

Collective modes of an anisotropic quark-gluon plasma

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We analyze the collective modes of high-temperature QCD in the case when there is an anisotropy in the momentum-space distribution function for the gluons. We perform a tensor decomposition of the gluon self-energy and solve the dispersion relations for both stable and unstable modes. Results are presented for a class of anisotropic distribution functions which can be obtained by stretching or squeezing an isotropic distribution function along one direction in momentum space. We find that there are three stable modes and either one or two unstable modes, depending on whether the distribution function is stretched or squeezed. The presence of unstable modes which have exponential growth can lead to a more rapid thermalization and isotropization of the soft modes in a quark gluon plasma and therefore may play an important role in the dynamical evolution of a quark-gluon plasma.

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I. INTRODUCTION

In the ongoing ultrarelativistic heavy collision experiments at the BNL Relativistic Heavy Ion Collider (RHIC) and the upcoming experiments at the CERN Large Hadron Collider (LHC) the behavior of nuclear matter under extreme conditions will be studied. The hope of these experiments is to create temperatures which are high enough for nuclear matter to undergo a phase transition to a quark-gluon plasma (QGP). The quark-gluon plasma, if generated, is expected to expand, cool, and then hadronize in the final stage of its evolution. In this context, an outstanding question faced by experimentalists and theorists is whether or not the system will “thermalize” fast enough to allow a thermodynamic description of the system during the central part of its evolution.

In this paper we study the role of the collective modes of finite-temperature QCD in the thermalization, particularly the isotropization, of a finite-temperature QGP with anisotropic momentum-space distribution functions. This question has been addressed in previous papers in which the existence of instabilities of a QGP were studied. In Refs. [1–3] Mrówczyński discussed the existence of instabilities to chromomagnetic fluctuations with a particular orientation of the chromoelectric field and wave vector. In those papers Mrówczyński showed that there existed an instability which was the equivalent of the Weibel or filamentation instability in electrodynamics [4]. Weibel showed in his original paper that, within electrodynamics, unstable transverse modes exist in plasmas with anisotropic momentum distributions and he also derived their rate of growth in linear response theory. These types of instabilities are potentially very important to QGP evolution at RHIC or LHC due to the large amount of momentum-space anisotropy in the gluon distribution functions at $\tau \sim 1$ fm/c.

Mrówczyński and Randrup have recently performed phenomenological estimates of the growth rate of the instabilities for two types of anisotropic distribution functions [5]. They found that the degree of amplification of the Weibel instability is not expected to dominate the dynamics of a QGP; instead it is comparable to the contribution from elas-

tic Boltzmann collisions. However, they did point out that since a large number of the unstable modes could be excited then it is possible that their combined effect on the overall dynamics could be significant. In this paper we perform a detailed study of the hard-thermal-loop resummed gluon self-energy including a complete tensor decomposition of the self-energy, and identification of all stable and unstable collective modes.

In Sec. II we set up the framework used to obtain the hard-thermal-loop self-energy in a system with an anisotropic momentum space distribution. In Sec. III we present a tensor decomposition of the self-energy and dielectric tensors. In Sec. IV we work out the details of the tensor decomposition and give expressions for the self-energy “structure functions.” In Sec. V we discuss the static limit of the variational self-energy structure functions. In Sec. VI we use the tensor decomposition of the dielectric tensor to determine dispersion relations for all stable and unstable modes. In Sec. VII we present analytic expressions for the self-energy structure functions in the small-anisotropy limit. Finally, in Sec. VIII we present conclusions and an outlook for the application of the results found here. We provide a summary of our notational conventions and expressions for the various self-energy structure functions in two Appendixes.

II. HARD-THERMAL-LOOP SELF-ENERGY

We begin by repeating some of the steps necessary to derive the hard-thermal-loop resummed gluon self-energy within semiclassical transport theory [1–3]. Within this approach partons are described by their phase-space densities and their time evolution is given by Vlasov-type transport equations [6,7]. In this paper we will concentrate on the physics at the soft scale, $k \sim gT \ll T$, which is the first scale at which collective motion appears. At this scale the magnitude of the field fluctuations is $A \sim \sqrt{gT}$ and derivatives are of the scale $\partial_x \sim gT$. With this power counting a systematic truncation of the terms contributing to the transport equations for soft momenta can be realized.

At leading order in the coupling constant the color current, J^μ , induced by a soft gauge field, A^μ , with four-

momentum $K=(\omega, \mathbf{k})$ can be obtained by performing a covariant gradient expansion of the quark and gluon Wigner functions in mean-field approximation. The result is

$$J_{\text{ind}}^{\mu,a}(X) = g \int \frac{d^3p}{(2\pi)^3} V^\mu [2N_c \delta N^a(p, X) + N_f (\delta n_+^a(p, X) - \delta n_-^a(p, X))], \quad (1)$$

where $V^\mu=(1, \mathbf{k}/\omega)$ is the gauge field four-velocity, $\delta N^a(p, X)$ is the fluctuating part of the gluon density, and $\delta n_+^a(p, X)$ and $\delta n_-^a(p, X)$ are the fluctuating parts of the quark and anti-quark densities, respectively. Note that δN^a transforms as a vector in the adjoint representation ($\delta N^a \equiv \delta N^a T^a$) and δn_\pm^a transforms as a vector in the fundamental representation ($\delta n_\pm^a \equiv \delta n_\pm^a t^a$).

The quark and gluon density matrices above satisfy the following transport equations:

$$[V \cdot D_X, \delta n_\pm(p, X)] = \mp g V_\mu F^{\mu\nu}(p, X) \partial_\nu n_\pm(\mathbf{p}), \quad (2)$$

$$[V \cdot D_X, \delta N(p, X)] = -g V_\mu F^{\mu\nu}(p, X) \partial_\nu N(\mathbf{p}), \quad (3)$$

where is $D_X = \partial_X + igA(X)$ is the covariant derivative.

Solving the transport equations (2) and (3) for the fluctuations δN and δn_\pm gives the induced current via Eq. (1):

$$J_{\text{ind}}^\mu(X) = g^2 \int \frac{d^3p}{(2\pi)^3} V^\mu V^\alpha \partial_{(p)}^\beta f(\mathbf{p}) \int d\tau U(X, X - V\tau) \times F_{\alpha\beta}(X - V\tau) U(X - V\tau, X), \quad (4)$$

where $U(X, Y)$ is a gauge parallel transporter defined by the path-ordered integral

$$U(X, Y) = \mathcal{P} \exp \left[-ig \int_X^Y dZ_\mu A^\mu(Z) \right], \quad (5)$$

$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha - ig[A_\mu, A_\nu]$ is the gluon field strength tensor, and

$$f(\mathbf{p}) = 2N_c N(\mathbf{p}) + N_f [n_+(\mathbf{p}) + n_-(\mathbf{p})]. \quad (6)$$

Neglecting terms of subleading order in g (implying $U \rightarrow 1$ and $F_{\alpha\beta} \rightarrow \partial_\alpha A_\beta - \partial_\beta A_\alpha$) and performing a Fourier transform of the induced current to momentum space we obtain

$$J_{\text{ind}}^\mu(K) = g^2 \int \frac{d^3p}{(2\pi)^3} V^\mu \partial_{(p)}^\beta f(\mathbf{p}) \left(g_{\gamma\beta} - \frac{V_\gamma K_\beta}{K \cdot V + i\epsilon} \right) A^\gamma(K), \quad (7)$$

where ϵ is a small parameter that has to be sent to zero in the end.

From this expression of the induced current the self-energy is obtained via

$$\Pi^{\mu\nu}(K) = \frac{\delta J_{\text{ind}}^\mu(K)}{\delta A_\nu(K)}, \quad (8)$$

which gives

$$\Pi^{\mu\nu}(K) = g^2 \int \frac{d^3p}{(2\pi)^3} V^\mu \partial_{(p)}^\beta f(\mathbf{p}) \left(g_{\nu\beta} - \frac{V_\nu K_\beta}{K \cdot V + i\epsilon} \right). \quad (9)$$

This tensor is symmetric, $\Pi^{\mu\nu}(K) = \Pi^{\nu\mu}(K)$, and transverse, $K^\mu \Pi^{\mu\nu}(K) = 0$. Note that the same result can be obtained using diagrammatic methods if one assumes that the distribution function is symmetric under $\mathbf{p} \rightarrow -\mathbf{p}$ [8].

In the linear approximation the equation of motion for the gauge fields can be obtained by expressing the induced current in terms of the self-energy

$$J_{\text{ind}}^\mu(K) = \Pi^{\mu\nu}(K) A^\nu(K), \quad (10)$$

and plugging this into Maxwell's equation

$$-iK_\mu F^{\mu\nu}(K) = J_{\text{ind}}^\nu(K) + J_{\text{ext}}^\nu(K), \quad (11)$$

to obtain

$$[K^2 g^{\mu\nu} - K^\mu K^\nu + \Pi^{\mu\nu}(K)] A_\nu(K) = -J_{\text{ext}}^\nu(K), \quad (12)$$

where J_{ext}^ν is an external current. Using the gauge invariance of the self-energy we can write this in terms of a physical electric field by specifying a particular gauge. In the temporal axial gauge defined by $A_0=0$ we obtain

$$[(k^2 - \omega^2) \delta^{ij} - k^i k^j + \Pi^{ij}(K)] E^j(K) = [\Delta^{-1}(K)]^{ij} E^j(K) = i\omega J_{\text{ext}}^i(K). \quad (13)$$

Inverting the propagator allows us to determine the response of the system to the external source

$$E^i(K) = i\omega \Delta^{ij}(K) J_{\text{ext}}^j(K). \quad (14)$$

The dispersion relations for the collective modes can be obtained by finding the poles in the propagator $\Delta^{ij}(K)$.

III. TENSOR DECOMPOSITION

In this section we develop a tensor basis for an anisotropic system in which there is only one preferred direction. As mentioned above the self-energy is symmetric and transverse. As a result not all components of $\Pi^{\mu\nu}$ are independent and we can restrict our considerations to the spatial part of $\Pi^{\mu\nu}$, denoted Π^{ij} . We therefore need to construct a basis for a symmetric 3-tensor that—apart from the momentum k^i —also depends on a fixed anisotropy three-vector n^i , with $n^2=1$. Following Ref. [9] we first define the projection operator

$$A^{ij} = \delta^{ij} - k^i k^j / k^2, \quad (15)$$

and use it to construct $\tilde{n}^i = A^{ij} n^j$ which obeys $\tilde{n} \cdot k = 0$. With this we can construct the remaining three tensors

$$B^{ij} = k^i k^j / k^2, \quad (16)$$

$$C^{ij} = \tilde{n}^i \tilde{n}^j / \tilde{n}^2, \quad (17)$$

$$D^{ij} = k^i \tilde{n}^j + k^j \tilde{n}^i. \quad (18)$$

Any symmetric 3-tensor \mathbf{T} can now be decomposed into the basis spanned by the four tensors $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and \mathbf{D}

$$\mathbf{T} = a \mathbf{A} + b \mathbf{B} + c \mathbf{C} + d \mathbf{D}. \quad (19)$$

Furthermore, the inverse of any such tensor is then given as

$$\mathbf{T}^{-1} = a^{-1} \mathbf{A} + \frac{(a+c)\mathbf{B} - a^{-1}(bc - \tilde{n}^2 k^2 d^2)\mathbf{C} - d\mathbf{D}}{b(a+c) - \tilde{n}^2 k^2 d^2}. \quad (20)$$

IV. SELF-ENERGY STRUCTURE FUNCTIONS

The spacelike components of the self-energy tensor can be written as

$$\Pi^{ij}(K) = -g^2 \int \frac{d^3 p}{(2\pi)^3} v^i v^j f(\mathbf{p}) \left(\delta^{ij} + \frac{v^j k^i}{K \cdot V + i\epsilon} \right). \quad (21)$$

At this point the distribution function $f(\mathbf{p})$ is completely arbitrary. In order to proceed we need to specify a form for the distribution function. In what follows we will assume that $f(\mathbf{p})$ can be obtained from an arbitrary isotropic distribution function by the rescaling of only one direction in momentum space. In practice this means that, given any isotropic distribution function $f_{\text{iso}}(\mathbf{p}^2)$, we can construct an anisotropic version by changing the argument

$$f(\mathbf{p}) = f_{\text{iso}}(\mathbf{p}^2 + \xi(\mathbf{p} \cdot \hat{\mathbf{n}})^2), \quad (22)$$

where $\hat{\mathbf{n}}$ is the direction of the anisotropy and $\xi > -1$ is a adjustable anisotropy parameter. Note that $\xi > 0$ corresponds to a contraction of the distribution in the $\hat{\mathbf{n}}$ direction whereas $-1 < \xi < 0$ corresponds to a stretching of the distribution in the $\hat{\mathbf{n}}$ direction. This assumption allows us to simplify Eq. (21) by performing a change of variables to \tilde{p}

$$\tilde{p}^2 = p^2 [1 + \xi(\mathbf{v} \cdot \mathbf{n})^2]. \quad (23)$$

After making this change of variables it is possible to integrate out the $|\tilde{p}|$ dependence, giving

$$\Pi^{ij}(K) = m_D^2 \int \frac{d\Omega}{4\pi} v^i v^j \frac{v^l + \xi(\mathbf{v} \cdot \mathbf{n})n^l}{(1 + \xi(\mathbf{v} \cdot \mathbf{n})^2)^2} \left(\delta^{ij} + \frac{v^j k^i}{K \cdot V + i\epsilon} \right), \quad (24)$$

where

$$m_D^2 = -\frac{g^2}{2\pi^2} \int_0^\infty dp p^2 \frac{df_{\text{iso}}(p^2)}{dp}. \quad (25)$$

We can then decompose the self-energy into four structure functions

$$\Pi^{ij} = \alpha A^{ij} + \beta B^{ij} + \gamma C^{ij} + \delta D^{ij}, \quad (26)$$

which are determined by taking the following contractions:

$$k^i \Pi^{ij} k^j = k^2 \beta,$$

$$\tilde{n}^i \Pi^{ij} k^j = \tilde{n}^2 k^2 \delta,$$

$$\tilde{n}^i \Pi^{ij} \tilde{n}^j = \tilde{n}^2 (\alpha + \gamma),$$

$$\text{Tr} \Pi^{ij} = 2\alpha + \beta + \gamma. \quad (27)$$

In Appendix B we collect the resulting integral expressions for the structure functions. All four structure functions depend on m_D , ω , k , ξ , and $\hat{\mathbf{k}} \cdot \hat{\mathbf{n}} = \cos \theta_n$. In the limit $\xi \rightarrow 0$ the structure functions α and β reduce to the isotropic hard-thermal-loop self-energies and γ and δ vanish

$$\alpha(K, 0) = \Pi_T(K),$$

$$\beta(K, 0) = \frac{\omega^2}{k^2} \Pi_L(K),$$

$$\gamma(K, 0) = 0,$$

$$\delta(K, 0) = 0, \quad (28)$$

with

$$\Pi_T(K) = \frac{m_D^2}{2} \frac{\omega^2}{k^2} \left[1 - \frac{\omega^2 - k^2}{2\omega k} \log \frac{\omega + k}{\omega - k} \right], \quad (29)$$

$$\Pi_L(K) = m_D^2 \left[\frac{\omega}{2k} \log \frac{\omega + k}{\omega - k} - 1 \right]. \quad (30)$$

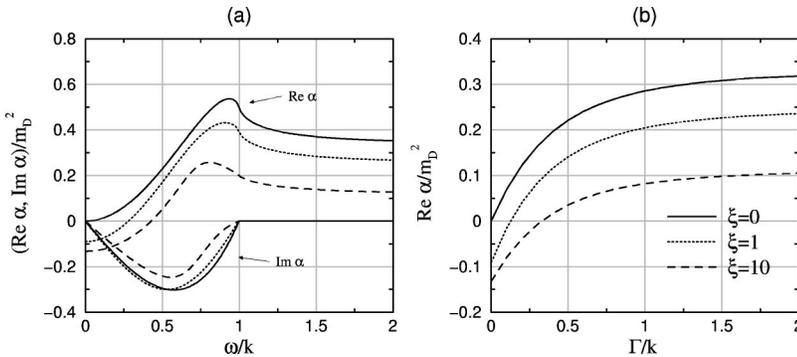


FIG. 1. Real and imaginary parts of α/m_D^2 as a function of real ω/k are shown in (a) and in (b) the real part of α/m_D^2 is shown for $\omega/k = i\Gamma/k$ with $\theta_n = \pi/4$ and $\xi = \{0, 1, 10\}$ in both cases.

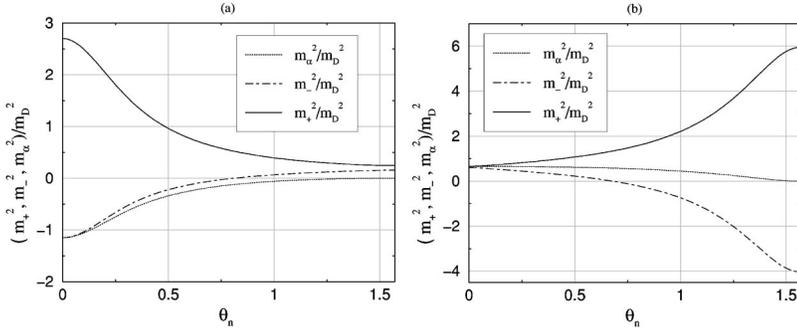


FIG. 2. Angular dependence of m_α^2 , m_+^2 , and m_-^2 at fixed (a) $\xi=10$ and (b) $\xi=-0.9$.

For finite ξ the analytic structure of the structure functions is the same as in the isotropic case. There is a cut in the complex ω plane which we can choose to run along the real ω axis from $-k < \omega < k$. For real valued ω the structure functions are complex for all $\omega < k$ and real for $\omega > k$. For imaginary valued ω all four structure functions are real valued. In Fig. 1 we plot the structure function α for real and imaginary values of ω , $\xi = \{0, 1, 10\}$, and $\theta_n = \pi/4$.

With these structure functions in hand we can construct the propagator $\Delta^{ij}(K)$ using the expressions from the previous section. Writing $\Delta^{-1}(K)$ in terms of our tensor basis

$$\Delta^{-1}(K) = (k^2 - \omega^2 + \alpha)\mathbf{A} + (\beta - \omega^2)\mathbf{B} + \gamma\mathbf{C} + \delta\mathbf{D} \quad (31)$$

and applying the inversion formula (20) we obtain an expression for the propagator

$$\Delta(K) = \Delta_A \mathbf{A} + (k^2 - \omega^2 + \alpha + \gamma)\Delta_G \mathbf{B} + [(\beta - \omega^2) \times \Delta_G - \Delta_A]\mathbf{C} - \delta\Delta_G \mathbf{D}, \quad (32)$$

with

$$\Delta_A^{-1}(K) = k^2 - \omega^2 + \alpha, \quad (33)$$

$$\Delta_G^{-1}(K) = (k^2 - \omega^2 + \alpha + \gamma)(\beta - \omega^2) - k^2 \tilde{n}^2 \delta^2. \quad (34)$$

Note that we can reorganize $\Delta(K)$ and write it as

$$\Delta(K) = \Delta_A [\mathbf{A} - \mathbf{C}] + \Delta_G [(k^2 - \omega^2 + \alpha + \gamma)\mathbf{B} + (\beta - \omega^2)\mathbf{C} - \delta\mathbf{D}]. \quad (35)$$

V. STATIC LIMIT

In order to see how the momentum-space anisotropy in the distribution functions affects the response to static electric and magnetic fluctuations we examine the limit $\omega \rightarrow 0$ of the propagators (33) and (34). Approaching along the real ω axis we find that to leading order $\alpha \sim \gamma \sim O(\omega^0)$, $\beta \sim O(\omega^2)$, and $\delta \sim O(i\omega)$.¹ We can therefore define four mass scales

$$m_\alpha^2 = \lim_{\omega \rightarrow 0} \alpha,$$

¹Identical results can be obtained by coming in along the imaginary axis with a suitable redefinition of m_δ^2 .

$$m_\beta^2 = \lim_{\omega \rightarrow 0} -\frac{k^2}{\omega^2} \beta,$$

$$m_\gamma^2 = \lim_{\omega \rightarrow 0} \gamma,$$

$$m_\delta^2 = \lim_{\omega \rightarrow 0} \frac{\tilde{n} k^2}{\omega} \text{Im } \delta. \quad (36)$$

Writing the static limit of the propagators (33) and (34) in terms of these masses gives

$$\Delta_A^{-1} = k^2 + m_\alpha^2 \quad (37)$$

$$\Delta_G^{-1} = -\frac{\omega^2}{k^2} [(k^2 + m_\alpha^2 + m_\gamma^2)(k^2 + m_\beta^2) - m_\delta^4]. \quad (38)$$

Δ_G^{-1} can be factorized into

$$\Delta_G^{-1} = -\frac{\omega^2}{k^2} (k^2 + m_+^2)(k^2 + m_-^2), \quad (39)$$

where

$$2m_\pm^2 = M^2 \pm \sqrt{M^4 - 4(m_\beta^2(m_\alpha^2 + m_\gamma^2) - m_\delta^4)}, \quad (40)$$

with

$$M^2 = m_\alpha^2 + m_\beta^2 + m_\gamma^2. \quad (41)$$

In the isotropic limit, $\xi \rightarrow 0$, $m_\alpha^2 = m_\gamma^2 = m_\delta^2 = m_-^2 = 0$ and $m_+^2 = m_D^2$. For finite ξ it is possible to evaluate all four masses defined above analytically. The results for m_α and m_β are listed in Appendix B. In Fig. 2 we plot the angular dependence of m_α^2 , m_+^2 , and m_-^2 at fixed $\xi=10$ and $\xi=-0.9$. In the case $\xi > 0$ (Fig. 2a) we see that for small θ_n the scale $m_+^2 \geq m_D^2$ and for θ_n near $\pi/2$, $m_+^2 \leq m_D^2$. For small θ_n the scales m_α^2 and m_-^2 are negative. The fact that m_α and m_- are non-vanishing is in agreement with the findings of Cooper *et al.* [10]; however, they neglected to consider the fact that these masses might be negative and would therefore not correspond to screening of the magnetic interaction. The fact that these quantities are negative indicates that for $\xi > 0$ the system possesses an instability to transverse and ‘‘mixed’’ external perturbations associated with m_α^2 and m_-^2 , respectively. The transverse instability is present for any $\theta_n \neq \pi/2$

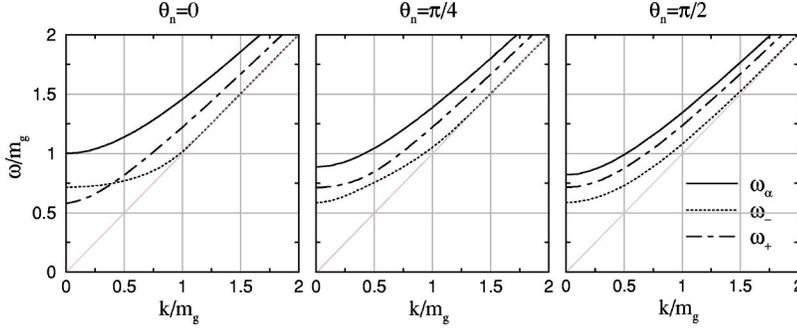


FIG. 3. Angular dependence of ω_α , ω_+ , and ω_- for $m_g = m_D/\sqrt{3}$, $\xi = 10$, and $\theta_n = \{0, \pi/4, \pi/2\}$.

while the mixed instability is only present for $\theta_n < \theta_c^{\text{mixed}}$ with θ_c^{mixed} depending on the value of ξ . In the case $\xi < 0$ (Fig. 2b) we see that for small θ_n the scale $m_+^2 \ll m_D^2$ and for θ_n near $\pi/2$, $m_+^2 \gg m_D^2$. For $\theta_n \gtrsim \pi/4$ the scale m_-^2 is negative again, signaling the presence of an instability in the system. In the next section we will discuss these instabilities in more detail.

VI. COLLECTIVE MODES

A similar factorization of Δ_G^{-1} can be achieved in the non-static case allowing us to determine the dispersion relations for all of the collective modes in the system.

A. Stable modes

First, let us consider the stable collective modes which have poles at real valued $\omega > k$. In this case we factorize Δ_G^{-1} as

$$\Delta_G^{-1} = (\omega^2 - \Omega_+^2)(\omega^2 - \Omega_-^2), \quad (42)$$

where

$$2\Omega_\pm^2 = \bar{\Omega}^2 \pm \sqrt{\bar{\Omega}^4 - 4[(\alpha + \gamma + k^2)\beta - k^2\tilde{n}^2\delta^2]}, \quad (43)$$

and

$$\bar{\Omega}^2 = \alpha + \beta + \gamma + k^2. \quad (44)$$

Note that the quantity under the square root in Eq. (43) can be written as $(\alpha - \beta + \gamma + k^2)^2 + 4k^2\tilde{n}^2\delta^2$, which is always positive for real $\omega > k$. Therefore there are at most two stable modes coming from Δ_G .

The remaining stable collective mode comes from the zero of Δ_A^{-1} . The dispersion relations for all of the collective modes can be determined by finding the solutions to

$$\omega_\pm^2 = \Omega_\pm^2(\omega_\pm), \quad (45)$$

$$\omega_\alpha^2 = k^2 + \alpha(\omega_\alpha). \quad (46)$$

In the isotropic limit (28) $\omega_\alpha = \omega_+ = \omega_T$ and $\omega_- = \omega_L$. For finite ξ there are three stable quasiparticle modes with dispersion relations which depend on the angle of propagation with respect to the anisotropy vector, θ_n . In Fig. 3 we plot the dispersion relations for all three modes for $m_D = \sqrt{3}$, $\xi = 10$, and $\theta_n = \{0, \pi/4, \pi/2\}$.

B. Unstable modes

For non-zero ξ the propagator also has poles along the imaginary ω axis.² The dispersion relation for these modes can be determined by taking $\omega \rightarrow i\Gamma$ with Γ real valued and solving for $\Gamma(k)$. In this case we factorize the inverse propagator as

$$\Delta_G^{-1} = (\Gamma^2 + \Omega_+^2)(\Gamma^2 + \Omega_-^2), \quad (47)$$

where Ω_\pm on the right-hand side are evaluated at $\omega = i\Gamma$. However, in contrast to the stable modes there is at most one solution in this case since numerically we find that $\Omega_+^2 > 0$ for all $\Gamma > 0$.

For $\xi > 0$ there is also an unstable mode present in Δ_A so that in this case there are two unstable modes in the system which can be found by solving

$$\Gamma_-^2 = -\Omega_-^2(i\Gamma_-), \quad (48)$$

$$\Gamma_\alpha^2 = -k^2 - \alpha(i\Gamma_\alpha). \quad (49)$$

Note that in both cases there are two solutions corresponding to modes with positive and negative growth rates. One of these corresponds to an exponentially growing solution and the other an exponentially decaying one. In Fig. 4a we plot $\Gamma_\alpha(k)$ and $\Gamma_-(k)$ with $\xi = 10$ and $\theta_n = \pi/8$. For $\xi < 0$ there is no longer an unstable mode coming from Δ_A and there is, therefore, only one unstable mode coming from Γ_- . In Fig. 4b we plot $\Gamma_-(k)$ with $\xi = -0.9$ and $\theta_n = \pi/2$.

VII. SMALL ξ EXPANSION

In the small- ξ limit it is possible to obtain analytic expressions for all of the structure functions order-by-order in ξ . To linear order in ξ

$$\alpha = \Pi_T(z) + \xi \left[\frac{z^2}{12} (3 + 5 \cos 2\theta_n) m_D^2 - \frac{1}{6} (1 + \cos 2\theta_n) m_D^2 + \frac{1}{4} \Pi_T(z) ((1 + 3 \cos 2\theta_n) - z^2 (3 + 5 \cos 2\theta_n)) \right],$$

²We have checked for poles at complex ω numerically but found none.

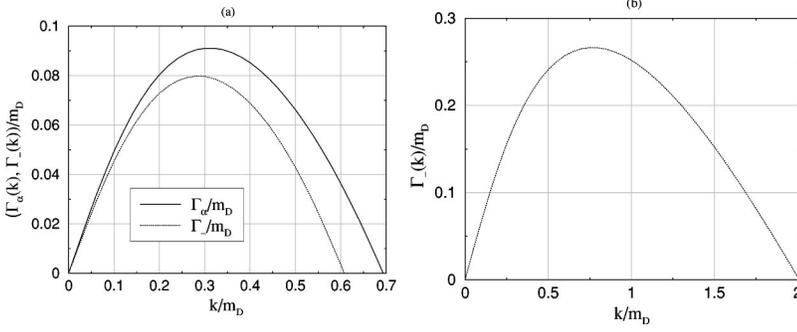


FIG. 4. $\Gamma_\alpha(k)$ and $\Gamma_-(k)$ as a function of k with (a) $\xi = 10$ and $\theta_n = \pi/8$ and (b) $\xi = -0.9$ and $\theta_n = \pi/2$.

$$z^{-2}\beta = \Pi_L(z) + \xi \left[\frac{1}{6}(1 + 3 \cos 2\theta_n)m_D^2 + \Pi_L(z) \right. \\ \left. \times \left(\cos 2\theta_n - \frac{z^2}{2}(1 + 3 \cos 2\theta_n) \right) \right],$$

$$\gamma = \frac{\xi}{3}(3\Pi_T(z) - m_D^2)(z^2 - 1)\sin^2\theta_n,$$

$$\delta = \frac{\xi}{3k}[4z^2m_D^2 + 3\Pi_T(z)(1 - 4z^2)]\cos\theta_n, \quad (50)$$

where $z = \omega/k$.

A. Static limit

Using the linear expansions and the fact that in the static limit $\Pi_L \rightarrow -m_D^2$ and $\Pi_T \rightarrow -i\pi\omega/(4k)$ we can write for the masses (36)

$$\hat{m}_\alpha^2 = -\frac{\xi}{6}(1 + \cos 2\theta_n), \\ \hat{m}_\beta^2 = 1 + \frac{\xi}{6}(3 \cos 2\theta_n - 1), \\ \hat{m}_\gamma^2 = \frac{\xi}{3}\sin^2\theta_n, \\ \hat{m}_\delta^2 = -\frac{\pi}{4}\sin\theta_n\cos\theta_n, \quad (51)$$

where $\hat{m}^2 = m^2/m_D^2$. Using these we can obtain small- ξ expressions for m_\pm defined in Eq. (40)

$$\hat{m}_+^2 = 1 + \frac{\xi}{6}(3 \cos 2\theta_n - 1), \\ \hat{m}_-^2 = -\frac{\xi}{3}\cos 2\theta_n. \quad (52)$$

B. Collective modes

Since δ is $O(\xi)$ it can be ignored in the expansion of Eq. (43) so that to linear order in ξ the collective modes satisfy

$$\Delta_A^{-1} = k^2 - \omega^2 + \alpha = 0$$

$$\Delta_G^{-1} = (k^2 - \omega^2 + \alpha + \gamma)(\beta - \omega^2) = 0, \quad (53)$$

where α , β , and γ are given by Eq. (50). Note again that there is only one unstable mode coming from Δ_G^{-1} since $\beta(i\Gamma) > 0$ for all $\Gamma > 0$.

VIII. CONCLUSIONS

In this paper we have derived a tensor basis for the gluon self-energy in a high-temperature quark-gluon plasma with an anisotropic momentum-space distribution. We then restricted the distribution function by requiring that it could be obtained from an isotropic distribution function by the rescaling of one direction specified by an anisotropy vector, $\hat{\mathbf{n}}$, and strength, ξ . Positive values of ξ correspond to a contraction of the isotropic distribution function along $\hat{\mathbf{n}}$ while negative values of ξ correspond to a stretching along $\hat{\mathbf{n}}$. Within this framework we could derive analytic forms for all of the structure functions associated with the tensor basis. Using these expressions we were then able to identify and determine the dispersion relations for the collective modes for both positive and negative ξ . We found that for $\xi > 0$ there were at most three stable and two unstable modes with dispersion relations which depended on the angle between the wave vector, \mathbf{k} , and the anisotropy vector. For $\xi < 0$ we found that there were also three stable modes but only one unstable mode. Additionally, we obtained analytic expressions for the structure functions in the limit of small ξ . These results should provide a reference point for the systematic study of the isotropization of a relativistic plasma.

The study of Mrówczyński and Randrup suggests that during heavy-ion collisions the rate of isotropization via collective modes is comparable with collisions and therefore cannot be ignored [5]. In this paper we have made no attempt to discuss the phenomenological rate for instability growth because there are a number of questions which would need to be addressed prior to making any definitive statements about the role of instabilities in plasma evolution and their expected contribution to observables. This is because we have only derived the self-energy in a linear expansion in the fluctuations and to leading-order in the coupling constant. Assuming that there is truly exponential growth of the fields in the direction of the anisotropy this means that the linear ap-

proximation will break down very quickly. In practice the non-linear terms in the transport equations will become important and regulate the growth of the modes which have become unstable.

Within electrodynamics the coupling constant is small and it is possible to experimentally study the Weibel instability [11]. However, with QCD the story is dramatically different since for experimentally realizable situations the coupling constant is large and the non-linear effects due to gluon self-interaction become important much sooner than any non-linear effects would for QED. Nevertheless, this does not diminish from the fact that these unstable modes exist and will therefore have a role to play in plasma evolution. In order to assess this role, however, detailed studies of the time evolution of anisotropic quark-gluon plasmas will need to be performed.

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APPENDIX A: NOTATION AND CONVENTIONS

We summarize here the notation and conventions which we use in the main body of the text.

Natural units: $\hbar = c = 1$.

Metric: $g^{\mu\nu} = g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

4-vectors: Indicated by Greek indices, e.g. $K^\mu = (\omega, \mathbf{k})$.

3-vectors: Indicated by lowercase Latin characters. Upper Latin indices like i, j, k use a Euclidean 3-metric, e.g. $\mathbf{k} = k^i$, $k^i k^i = \mathbf{k}^2$.

Fourier transform:

$$j^\mu(K) = \int d^4X e^{iK \cdot X} j^\mu(X)$$

$$j^\mu(X) = \int \frac{d^4K}{(2\pi)^4} e^{-iK \cdot X} j^\mu(K).$$

APPENDIX B: ANALYTIC EXPRESSIONS FOR STRUCTURE FUNCTIONS

In this appendix we collect the integral and analytic expressions for the structure functions α , β , γ , and δ . Choosing $\mathbf{n} = \hat{\mathbf{z}}$ and \mathbf{k} to lie in the x - z plane ($k_x/k_z = \tan \theta_n$) we have $\mathbf{v} \cdot \mathbf{n} = \cos \theta$ and $\mathbf{v} \cdot \mathbf{k} = k_x \cos \phi \sin \theta + k_z \cos \theta$. Using this parametrization the ϕ integration in all four structure functions defined by the contractions in Eq. (27) can be performed analytically:

$$\alpha(K, \xi) = \frac{m_D^2}{k^2 \tilde{n}^2} \int \frac{d(\cos \theta)}{2} \frac{\omega + \xi k_z \cos \theta}{[1 + \xi \cos^2 \theta]^2} \left\{ \omega - k_z \cos \theta + k^2 \left[s^2 - \left(\cos \theta - \frac{\omega k_z}{k^2} \right)^2 \right] \times R(\omega - k_z \cos \theta, k_x \sin \theta) \right\}, \quad (\text{B1})$$

$$\beta(K, \xi) = -\frac{m_D^2 \omega^2}{k^2} \int \frac{d(\cos \theta)}{2} \frac{1}{[1 + \xi \cos^2 \theta]^2} \times [1 - (\omega + \xi k_z \cos \theta) R(\omega - k_z \cos \theta, k_x \sin \theta)], \quad (\text{B2})$$

$$\gamma(K, \xi) = m_D^2 \int \frac{d(\cos \theta)}{2} \frac{1}{k^2 [1 + \xi \cos^2 \theta]^2} \left\{ \omega^2 + \xi k^2 \cos^2 \theta - 2 \frac{k^2}{k_x^2} (\omega^2 - \xi k_z^2 \cos^2 \theta) + \frac{(\omega + \xi k_z \cos \theta) k^4}{k_x^2} \left[2 \left(\cos \theta - \frac{\omega k_z}{k^2} \right)^2 - s^2 \right] \times R(\omega - k_z \cos \theta, k_x \sin \theta) \right\}, \quad (\text{B3})$$

$$\delta(K, \xi) = \frac{m_D^2 \omega}{k^4 \tilde{n}^2} \int \frac{d(\cos \theta)}{2} \frac{\omega + \xi k_z \cos \theta}{[1 + \xi \cos^2 \theta]^2} [k_z + (k^2 \cos \theta - \omega k_z) R(\omega - k_z \cos \theta, k_x \sin \theta)], \quad (\text{B4})$$

where $s^2 = (1 - \omega^2/k^2)(k_x^2/k^2)$ and

$$R(a, b) = \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{1}{a - b \cos \phi + i\epsilon} = \frac{1}{\sqrt{a+b+i\epsilon} \sqrt{a-b+i\epsilon}}. \quad (\text{B5})$$

When a and b are real valued R can be simplified to

$$R(a, b) = \frac{\text{sgn}(a) \Theta(a^2 - b^2)}{\sqrt{a^2 - b^2}} - \frac{i \Theta(b^2 - a^2)}{\sqrt{b^2 - a^2}}, \quad (\text{B6})$$

with $\Theta(x)$ being the usual step function. Note that the remaining integration over θ can also be done analytically but the results are rather unwieldy so we do not list them here.

Static limit

In the limit $\omega \rightarrow 0$ it is possible to obtain analytic expressions for all four structure functions. The results for m_α and m_β defined in Eq. (36) are

$$m_\alpha^2 = -\frac{m_D^2}{2k_x^2\sqrt{\xi}} \left(k_z^2 \arctan \sqrt{\xi} - \frac{k_z k^2}{\sqrt{k^2 + \xi k_x^2}} \arctan \frac{\sqrt{\xi} k_z}{\sqrt{k^2 + \xi k_x^2}} \right), \quad (\text{B7})$$

$$m_\beta^2 = m_D^2 \frac{[\sqrt{\xi} + (1 + \xi) \arctan \sqrt{\xi}](k^2 + \xi k_x^2) + \xi k_z \left(k_z \sqrt{\xi} + \frac{k^2(1 + \xi)}{\sqrt{k^2 + \xi k_x^2}} \arctan \frac{\sqrt{\xi} k_z}{\sqrt{k^2 + \xi k_x^2}} \right)}{2\sqrt{\xi}(1 + \xi)(k^2 + \xi k_x^2)}, \quad (\text{B8})$$

with similar results for m_γ^2 and m_δ^2 .

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