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Petite unification of quarks and leptons: Twenty-two years after

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A recent surge of interest in the novel ideas of large extra dimensions and their implications, such as the *early* unification of quarks and leptons, has prompted us to revive a paper written 22 years ago. In that paper, we provided a general discussion of quark-lepton unification characterized by the gauge group $G_S \otimes G_W$ with two couplings g_S and g_W and by the unification mass scales $M=10$ TeV–1000 TeV. The constraint from $\sin^2\theta_W$ restricts the choices for G_W and our favorite model for petite unification was chosen to be $SU(4)_{\rm PS} \otimes SU(2)^4$. In the present paper, we review the main results of our earlier paper and propose two new models based on the groups $SU(4)_{\rm PS}\otimes SU(2)^3$ and $SU(4)_{\rm PS}\otimes SU(3)^2$, for which the consistency with the measured value of $\sin^2\theta_W(M_Z^2)$ determines the unification scale to be roughly 1 TeV and 3–10 TeV, respectively. The implication of this very early unification is the existence of new quarks and leptons with charges up to 4/3 (for quarks) and 2 (for leptons) and masses $O(250$ GeV$)$. Interestingly, in these models the rare decay $K_L\to\mu e$ is automatically absent at the tree level and the one-loop contributions are consistent with the experimental upper bound for this decay. On the other hand, the original $SU(4)_{\rm PS}\otimes SU(2)^4$ model can be made consistent with the measured value of $\sin^2\theta_W(M_Z^2)$ and the unification scale $M=O(1$ TeV$)$ only provided there exist at least nine ordinary quark and lepton generations, with four generations in the case of the supersymmetric version. Moreover, the solution to the $K_L\to\mu e$ problem is not as natural as in the two other scenarios. We comment on the recent papers on early unification in the context of large extra dimensions.

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I. INTRODUCTION

Twenty-two years ago, we proposed alternatives [1] to popular grand unified models such as $SU(5)$ [2,3] or $SO(10)$ [4,5], based on a less ambitious program which aimed at unifying quarks and leptons at some energy scale M that is not too much greater than the electroweak scale [1]. We assumed that the standard model (SM) $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$, which has three independent couplings g_3, g_2, and g', is embedded into a gauge theory $G_S\otimes G_W$, which is characterized by two independent couplings g_S and g_W, at a "petite unification" scale M, which can be as small as $M=10^{5\pm1}$ GeV, namely, the TeV region. We further assumed that G_S and G_W are either simple or pseudosimple (a direct product of simple groups with identical couplings). Our approach was a "bottom up" one; that is to say, we used the available inputs from the "low energy" to constrain the choices of G_S and G_W. We used $\sin^2\theta_W$ and the known fermion representations as inputs. It turned out that the choices of G_W are quite restricted. Furthermore, if G_S is chosen to be $SU(4)$ in the manner of Pati and Salam [6], this restriction is even stronger, with the minimal choice for G_W being $[SU(2)]^4$ and the corresponding petite unification theory (PUT)

$$\mathrm{PUT}_0 = SU(4)_{\rm PS}\otimes SU(2)_L\otimes SU(2)_R\otimes S\tilde{U}(2)_L\otimes S\tilde{U}(2)_R. \tag{1}$$

This minimal model was discussed at length in our paper.

In the $SU(4)_{\rm PS}\otimes[SU(2)]^4$ model the value of $\sin^2\theta_W$ at the unification scale $M \gg M_Z$ turns out to be $\sin^2\theta_W^0=1/4$, very close to its experimental value, which is now very precisely known: $\sin^2\theta_W(M_Z^2)=0.23113(15)$ [7]. For $M=100$ TeV the inclusion of $O(\alpha)$ corrections and the renormalization group evolution led in 1981 to $\sin^2\theta_W(M_Z^2)\approx 0.22$, still consistent with the data then available. As we will show below, with the present value of the QCD coupling constant $\alpha_s(M_Z^2)$, consistency with the measured, very precise, value of $\sin^2\theta_W(M_Z^2)$ requires the unification scale M in this model to be as low as 330 GeV. This is clearly unacceptable as the lower bound on the right-handed gauge boson mass is $M_{W_R}\geq 800$ GeV [7]. The scale M can be raised to 1 TeV by adding six additional standard fermion generations with masses $O(250$ GeV$)$ or making the model supersymmetric, in which case two new fermion generations suffice. However, in the simplest version of this model the rare decay $K_L\to\mu e$ proceeds at the tree level and its rate with $M=1$ TeV exceeds the experimental upper bound by many orders of magnitude. A possible solution to these difficulties, as advocated recently in [8], is to introduce one large extra dimension to obtain acceptable values for $\sin^2\theta_W(M_Z^2)$ and $BR(K_L\to\mu e)$ with $M=O(1$–10 TeV$)$ and the usual three fermion generations. We will discuss other alternatives in this paper.

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In the present paper we would like to propose two possibly more attractive PUT groups:

$$\mathrm{PUT}_1 = SU(4)_{\mathrm{PS}} \otimes SU(2)_L \otimes SU(2)_H \otimes SU(2)_R \qquad (2)$$

and

$$\mathrm{PUT}_2 = SU(4)_{\mathrm{PS}} \otimes SU(3)_L \otimes SU(3)_H, \qquad (3)$$

which were listed in our PUT classification of 1981 but were not analyzed by us in detail. In these models $\sin^2\theta_W^0$ equals 1/3 and 3/8, respectively, but a very fast renormalization group evolution allows one to obtain the correct $\sin^2\theta_W(M_Z^2)$ with $M = 1$ TeV and $M = 3.3$ TeV, respectively, when the spontaneous breakdown of the PUT groups to the standard model group proceeds in one step. Moreover, the fast renormalization group evolution combined with the very precise experimental value for $\sin^2\theta_W(M_Z^2)$ determines these unification scales within 10–15 %. If the breakdowns of $SU(4)_{\mathrm{PS}}$ and of G_W are allowed to appear at two different scales M and $\tilde{M} < M$, these two scales have to be close to 1 TeV in the case of PUT_1 but can differ by up to an order of magnitude in the case of PUT_2, with roughly $3 \leq M \leq 10$ TeV and $0.8 \leq \tilde{M} \leq 3$ TeV.

These two scenarios for early unification of quark and leptons have three interesting properties.

(1) In addition to the standard three generations of quarks and leptons, three new generations of unconventional quarks and leptons with charges up to 4/3 (for quarks) and 2 (for leptons) and masses $O(250$ GeV$)$ are automatically present. The horizontal groups $SU(2)_H$ and $SU(3)_H$ connect the standard fermions with the unconventional ones.

(2) The placement of the ordinary quarks and leptons in the fundamental representation of $SU(4)_{\mathrm{PS}}$ is such that there are *no tree-level* transitions between ordinary quarks and leptons mediated by the $SU(4)_{\mathrm{PS}}$ gauge bosons. This prevents rare decays such as $K_L \to \mu e$ from acquiring large rates, even when the masses of these gauge bosons are in the few TeV range.

(3) There are new contributions to flavor changing neutral current (FCNC) processes involving standard quarks and leptons that are mediated by the horizontal $SU(2)_H$ and $SU(3)_H$ weak gauge bosons and the new unconventional quarks and leptons. However, they appear first at the one-loop level and can be made consistent with the existing experimental bounds.

Our paper is organized as follows. In Sec. II, we review the steps that lead to the three choices for G_W mentioned above and summarize the most important formulas. In particular, we derive the general expression for $\sin^2\theta_W^0$ and discuss its relation to $\sin^2\theta_W(M_Z^2)$. In Sec. III we present in detail the fermion content of the selected groups. The results of the renormalization group analysis of $\sin^2\theta_W$, in the scenarios in question, are presented in Sec. IV, and in Sec. V the rare decay $K_L \to \mu e$ is briefly discussed. Here we emphasize that, while in the $SU(4)_{\mathrm{PS}} \otimes SU(2)^4$ scenario it is very difficult to satisfy the experimental bound on $K_L \to \mu e$ when $M = O(1$ TeV$)$, the presence of a Glashow-Iliopoulos-Maiani- (GIM-)like mechanism in the remaining two scenarios allows this bound to be satisfied without any unnatural conditions on the mass spectrum of the new quarks and leptons and related Cabibbo-Kobayashi-Maskawa- (CKM-)like mixing matrix. Similar comments apply to FCNC processes.

In Sec. VI we compare our work of 1981 and that presented here with recent papers on the early unification of quarks and leptons in the context of large extra dimensions [8,9]. As a matter of fact, the $SU(3)_W$ model of Dimopoulos and Kaplan [9] is just one of the cases considered by us in [1] and the analysis in [8] is the generalization of our $SU(4)_{\mathrm{PS}} \otimes [SU(2)]^4$ model to extra dimensions. Finally, in Sec. VII we summarize the main results of our paper and offer some perspectives for future work. A detailed analysis of $K_L \to \mu e$ and other phenomenological implications of the PUT groups discussed here will be presented elsewhere.

II. PETITE UNIFICATION REEXAMINED

A. Preliminaries

The objective, then and now, is to unify quarks and leptons at an intermediate scale in the TeV range. We assume, then and now, that $SU(3) \otimes SU(2)_L \otimes U(1)_Y$ is embedded in $G = G_S(g_S) \otimes G_W(g_W)$, where g_S and g_W denote the corresponding couplings. Furthermore, G_S and G_W are assumed to be either simple or pseudosimple, i.e., a direct product of simple groups with identical couplings. The pattern of symmetry breaking is assumed to be

$$G \xrightarrow{M} G_1 \xrightarrow{\tilde{M}} G_2 \xrightarrow{M_Z} SU(3)_c \otimes U(1)_{\mathrm{EM}}, \qquad (4)$$

where

$$G_1 = SU(3)_c(g_3) \otimes \tilde{G}_S(\tilde{g}_S) \otimes G_W(g_W) \qquad (5)$$

and

$$G_2 = SU(3)_c(g_3) \otimes SU(2)_L(g_2) \otimes U(1)_Y(g'). \qquad (6)$$

We assume $M_Z < \tilde{M} \leq M$. In principle, G can be broken down directly to G_2, but to be more general the pattern (4) was assumed in [1]. Furthermore, in accordance with our petite-unification idea, we require M and $\tilde{M}$ to be at most a few orders of magnitude larger than M_Z, the weak hypercharge $U(1)_Y$ group to merge into both $\tilde{G}_S$ and G_W at $\tilde{M}$, and $SU(3)_c$ and $\tilde{G}_S$ to be unbroken subgroups of G_S so that their generators are unbroken generators of G_S. The second requirement allows us to put quarks and leptons into identical representations of the weak group G_W and consequently make the quarks and leptons indistinguishable when the strong interactions are turned off. The last requirement implies that

$$g_3(M^2) = \tilde{g}_S(M^2) = g_S(M^2). \qquad (7)$$

B. $\sin^2\theta_W$ and the choices of G_W

We will next summarize the salient points of our earlier paper concerning the restrictions imposed on G_W from the value of $\sin^2\theta_W$. We will focus, in particular, on the case where $G_W=[SU(N)]^k$ and use $\sin^2\theta_W$ to constrain the pair (N,k). Furthermore, we argued in [1] that the most economical choice for G_S is $SU(4)$ following Pati and Salam, although we presented a more general discussion there. In the following we shall then deal principally with the groups

$$G=SU(4)_{\rm PS}\otimes[SU(N)]^k, \quad \tilde{G}_S=U(1)_{\rm S}. \tag{8}$$

To derive $\sin^2\theta_W$, we write the generators T_{3L} and T_0 of $SU(2)_L$ and $U(1)_Y$, respectively, in terms of the generators of G_S and G_W. As usual, one has for the electric charge generator Q

$$Q=T_{3L}+T_0, \tag{9}$$

where T_{3L} and T_0 are diagonal generators of $SU(2)_L$ and $U(1)_Y$, respectively. They can be written as

$$T_{3L}=\sum_\alpha C'_{\alpha W}T^0_{\alpha W} \tag{10}$$

and

$$T_0=\sum_\alpha C_{\alpha W}T^0_{\alpha W}+C_ST_{15}, \tag{11}$$

where $T^0_{\alpha W}$ and T_{15} are the diagonal generators of G_W and $SU(4)_{\rm PS}$, respectively, with $T^0_{\alpha W}$ being the generators of the $SU(2)$ disjoint subgroups of G_W. Also, $C'_{\alpha W}$ and $C_{\alpha W}$ are orthogonal to each other.

Equations (10),(11) form the basis for the derivation of $\sin^2\theta_W$. In [1], we discussed two cases, which were called (a) the "unlocked standard model" where the generators of $SU(2)_L$ are the unbroken generators of G_W, and (b) the "locked standard model" where the generators of $SU(2)_L$ are the unbroken combination of generators belonging to several disjoint $SU(2)$ subgroups of G_W. We showed that case (a) (the unlocked standard model) is the most economical one and this is the one we choose to concentrate on in the present paper. The reader is encouraged to consult [1] for a more general discussion. Therefore, for case (a), one has

$$T_{3L}=T^0_{3W}, \tag{12}$$

where T^0_{3W} is a diagonal generator of one of the $SU(2)$ subgroups of G_W. This implies that $C'_{3W}=1$ with all other coefficients in Eq. (10) equal to zero. In consequence, in the unlocked standard model scenario, one is now in a position to derive $\sin^2\theta_W$, taking into account the pattern (4). First, we present a formula for the renormalized value of $\sin^2\theta_W$ at the one-loop level. We will comment on its generalization to two loops in Sec. IV. From

$$\frac{1}{e^2(M_Z^2)}=\frac{1}{[g_2(M_Z^2)]^2}+\frac{1}{[g'(M_Z^2)]^2}, \tag{13}$$

$$g_2(\tilde{M}^2)=g_W(\tilde{M}^2), \tag{14}$$

$$\frac{1}{[g'(\tilde{M}^2)]^2}=\frac{\sum_\alpha C^2_{\alpha W}}{[g_W(\tilde{M}^2)]^2}+\frac{C_S^2}{[\tilde{g}_S(\tilde{M}^2)]^2}, \tag{15}$$

and using the modified minimal subtraction ($\overline{\rm MS}$) definition for $\sin^2\theta_W$, namely,

$$\sin^2\theta_W(M_Z^2)=\frac{e^2(M_Z^2)}{g_2^2(M_Z^2)}, \tag{16}$$

one obtains the master formula [1]

$$\sin^2\theta_W(M_Z^2)=\sin^2\theta_W^0\left[1-C_S^2\frac{\alpha(M_Z^2)}{\alpha_S(M_Z^2)}-8\pi\alpha(M_Z^2)\times\left(K\ln\frac{\tilde{M}}{M_Z}+K'\ln\frac{M}{\tilde{M}}\right)\right], \tag{17}$$

where

$$\alpha(M_Z^2)\equiv\frac{e^2(M_Z^2)}{4\pi}, \quad \alpha_S(M_Z^2)\equiv\frac{g_3^2(M_Z^2)}{4\pi}, \tag{18}$$

$$\sin^2\theta_W^0=\frac{1}{1+C_W^2}, \tag{19}$$

with $C_W^2=\Sigma_\alpha C^2_{\alpha W}$, and

$$K=b_1-C_W^2b_2-C_S^2b_3, \tag{20}$$

$$K'=C_S^2(\tilde{b}-\tilde{b}_3). \tag{21}$$

Here, b_1, b_2, b_3 ($\tilde{b}_3$), and $\tilde{b}$ are the one-loop coefficients of the beta functions for $U(1)_Y$, $SU(2)_L$, $SU(3)_c$, and $U(1)_S$, respectively, with $\tilde{b}_3\neq b_3$ due to possible contributions of new particles with masses larger than $\tilde{M}$. Explicit expressions for these coefficients are given in Sec. IV. We will see there that in the case of the new groups in Eqs. (2) and (3) the presence of new particles with masses $O(250\ {\rm GeV})$ will require the introduction of the appropriate threshold corrections in K.

Neglecting the contributions of new particles to K and K' for a moment and using the $\overline{\rm MS}$ values [7]

$$1/\alpha(M_Z^2)=127.934(27), \quad \alpha_S(M_Z^2)=0.1172(20), \tag{22}$$

we find

$$\sin^2\theta_W(M_Z^2)=R\sin^2\theta_W^0 \tag{23}$$

where

$$R=1-0.067C_S^2-0.014C_S^2\ln\frac{M}{\tilde{M}}-(0.009+0.004C_W^2+0.009C_S^2)\ln\frac{\tilde{M}}{M_Z}. \qquad (24)$$

We observe that $\sin^2\theta_W(M_Z^2)$ is a sensitive function of C_W^2, present in particular in $\sin^2\theta_W^0$, and of C_S^2 in the term $C_S^2\alpha(M_Z^2)/\alpha_S(M_Z^2)$ and in the renormalization group corrections. The renormalization of $\sin^2\theta_W$ increases with increasing C_S^2 but of course also depends strongly on the values of b_i', which in turn depend on the content of the fermion representations and their weak and strong charges.

As $\sin^2\theta_W(M_Z^2)$ is known with very high precision,

$$\sin^2\theta_W(M_Z^2)|_{\rm exp}=0.23113(15), \qquad (25)$$

and C_W and C_S in the case of $SU(4)\otimes[SU(N)]^k$ can take only special values, only certain pairs (C_W,C_S) are allowed if we are interested in the unification scales $M\leqslant 1000$ TeV and in particular $M\leqslant 10$ TeV. We will now briefly describe the steps that led us in [1] to the acceptable choices of (C_W,C_S).

The crucial quantity to be considered first is $\sin^2\theta_W^0$, which is determined at the petite unification scale M. For $G_W=[SU(N)]^k$, a given pair of (N,k) will determine C_W and hence $\sin^2\theta_W^0$ through Eq. (19), which can also be written as

$$\sin^2\theta_W^0=\frac{1}{1+C_W^2}=\left[\frac{{\rm Tr}\,T_{3L}^2}{{\rm Tr}\,Q^2}\right]_{\rm adjoint}, \qquad (26)$$

where the last term in Eq. (26) reflects the fact that the adjoint representation of G_W is a singlet of G_S. It is then sufficient to evaluate Eq. (26) by simply examining the adjoint representation.

Since quarks and leptons are assumed to be in separate (but identical) representations of G_W, the gauge bosons of G_W have integer charges. Assuming next a permutation symmetry among the $SU(N)$'s in G_W, and allowing for arbitrary integer charges for the gauge bosons, one finds [1]

$$\sin^2\theta_W^0=\frac{N}{k\,{\rm Tr}(Q_W^2)|_{\rm adjoint}}, \quad {\rm Tr}(Q_W^2)|_{\rm adjoint}=\sum_{i=1}^{\alpha} i^2 n_i, \qquad (27)$$

where ${\rm Tr}(Q_W^2)|_{\rm adjoint}$ is for each $SU(N)$, n_i is the number of gauge bosons with $|Q|=i$, and α is the maximal gauge-boson charge involved. Since the adjoint representation can be constructed from the product of the fundamental representation N and its conjugate $\overline{N}$, one can compute n_i by looking at the charge distribution of the fundamental representation, namely,

$$[\underbrace{\tilde{Q}_W,\ldots,\tilde{Q}_W}_{r_0},\underbrace{\tilde{Q}_W-1,\ldots,\tilde{Q}_W-1}_{r_1},\underbrace{\tilde{Q}_W-\alpha,\ldots,\tilde{Q}_W-\alpha}_{r_\alpha}], \qquad (28)$$

where $\tilde{Q}_W$ is an eigenvalue of Q_W.

The detailed analysis in [1] showed that gauge bosons with charges ± 3 or higher corresponding to $N\geqslant 4$ are excluded since one can derive the inequality $\sin^2\theta_W^0\leqslant 1/[12-(8/N)]\leqslant 1/10$ which rules out this case. Further, for doubly charged gauge bosons, the maximal allowed number is 2 (for ± 2), leading to ${\rm Tr}(Q_W^2)=4N$ for any $SU(N)$ with $N\geqslant 3$. For $k=1$ this gives $\sin^2\theta_W^0=1/4$. However, as shown in [1], in this case $C_S^2=8/3$, implying through Eq. (24) $\sin^2\theta_W(M_Z^2)=0.205$ even without including the renormalization group effects that decrease it even further. As a consequence, scenarios with $G_W=SU(3),SU(4),\ldots,$ having two doubly charged gauge bosons are inconsistent with the data.

We thus obtain an important result: The only charges of weak gauge bosons that are consistent with the measured value of $\sin^2\theta_W(M_Z^2)$ within the petite unification framework with the gauge group $SU(4)\otimes[SU(N)]^k$ are 0 and ± 1.

Consequently, the formula (27) simplifies to

$$\sin^2\theta_W^0=\frac{N}{kn_1}, \qquad (29)$$

where n_1 is the number of weak gauge bosons with $Q=\pm 1$ in $SU(N)$.

In order to find n_1 let us consider first the class (i) of fermion representations that transform under G_W as

$$(f,1,1,\ldots,1),\ (1,f,1,\ldots,1). \qquad (30)$$

Each entry in (30) corresponds to the group $\tilde{G}$ in the product $G_W=\tilde{G}\otimes\tilde{G}\otimes\cdots\otimes\tilde{G}$. That is, quarks and leptons transform nontrivially under one of the groups $\tilde{G}$ and are singlets under the rest. The fundamental representation for the group $\tilde{G}$ has then a charge distribution

$$[\underbrace{\tilde{Q}_W,\ldots,\tilde{Q}_W}_{r_0},\underbrace{\tilde{Q}_W-1,\ldots,\tilde{Q}_W-1}_{r_1}], \qquad (31)$$

with $r_0+r_1=N$. The tracelessness condition for the charge operator Q_W gives the eigenvalues

$$\tilde{Q}_W=1-\frac{r_0}{N}, \quad \tilde{Q}_W-1. \qquad (32)$$

Moreover, we find

$$n_1=2r_0r_1=2r_0(N-r_0) \qquad (33)$$

TABLE I. The values of $\sin^2\theta^0$ for the weak groups $G_W = SU(N)^k$ and different fermion representations.

G_W	r_0	$\sin^2\theta_W^0$	$(f,1)+(1,\overline{f})$ $\tilde{Q}_W^i$	$(f,\overline{f})$ $\tilde{Q}_W^i$
$[SU(2)]^3$	1	0.333	$\pm 1/2$	$0,\pm 1$
$[SU(2)]^4$	1	0.250	$\pm 1/2$	$0,\pm 1$
$[SU(3)]^2$	1	0.375	$2/3,-1/3$	$0,\pm 1$
$[SU(3)]^3$	1	0.250	$2/3,-1/3$	$0,\pm 1$
$[SU(4)]^2$	2	0.250	$\pm 1/2$	$0,\pm 1$
$[SU(5)]^2$	1	0.313	$4/5,\ -1/5$	$0,\pm 1$
$[SU(6)]^2$	1	0.300	$5/6,-1/6$	$0,\pm 1$
$SU(7)$	3	0.292	$4/7,-3/7$	
$[SU(7)]^2$	1	0.292	$6/7,-1/7$	$0,\pm 1$
$SU(8)$	3	0.267	$5/8,-3/8$	
$SU(8)$	4	0.250	$\pm 1/2$	

and consequently the very useful formula

$$\sin^2\theta_W^0 = \frac{N}{2kr_0(N-r_0)} = \frac{1}{1+C_W^2}, \tag{34}$$

which can be used to calculate $\sin^2\theta_W^0$ and C_W^2 for given N, k, and r_0. This formula is equivalent to the formulas given in [1] but is more transparent. The results for $\sin^2\theta_W^0$ are given in Table I, where we also give the values of the charges $\tilde{Q}_W^i$ in the fundamental representation obtained by means of Eq. (32). We observe a correlation between the values of $\sin^2\theta_W^0$ for given (N,k) and the weak charges of quarks and leptons. This correlation implies eventually the correlation between $\sin^2\theta_W^0$ and electric charges of quarks and leptons that follows from

$$Q = Q_S + Q_W = C_S T_{15} + Q_W, \tag{35}$$

where T_{15} is the diagonal generator of $SU(4)_{\rm PS}$ that commutes with $SU(3)_c$. We will return to this correlation below.

If fermions transform as [class (ii)] $(f,\overline{f})$ under any pair $\tilde{G}\otimes\tilde{G}$ in G_W and are singlets under the rest, that is, in the symbolical notation of (30) one has

$$(f,\overline{f},1,\ldots,1), \tag{36}$$

the charge distribution is an $N\times N$ matrix with $r_0+r_1=N$ columns and $r_0'+r_1'=N$ rows [see Eq. (4.10) of [1]]. This matrix looks like

$$\begin{pmatrix} \tilde{Q}_W & \ldots & \tilde{Q}_W & \tilde{Q}_W-1 & \ldots & \tilde{Q}_W-1 \\ & \vdots & & & & \\ \tilde{Q}_W & \ldots & \tilde{Q}_W & \tilde{Q}_W-1 & \ldots & \tilde{Q}_W-1 \\ \tilde{Q}_W+1 & \ldots & \tilde{Q}_W+1 & \tilde{Q}_W & \ldots & \tilde{Q}_W \\ & \vdots & & & & \\ \tilde{Q}_W+1 & \ldots & \tilde{Q}_W+1 & \tilde{Q}_W & \ldots & \tilde{Q}_W \end{pmatrix}, \tag{37}$$

where the rows refer to f and the columns to $\overline{f}$. The eigenvalues of Q_W are now [1]

$$\tilde{Q}_W = \frac{r_0'-r_0}{N}, \quad \tilde{Q}_W \pm 1. \tag{38}$$

It turns out that from the point of view of $\sin^2\theta_W$ only the cases $r_0'=r_0$ and consequently $r_1'=r_1=N-r_0$ are of interest to us, implying $\tilde{Q}_W=0,\pm 1$ as shown in Table I. Moreover, the formula (34) also applies here.

Whether the groups listed in Table I give an acceptable $\sin^2\theta_W(M_Z^2)$ depends also on C_S^2 as discussed before. In fact, it was shown in [1] that if G_S is chosen to be the Pati-Salam $SU(4)$ with each standard quark $SU(3)_c$ triplet put with a lepton into the same fundamental representation of $SU(4)$ and the electric charges of quarks and leptons are restricted to

$$Q_q = \frac{d}{3}+n, \quad Q_l = n', \quad n,n' \text{ integer}, \quad d=1,2, \tag{39}$$

then many of the possibilities given in Table I can be eliminated. The choice in Eq. (39) allows inclusion of at least quarks and leptons with ordinary charges. Indeed, under the latter assumption one can show that $\tilde{Q}_W^i$ should be multiples of 1/4, in fact,

$$\tilde{Q}_W^i = \frac{1}{4}(3Q_q^i + Q_l^i). \tag{40}$$

Consequently a number of possibilities listed in Table I can be eliminated by this requirement alone. For the remaining cases that satisfy Eq. (40) we find using

$$Q_q^i = \frac{C_S}{2\sqrt{6}} + \tilde{Q}_W^i, \quad Q_l^i = -\frac{3C_S}{2\sqrt{6}} + \tilde{Q}_W^i \tag{41}$$

the expression for C_S^2 in terms of quark and lepton electric charges:

$$C_S^2 = \frac{1}{6}(3Q_q^i - 3Q_l^i)^2. \tag{42}$$

One word of caution is in order here. The previous statements related to Eq. (39) refer only to scenarios in which the only representations present are of a single class, i.e., (i) or

TABLE II. The values of lepton (Q_l^i) and quark (Q_q^i) electric charges corresponding to the weak charges ($\tilde{Q}_W^i$) discussed in the text. The values of C_S^2 were obtained from Eq. (42).

$\tilde{Q}_W^i$	Q_l^i	Q_q^i	C_S^2
1/2	0	2/3	
−1/2	−1	−1/3	
1/2	1	1/3	2/3
−1/2	0	−2/3	
1	0	4/3	
0	−1	1/3	
−1	−2	−2/3	8/3
1	2	2/3	
0	1	−1/3	
−1	0	−4/3	
5/4	1	4/3	
1/4	0	1/3	1/6
−3/4	−1	−2/3	

(ii). In the case where both classes are needed, as will be the case of PUT_1, we should broaden the restriction (39) in the following sense. First, the value of C_S^2 should be chosen judiciously depending on $\sin^2\theta_W^0$. Once it is chosen, the charges of the fermions are determined depending on their representations under G_W and are given by Eq. (35), namely, $Q=C_S T_{15}+Q_W$. As we discussed earlier and show in Table I, representations $(f,1,1,\dots)$ have $Q_W=\pm 1/2$ and representations $(f,\bar{f},1,\dots)$ have $Q_W=0,\pm 1$. Obviously, when a scenario contains both classes of representations, it will be unavoidable to have quarks and leptons with ''funny'' charges in addition to the familiar ones. As we will discuss below in the context of PUT_1, as long as some of these ''funny'' fermions belong to a vectorlike representation of one of the G_W gauge groups, they can be very massive, in the sense that their masses are not proportional to the SM electroweak scale. The obvious caution that one has to take is that, in a mixed case, at least one of the representations has to contain SM fermions.

With the condition on $Q_{q,l}^i$ in Eq. (39) the lowest values for C_S^2 are found to be

$$C_S^2=\frac{1}{6},\ \frac{2}{3},\ \frac{8}{3}. \tag{43}$$

The next value $C_S^2=25/6$ and higher values would require very small $\sin^2\theta_W(M_Z^2)$ and rather high quark and lepton charges. In Table II we list the $\tilde{Q}_W^i$ of Table I that satisfy Eq. (40) along with the corresponding quark and lepton charges, as well as the values of C_S^2. Although, for completeness, we also list the case $C_S^2=1/6$ in Table II, it was shown in [1] that it corresponds to a weak group $SU(4)_1\otimes SU(4)_2$ which has $\sin^2\theta_W^0=0.286$. Because of the low value of C_S^2, one needs $M>10^6$ GeV in order to obtain the correct value of $\sin^2\theta_W(M_Z^2)$, and consequently this scenario does not fit into our framework.

We now classify the G_W groups listed in Table I in terms of their possible agreement with $\sin^2\theta_W(M_Z^2)$. As seen from Tables I and II only the values $C_S^2=2/3,8/3$ have to be considered. We can then make the following observations.

(a) Groups that can have $C_S^2=2/3$ are those for which $\tilde{Q}_W^i=\pm 1/2$, which corresponds to representations that contain only conventionally charged quarks and leptons, as can be seen from Table II. From Table I, these weak groups are $[SU(2)]^3$, $[SU(2)]^4$, $[SU(4)]^2$, and $SU(8)$, with $\sin^2\theta_W^0=0.333,0.25,0.25,0.25$, respectively. For $[SU(2)]^3$, one would need a petite unification scale substantially larger than 1000 TeV because $C_S^2=2/3$ is too small to bring $\sin^2\theta_W^0=0.333$ down to $\sin^2\theta_W(M_Z^2)\sim 0.23$. (We shall, however, come back to this group in the discussion below.) The promising groups in this class of models are, in order of complexity, $[SU(2)]^4$, $[SU(4)]^2$, and $SU(8)$, all of which have $\sin^2\theta_W^0=0.25$. In particular, the group $[SU(2)]^4$ was our favorite choice in [1]. The renormalization group (RG) analysis of these models will be discussed in Sec. IV.

(b) Groups that have $C_S^2=8/3$ are those with $\tilde{Q}_W^i=0,\pm 1$, which corresponds to representations having quark charges as high as $\pm 4/3$ and lepton charges as high as ± 2 in addition to the standard charges. Because of the high value for C_S^2, we need those groups for which $\sin^2\theta_W^0>0.3$. From Table II, one can see that only three groups satisfy this criterion: $[SU(2)]^3$, $[SU(3)]^2$, and $[SU(5)]^2$, with $\sin^2\theta_W^0=0.333,0.375,0.313$, respectively. The implications of the first two of these models through a RG analysis will be discussed in Sec. VI.

In summary, we have arrived at two classes of weak gauge groups G_W which with $G_S=SU(4)_{\mathrm{PS}}$ might satisfy the experimental constraint on $\sin^2\theta_W(M_Z^2)$:

$$[SU(2)]^4,\quad [SU(4)]^2,\quad SU(8), \tag{44}$$

which have only conventionally charged quarks and leptons in the fundamental representations in Eq. (30), $C_S^2=2/3$ and $\sin^2\theta_W^0=0.25$; and

$$[SU(2)]^3,\quad [SU(3)]^2,\quad [SU(5)]^2, \tag{45}$$

which contain extra quarks and leptons with higher charges ($\pm 4/3$ and ± 2) placed together with the standard quarks and leptons in the representations (36). See also Table II. These groups have higher initial $\sin^2\theta_W^0=0.333,\ 0.375,\ 0.313$, respectively, and $C_S^2=8/3$.

III. FERMION CONTENT OF SELECTED GROUPS

A. Preliminaries

In this section we will present in detail the fermion content of three groups, PUT_0, PUT_1, and PUT_2 as defined in Eqs. (1), (2), and (3), respectively. As we shall see in the next section, these three groups seem to be the best candidates for a successful petite unification consistent with the measured value of $\sin^2\theta_W$. The values for $\sin^2\theta_W^0$ in these three scenarios are 1/4, 1/3, and 3/8, respectively, with the last being very reminiscent of the quintessential $SU(5)$ value. Our analysis of the previous section implies then that the only chance to satisfy the $\sin^2\theta_W$ constraint is to choose for these three groups C_S^2 equal to 2/3, 8/3, and 8/3, respectively. In other words, as one can deduce from Table II, we should have class (i) representation, i.e., (4,2,1,1,1), (4,1,2,1,1), (4,1,1,2,1), (4,1,1,1,2), for $SU(4)_S\otimes[SU(2)]^4$, and class (ii) representation, i.e., $(4,3,\overline{3})$, for $SU(4)_S\otimes[SU(3)]^2$. On the other hand, we will show that, for $SU(4)_S\otimes[SU(2)]^3$, both classes are involved.

While the value of C_S^2 is an important ingredient in the relation between $\sin^2\theta_W^0$ and $\sin^2\theta_W(M_Z^2)$, the values of the renormalization group coefficients b_i that enter K and K' in Eqs. (20) and (21) are equally important. In order to find these values in the scenarios considered, it is necessary to identify the fermion representations and the relevant charges with respect to the SM group and $U(1)_S$. This is what we intend to do next.

B. $SU(4)_{PS}\otimes[SU(2)]^4$

This scenario has already been worked out in detail in [1], and we will recall only the most important points. The weak group

$$[G_W]_0=SU(2)_L\otimes SU(2)_R\otimes S\tilde{U}(2)_L\otimes S\tilde{U}(2)_R \tag{46}$$

consists of the standard weak gauge group of the Pati-Salam model and its "mirror group" $S\tilde{U}(2)_L\otimes S\tilde{U}(2)_R$, necessary to obtain the correct $\sin^2\theta_W$. In the original Pati-Salam model [6] one has $\sin^2\theta_W^0=1/2$, which is much too high for an early unification with $C_S^2=2/3$. We will return to it in Sec. IV.

Let us denote by l_L the usual left-handed lepton $SU(2)_L$ doublet, and by q_L the left-handed quark doublet. The $SU(2)_R$ doublets are denoted by l_R and q_R. Similarly, the $S\tilde{U}(2)_{L,R}$ doublets will be denoted by $\tilde{l}_{L,R}$ and $\tilde{q}_{L,R}$. Consequently, each generation of $SU(4)_{PS}\otimes[SU(2)]^4$ can be written as

$$\Psi_L=(q_L,l_L)=(4,2,1,1,1)_L, \tag{47}$$

$$\Psi_R=(q_R,l_R)=(4,1,2,1,1)_R, \tag{48}$$

$$\tilde{\Psi}_L=(\tilde{q}_L,\tilde{l}_L)=(4,1,1,2,1)_L, \tag{49}$$

$$\tilde{\Psi}_R=(\tilde{q}_R,\tilde{l}_R)=(4,1,1,1,2)_R. \tag{50}$$

$\tilde{\Psi}_L$ and $\tilde{\Psi}_R$ are what we call "mirror fermions."

Note that in this scenario the weak charges in each $SU(2)$ representation are

$$Q_W=(1/2,-1/2) \tag{51}$$

and with $C_S^2=2/3$

$$Q_q^i=\frac{1}{6}+Q_W^i, \quad Q_l^i=-\frac{1}{2}+Q_W^i. \tag{52}$$

Consequently, only conventional electric charges are present and they are the same for the ordinary and mirror fermions. However, the latter are $SU(2)_L$ [as well as $SU(2)_R$] singlets.

Now, in order to have a "petite unification" with only two independent couplings g_S and g_W, the four gauge couplings of $[SU(2)]^4$ have to be equal to each other above the scale $\tilde{M}$. Consequently, the mirror fermions have to be *lighter* than $\tilde{M}$. Below $\tilde{M}$, the masses of mirror fermions and possible extra generations are unconstrained, however, although the detailed spectrum depends on the Higgs system used to generate the fermion masses. As discussed in [1], the appropriate Higgs scalars that could give masses to the normal and mirror fermions can transform as (1,2,2,1,1) and (1,1,1,2,2), respectively. We refer for details to [1], where a possible breakdown mechanism for the gauge group $SU(4)_{PS}\otimes[SU(2)]^4$ is discussed. Needless to say, it is a quite complicated task to generate fermion masses in general, and we leave it for the future.

Experimentally, it is safe to assume that any long-lived new quarks, if they exist, should have a mass larger than 200 GeV [10,11]. For new leptons, the experimental lower bounds are weaker (45 and 90 GeV for stable and unstable neutral heavy leptons, respectively, and 100 GeV for charged leptons [7]).

Now, the possible extra generations of ordinary fermions couple to the SM Higgs field. This normally means that they cannot be much heavier than, say, 200 GeV, and the SU(2) doublet partners have to be approximately degenerate in mass to be consistent with the electroweak precision studies. We will assume that they have masses $O(250\text{ GeV})$. On the other hand, as the mirror fermions and the relevant Higgs system are singlets under $SU(2)_L$, the latter restriction is absent. In fact as already found in [1], it is more favorable from the point of view of the RG analysis that the mirror fermion masses are close to $\tilde{M}$, so that their contributions to K in Eq. (20) can be neglected.

Finally, let us recall that in this model the ordinary quarks and leptons are coupled to each other by the heavy PS gauge bosons with masses $O(M)$ and electric charges $\pm 2/3$. The detailed presentation of the $SU(4)_{PS}$ gauge boson sector can be found in [1], where the implications of these quark-lepton couplings for very rare or forbidden decays are also analyzed. We will update this analysis in Sec. V.

C. $SU(4)_{PS}\otimes[SU(2)]^3$

From Table I, we see that $\sin^2\theta_W^0=1/3$ in this case and one should have $C_S^2=8/3$. What are the appropriate fermion representations? As usual, the requirements are simply that these representations are anomaly-free under $SU(4)_{PS}\otimes[SU(2)]^3$, and that they appear in a sufficient number so as to ensure the equality of the three "weak" couplings above $\tilde{M}$. The most economical way to satisfy these requirements is to have the following fermion content for each generation, which also gives a rather interesting physical interpretation of $[SU(2)]^3$:

(a) $(4,2,2,1)_L$,
(b) $(4,1,2,2)_R$,

(c) $(4,2,1,1)_L$, $(4,2,1,1)_R$,
(d) $(4,1,1,2)_L$, $(4,1,1,2)_R$.

This is clearly a situation in which one has mixed representations of classes (i) and (ii). Before addressing the issues of charges, let us first verify whether (a)–(d) are anomaly-free. If (a) and (b) represent the same particles but with opposite chiralities, then they are anomaly-free when combined. Also, (c) and (d) are separately anomaly-free. In addition, the number of degrees of freedom for (a)–(d) combined is exactly what one needs to guarantee the equality of the G_W couplings above $\tilde{M}$.

The physical interpretation of $[SU(2)]^3$ is now clear, namely,

$$[G_W]_1 = SU(2)_L \otimes SU(2)_H \otimes SU(2)_R. \tag{53}$$

As we will show below, $SU(2)_H$ is the "horizontal" gauge group which links conventionally charged SM fermions to the unconventionally charged ones. To clearly see these features, let us write down explicitly the charge structure of the fermions in (a)–(d). First we look at (a) and (b).

In accordance with Eq. (37), Q_W for (a) and (b) is simply given by

$$Q_W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{54}$$

with the columns and the rows representing $SU(2)_{L,R}$ and $SU(2)_H$ doublets, respectively.

With $C_S^2 = 8/3$, the electric charges of the quarks and leptons are then given by

$$Q_q^i = 1/3 + \tilde{Q}_W^i, \quad Q_l^i = -1 + \tilde{Q}_W^i, \tag{55}$$

and consequently with Eq. (54) these charges are

$$Q_q = \begin{pmatrix} 1/3 & 4/3 \\ -2/3 & 1/3 \end{pmatrix} \tag{56}$$

for the quarks and

$$Q_l = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} \tag{57}$$

for the leptons. Notice that one now has quarks and leptons with unconventional charges, 4/3 and 2.

For (c) and (d), one has $Q_W = \pm 1/2$ as in Eq. (51). But since the charges of fermions are still given by Eq. (55), one now has the following charge assignments for the vectorlike quarks and leptons: $5/6, -1/6$ for the quarks and $-1/2$, $-3/2$ for the leptons. These are the "funny" charges mentioned in the previous section. Let us remember that these are vectorlike fermions and, therefore, can possess large masses which are not connected to the electroweak scale, nor to the scale of $SU(2)_R$ breaking. We shall come back to this point in the RG analysis.

To facilitate the discussion, we now present the following notation for the above quarks and leptons, for each generation. We have (with the electric charges shown in parentheses)

$$\psi^q_{L,R} = \begin{pmatrix} u(2/3) \\ d(-1/3) \end{pmatrix}_{L,R}, \tag{58a}$$

$$\tilde{Q}_{L,R} = \begin{pmatrix} \tilde{U}(4/3) \\ \tilde{D}(1/3) \end{pmatrix}_{L,R}, \tag{58b}$$

$$\psi^l_{L,R} = \begin{pmatrix} \nu(0) \\ l(-1) \end{pmatrix}_{L,R}, \tag{58c}$$

$$\tilde{L}_{L,R} = \begin{pmatrix} \tilde{l}_u(-1) \\ \tilde{l}_d(-2) \end{pmatrix}_{L,R}, \tag{58d}$$

$$\tilde{Q}'^{,\prime\prime}_{L,R} = \begin{pmatrix} \tilde{U}'^{,\prime\prime}(5/6) \\ \tilde{D}'^{,\prime\prime}(-1/6) \end{pmatrix}_{L,R}, \tag{58e}$$

$$\tilde{L}'^{,\prime\prime}_{L,R} = \begin{pmatrix} \tilde{l}'^{,\prime\prime}_u(-1/2) \\ \tilde{l}'^{,\prime\prime}_d(-3/2) \end{pmatrix}_{L,R}. \tag{58f}$$

In order to put these $SU(2)$ doublets into representations (a)–(d), we note that the following field transforms like a $\overline{2}$, which is equivalent to a 2 of $SU(2)_L$:

$$i\tau_2 \psi^{q,*}_{L,R} = \begin{pmatrix} d^*(1/3) \\ -u^*(-2/3) \end{pmatrix}_{L,R}, \tag{59}$$

with τ_2 being an $SU(2)_{L,R}$ generator.

Using the above definitions, one can write

$$(4,2,2,1)_L = [(i\tau_2\psi^{q,*}_L, \tilde{Q}_L), (\tilde{L}_L, \psi^l_L)], \tag{60}$$

$$(4,1,2,2)_R = [(i\tau_2\psi^{q,*}_R, \tilde{Q}_R), (\tilde{L}_R, \psi^l_R)] \tag{61}$$

and

$$(4,2,1,1)_{L,R} = [\tilde{Q}'_{L,R}, \tilde{L}'_{L,R}], \tag{62}$$

$$(4,1,1,2)_{L,R} = [\tilde{Q}''_{L,R}, \tilde{L}''_{L,R}]. \tag{63}$$

Three remarks are in order here.

First, the fermions in Eqs. (62),(63) are vectorlike and, in consequence, can have gauge-invariant bare masses which can be much larger than the electroweak scale.

Second, the placement of the quarks and leptons in Eqs. (60),(61) is such that there are *no tree-level* transitions between ordinary quarks and leptons mediated by the $SU(4)_{PS}$ gauge bosons. Indeed, in contrast to the previous scenario the electric charges of the PS gauge bosons are now $\pm 4/3$ and, as seen for instance in Eqs. (60), (56), and (57), these gauge bosons couple a left-handed ordinary anti-down-quark with charge 1/3 to a new heavy -1 charge lepton and a left-handed ordinary charged lepton with charge -1 to a new heavy 1/3 charge quark. Analogous comments apply to anti-up-quarks and neutrinos.

Third, as seen explicitly in Eqs. (56) and (57), the horizontal $SU(2)_H$ weak gauge bosons couple the ordinary quarks and leptons to new heavy quarks and leptons, respectively, and consequently there are no dangerous tree-level flavor changing neutral current transitions between the ordinary quarks and between the ordinary leptons mediated by the $SU(2)_H$ bosons.

As we shall see, the second property will prevent rare decays such as $K_L \to \mu e$ from acquiring large rates, even for masses of the PS gauge bosons as low as 1 TeV. Similar comments apply to horizontal $SU(2)_H$ gauge bosons with respect to FCNC transitions.

D. $SU(4)_{PS} \otimes [SU(3)]^2$

In this scenario the weak gauge group is

$$[G_W]_2 = SU(3)_L \otimes SU(3)_H \tag{64}$$

with the SM $SU(2)_L$ group being the subgroup of $SU(3)_L$. As we will show below the "horizontal" gauge group $SU(3)_H$ [similarly to $SU(2)_H$ in the previous scenario] links conventionally charged SM fermions to the unconventionally charged ones.

As we discussed above, $\sin^2\theta_W^0 = 3/8$ in this model and $C_S^2 = 8/3$ is required. The appropriate fermion representations that are together anomaly-free are then $(4,3,\bar{3})$ and $(4,\bar{3},3)$. The "weak charge" matrices are now written as

$$Q_W = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{65}$$

for $(4,3,\bar{3})$, and

$$Q_W = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{66}$$

for $(4,\bar{3},3)$, both with eigenvalues $0,\pm 1$. The charges for the fermions are given by Eq. (55) as in the previous scenario, but as only representation of class (ii) are present the fermions with "funny" charges are absent. We will soon see that the rows in Eqs. (65) and (66) correspond to $SU(3)_L$ triplets with the $SU(2)_L$ doublets occupying the first two entries in these triplets. The columns in Eqs. (65) and (66) correspond to $SU(3)_H$ triplets.

From Eqs. (31),(65), the three fundamental representations of $SU(3)_L$ have the weak charge distributions (0,1,1), $(-1,0,0)$, and $(-1,0,0)$. This corresponds to the electric charge distributions (1/3,4/3,4/3), $(-2/3,1/3,1/3)$, and $(-2/3,1/3,1/3)$ for the quarks and $(-1,0,0)$, $(-2,-1,-1)$, and $(-2,-1,-1)$ for the leptons. In short, each $(4,3,\bar{3})$ representation will have the following fermion content:

$$\begin{aligned}\Psi_1 = (&[(1/3,4/3,4/3),(-1,0,0)],\\ &[(-2/3,1/3,1/3),(-2,-1,-1)],\\ &[(-2/3,1/3,1/3),(-2,-1,-1)]).\end{aligned} \tag{67}$$

Similarly, each $(4,\bar{3},3)$ representation has the following fermion content:

$$\begin{aligned}\Psi_2 = (&[(1/3,-2/3,-2/3),(-1,-2,-2)],\\ &[(4/3,1/3,1/3),(0,-1,-1)],\\ &[(4/3,1/3,1/3),(0,-1,-1)]).\end{aligned} \tag{68}$$

To appreciate the physical meaning of Ψ_1 and Ψ_2, it is best to express them explicitly in terms of various particles. In particular, we would like to clearly distinguish fields which represent SM particles and those which represent new kinds of particles. For that purpose, we introduce left-handed Weyl fields grouped together as $SU(2)_L$ doublets or singlets. The electric charges are given in the parentheses. For the SM particles, we require, for each family, a left-handed lepton doublet, a left-handed quark doublet, a right-handed charged lepton, a right-handed up quark, and a right-handed down quark.

Since it is convenient to put into a given representation particles of the same chirality, we will make use, in subsequent discussions, of the usual definition of a charge conjugate field:

$$\psi^c_{L,R} \equiv \mathcal{C}\psi^q_{L,R}\mathcal{C}^{-1} = C\overline{\psi}^T_{R,L}, \tag{69}$$

where $C = i\gamma^2\gamma^0$.

First, we start with the $(4,3,\bar{3})$ representation. We shall first list normal quarks and leptons, followed by those that possess unusual electric charges. The notations used below should not be confused with the ones used in Sec. III C. One has

$$\psi^q_L = \begin{pmatrix} u(2/3) \\ d(-1/3) \end{pmatrix}_L, \quad d^c_L(1/3) = C\bar{d}^T_R, \tag{70a}$$

$$\psi^l_L = \begin{pmatrix} \nu(0) \\ l(-1) \end{pmatrix}_L, \quad \nu^c_L = C\bar{\nu}^T_R, \tag{70b}$$

$$Q_L = \begin{pmatrix} U(-1/3) \\ D(-4/3) \end{pmatrix}_L, \quad D^c_L(4/3) = C\bar{D}^T_R, \tag{70c}$$

$$L_{1L} = \begin{pmatrix} l_{u1}(2) \\ l_{d1}(1) \end{pmatrix}_L, \quad l^c_{d1,L}(-1) = C\bar{l}^T_{d1,R}, \tag{70d}$$

$$L_{2L} = \begin{pmatrix} l_{u2}(2) \\ l_{d2}(1) \end{pmatrix}_L, \tag{70e}$$

$$\tilde{\psi}_L^q = \begin{pmatrix} \tilde{u}(2/3) \\ \tilde{d}(-1/3) \end{pmatrix}_L, \quad d'_L(-1/3), \tag{70f}$$

$$l'_L(+1). \tag{70g}$$

In the above, we have put particles in $SU(2)_L$ doublets and singlets. To put these fields into the representation $(4,3,\overline{3})$, we shall need the following $SU(2)_L$ doublets obtained from above:

$$i\tau_2 L_{1L}^* = \begin{pmatrix} l_{d1}^*(-1) \\ -l_{u1}^*(-2) \end{pmatrix}_L, \quad i\tau_2 Q_L^* = \begin{pmatrix} D^*(4/3) \\ -U^*(1/3) \end{pmatrix}_L,$$

$$i\tau_2 \psi_L^{q,*} = \begin{pmatrix} d^*(1/3) \\ -u^*(-2/3) \end{pmatrix}_L, \quad i\tau_2 \tilde{\psi}_L^{q,*} = \begin{pmatrix} \tilde{d}^*(1/3) \\ -\tilde{u}^*(-2/3) \end{pmatrix}_L, \tag{71}$$

where τ_2 is a generator of $SU(2)_L$. One can now write $(4,3,\overline{3})$ in terms of specific fields, namely,

$$\Psi_1 = ([(i\tau_2 Q_L^*, D_L^c),(\psi_L^l, \nu_L^c)],$$
$$[(i\tau_2 \psi_L^{q,*}, d_L^c),(i\tau_2 L_{1L}^*, l_{d1,L}^c)],$$
$$[(i\tau_2 \tilde{\psi}_L^{q,*}, d_L^{\prime,*}),(i\tau_2 L_{2L}^*, l_L^{\prime,*})]). \tag{72}$$

From Eq. (72), one can identify the SM fields, namely, $\psi_L^l, i\tau_2\psi_L^{q,*}, \nu_L^c, d_L^c$. However, this representation is incomplete in that the right-handed charged lepton and up-quark fields are missing. This is where the $(4,\overline{3},3)$ representation comes in. The meaning of the non-SM fields appearing in Eq. (72) will be elucidated below.

For the $(4,\overline{3},3)$ representation, one can look at Eq. (68) to find the appropriate fields. To this end, let us introduce

$$\tilde{\psi}_{L,R}^l = \begin{pmatrix} \tilde{\nu}(0) \\ \tilde{l}(-1) \end{pmatrix}_{L,R}, \quad l_L^c(+1) = C\overline{l}_R^T, \tag{73a}$$

$$l'_R(+1), \tag{73b}$$

$$\tilde{\psi}_R^q = \begin{pmatrix} \tilde{u}(+2/3) \\ \tilde{d}(-1/3) \end{pmatrix}_R, \tag{73c}$$

$$L_{2R} = \begin{pmatrix} l_{u2}(2) \\ l_{d2}(1) \end{pmatrix}_R, \quad l_{u1,L}^c(-2) = C\overline{l}_{u1,R}^T, \tag{73d}$$

$$Q'_{L,R} = \begin{pmatrix} U'(-1/3) \\ D'(-4/3) \end{pmatrix}_{L,R}, \quad U_L^c(1/3) = C\overline{U}_R^T, \tag{73e}$$

$$d'_R(-1/3). \tag{73f}$$

From the above equations, one can immediately identify the following vectorlike fields: $L_{2L,R}$, $Q'_{L,R}$, $\tilde{\psi}_{L,R}^l$, $\tilde{\psi}_{L,R}^q$, $l'_{L,R}$, and $d'_{L,R}$.

Next, in order to match the charge assignments of Eq. (68), we define the following $SU(2)_L$ doublets, using the ones defined in Eq. (71):

$$\tilde{\psi}_L^{l,c} = C\overline{\tilde{\psi}}_R^{l,T} = \begin{pmatrix} \tilde{\nu}_L^c(0) \\ \tilde{l}_L^c(+1) \end{pmatrix}, \tag{74a}$$

$$i\tau_2 L_{2L}^c = i\tau_2 C\overline{L}_{2R}^T = \begin{pmatrix} l_{d2,L}^c(-2) \\ -l_{u2,L}^c(-1) \end{pmatrix}, \tag{74b}$$

$$i\tau_2 Q_L^{\prime,c} = i\tau_2 C\overline{Q}'^T_R = \begin{pmatrix} D_L^{\prime,c}(4/3) \\ -U_L^{\prime,c}(1/3) \end{pmatrix}, \tag{74c}$$

$$i\tau_2 Q_L^{\prime,*} = \begin{pmatrix} D_L^{\prime,*}(4/3) \\ -U_L^{\prime,*}(1/3) \end{pmatrix}, \tag{74d}$$

$$i\tau_2 \tilde{\psi}_L^{q,c} = i\tau_2, C\overline{\tilde{\psi}}_R^{q,T} = \begin{pmatrix} \tilde{d}_L^c(1/3) \\ -\tilde{u}_L^c(-2/3) \end{pmatrix}. \tag{74e}$$

The representation $(4,\overline{3},3)$ can now be written explicitly as

$$\Psi_2 = ([(i\tau_2 \tilde{\psi}_L^{q,c}, u_L^c),(i\tau_2 L_{2L}^c, l_{u1,L}^c)],$$
$$[(i\tau_2 Q_L^{\prime,c}, U_L^c),(\tilde{\psi}_L^{l,c}, l_L^c)],$$
$$[(i\tau_2 Q_L^{\prime,*}, d_L^{\prime,c}),(\tilde{\psi}_L^{l,*}, l_L^{\prime,c})]). \tag{75}$$

Several remarks are in order here. First, the $(4,3,\overline{3})$ and $(4,\overline{3},3)$ representations, as described by Ψ_1 and Ψ_2, together form an anomaly-free representation of the group $SU(4)_S \otimes [SU(3)]^2$. Second, the particle content described in Eqs. (70) and (73) has the following features.

(1) There are two types of families with SM transformations under $SU(2)_L$, i.e., left-handed doublets and right-handed singlets: one contains the SM quarks and leptons and the other one contains unconventional quarks and leptons with charges up to 4/3 (for the quarks) and 2 (for the leptons). The unconventional fields are Q_L, D_L^c, U_L^c, L_{1L}, $l_{d1,L}^c$, and $l_{u1,L}^c$. The (normal and unconventional) quarks and leptons couple to the SM Higgs field. This normally means that their masses cannot be much heavier than, say, 200 GeV.

(2) There are, in addition, two families of quarks and leptons, $(\tilde{\psi}^q,\tilde{\psi}^l)_{L,R}$ and $(Q',L_2)_{L,R}$, with normal and unconventional charges which are *vectorlike* under $SU(2)_L$. This means that their masses come from sources other than the SM Higgs field and they can be much heavier than the first two types of families mentioned above.

(3) Next, there are two *vectorlike* $SU(2)_L$ singlets with charge $+1$ for the leptonlike color singlet ($l'_{L,R}$) and charge $-1/3$ for the quarklike color triplet ($d'_{L,R}$). They also can acquire large masses.

Finally as in the previous scenario we have two phenomenologically very relevant properties that can be clearly seen in Eqs. (72),(75).

(1) The placement of the quarks and leptons in Eqs. (72), (75) is such that there are *no tree-level* transitions between ordinary quarks and leptons mediated by the $SU(4)_{\rm PS}$ gauge bosons. Also here the electric charges of the PS gauge bosons are $\pm 4/3$.

(2) The horizontal $SU(3)_H$ weak gauge bosons couple the ordinary quarks and leptons to new heavy quarks and leptons, respectively, and consequently there are no dangerous tree-level flavor changing neutral current transitions between the ordinary quarks and between the ordinary leptons mediated by the $SU(3)_H$ bosons.

IV. RG ANALYSIS OF $\sin^2\theta_W$

A. Preliminaries

In 1981 the values of $\sin^2\theta_W(M_Z^2)$ and $\alpha_s(M_Z^2)$ were rather poorly known. As of 2003 we know them with a very high precision as given in Eqs. (22) and (25) with $\alpha_s(M_Z^2)$ substantially smaller than in 1981 so that the $O(\alpha/\alpha_s)$ correction in Eq. (17) now plays a bigger role. In this section we will update our 1981 renormalization group analysis of PUT_0 and generalize it to the additional scenarios considered in the previous section.

The master formula for $\sin^2\theta_W(M_Z^2)$ in Eq. (17) was obtained in the one-loop approximation, whereas the values of $\sin^2\theta_W(M_Z^2)_{\rm expt}$, $\alpha_s(M_Z^2)$, and $\alpha(M_Z^2)$ were extracted from various data including higher order QCD and electroweak corrections. Strictly speaking we should then generalize Eq. (17) to include two-loop contributions. This would be indispensible in the case of grand unified theories where μ varies from M_Z to 10^{16} GeV and the change of the gauge couplings in this range is substantial. On the other hand, in the case of early unification, the changes of the couplings between M_Z and $(\tilde{M},M)$ that are in the TeV range are rather small and the two-loop contributions to Eq. (17) are insignificant. In what follows we will therefore use the one-loop formula (17), relegating the RG analysis at two-loop level to a future paper.

While M and $\tilde{M}$ differ in principle from each other, with $M\geqslant\tilde{M}$, we will first set $\tilde{M}=M$. Consequently, the last term in Eq. (17) is absent and only the coefficient K has to be calculated. On the other hand, in the scenarios considered, there are new particles with masses below M and their contributions to Eq. (17) have to be taken into account. Now, as discussed in the previous section, all new particles with nontrivial properties under $SU(2)_L$ that are not vectorlike cannot have masses much larger than 200 GeV. In the RG analysis we will set all these masses to be equal to a single scale M_F with

$$M_F=250\pm50\ \ \mathrm{GeV}, \tag{76}$$

and we will assume that all the remaining new particles have masses very close to M so that their contributions to Eq. (17) can be neglected.

Under these assumptions, the following replacement should be made in Eq. (17):

$$K\ln\frac{\tilde{M}}{M_Z}\to K_{n_G=3}\ln\frac{M_F}{M_Z}+K_{\rm total}\ln\frac{M}{M_F} \tag{77}$$

where

$$K_{n_G=3}=[b_1-C_W^2b_2-C_S^2b_3]_{n_G=3}, \tag{78}$$

with the b_i's receiving only contributions from the three ordinary generations (n_G) of quarks and leptons and the SM Higgs doublet. On the other hand,

$$K_{\rm total}=[b_1-C_W^2b_2-C_S^2b_3]_{\rm total} \tag{79}$$

includes all particles with masses below M.

With $\tilde{M}=M$, M_F given in Eq. (76), $\alpha_s(M_Z^2)$ and $\alpha(M_Z^2)$ known experimentally, and C_S^2, C_W^2, and b_i fixed (see below) in each scenario we can determine the value of M that is consistent with the experimental value $\sin^2\theta_W(M_Z^2)_{\rm exp}$ in Eq. (25). This is what we will do first. Subsequently we will analyze the general case with $\tilde{M}\leqslant M$. In the next section we will investigate whether the values of M determined here are consistent with bounds on rare decays.

B. $SU(4)_{\rm PS}\otimes[SU(2)]^4$

In this scenario

$$\sin^2\theta_W^0=\frac{1}{4},\quad C_W^2=3,\quad C_S^2=\frac{2}{3}, \tag{80}$$

and

$$b_1=\frac{1}{48\pi^2}\left[\frac{20}{3}n_G+\frac{1}{2}\right], \tag{81}$$

$$b_2=\frac{1}{48\pi^2}\left[4n_G+\frac{1}{2}-22\right], \tag{82}$$

$$b_3=\frac{1}{48\pi^2}[4n_G-33] \tag{83}$$

with $n_G=3$ in $K_{n_G=3}$ and $n_G\geqslant3$ in $K_{\rm total}$. The "1/2" is the contribution of the Higgs doublet.

We find then

$$M\leqslant330\ \ \mathrm{GeV},\quad n_G=3, \tag{84}$$

which is clearly excluded. Including new generations of ordinary fermions with masses $O(M_F)$ allows us to increase M as seen in the following formula:

$$\sin^2\theta_W(M_Z^2)=0.2389-0.0065\ln\frac{M_F}{M_Z}-0.0001P\ln\frac{M}{M_F},\tag{85}$$

where

$$P=87-8n_G.\tag{86}$$

As the coefficient in front of the last logarithm in Eq. (85) must be very small in order to obtain the correct $\sin^2\theta_W(M_Z^2)$, the result for M in this scenario is rather sensitive to the input parameters, in particular n_G and M_F. However, requiring $M_F\geqslant 200$ GeV and $M\geqslant 800$ GeV, we find the lowest acceptable value for n_G to be $n_G=9$.

On the other hand, making the model supersymmetric (SUSY) and setting as an example the masses of all SUSY particles equal to M_F, one finds

$$[b_1]_{\rm total}=\frac{1}{48\pi^2}[10n_G+3],\tag{87}$$

$$[b_2]_{\rm total}=\frac{1}{48\pi^2}[6n_G+3-18],\tag{88}$$

$$[b_3]_{\rm total}=\frac{1}{48\pi^2}[6n_G-27].\tag{89}$$

This gives the formula (85) with

$$P=66-12n_G\tag{90}$$

and the lowest acceptable value for n_G is $n_G=4$. For $n_G=3$ we find $M\leqslant 550$ GeV, which is excluded.

Whether this model is supersymmetric or not, the compatibility of this scenario with the experimental value of $\sin^2\theta_W(M_Z^2)_{\rm expt}$ requires, for $M\geqslant 800$ GeV, many new particles around the M_F scale.

The RG analysis of $SU(4)^2$ and $SU(8)$ proceeds in a similar manner but as these groups are very large we will not consider them further.

C. $SU(4)_{\rm PS}\otimes[SU(2)]^3$

In this scenario

$$\sin^2\theta_W^0=\frac{1}{3},\quad C_W^2=2,\quad C_S^2=\frac{8}{3},\tag{91}$$

and $[b_i]_{n_G=3}$ are simply given by Eqs. (81)–(83). Above M_F new generations of quarks and leptons with unconventional electric charges contribute and we find

$$[b_1]_{\rm total}=\frac{1}{48\pi^2}\left[\frac{20}{3}n_G+\frac{1}{2}+\frac{116}{3}n_G^{\rm new}\right],\tag{92}$$

$$[b_2]_{\rm total}=\frac{1}{48\pi^2}\left[4(n_G+n_G^{\rm new})+\frac{1}{2}-22\right],\tag{93}$$

$$[b_3]_{\rm total}=\frac{1}{48\pi^2}[4(n_G+n_G^{\rm new})-33]\tag{94}$$

with

$$n_G^{\rm new}=n_G.\tag{95}$$

We note in particular the large contribution of the new fermions to b_1, which is related to the high charges of these fermions. This gives for $n_G=3$

$$\sin^2\theta_W(M_Z^2)=0.2740-0.0132\ln\frac{M_F}{M_Z}-0.0215\ln\frac{M}{M_F}.\tag{96}$$

We observe that the coefficients of the logarithms are much larger than in the previous scenario and the correct value of $\sin^2\theta_W(M_Z^2)$ can be found with low unification scale and $n_G=3$ in spite of the much higher value of $\sin^2\theta_W^0$. Scanning $\alpha_s(M_Z^2)$ and M_F in the ranges of Eqs. (22) and (76), respectively, and requiring (at the 2σ level)

$$0.23083\leqslant\sin^2\theta_W(M_Z^2)\leqslant 0.23143,\tag{97}$$

we find

$$M=1.00\pm0.14\ \text{TeV},\quad n_G=3,\tag{98}$$

with *lower* values for $n_G>3$. Thus in this scenario additional generations of ordinary quarks and leptons are disfavored although $n_G=5$ would still give $M\geqslant 800$ GeV.

D. $SU(4)_{\rm PS}\otimes[SU(3)]^2$

In this scenario

$$\sin^2\theta_W^0=\frac{3}{8},\quad C_W^2=\frac{5}{3},\quad C_S^2=\frac{8}{3},\tag{99}$$

and the b_i coefficients are the same as in the last scenario. In this case Eq. (96) is replaced by

$$\sin^2\theta_W(M_Z^2)=0.3083-0.0144\ln\frac{M_F}{M_Z}-0.0243\ln\frac{M}{M_F}\tag{100}$$

and we find

$$M=(3.30\pm0.47)\ \text{TeV},\quad n_G=3,\tag{101}$$

with *lower* values for $n_G>3$. For instance for $n_G=4$ and $n_G=5$, M is found for the central values of the input parameters in the ballpark of 3.0 TeV and 2.6 TeV, respectively.

E. $SU(4)_{\rm PS}\otimes[SU(2)]^2$

Finally, let us consider the original Pati-Salam model [6]. Here

$$\sin^2\theta_W^0=\frac{1}{2},\quad C_W^2=1,\quad C_S^2=\frac{2}{3},\tag{102}$$

and the b_i coefficients are the same as in the $SU(4)_{\rm PS} \otimes [SU(2)]^4$ scenario. This gives

$$M \approx 5 \times 10^{10}\ \text{TeV}, \quad n_G = 3, \tag{103}$$

with higher values for $n_G > 3$. Clearly this model is not an early unification model.

F. The case of $\tilde{M} \neq M$

Let us finally consider the general case $\tilde{M} \leqslant M$ with $\tilde{M} \geqslant 800$ GeV as required by the lower limit of right-handed gauge boson masses in the case of $[SU(2)]^4$ and $[SU(2)]^3$ scenarios. The latter restriction is absent in the case of $SU(3)^2$ but as we will see below in this case $\tilde{M}$ has to be above 1 TeV if we want $M \leqslant 10$ TeV.

For $\tilde{M} \leqslant M$ the last logarithm in Eq. (77) is replaced as follows:

$$K_{\rm total} \ln \frac{M}{M_F} \rightarrow K_{\rm total} \ln \frac{\tilde{M}}{M_F} + K' \ln \frac{M}{\tilde{M}} \tag{104}$$

with K' defined in Eq. (21).

Now, the values of $\tilde{b}$ and of $\tilde{b}_3$ relevant for the evolution of the couplings $\tilde{g}_S$ and g_3 for scales above $\tilde{M}$ include contributions from all fermions present in the model, that is, the vectorlike ones also. However, as $SU(3)_c$ and $U(1)_S$ are subgroups of $SU(4)_{\rm PS}$, the contributions of all fermions to $\tilde{b}$ and of $\tilde{b}_3$ are equal to each other at the one-loop level and consequently we find

$$K' = C_S^2 \frac{33}{48\pi^2} \tag{105}$$

for all nonsupersymmetric scenarios considered here with 33 replaced by 27 in the case of supersymmetry.

In the case of PUT_0 the factor $C_S^2 33 = 22$ in K' should be compared with 15 present in $K_{\rm total}$ for $n_G = 9$. Consequently the evolution between $\tilde{M}$ and M is essentially the same as between M_F and $\tilde{M}$ and making $\tilde{M} \neq M$ will not help to increase the value of M. It will even lower it.

In the case of PUT_1 the factor $C_S^2 33 = 88$ in K' should be compared with 311/2 present in $K_{\rm total}$. Therefore lowering $\tilde{M}$ to 800 GeV allows for central values of all parameters to increase M from 1.0 TeV in Eq. (98) to approximately 1.2 TeV.

In the case of PUT_2 the factor $C_S^2 33 = 88$ in K' should be compared with 493/3 present in $K_{\rm total}$. Therefore lowering $\tilde{M}$ to 800 GeV allows for central values of all parameters to increase M from 3.3 TeV in Eq. (101) to as high as 9.9 TeV.

In Fig. 1 we show the allowed regions in the space $(\tilde{M}, M)$ that have been obtained by varying $\alpha_S(M_Z^2)$, M_F, and $\sin^2\theta_W(M_Z^2)$ in the ranges of Eqs. (22), (76), and (97), respectively. For a given $\tilde{M}$, the maximal value of M is found for the minimal $\sin^2\theta_W(M_Z^2)$ and maximal values of M_F and $\alpha_S(M_Z^2)$. The minimal value of M is found for the

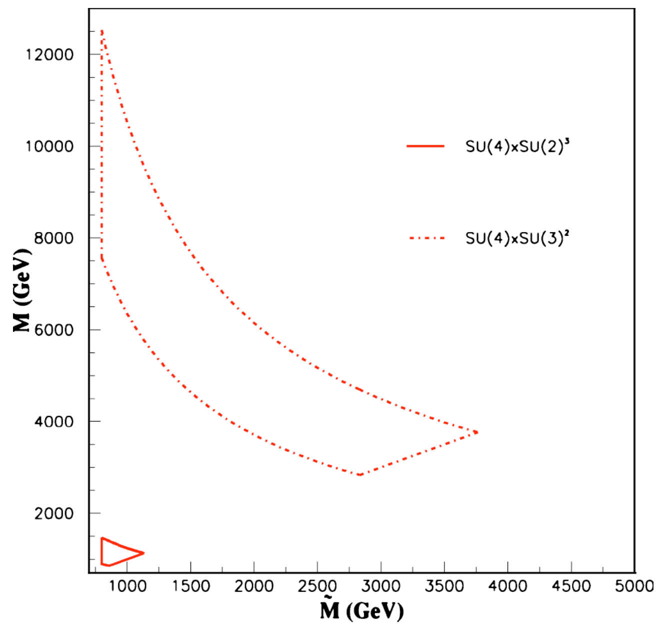

FIG. 1. The allowed ranges for the $SU(2)^3$ and $SU(3)^2$ scenarios as discussed in the text.

maximal $\sin^2\theta_W(M_Z^2)$ and minimal values of M_F and $\alpha_S(M_Z^2)$. The vertical boundary lines at $\tilde{M} = 800$ GeV have been set as discussed above and the boundary lines on the right represent the case $\tilde{M} = M$ considered previously. See the ranges in Eqs. (98) and (101).

We observe that even when $\tilde{M} \neq M$ the two scales have to be rather close to 1 TeV in the $SU(2)^3$ scenario. On the other hand, a much larger allowed region is obtained in the case of the $SU(3)^2$ scenario, where $\tilde{M}$ and M can differ by even an order of magnitude. However, we find that if M is required to be less than 10 TeV, the scale $\tilde{M}$ has to be larger than ~ 1.1 TeV.

G. Summary

We observe that, whereas the $SU(2)^4$ scenario requires new generations of ordinary quarks and leptons in order to be consistent with the experimental value of $\sin^2\theta_W(M_Z^2)$ and $M > 800$ GeV, in the case of the scenarios $SU(2)^3$ and $SU(3)^2$, the correct value of $\sin^2\theta_W(M_Z^2)$ in the case of $\tilde{M} = M$ can be obtained with $n_G = 3$ for $M \approx 1$ TeV and $M \approx 3.3$ TeV, respectively. In Fig. 2 we show $\sin^2\theta_W(M_Z^2)$ as a function of M for the $SU(2)^4$ scenario with $n_G = 9$ and for the scenarios $SU(2)^3$ and $SU(3)^2$ with $n_G = 3$. To this end we have set $\alpha_s(M_Z^2)$ and M_F to their central values. The curve for the supersymmetric scenario $SU(2)^4$ with $n_G = 4$ is rather similar to the nonsupersymmetric case with $n_G = 9$ shown in the figure. The large sensitivity to M_F in the case of the $SU(2)^4$ scenario is shown by the curve with $M_F = 200$ GeV.

Removing the equality $\tilde{M} = M$ and lowering $\tilde{M}$ to 800 GeV, has essentially no impact on the value of M in the case of the $SU(2)^4$ scenario. An increase of M by at most

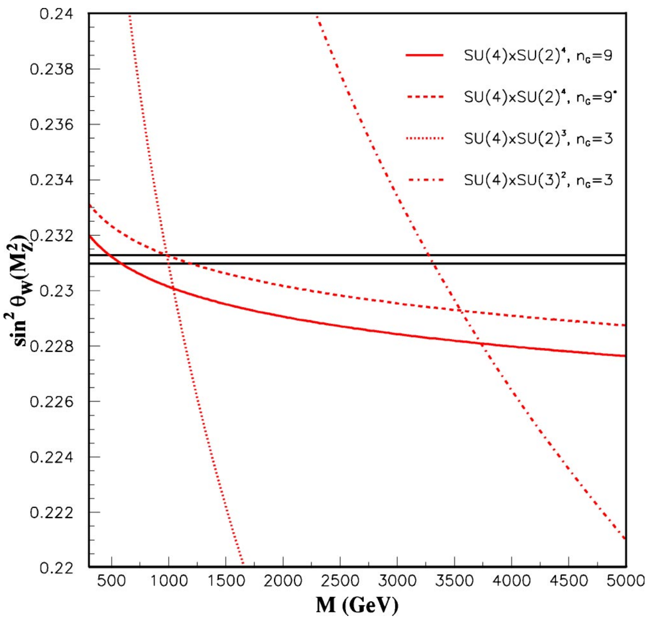

FIG. 2. $\sin^2\theta_W(M_Z^2)$ as a function of M in various scenarios. The horizontal band represents the experimental value. The dashed curve ($n_G=9*$) is obtained by using $M_F=200$ GeV, while the other three curves are obtained by using $M_F=250$ GeV.

300 GeV is found in the case of the $SU(2)^3$ scenario, implying that in this model M and $\tilde{M}$ are forced to be of the same order of magnitude and in the ballpark of 1 TeV. On the other hand in the $SU(3)^2$ scenario M can be by an order of magnitude larger than $\tilde{M}$ and be as high as 12 TeV. The allowed regions are shown in Fig. 1.

V. ON $K_L \to \mu e$

A. Preliminaries

In our choice of $SU(4)_{\mathrm{PS}}$ as the strong group, we had already noticed in [1] that the heavy PS gauge bosons which connect quarks to leptons can, in principle, induce the rare decay process $K_L \to \mu e$. In the most naive version of the process, $K_L \to \mu e$ can occur at the tree level (only in the $SU(4)_{\mathrm{PS}} \otimes [SU(2)]^4$ case) if one assumes, as we did in [1], some kind of "kinship" hypothesis such as $d \leftrightarrow e$ and $s \leftrightarrow \mu$. That is, no generation mixing. With this hypothesis, we obtained an effective Lagrangian for the subprocess $d+\mu \to e+s$ of the form

$$\mathcal{L}_{\mathrm{eff}}^{d\mu \to es} = \sqrt{2} G_S \sum_{i=1}^{3} (\overline{d}_i \gamma_\mu e \overline{\mu} \gamma^\mu s_i + \mathrm{H.c.}), \tag{106}$$

where the sum is over color and where

$$g_S^2/2m_G^2 = \sqrt{2} G_S. \tag{107}$$

In Eq. (107), the quantity m_G represents a typical mass of the PS gauge bosons and is comparable to the scale M.

In [1] we made the estimate of the branching ratio for $K_L \to \mu^\pm e^\mp$ by comparing this decay with $K_L \to \mu\overline{\mu}$. However, it will be more convenient to calculate $BR(K_L \to \mu^\pm e^\mp)$ directly. Making the Fierz transformation in Eq. (106) and neglecting the axial-vector-current contribution as in [1] we find the amplitude

$$A(K_L \to \mu^\pm e^\mp) = i F_K G_S \frac{m_K^2}{m_s + m_d} [(\overline{\mu} \gamma_5 e) + (\overline{e \gamma_5 \mu})] \tag{108}$$

where m_K is the kaon mass, F_K the kaon decay constant, and $m_{s,d}$ the current quark masses. Neglecting the electron mass, we find

$$BR(K_L \to \mu^\pm e^\mp) = \frac{\pi}{2} \frac{\alpha^2}{m_G^4} m_K F_K^2 \tau(K_L) \sqrt{1 - \frac{m_\mu^2}{m_K^2}} \left[\frac{m_K^2}{m_s + m_d}\right]^2. \tag{109}$$

Using $F_K = 160$ MeV, $m_s + m_d = 140$ MeV, and the values for m_K, $\tau(K_L)$, and m_μ from [7], we find

$$BR(K_L \to \mu^\pm e^\mp) = 4.7 \times 10^{-12} \left(\frac{\alpha_S(m_G)}{0.1}\right)^2 \left[\frac{1.8 \times 10^3 \ \mathrm{TeV}}{m_G}\right]^4 \tag{110}$$

to be compared with the experimental bound [7]

$$BR(K_L \to \mu e) < 4.7 \times 10^{-12}. \tag{111}$$

Now, $\alpha_S(m_G) = \alpha_3(m_G)$ and as the presence of new particles at scales lower than m_G slows down the running of the QCD coupling constant, $\alpha_3(m_G)$ with $m_G = O(1$ TeV) is not significantly different from 0.1. We conclude then that in a scenario with no generation mixing and tree-level contributions, the branching ratio $BR(K_L \to \mu^\pm e^\mp)$ with $m_G = O(1$ TeV) violates the experimental bound by at least 13 orders of magnitude.

Let us then consider the presence of possible mixing among generations. To be correct, we first denote the $T_{3L} = -1/2$ quarks by $D_0 = (d_0, s_0, b_0)$ and similarly by $L_0 = (e_0, \mu_0, \tau_0)$ for the leptons, with the subscripts 0 referring to the eigenstates before mass mixing. A typical $SU(4)/[SU(3) \otimes U(1)_{B\text{-}L}]$ current would be of the form $J_{LQ}^\mu = \overline{D}_0 \gamma^\mu L_0$. Notice that this discussion applies only to the case $SU(4)_{\mathrm{PS}} \otimes [SU(2)]^4$ where tree-level SM leptoquark transitions can occur. If we now diagonalize the mass matrices for the down quark and for the charged lepton sectors, we can express D_0 and L_0 in terms of the mass eigenstates as follows: $D = U_D D_0$ and $L = U_L L_0$. The above current can be rewritten as $J_{LQ}^\mu = \overline{D} \gamma^\mu U_D U_L^{-1} L$. One now has the quark-lepton mixing matrix $V_{LQ} = U_D U_L^{-1}$ involved in all quark-lepton transitions. In consequence, what should appear on the right-hand sides of Eqs. (106) and (109) are extra factors $V_{ed}^* V_{\mu s}$ and $|V_{ed}^* V_{\mu s}|^2$, respectively. Here V_{ed} and $V_{\mu s}$ are matrix elements of V_{LQ}.

In the absence of a convincing model of fermion masses, there is no reason to rule out the possibility that the mixing coefficient $|V_{ed}V_{\mu s}|^2$ could be of order 10^{-13}, but such a very strong suppression appears rather strange and unnatural. Moreover, as V_{LQ} is a unitary matrix, not all of its elements can be set to zero, and consequently even if the $K_L\rightarrow\mu e$ bound can be satisfied in this manner, other elements of V_{LQ} that are relevant for lepton flavor violation in B decays could be too large. Clearly, the presence of more than three generations, and consequently of many free parameters in V_{LQ}, could help, but such a fine-tunning in essentially all processes is rather *ad hoc*.

We conclude therefore that an early unification of quark and leptons requires either the absence of tree-level contributions to $K_L\rightarrow\mu e$ and to analogous very rare decays or the presence of a new suppression mechanism in addition to $|V^*_{ed}V_{\mu s}|^2$ considered above.

We shall now discuss the implication of these findings for the three candidates presented in the previous section, namely, $SU(4)_{\mathrm{PS}}\otimes[SU(2)]^4$, $SU(4)_{\mathrm{PS}}\otimes[SU(2)]^3$, and $SU(4)_{\mathrm{PS}}\otimes[SU(3)]^2$.

B. $SU(4)_{\mathrm{PS}}\otimes[SU(2)]^4$

In this scenario, the decay $K_L\rightarrow\mu e$ takes place at the tree level and the RG analysis above has shown that the PUT scale is typically around 1 TeV or less in order to agree with the experimental value for $\sin^2\theta_W(M_Z^2)$. Consequently, as just discussed, this scenario is ruled out unless additional suppression mechanisms in addition to $|V^*_{ed}V_{\mu s}|^2$ can be invoked.

This could come from aspects of physics of large extra dimensions for example. One could add, for instance, an extra spatial dimension (for the purpose at hand) and denote it, for simplicity, by y. It has been shown that the compactification of this extra dimension on an orbifold S_1/Z_2 gives rise to chiral zero modes in four dimensions [12]. In [8], it was proposed that $SU(4)_{\mathrm{PS}}$ is broken by boundary conditions. As a consequence, a quartet that contains a quark and a lepton can have only one chiral zero mode, which could be either a quark or a lepton, with the other one being a heavy partner. Since SM particles are supposed to be chiral zero modes in four dimensions, they cannot belong to the same quartet. Therefore there is no transition between SM quarks and leptons via the PS gauge bosons at the tree level, and M can be as low as a few TeV. Another possibility is the following scenario. The interaction of these chiral zero modes with a background scalar field which has a kink solution along the extra dimension has the effect of localizing these chiral zero modes at various locations along y. These chiral zero modes would represent the quarks and leptons of the SM. An effective interaction in four dimensions which involves a quark and a lepton, such as the leptoquark transition generated by the PS gauge bosons, will contain a factor

$$C_{ql}=\int\xi_q(y)\xi_l(y)dy \tag{112}$$

in the effective coupling, where $\xi_q(y)$ and $\xi_l(y)$ represent the wave functions along y of the quark and lepton chiral zero modes, respectively. When the quarks and leptons are localized far away from each other along y, the factor C_{ql} can be exponentially small [13]. If this scenario is correct then the bound (111) can easily be satisfied for this model if $|V_{ed}V_{\mu s}C_{de}C_{s\mu}|^2$ is of order 10^{-13}. Even if $|V_{ed}V_{\mu s}|^2$ were of the order of unity, it is not hard to arrange for $|C_{de}C_{s\mu}|^2$ to be of order 10^{-13}, i.e., for $|C_{de}C_{s\mu}|\sim10^{-6}$.

We observe then that the constraint from $K_L\rightarrow\mu e$ has severe implications on the $SU(4)_S\otimes[SU(2)]^4$ model because of the low PUT scale as required by the fit to the value of $\sin^2\theta_W(M_Z^2)$. It implies either or both of the following scenarios: (1) The mass matrices are such that $|V_{ed}V_{\mu s}|^2$ is very small; and/or (2) A suppression mechanism exists coming from the physics of large extra dimensions.

C. $SU(4)_{\mathrm{PS}}\otimes[SU(2)]^3$

As we saw in Sec. III, the particle content of this group is rather interesting. The SM fermions belong to $(4,2,2,1)_L=[(i\tau_2\psi_L^{q,*},\tilde{Q}'_L),(\tilde{L}_L,\psi_L^l)]$ and $(4,1,2,2)_R=[(i\tau_2\psi_R^{q,*},\tilde{Q}'_R),(\tilde{L}_R,\psi_R^l)]$. From this fermion content, one can see that the $SU(4)/[SU(3)\otimes U(1)_{B\text{-}L}]$ gauge bosons with electric charges $\pm4/3$ link the normal quarks $i\tau_2\psi_{L,R}^{q,*}$ with the higher charged leptons $\tilde{L}_{L,R}$, and the normal leptons $\psi_{L,R}^l$ with the higher charged quarks $\tilde{Q}'_{L,R}$. What this implies is that, at the tree level, there is no transition between normal quarks and normal leptons. However, it can occur at the one-loop level through a box diagram with two PS boson exchanges $[M_{\mathrm{PS}}=O(M)]$ and new heavy quarks $(\tilde{Q})$ and new heavy leptons $(\tilde{L})$ that have masses $O(M_F)$ with M_F given in Eq. (76). $\tilde{Q}$ and $\tilde{L}$ appear in three generations and the mixing between these generations is given by 3×3 matrices to be denoted by U and V, respectively. In the case of degenerate masses of $\tilde{Q}_i$ and $\tilde{L}_i$ the GIM mechanism is at work and the decay $K_L\rightarrow\mu e$ is absent. However, the GIM mechanism remains powerful also when the masses are nondegenerate but all in the range 200–300 GeV. In this case it provides a suppression factor of $O(10^{-4})$ at the level of the branching ratio. With the typical loop factor $(16\pi^2)^{-2}\approx4\times10^{-5}$, the upper bound on the relevant mixing factors $|V_{id}V^*_{is}|^2|U_{jd}U^*_{js}|^2$ coming from $K_L\rightarrow\mu e$ amounts roughly to $O(10^{-4})$ and can be easily satisfied.

A detailed presentation of this calculation and the analysis of FCNC processes mediated by the $SU(2)_H$ bosons is beyond the scope of this paper and will be presented elsewhere, but this discussion shows that, in this scenario, the low unification scale required by the value of $\sin^2\theta_W(M_Z^2)$ is consistent with the present upper bound on $K_L\rightarrow\mu e$ and does not pose any problems for FCNC transitions at present.

D. $SU(4)_{\mathrm{PS}}\otimes[SU(3)]^2$

The constraint coming from $K_L\rightarrow\mu e$ in this model is very similar to the previous one. A look at the fermion content, as shown in Eqs. (72),(75), reveals that the PS gauge bosons once more link normal quarks and leptons to their higher charged counterparts. As a result, there is no tree-level contribution to $K_L\rightarrow\mu e$. Again, this process will occur at one loop, with an analysis similar to the one mentioned above.

VI. COMPARISON WITH THE LITERATURE

In order to make an assessment of our work and compare it with recent attempts at ''low scale'' unification, we summarize below the essential results that were presented above. The three ''simplest'' candidates for petite unification—a possible nickname could be ''Tevunification''—are $SU(4)_{\mathrm{PS}}\otimes[SU(2)]^4$, $SU(4)_{\mathrm{PS}}\otimes[SU(2)]^3$, and $SU(4)_{\mathrm{PS}}\otimes[SU(3)]^2$. As mentioned at various places in the paper, the philosophy of our petite unification is to have a unification scale $M\leqslant 1000$ TeV and preferably $M\leqslant 10$ TeV.

$PUT_0 = SU(4)_{\mathrm{PS}} \otimes SU(2)_L \otimes SU(2)_R \otimes S\tilde{U}(2)_L \otimes S\tilde{U}(2)_R$. This is the favorite scenario in our 1981 paper [1]. This model has only quarks and leptons (including possible new ones) having standard electric charges. In our update of various numerical results, the conclusions drawn from our analysis can be summarized as follows. In order to obtain the correct value of $\sin^2\theta_W(M_Z^2)$ and requiring that $M\sim 1$ TeV, our RG analysis (assuming $\tilde{M}=M$) reveals that we need at least nine generations ($n_G=9$), with the new generations having masses of order 250 GeV, or $n_G=4$ if we include supersymmetry. In our RG analysis, the main important assumption which is made is that the masses of all new particles are taken to be of order 250 GeV. No additional assumptions are made about extra new physics other than petite unification above the scale M at this stage.

However, this scenario with a PUT scale of order 1 TeV suffers from the problem with the branching ratio for the process $K_L\to\mu e$, which in this scenario can occur at the tree level. Several possible remedies were discussed above, in particular in the context of the physics of large extra dimensions.

$PUT_1=SU(4)_{\mathrm{PS}}\otimes SU(2)_L\otimes SU(2)_H\otimes SU(2)_R$. In this model the PUT scale is required to be $M\sim 1$ TeV. In addition to the standard three generations of quark and leptons, three new generations of unconventional quarks and leptons with charges up to 4/3 (for quarks) and 2 (for leptons) and masses $O(250$ GeV$)$ are automatically present. The horizontal group $SU(2)_H$ connects the standard fermions with the unconventional ones. In addition, there are also very heavy vectorlike particles, which, however, are irrelevant to the phenomenology discussed in this paper. Furthermore, in this model, the process $K_L\to\mu e$ is forbidden at the tree level and appears only at the one-loop level. In consequence, despite the appearance of a low PUT scale, the constraint from $K_L\to\mu e$ can easily be satisfied, in contrast to the $SU(2)^4$ scenario. No additional new physics such as large extra dimensions is needed at this stage.

$PUT_2=SU(4)_{\mathrm{PS}}\otimes SU(3)_L\otimes SU(3)_H$. In this model the PUT scale is required to be in the range $M\sim 3.3$–10 TeV. Here, the horizontal group $SU(3)_H$ connects the standard fermions with the unconventional ones. It also contains new higher charged quarks and leptons with masses as in the $SU(2)^3$ scenario. Also, the process $K_L\to\mu e$ occurs only at one loop, and the experimental bound for this decay can be easily satisfied as well. Again, no additional new physics is needed at this stage.

In summary, PUT$_1$ and PUT$_2$ are able to predict $\sin^2\theta_W(M_Z^2)$ and to satisfy the constraint on $K_L\to\mu e$ *within the perturbative regime*. The offshoot of this is the prediction of the existence of three generations of unconventional quarks and leptons with charges up to 4/3 (for quarks) and 2 (for leptons) and masses $O(250$ GeV$)$.

Having briefly summarized the results of our three ''favorite'' scenarios, we are now ready to make a comparison with the literature (surely an incomplete task). In particular, we would like to compare our results with those of [8] and [9], whose main focus was to derive $\sin^2\theta_W$.

Reference [8] basically generalized our $SU(4)_{\mathrm{PS}}\otimes[SU(2)]^4$ model of 1981 to large extra dimensions. This paper was motivated by the possibility of a TeV scale unification. The first goal there was to obtain a reasonable estimate for $\sin^2\theta_W(M_Z^2)$ for a unification scale of $O(1$ TeV$)$. The second goal was to prevent the process $K_L\to\mu e$ from acquiring a large branching ratio due to the low unification scale. To reach the first goal, a number of assumptions were made: the size of the cutoff scale where the regime of strong couplings set in (one might wonder whether or not the leading log approximation is still valid), the size of the tree-level boundary corrections, and the contribution from the relative running of the $SU(2)$ gauge couplings above the compactification scale. This last assumption, in particular, which is very model dependent, is crucial in obtaining an agreement with data. We have checked that when supersymmetric contributions to the running of coupling constants are switched on only above 200 GeV and not at M_Z as was done in [8] it is not possible to obtain acceptable solutions for the situation in which the $SU(2)$ gauge couplings run parallel to each other, as the correct value of the weak mixing angle would require with $n_G=3$ a compactification scale significantly lower than 1 TeV. On the other hand, in a model in which the breakdown of gauge symmetries is accomplished by using boundary conditions, the authors of [8] find a positive contribution to $\sin^2\theta_W(M_Z^2)$ from scales higher than the compactification scale, and the correct value of the mixing angle can be found for the compactification scale $O(2$ TeV$)$. In summary, the actual ''prediction'' for $\sin^2\theta_W(M_Z^2)$ in this model depends crucially on the assumptions made about various details of the physics of large extra dimensions. The second goal mentioned above is achieved by the orbifold boundary conditions which split a quartet of $SU(4)_{\mathrm{PS}}$ into zero and nonzero modes. Since the SM particles are supposed to be surviving zero modes in four dimensions, ordinary quarks and leptons cannot be in the same quartet, similarly to the case of the $SU(2)^3$ and $SU(3)^2$ models considered here. Consequently, there are no tree-level transitions between SM quarks and leptons, and the $SU(4)/[SU(3)\otimes U(1)_{B\text{-}L}]$ gauge bosons can be relatively ''light'' $[O(1$ TeV$)]$ without violating the upper bound on the rate of $K_L\to\mu e$. This model predicts heavy copies of the SM particles with masses of $O(1$ TeV$)$.

Reference [9] proposed to extend the standard model $SU(2)_L\otimes U(1)_Y$ to $SU(3)\otimes SU(2)\otimes U(1)$ at some scale M of $O(1$ TeV$)$. In this model, $SU(3)\otimes SU(2)\otimes U(1)\to SU(2)_L\otimes U(1)_Y$ at M, which gives the following relations between the couplings of the SM and its parent group:

$$\frac{1}{g_2^2}=\frac{1}{g_3^2}+\frac{1}{\tilde{g}^2}, \quad \frac{1}{g'^{,2}}=\frac{3}{g_3^2}+\frac{1}{\tilde{g}'^{,2}}, \tag{113}$$

where the couplings on the right-hand sides of these equations belong to those of the parent group while those on the left-hand sides are those of the SM. In the limit $\tilde{g},\tilde{g}'\to\infty$ [the exact $SU(3)$ limit], one can easily derive $\sin^2\theta_W^0=1/4$. Using the RG equations for g_2 and g' to match the value of $\sin^2\theta_W$ at M_Z, Dimopoulos and Kaplan obtained a value for the unification scale $M_0=3.75$ TeV in the limit $\tilde{g},\tilde{g}'\to\infty$. As mentioned in [8], this prediction is not precise because of these assumptions. Once more, one is facing the problem with strong couplings. Furthermore, unlike the case with the Pati-Salam group or with the quintessential grand unified theories, there is no charge quantization in this scenario. However, it is similar in spirit to our 1981 paper [1] in that $\sin^2\theta_W^0$ is determined entirely from the weak group although two of the groups in [9] are not so weak after all. Notice that the exact $SU(3)$ limit of [9] giving $\sin^2\theta_W^0=1/4$ is similar to our case of $G_W=SU(3)$ (with two doubly charged gauge bosons) as discussed in [1] and mentioned in Sec. II B. In our case, this is ruled out by $\sin^2\theta_W(M_Z^2)$.

Finally, in addition to [1], there are two other papers within the past three years which dealt with $SU(3)\otimes SU(3)^2$ [14] and $SU(4)\otimes SU(2)^3$ [15] in a very different context.

VII. CONCLUSIONS

We have revived our previous paper [1], which provided a general discussion of an early quark-lepton unification characterized by the gauge group $G_S\otimes G_W$. As a by-product we have presented the simple formula (34) for $\sin^2\theta_W^0$ in the case of $G_W=SU(N)^k$ that is equivalent to the formula in [1] but is more transparent.

During the last twenty-two years the experimental value for $\sin^2\theta_W(M_Z^2)$ became very precise and the value of $\alpha_s(M_Z^2)$ became not only more precise but also significantly smaller. As a result of these changes, our favorite 1981 scenario, $SU(4)_{PS}\otimes[SU(2)]^4$, cannot be made consistent simultaneously with the data for $\alpha_s(M_Z^2)$ and the lower bound on the masses of right-handed gauge bosons unless six new generations of ordinary quarks and leptons are present. However, with the very low unification scale $O(1\ \text{TeV})$, the improved experimental upper bound on $K_L\to\mu e$ is violated in this model by many orders of magnitude unless new, not always natural, strong suppression factors are invoked.

Fortunately, we have found two new petite unification models for which the situation is much more favorable. These are the models based on the groups $SU(4)_{PS}\otimes[SU(2)]^3$ and $SU(4)_{PS}\otimes[SU(3)]^2$, of which the first one is more appealing in view of its simpler fermion content. The interesting properties of these models, described already briefly in Sec. I and in detail in Secs. III–IV, are as follows.

(1) The correct value of $\sin^2\theta_W(M_Z^2)$ with the unification scale in the ballpark of 1 TeV and 3–10 TeV, respectively.

(2) The absence of tree-level lepton flavor violation and of tree-level FCNC processes. These transitions are generated at one loop through the exchanges of the heavy PS gauge bosons, new heavy quarks, and leptons with unconventional electric charges (up to 4/3 for quarks and 2 for leptons), and through the exchanges of "horizontal" weak gauge bosons that couple the ordinary quarks and leptons with these new heavy fermions. Due to the GIM–like mechanism, the bound on $K_L\to\mu e$ can easily be satisfied and the FCNC processes brought under control.

The rich phenomenology resulting in these two new scenarios will be presented in detail in a forthcoming paper.

Finally, we would like to stress the fact that the physics of our two scenarios PUT_1 and PUT_2 stands on its own regardless of whether or not TeV-scale large extra dimensions exist. Even if they do exist, the predictions of PUT_1 and PUT_2 would be *independent* of the details of the physics of large extra dimensions.

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