{C}}$		$D=2$ $A_2^1=\frac{1}{2}\sqrt{\frac{S}{C}}$	
α	Theory	Simulation	Theory	Simulation
0	$1.91\eta^{-1}$	$0.88(0.14)\cdot\eta^{-1.00(0.03)}$	$1.11\eta^{-1}$	$0.77(0.23)\cdot\eta^{-0.99(0.03)}$
1	$2.02\eta^{-1}$	$0.93(0.13)\cdot\eta^{-0.99(0.01)}$	$1.18\eta^{-1}$	$0.93(0.17)\cdot\eta^{-1.00(0.02)}$
2	$2.16\eta^{-1}$	$0.96(0.12)\cdot\eta^{-1.00(0.01)}$	$1.24\eta^{-1}$	$1.15(0.23)\cdot\eta^{-0.99(0.01)}$

$$\frac{S}{C}=\begin{cases}\dfrac{1}{\eta^2}\dfrac{D(D+2+\beta)(1+\beta)}{2(D-1)\beta} & (N=1)\\[2ex] \dfrac{D[3(D+2)+2\beta](1+\beta)}{4(D-2)\beta} & (N=2).\end{cases} \tag{104}$$

It was shown in Sec. V E that the projected area density is proportional to $(S/C)^{N/2}$, and therefore classical scaling behavior for all defects is predicted, unless $\beta=0$. By classical scaling, we mean that the area density goes in proportion to conformal time as naive dimensional analysis would predict: a p-dimensional area density in D dimensions should be proportional to η^{-N}, as indeed it is in this theory. When $\beta=0$, as is the case for $(D-2)$-branes in $D=3$ (strings) in Minkowski space, and for $(D-1)$-branes in curvature-dominated FRW backgrounds, logarithmic violations to naive scaling appear.

We are also able to compute the scaling *amplitudes*, the coefficients of the relations between the area density and the appropriate power of time. These can then be compared with numerical simulations. The scaling projected comoving area densities for walls and strings in the radiation and matter eras are displayed in Tables V and VI. Note that in Table VI, the results for strings in matter and radiation-dominated universes have been taken from [22], who give *proper* area densities. These have been converted to projected area densities by dividing by $\langle(1-v^2)^{-1/2}\rangle$, where v is the average speed of the string. While not strictly the correct procedure, it gives a good enough answer given the uncertainty.

TABLE VI. Comparison between theoretical and numerical simulation values of the string scaling density in Minkowski space, FRW radiation, and FRW matter-dominated universes ($\alpha=0,1,2$, respectively) in three dimensions. The numerical values are taken from [21] and [22], with the latter converted from proper to projected area densities. The numerical fits in Minkowski space did not look for a logarithmic scaling violation.

	$D=3$ $A_3^1=\frac{S}{\pi C}$	
α	Theory	Simulation
0	$3.6\eta^{-2}\log(\eta\Lambda)$	$(11\pm1)\eta^{-2}$
1	$6.8\eta^{-2}$	$(18\pm6)\eta^{-2}$
2	$7.1\eta^{-2}$	$(14\pm4)\eta^{-2}$

To convert between comoving and physical areas, one uses the formula $A_{\rm phys}(t)=a^{-N}A(\eta)$, and the fact that $a(\eta)\eta=(1+\alpha)t$.

The scaling amplitudes differ from those obtained in numerical simulations of ϕ^4 theory [25] and of the Abelian-Higgs model [21,22], by a factor of about 2. However, it should be noted that there are large errors on the central value. The authors of Refs. [21,22] did not look for logarithmic scaling violations in the area density for strings in Minkowski space, choosing instead to fit to a simple power law. Finding such a violation is numerically very demanding, as a large dynamic range is required.

G. Biased initial conditions

One may also ask how the network behaves when a small bias is introduced into the initial conditions, that is, if $\langle u^A(x_i)\rangle=U^A$. In numerical experiments simulating biased initial conditions for strings [52] it is found that as the bias is increased the string passes through a transition from a phase with a finite fraction of percolating ‘‘infinite’’ string and with a power-law size distribution of loops, to one without infinite string, and with an exponential size distribution for the loops. In numerical simulations of domain walls [42,43], it is found that even for very small initial biases, for which the walls percolate, the system still evolves away from the percolating state and eventually the large walls break up and disappear. Similar behavior is well known in the study of quenches of condensed matter systems with a nonconserved order parameter [38,53–55].

The theoretical description of this behavior is fairly straightforward. Introducing a bias into the initial conditions for walls alters the Gaussian average of Sec. V C to

$$\langle A_D^1\rangle=\frac{1}{(2\pi)^{d/2}}\sqrt{\frac{S}{C}}\int du d^D\pi\,\delta(u)|\pi_i| \times e^{-1/2(u-U/\sqrt{C})^2-(1/2)\pi_i\delta^{ij}\pi_j}, \tag{105}$$

and hence

$$\langle A_D^1\rangle=\sqrt{\frac{S}{\pi C}}\frac{\Gamma[(D+1)/2]}{\Gamma(D/2)}e^{-(1/2)U^2/C}. \tag{106}$$

It is clear that this form is common to all defects in all dimensions: if $\langle A_D^1\rangle_0$ is the unbiased average area density, then the result of including a bias is

$$\langle A_D^1\rangle=\langle A_D^1\rangle_0 e^{-(1/2)U^2/C}, \tag{107}$$

with an obvious generalization to $N>1$. If the system is close to being self-similar at some initial time $\eta_{\rm i}$ when the magnitude of the bias is U and the fluctuation around that value $C(\eta_{\rm i})$, then one can predict that the area density goes as

$$\mathcal{A}\sim\eta^{-N/2}\exp(-cU^2\eta^D), \tag{108}$$

where c is a constant. One can also show that the time η_c at which the defect density falls to a fraction e^{-1} of its scaling value is

$$\eta_c=\eta_i[U^2/2C(\eta_i)]^{-1/D}. \tag{109}$$

The simulations by Larsson and White are consistent with Eqs. (106) and (109) in $D=2$, but do not have sufficiently good statistics in $D=3$ [43]. Coulson *et al.* [42] did not attempt a fit of the form (106) to their simulations.

VI. SCALING AND ENERGY LOSS

There is an apparent inconsistency in our conclusions for topological defect networks. We started by establishing that one could find approximate solutions to the field equations by finding extremal surfaces in space-time, and then constructing static solutions in coordinates which moved with the surface. We then showed that one could construct random surfaces in FRW space-times which are on average extremal, whose average area density obeyed a classical scaling law with conformal time η. The assumption is that there are defectlike solutions which are somehow close to static solutions centered on these random surfaces.

There is a problem with this picture: the defect area density decreases with time and therefore the energy in the form of defects also decreases. This energy must go somewhere, and an obvious channel is into propagating modes of the fields, or radiation. However, it is difficult to reconcile the idea that the network energy is lost into radiation with the perturbative approach to finding curved defect solutions, which assumes that the deviation from the comoving static solution decreases with the curvature of the defect.

Indeed, there is good numerical evidence that the perturbative approach works in certain cases [22,40]. The configurations where it has been tested are colliding traveling waves, either sinusoidal [22] or more complex [40]. When traveling waves are correctly prepared to the recipe laid down by Vachaspati [47], the collision does produce perturbations in the form of radiation, which is, however, exponentially suppressed with decreasing curvature.

It should be noted, however, that pure traveling waves are obtained from very special initial conditions. A random defect network is not prepared so carefully and it appears that it does radiate by an as yet poorly understood mechanism [21,22]. The radiation shows no sign of being exponentially suppressed with increasing curvature. What is clear is that one or more of the assumptions implicit in the perturbative approach to finding curved defect solutions must be violated. Two possibilities are that the extrinsic curvature is much larger than ξ^{-1}, maybe due to kinks, or there are nonlinear radiative processes, perhaps involving the breather modes [56].

VII. SUMMARY AND CONCLUSIONS

To summarize, this paper describes a new analytic technique for describing the dynamics of a random network of branes or topological defects, applicable to the brane gas universe or a cosmological phase transition. It is a relativistic version of a well-known approach in condensed matter physics, due to Ohta, Jasnow, and Kawasaki [38], which uses a mean-field approach to find approximate solutions to the Allen-Cahn equation for the motion of a surface representing a phase boundary. In the relativistic version, the surfaces are branes or defects obeying the Nambu-Goto equation (i.e., they have zero extrinsic curvature), but the condensed matter analogues can be obtained as a certain limit (see Appendixes D 1 and E 1), which acts as a check. In rederiving these condensed matter results, an expression for the length density of strings due to Toyoki and Honda [50] has been corrected (see Sec. V D).

In most cases the prediction is that the (generalized) area density of a p-dimensional defect in D dimensions should scale with conformal time as $\eta^{-(D-p)}$, with a scaling amplitude of $O(1)$. This appears to agree quantitatively with numerical simulations of domain walls [25,43]. In certain cases, such as strings in $D=3$, there is a prediction of a logarithmic violation of the naive scaling law. There are further predictions for defects with biased initial conditions, for strings in 3D, and for $(D-1)$- and $(D-2)$-branes which would be interesting to test.

From the point of view of the brane gas universe, it would be interesting to look at 1-, 2-, and 5-branes in higher dimensions. One of the most interesting features of the brane gas scenario is that it offers and explanation of why the Universe has three large dimensions: strings do not generically interact with each other in more than three dimensions, and so winding modes can never decay. It is only a three-dimensional subspace, where the winding modes can interact with each other and annihilate, which can expand and become large. It follows from this idea that strings cannot scale in more than three space dimensions, as there is no opportunity for the initial winding modes to break up into closed loops in the conventional picture of energy loss by a string network. It is therefore important to see whether the theoretical techniques presented in this paper predict scaling for strings in higher dimensions.

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APPENDIX A: PRELIMINARIES

Define a projector onto the rank N antisymmetric tensors (which is also an identity operator for those tensors),

$$\delta^{\nu_1\cdots\nu_N}_{\mu_1\cdots\mu_N}=\frac{1}{N!}(\delta^{\nu_1}_{\mu_1}\ldots\delta^{\nu_N}_{\mu_N}+\text{signed perms on }\ \nu_i). \tag{A1}$$

This projector has the properties

$$\delta^{\nu_1\cdots\nu_N}_{\mu_1\cdots\mu_N}=\frac{1}{N!(d-N)!}\epsilon_{\mu_1\cdots\mu_N\mu_{N+1}\cdots\mu_d}\times\epsilon^{\nu_1\cdots\nu_N\mu_{N+1}\cdots\mu_d}, \tag{A2}$$

$$\delta^{\nu_1\cdots\nu_N}_{\mu_1\cdots\mu_N}\delta^{\rho_1\cdots\rho_N}_{\nu_1\cdots\nu_N}=\delta^{\rho_1\cdots\rho_N}_{\mu_1\cdots\mu_N}, \tag{A3}$$

$$\delta^{\mu_1\cdots\mu_N}_{\mu_1\cdots\mu_N}=\frac{d!}{N!(d-N)!}. \tag{A4}$$

Define the matrix $h^{AB}=\partial_\mu u^A\partial_\nu u^B g^{\mu\nu}$. Then

$$\det h=\frac{1}{N!}g^{\mu_1\nu_1}\cdots g^{\mu_N\nu_N}\partial_{\mu_1}u^{A_1}\partial_{\nu_1}u^{B_1}\cdots\partial_{\mu_N}\times u^{A_N}\partial_{\nu_N}u^{B_N}\epsilon_{A_1\cdots A_N}\epsilon_{B_1\cdots B_N}. \tag{A5}$$

Define the antisymmetric rank N tensor

$$F_{\mu_1\cdots\mu_N}=\partial_{\mu_1}u^{A_1}\cdots\partial_{\mu_N}u^{A_N}\epsilon_{A_1\cdots A_N}. \tag{A6}$$

Then we may write

$$\det h=\frac{1}{N!}F_{\nu_1\cdots\nu_N}F^{\nu_1\cdots\nu_N}. \tag{A7}$$

Note that

$$\frac{\partial}{\partial g^{\mu\nu}}\det h=\frac{\partial_\mu u^{A_1}\partial_\nu u^{B_1}}{(N-1)!}(\partial u^{A_2}\cdot\partial u^{B_2}\cdots\partial u^{A_N}\cdot\partial u^{B_N}\times\epsilon_{A_1\cdots A_N}\epsilon_{B_1\cdots B_N})$$
$$=\partial_\mu u^{A_1}\partial_\nu u^{B_1}\cdot h_{A_1B_1}\det h. \tag{A8}$$

APPENDIX B: AVERAGING THE EXTREMAL SURFACE EQUATION

In a general space-time, the equation for a D-dimensional surface with zero extrinsic curvature is

$$(g^{\mu\nu}-h_{AB}\partial^\mu u^A\partial^\nu u^B)(\partial_\mu\partial_\nu u^C-\Gamma^\tau_{\mu\nu}\partial_\tau u^C)=0, \tag{B1}$$

where

$$h_{AB}=(h^{AB})^{-1}=\frac{1}{(N-1)!}\frac{1}{\det h}(\partial u^{A_2}\cdot\partial u^{B_2}\cdots\partial u^{A_N}\cdot\partial u^{B_N}\times\epsilon_{A_1\cdots A_N}\epsilon_{B_1\ldots B_N}).$$

The surfaces of constant u^C satisfying this equation have $K^C=0$. Note that the following is the projector onto the tangent space of the surface of constant u^C:

$$P^{\mu\nu}_{\parallel}=g^{\mu\nu}-h_{AB}\partial^\mu u^A\partial^\nu u^B. \tag{B2}$$

Thus, if we write $v^C=\partial u^C$ as the coordinate vectors normal to the surfaces of constant u^C, we can express the equation as

$$P^{\mu\nu}_{\parallel}\nabla_\mu v^C_\nu=0. \tag{B3}$$

Recalling the identity (A8) we see that the following equation holds:

$$\left(g^{\mu\nu}-\frac{\partial}{\partial g_{\mu\nu}}\right)[\det h(\partial_\mu\partial_\nu u^C-\Gamma^\tau_{\mu\nu}\partial_\tau u^C)]=0. \tag{B4}$$

Hence, in order to obtain the equations for surfaces whose *average* extrinsic curvature is zero, we need to average the quantities $\det h\,\partial_\mu\partial_\nu u^C$ and $\det h\,\partial_\tau u^C$.

1. The Gaussian average $\langle\det h\ \partial_\mu\partial_\nu u^C\rangle$

Exploiting its antisymmetry, we may rewrite the tensor $F_{\mu_1\ldots\mu_N}$ as

$$F_{\mu_1\cdots\mu_N}=N!\,\delta^{\nu_1\cdots\nu_N}_{\mu_1\cdots\mu_N}\partial_{\nu_1}u^1\cdots\partial_{\nu_N}u^N. \tag{B5}$$

Hence the determinant becomes

$$\det h=N!\,\delta^{\nu_1\cdots\nu_N}_{\mu_1\cdots\mu_N}\partial^{\mu_1}u^1\partial_{\nu_1}u^1\cdots\partial^{\mu_N}u^N\partial_{\nu_N}u^N. \tag{B6}$$

We introduce $m^A_{\mu\nu}=\partial_\mu u^A\partial_\nu u^A$ (with no implied summation), which is an unnormalized projector orthogonal to the surfaces of constant u^A. Then

$$\det h=N!\,\delta^{\mu_1\cdots\mu_N\nu_1\cdots\nu_N}m^1_{\mu_1\nu_1}\cdots m^N_{\mu_N\nu_N}. \tag{B7}$$

Hence

$$\langle\det h\partial_\mu\partial_\nu u^C\rangle=\langle\det h\rangle\partial_\mu\partial_\nu u^C+N!\,\delta^{\mu_1\cdots\mu_N\nu_1\cdots\nu_N}\times\langle m^1_{\mu_1\mu_2}\cdots\widehat{m^C_{\mu_C\nu_C}}\cdots m^N_{\mu_N\nu_N}\rangle\times\langle m^C_{\mu_C\nu_C}\partial_\mu\partial_\nu u^C\rangle,$$

with no implied summation on the index C, and the wide hat symbol is used to denote a term removed from the product inside the angle brackets. We now use the relations

$$\langle\partial_\mu u^A\partial_\nu u^B\rangle=M_{\mu\nu}\delta^{AB}, \tag{B8}$$

$$\langle\partial_\rho u^A\partial_\mu\partial_\nu u^B\rangle=\gamma_{\rho\mu\nu}\delta^{AB}, \tag{B9}$$

$$\langle m^C_{\mu_1\nu_1}\partial_\mu\partial_\nu u^C\rangle=\gamma_{\mu_1\mu\nu}\partial_{\nu_1}u^C+\gamma_{\nu_1\mu\nu}\partial_{\mu_1}u^C \tag{B10}$$

from which we can immediately derive

$$\langle\det h\rangle=N!\,\delta^{\mu_1\cdots\mu_N\nu_1\cdots\nu_N}M_{\mu_1\nu_1}\cdots M_{\mu_N\nu_N} \tag{B11}$$

and

$$N!\,\delta^{\mu_1\cdots\mu_N\nu_1\cdots\nu_N}\langle m^1_{\mu_1\nu_1}\cdots\widehat{m^C_{\mu_C\nu_C}}\cdots m^N_{\mu_N\nu_N}\rangle \times\langle m^C_{\mu_C\nu_C}\partial_\mu\partial_\nu u^C\rangle =\frac{2}{N}\frac{\partial}{\partial M_{\rho\sigma}}\langle \det h\rangle\gamma_{\rho\mu\nu}\partial_\sigma u^C. \tag{B12}$$

2. The Gaussian average $\langle \det h\,\partial_\tau u^C\rangle$

It follows from the previous section that

$$\langle \det h\,\partial_\tau u^C\rangle=\langle \det h\rangle\partial_\tau u^C+2N!\,\delta^{\mu_1\cdots\mu_N\nu_1\cdots\nu_N}\times M_{\mu_1\nu_1}\cdots\widehat{M_{\mu_C\nu_C}}\cdots M_{\mu_N\nu_N}M_{\mu_C\tau}\partial_{\nu_C}u^C. \tag{B13}$$

3. More definitions

We define a kind of determinant $\overline{\det}$ through the relation

$$\overline{\det}\,M=N!\,\delta^{\mu_1\cdots\mu_N\nu_1\cdots\nu_N}M_{\mu_1\nu_1}\cdots M_{\mu_N\nu_N}=\langle \det h\rangle. \tag{B14}$$

We can therefore define a cofactor for $M_{\mu\nu}$, which we denote $\overline{M}^{\mu\nu}$, through

$$\overline{M}^{\mu\nu}=N!\,\delta^{\mu\mu_2\cdots\mu_N\nu\nu_2\cdots\nu_N}M_{\mu_2\nu_2}\cdots M_{\mu_N\nu_N}/\overline{\det}\,M. \tag{B15}$$

Thus we may write

$$\langle \det h\,\partial_\mu\partial_\nu u^C\rangle=\overline{\det}\,M\left(\partial_\mu\partial_\nu u^C+\frac{2}{N}\overline{M}^{\rho\sigma}\gamma_{\rho\mu\nu}\partial_\sigma u^C\right) \tag{B16}$$

and

$$\langle \det h\,\partial_\mu\partial_\nu u^C\rangle=\overline{\det}\,M\left(\partial_\mu\partial_\nu u^C+\frac{2}{N}\overline{M}^{\rho\sigma}M_{\rho\tau}\partial_\sigma u^C\right). \tag{B17}$$

APPENDIX C: THE MEAN-FIELD ZERO CURVATURE EQUATION

Putting the results of Appendix B together, we find that the Gaussian averaged equations for zero extrinsic curvature surfaces is

$$\left(g^{\mu\nu}-\frac{\partial}{\partial g_{\mu\nu}}\right)\overline{\det}\,M\left[\partial_\mu\partial_\nu u^C+\frac{2}{N}\overline{M}^{\kappa\tau}\gamma_{\kappa\mu\nu}\partial_\tau u^C -\Gamma^\tau_{\mu\nu}\left(\partial_\tau u^C+\frac{2}{N}\overline{M}^{\kappa\lambda}M_{\lambda\tau}\partial_\lambda u^C\right)\right]=0. \tag{C1}$$

For future convenience we will break this equation down into four terms:

$$T_A=\left(g^{\mu\nu}-\frac{\partial}{\partial g_{\mu\nu}}\right)\overline{\det}\,M\,\partial_\mu\partial_\nu u^C, \tag{C2}$$

$$T_B=\frac{2}{N}\left(g^{\mu\nu}-\frac{\partial}{\partial g_{\mu\nu}}\right)\overline{\det}\,M\overline{M}^{\kappa\lambda}\gamma_{\kappa\mu\nu}\partial_\lambda u^C, \tag{C3}$$

$$T_C=\left(g^{\mu\nu}-\frac{\partial}{\partial g_{\mu\nu}}\right)\overline{\det}\,M\Gamma^\lambda_{\mu\nu}\partial_\lambda u^C, \tag{C4}$$

$$T_D=\frac{2}{N}\left(g^{\mu\nu}-\frac{\partial}{\partial g_{\mu\nu}}\right)\overline{\det}\,M\overline{M}^{\kappa\tau}M_{\kappa\lambda}\Gamma^\lambda_{\mu\nu}\partial_\tau u^C. \tag{C5}$$

Before reducing this equation further in cases of definite N we will need the following explicit expressions for the correlation functions $M_{\mu\nu}$ and $\gamma_{\kappa\mu\nu}$, consistent with the spatial $O(D)$ symmetry:

$$M_{\mu\nu}=\begin{pmatrix} T & 0\\ 0 & S\delta_{mn}\end{pmatrix}, \tag{C6}$$

$$\gamma_{000}=\tfrac{1}{2}\dot{T},\qquad \gamma_{0mn}=-\tfrac{1}{2}\dot{S}\delta_{mn},\qquad \gamma_{m0n}=\gamma_{mn0}=\tfrac{1}{2}\dot{S}\delta_{mn}. \tag{C7}$$

Note that

$$\gamma_{\mu\nu\rho}=\tfrac{1}{2}(M_{\mu\nu,\rho}+M_{\mu\rho,\nu}-M_{\nu\rho,\mu}). \tag{C8}$$

We also need the Christoffel symbol for a flat FRW background, which has metric $g_{\mu\nu}=a^2(\eta)\eta_{\mu\nu}$, where $\eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$ is the Minkowski metric. It is

$$\Gamma^\lambda_{\mu\nu}=\frac{\dot a}{a}(\delta^\lambda_\mu\delta^0_\nu+\delta^\lambda_\nu\delta^0_\mu-\eta_{\mu\nu}\eta^{\lambda 0}). \tag{C9}$$

APPENDIX D: ZERO CURVATURE FOR $N=1$

The simplest case is with one coordinate field u, which is appropriate for domain walls in three dimensions. Here we have

$$\overline{\det}\,M=g^{\rho\sigma}M_{\rho\sigma}=M, \tag{D1}$$

$$\overline{\det}\,M\overline{M}^{\kappa\lambda}=g^{\kappa\lambda}, \tag{D2}$$

$$\overline{\det}\,M\overline{M}^{\kappa\lambda}M_{\kappa\tau}=M^\lambda{}_\tau. \tag{D3}$$

The required derivatives with respect to the metric are also easily found:

$$\frac{\partial}{\partial g_{\mu\nu}}\overline{\det}\,M=M^{\mu\nu}, \tag{D4}$$

$$\frac{\partial}{\partial g_{\mu\nu}}\overline{\det}\,M\overline{M}^{\kappa\lambda}=g^{\mu\kappa}g^{\nu\lambda}. \tag{D5}$$

Using the explicit form of $M_{\mu\nu}$ [Eq. (C6)] we can also write down

$$M=a^{-2}(-T+DS). \tag{D6}$$

Term T_A. Using Eqs. (D1) and (D4) we find

$$T_A=(g^{\mu\nu}M-M^{\mu\nu})\partial_\mu\partial_\nu u. \tag{D7}$$

Using Eqs. (C6) and (D6), this simplifies to

$$T_A=-a^{-4}DS\left[\ddot{u}-\left(\frac{D-1}{D}-\frac{T}{DS}\right)\nabla^2 u\right]. \tag{D8}$$

Term T_B. Using Eqs. (D2) and (D4), we have that

$$T_B=2(g^{\mu\nu}g^{\kappa\lambda}-g^{\mu\kappa}g^{\nu\lambda})\gamma_{\kappa\mu\nu}\partial_\lambda u. \tag{D9}$$

Using Eq. (C7) one can quickly show that

$$T_B=2a^{-4}D\dot{S}\dot{u}. \tag{D10}$$

Term T_C. Using Eqs. (D1), (D4), and (C9), one finds

$$\begin{aligned}T_C&=(g^{\mu\nu}M-M^{\mu\nu})\Gamma^\lambda_{\mu\nu}\partial_\lambda u\\&=[M(1-D)g^{0\lambda}-2M^{0\lambda}+Mg^{0\lambda}]\partial_\lambda u\\&=a^{-4}DS[D-2-(T/S)]\dot{u}.\end{aligned} \tag{D11}$$

Term T_D. Using Eqs. (D3) and (D5), we find that

$$\begin{aligned}T_D&=2(g^{\mu\nu}g^{\kappa\tau}-g^{\mu\kappa}g^{\nu\tau})M_{\kappa\lambda}\Gamma^\lambda_{\mu\nu}\partial_\tau u\\&=2[(2-D)g^{0\lambda}g^{\kappa\tau}-g^{\kappa\lambda}g^{0\tau}-g^{\tau\lambda}g^{0\kappa}]M_{\kappa\lambda}\left(\frac{\dot{a}}{a}\right)\partial_\tau u\\&=2[(1-D)M^{0\tau}-g^{0\tau}M]\left(\frac{\dot{a}}{a}\right)\partial_\tau u.\end{aligned} \tag{D12}$$

Substituting the known forms of $M^{0\tau}$ and M, we arrive at

$$T_D=2a^{-4}DS[1-(T/S)]\dot{u}. \tag{D13}$$

The equation for *u*

We can now construct the Gaussian averaged or "mean-field" equations of motion satisfied by the coordinate function u, which is applicable to domain walls when $D=3$. The equations are made from the four terms we calculated in the previous section: $T_A+T_B-T_C-T_D=0$. Putting them all together, and dividing by the factor $a^{-4}DS$, we find

$$\ddot{u}^c+\frac{\mu}{\eta}\dot{u}^C-v^2\nabla^2u^C=0, \tag{D14}$$

with

$$\mu=[D-3(T/S)]\left(\eta\frac{\dot{a}}{a}\right)-2\left(\eta\frac{\dot{S}}{S}\right), \tag{D15}$$

$$v^2=[(D-1)-(T/S)]/D. \tag{D16}$$

We can recover the well-known Allen-Cahn equation for the overdamped motion of domain walls by identifying the damping constant $\Gamma=a/\dot{a}$, and neglecting T/S, $\eta\dot{S}/S$, and the second-order time derivative of u:

$$\dot{u}=\Gamma\frac{D-1}{D^2}\nabla^2 u. \tag{D17}$$

APPENDIX E: ZERO CURVATURE FOR $N=2$

When $N=2$, the expressions for the various quantities involving M in the equations of motion are still straightforward to evaluate:

$$\begin{aligned}\overline{\det}M&=(g^{\mu_1\nu_1}g^{\mu_2\nu_2}-g^{\mu_1\nu_2}g^{\mu_2\nu_1})M_{\mu_1\nu_1}M_{\mu_2\nu_2}\\&=(M^2-M^{\mu\nu}M_{\mu\nu}),\end{aligned} \tag{E1}$$

$$\begin{aligned}\overline{\det}M\overline{M}^{\kappa\lambda}&=(g^{\kappa\lambda}g^{\mu_2\nu_2}-g^{\kappa\nu_2}g^{\mu_2\lambda})M_{\mu_2\nu_2}\\&=g^{\kappa\lambda}M-M^{\kappa\lambda},\end{aligned} \tag{E2}$$

$$\begin{aligned}\overline{\det}M\overline{M}^{\kappa\lambda}M_{\kappa\tau}&=(g^{\kappa\lambda}g^{\mu_2\nu_2}-g^{\kappa\nu_2}g^{\mu_2\lambda})M_{\mu_2\nu_2}M_{\kappa\tau}\\&=M^\lambda_\tau M-M^{\kappa\lambda}M_{\kappa\tau}.\end{aligned} \tag{E3}$$

We also need to differentiate two of these expressions with respect to the metric $g_{\mu\nu}$:

$$\frac{\partial}{\partial g_{\mu\nu}}\overline{\det}M=2(M^{\mu\nu}M-M^{\mu\lambda}M^\nu_\lambda), \tag{E4}$$

$$\begin{aligned}\frac{\partial}{\partial g_{\mu\nu}}\overline{\det}M\overline{M^{\kappa\lambda}}&=(g^{\mu\kappa}g^{\nu\lambda}M+g^{\kappa\lambda}M^{\mu\nu}\\&\quad-g^{\mu\kappa}M^{\nu\lambda}-g^{\nu\lambda}M^{\mu\kappa}).\end{aligned} \tag{E5}$$

Introducing a further piece of notation, that $M\cdot M=M^\mu{}_\nu M^\nu{}_\mu$, we can show that

$$M\cdot M=a^{-4}(T^2+DS^2), \tag{E6}$$

$$\begin{aligned}\overline{\det}M&=M^2-M\cdot M\\&=a^{-4}DS^2[(D-1)-2T/S)],\end{aligned} \tag{E7}$$

$$M^{\mu\nu}M-M^{\mu\lambda}M_\lambda{}^\nu=-a^{-6}S^2\times\begin{pmatrix}DT/S & 0\\ 0 & [(D-1)-(T/S)]\delta_{mn}\end{pmatrix} \tag{E8}$$

$$g^{\mu\nu}\gamma_{\kappa\mu\nu}=\tfrac{1}{2}a^{-2}\delta^0_\kappa(\dot{T}+D\dot{S}). \tag{E9}$$

Term T_A. Using Eq. (E4), we find

$$T_A=(M^2-M\cdot M)\partial^2u^C-2(M^{\mu\nu}M-M^{\mu\lambda}M_\lambda{}^\nu)\partial_\mu\partial_\nu u^C. \tag{E10}$$

Hence, using Eqs. (E7) and (E8),

$$T_A=a^{-6}D(D-1)S^2\left[\ddot{u}^C-\left(\frac{D-2}{D}-\frac{2}{D}\frac{T}{S}\right)\nabla^2u^C\right]. \tag{E11}$$

Term T_B. Using Eqs. (E2) and (E9), we find first that

$$g^{\mu\nu}\overline{\det} M\overline{M}^{\kappa\lambda}\gamma_{\kappa\mu\nu}\partial_\lambda u^C = -\tfrac{1}{2}a^{-6}DS(\dot{T}+D\dot{S})\dot{u}^C. \tag{E12}$$

Using Eqs. (E5) and (E9), we find

$$\frac{\partial}{\partial g_{\mu\nu}}\overline{\det} M\overline{M}^{\kappa\lambda}\gamma_{\kappa\mu\nu}\partial_\lambda u^C = \frac{1}{2}a^{-6}DS[(D-2)\dot{S}-\dot{T}]\dot{u}^C. \tag{E13}$$

Putting the two expressions together, we find

$$T_B = -a^{-6}D(D-1)S^2\left(\frac{\dot{S}}{S}\right)\dot{u}^C. \tag{E14}$$

Term T_C. From Eqs. (E1) and (E4), we can immediately write down

$$T_C = [g^{\mu\nu}(M^2 - M\cdot M) - 2M^{\mu\nu}M + 2M^{\mu\kappa}M_\kappa{}^\nu]\Gamma^\lambda_{\mu\nu}\partial_\lambda u^C. \tag{E15}$$

Using Eq. (C9), we find

$$T_C = a^{-2}[(1-D)(M^2 - M\cdot M) + 4M^{00}(M^{00}-M) + 2(M^2 - M\cdot M)]\left(\frac{\dot{a}}{a}\right)\dot{u}^C. \tag{E16}$$

With Eqs. (E1) and (E8), we arrive at

$$T_C = -a^{-6}D(D-1)S^2[(D-3) - 2(T/S)]\left(\frac{\dot{a}}{a}\right)\dot{u}^C. \tag{E17}$$

Term T_D. For this last term, we begin with

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = (1-D)\left(\frac{\dot{a}}{a}\right)g^{0\lambda}. \tag{E18}$$

Hence from Eq. (E3) we see that

$$g^{\mu\nu}\ \overline{\det} M\overline{M}^{\kappa\tau}M_{\kappa\lambda}\Gamma^\lambda_{\mu\nu}\partial_\lambda u^C = a^{-2}(g^{00}M - M^{00})M^0_0(1-D)\left(\frac{\dot{a}}{a}\right)\dot{u}^C = a^{-6}D(D-1)ST\left(\frac{\dot{a}}{a}\right)\dot{u}^C. \tag{E19}$$

The second term in expression (C5) is more complicated. From Eqs. (E5) and (C9), we have

$$g^{\mu\rho}g^{\nu\sigma}\frac{\partial}{\partial g^{\rho\sigma}}\overline{\det} M\overline{M}^{\kappa\tau}M_{\kappa\lambda}\Gamma^\lambda_{\mu\nu} = (g^{\mu\kappa}g^{\nu\tau}M + g^{\kappa\tau}M^{\mu\nu} - g^{\mu\kappa}M^{\nu\tau} - g^{\nu\tau}M^{\mu\kappa})M_{\kappa\lambda}\Gamma^\lambda_{\mu\nu}. \tag{E20}$$

After some algebra, we find

$$\frac{\partial}{\partial g_{\mu\nu}}\ \overline{\det} M\overline{M}^{\kappa\tau}M_{\kappa\lambda}\Gamma^\lambda_{\mu\nu} = [g^{0\tau}(M^2 - M\cdot M) - 2M^{0\lambda}M + 2M^\tau{}_\lambda M^{0\lambda}]\left(\frac{\dot{a}}{a}\right) = a^{-6}D(D-1)S^2\delta^\tau_0\left(\frac{\dot{a}}{a}\right). \tag{E21}$$

Subtracting Eq. (E21) multiplied by $\partial_\tau u^C$ from Eq. (E19), we arrive at

$$T_D = -a^{-6}D(D-1)S^2[1-(T/S)]\left(\frac{\dot{a}}{a}\right)\dot{u}^C. \tag{E22}$$

1. The equation for $u^C(N=2)$

We can now construct the Gaussian averaged or ''mean field'' equations of motion satisfied by the coordinate functions u^C in the case $N=2$, appropriate for strings in three spatial dimensions. The equations are made from the four terms we calculated in the previous section: $T_A + T_B - T_C - T_D = 0$. Putting them all together, and dividing by the common factor $a^{-6}D(D-1)S^2$, we find

$$\ddot{u}^c + \frac{\mu}{\eta}\dot{u}^C - v^2\nabla^2 u^C = 0, \tag{E23}$$

with

$$\mu = [D-2-3(T/S)]\left(\eta\frac{\dot{a}}{a}\right) - \left(\eta\frac{\dot{S}}{S}\right), \tag{E24}$$

$$v^2 = [(D-2) - 2(T/S)]/D. \tag{E25}$$

We can recover the results of Toyoki and Honda for the motion of overdamped strings in $D=3$ by setting their diffusion constant $\Gamma = a/\dot{a}$, and neglecting T/S and $\eta\dot{S}/S$. In this case, we get

$$\dot{u}^C = \frac{\Gamma}{3}\nabla^2 u^C, \tag{E26}$$

which is identical to their equation (3.10).

APPENDIX F: PROBABILITY DISTRIBUTION FOR F_{ij}

The definition of the antisymmetric tensor F_{ij} is

$$F_{ij} = \partial_i u^A \partial_j u^B \epsilon_{AB}. \tag{F1}$$

The probability distribution for F_{ij} is therefore constructed from the Gaussian probability distribution of $\partial_i u^A$. F_{ij} is antisymmetric, so we need only consider half of the nonzero elements, e.g., by imposing $i<j$. Moreover, it is convenient to scale out the variance of $\partial_i u^A$, defining variables π^A_i and f_{ij} as follows:

$$\partial_i u^A = \sqrt{S}\pi^A_i, \quad F_{ij} = Sf_{ij}, \tag{F2}$$

where

$$\langle \partial_i u^A(x) \partial_j u^B(x) \rangle = S(t) \delta_{ij} \delta^{AB}. \tag{F3}$$

Hence, the probability distribution for f_{ij} is

$$P(f_{ij})|_{i<j} = \int \prod_A \frac{d^D \pi_i^A}{(2\pi)^{D/2}} e^{-(1/2)\pi_i^A \pi_i^A} \times \delta(f_{ij} - \pi_i^A \pi_j^B \epsilon_{AB})|_{i<j}. \tag{F4}$$

Using the Fourier representation of the δ function,

$$P(f_{ij})|_{i<j} = \int \frac{d^P k}{(2\pi)^P} \int \prod_A \frac{d^D \pi_i^A}{(2\pi)^{D/2}} \times e^{-(1/2)\pi_i^A \pi_i^A + i\Sigma_{i<j} k^{ij}(f_{ij} - \pi_i^A \pi_j^B \epsilon_{AB})}, \tag{F5}$$

where $P = D(D-1)/2$ is the dimension of k_{ij}.

We now do the π_i^A integrations in turn, starting with the highest A. First, note that

$$P(f_{ij})|_{i<j} = \int \frac{d^P k}{(2\pi)^P} e^{i\Sigma_{i<j} k^{ij} f_{ij}} \int \prod_A \frac{d^D \pi_i^A}{(2\pi)^{D/2}} \times e^{-(1/2)\pi_i^A \pi_i^A - ik^{ij}\pi_i^1 \pi_j^2}, \tag{F6}$$

where there is now no restriction on the sum over i,j in the second exponential. Second, define the variable $q^j = k^{ij}\pi_i^1$. Then we have to evaluate the integral

$$I(k_{ij}) = \int \prod_A \frac{d^D \pi_i^A}{(2\pi)^{D/2}} e^{-(1/2)\pi_i^A \pi_i^A - iq^j \pi_j^2}. \tag{F7}$$

Doing the π_i^2 integral first, this is

$$I(k_{ij}) = \int \frac{d^D \pi_i^1}{(2\pi)^{D/2}} e^{-(1/2)\pi_i^1 M_{ij} \pi_j^1}, \tag{F8}$$

$$= \det{}^{-1/2} M, \tag{F9}$$

where

$$M_{ij} = \delta_{ij} + k_{ik} k_{jk}. \tag{F10}$$

At this point we specialize to 3D, where we can write

$$k_{ij} = \epsilon_{ijk} p_k. \tag{F11}$$

Hence

$$M_{ij} = \delta_{ij}(1 + p^2) - p_i p_j. \tag{F12}$$

The eigenvalues of this matrix are $1 + p^2$ (twice) and 1, so

$$\det{}^{-1/2} M = (1 + p^2)^{-1}. \tag{F13}$$

Thus the probability distribution of f_{ij} is

$$P(f_{ij})|_{i<j} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{1 + p^2} e^{i\Sigma_{i<j} \epsilon^{ijk} f_{ij} p_k}. \tag{F14}$$

In 3D we can replace f_{ij} by $\phi_k = \Sigma_{i<j} \epsilon^{ijk} f_{ij}$, and the integral may be easily evaluated to give

$$P(\phi_k) = \frac{1}{4\pi\phi} e^{-\phi}, \tag{F15}$$

where $\phi^2 = \phi_k \phi_k$.

APPENDIX G: INTEGRAL FORMULAS AND CORRELATION FUNCTIONS

In this appendix, we perform the integrations necessary to evaluate the functions C, S, and T, defined in Sec. IV B, which we repeat here for convenience,

$$\delta^{AB} C(\eta) = \langle u^A(x) u^B(x) \rangle,$$

$$\delta^{AB} M_{\mu\nu}(\eta) = \langle \partial_\mu u^A(x) \partial_\nu u^B(x) \rangle,$$

with

$$M_{\mu\nu} = \begin{pmatrix} T(\eta) & 0 \\ 0 & \delta_{mn} S(\eta) \end{pmatrix}. \tag{G1}$$

We shall also evaluate $\dot{C}$ for the mixed correlator $\langle \partial_\mu u^A u^B \rangle$. We recall from Eq. (73) that the linearized solution for u^A with the correct boundary conditions is

$$u_{\mathbf{k}}^C(\eta) = A_{\mathbf{k}}^C \left(\frac{\eta}{\eta_{\rm i}} \right)^{(1-\mu)/2+\nu} \frac{J_\nu(kv\eta)}{(kv\eta)^\nu}, \tag{G2}$$

with $\nu = \pm(1-\mu)/2$. If we demand regularity and convergent integrals as $\eta \to 0$, we must take the negative sign here, as it will turn out that $\mu > 1$.

In order to calculate the two-point functions it is useful to define the following integral:

$$I(\rho, \sigma, \tau) \equiv \int_0^\infty dz z^{-\rho} J_\sigma(z) J_\tau(z), \tag{G3}$$

which has the value [57]

$$I(\rho, \sigma, \tau) = \frac{1}{2^\rho} \frac{\Gamma(\rho)\Gamma\left(\frac{\sigma+\tau-\rho+1}{2}\right)}{\Gamma\left(\frac{\rho-\sigma+\tau+1}{2}\right)\Gamma\left(\frac{\rho+\sigma+\tau+1}{2}\right)\Gamma\left(\frac{\rho+\sigma-\tau+1}{2}\right)}, \tag{G4}$$

provided $\mathrm{Re}(\sigma+\tau+1)>\mathrm{Re}(\rho)>0$. The first inequality comes from the condition that the integral be defined as $z\to 0$, and the second from requiring that it converge as $z\to\infty$. There is a simple pole at $\rho=0$. We can see that this comes from the $z^{-1/2}$ behavior of the Bessel functions as $z\to\infty$, and corresponds to a logarithmically divergent integral.

Defining the Fourier transform of the correlator C in the usual way through

$$C(\eta)=\int\frac{d^Dk}{(2\pi)^D}C_{\boldsymbol{k}}(\eta), \tag{G5}$$

we see from the solutions for u^A that

$$C(\eta)=\frac{1}{(v\eta)^D}\frac{\Omega_D}{(2\pi)^D}\int dz z^{D-1-2\nu}J_\nu^2(z)P_A(\boldsymbol{k}), \tag{G6}$$

where $z=kv\eta$, and $\Omega_D=2\pi^{D/2}/\Gamma(D/2)$ is the volume element of a $(D-1)$-sphere. We assume a power-law form for the power spectrum of $A_{\boldsymbol{k}}^A$,

$$P_A(\boldsymbol{k})=\frac{\sigma_{\mathrm{i}}(2\pi)^D}{\Omega_D\Lambda^D\Gamma(D+q)}\left(\frac{k}{\Lambda}\right)^q e^{-k/\Lambda}, \tag{G7}$$

where Λ is a high wave-number cutoff, satisfying $\Lambda v\eta\gg 1$ for all η of interest, and σ_i is the variance. Hence, defining $\beta=2\nu-D-1-q$,

$$C(\eta)=\frac{v^q}{(\Lambda v\eta)^{D+q}}\frac{\sigma_{\mathrm{i}}}{\Gamma(D+q)}I(2+\beta,\nu,\nu). \tag{G8}$$

Let us now calculate S from

$$DS(\eta)=\int\frac{d^Dk}{(2\pi)^D}k^2C_{\boldsymbol{k}}(\eta). \tag{G9}$$

One can straightforwardly show that

$$DS=\frac{v^q}{(\Lambda v\eta)^{D+q}}\frac{1}{(v\eta)^2}\frac{\sigma_{\mathrm{i}}}{\Gamma(D+q)}I(\beta,\nu,\nu). \tag{G10}$$

The correlation function T is obtained from

$$\delta^{AB}T=\int\frac{d^Dk}{(2\pi)^D}\langle\dot{u}_{\boldsymbol{k}}^A(\eta)\dot{u}_{-\boldsymbol{k}}^B(\eta)\rangle. \tag{G11}$$

Given the identity [57]

$$\frac{d}{dz}\left(\frac{J_\nu(z)}{z^\nu}\right)=-\frac{J_{\nu+1}(z)}{z^\nu}, \tag{G12}$$

one can show that

$$T=\frac{v^q}{(\Lambda v\eta)^{D+q}}\frac{1}{\eta^2}\frac{\sigma_{\mathrm{i}}}{\Gamma(D+q)}I(\beta,\nu+1,\nu+1). \tag{G13}$$

Note that the ratios S/C and T/S depend on the initial conditions only though the power q, which appears in β,

$$\frac{S}{C}=\frac{1}{D(v\eta)^2}\frac{I(\beta,\nu,\nu)}{I(2+\beta,\nu,\nu)}, \tag{G14}$$

$$\frac{T}{S}=Dv^2\frac{I(\beta,\nu+1,\nu+1)}{I(\beta,\nu,\nu)}. \tag{G15}$$

A little more algebra shows that

$$\frac{S}{C}=\frac{1}{\eta^2}\frac{D+2+\beta}{4v^2}\frac{\beta+1}{\beta}, \tag{G16}$$

$$\frac{T}{S}=Dv^2\frac{D+2}{D+2+2\beta}. \tag{G17}$$

Note that the ratio S/C appears to have a simple pole at $\beta=0$ [58]: however, when the cutoff is in place this is replaced by a logarithm, with

$$\frac{S}{C}\sim\frac{1}{\eta^2}\log(\Lambda v\eta). \tag{G18}$$

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