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Power suppressed operators and gauge invariance in soft-collinear effective theory

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The form of collinear gauge invariance for power suppressed operators in the soft-collinear effective theory (SCET) is discussed. Using a field redefinition we show that it is possible to make any power suppressed ultrasoft-collinear operators invariant under the original leading order gauge transformations. Our manipulations avoid gauge fixing. The Lagrangians to $\mathcal{O}(\lambda^2)$ are given in terms of these new fields. We then give a simple procedure for constructing power suppressed soft-collinear operators in $\mathrm{SCET_{II}}$ by using an intermediate theory $\mathrm{SCET_I}$.

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I. INTRODUCTION

The soft-collinear effective theory (SCET) has been proposed as a systematic approach for separating hard and soft scales in processes with energetic quarks and gluons [1–4]. The infrared physics is described in the effective theory in terms of collinear, soft, and ultrasoft fields with well defined momentum scaling. These fields are used to construct operators such as Lagrangians and currents that describe long distance effects, while hard corrections are contained in Wilson coefficients. This formalism builds on and extends earlier techniques used for discussing factorization [5].

The degrees of freedom in SCET include collinear quarks ξ_n and gluons A_n^μ with momentum scaling $p_c^\mu = (n\cdot p, \bar{n}\cdot p, p_\perp) \sim Q(\lambda^2,1,\lambda)$, soft modes q_s, A_s^μ with momenta $p_s^\mu \sim Q\lambda$, and ultrasoft (usoft) modes q_{us}, A_{us}^μ with momenta $p_{\rm us}^\mu \sim Q\lambda^2$. Here Q is the hard scale, $\lambda \ll 1$ is the expansion parameter, and $n_\mu, \bar{n}_\mu$ are two light-cone unit vectors satisfying $n^2 = \bar{n}^2 = 0$ and $n\cdot\bar{n} = 2$. The explicit set of required fields may differ depending on the relevant scales in a given process. For instance, in the Drell-Yan process it is useful to have collinear fields for two light-like directions and for multijet-production more than two directions are required [6,7].

In many exclusive heavy meson decays to energetic light hadrons there are important effects at the scales Q^2, $Q\Lambda$, and Λ^2, where $\Lambda \sim 0.5$ GeV is a hadronic scale. To correctly account for these effects, a sequence of two effective theories, $\mathrm{SCET_I}$ and $\mathrm{SCET_{II}}$, can be used [8].[1] One thus distinguishes between

$$\mathrm{SCET_I}:\ \text{collinear fields with } (p_c^+, p_c^-, p_c^\perp) \sim Q(\lambda^2,1,\lambda)$$

$$\text{and usoft fields with } p_{us}^\mu \sim Q\lambda^2 \text{ where } \lambda \sim \sqrt{\Lambda/Q}$$

$$\mathrm{SCET_{II}}:\ \text{collinear fields with} (p_c^+, p_c^-, p_c^\perp) \sim Q(\eta^2,1,\eta)$$

$$\text{and soft fields with } p_s^\mu \sim Q\eta \text{ where } \eta \sim \Lambda/Q.$$

For clarity the power counting parameter η is used for $\mathrm{SCET_{II}}$ rather than λ. In exclusive processes the energetic/soft hadrons are described by collinear/soft fields in $\mathrm{SCET_{II}}$. Both fields have $p_\perp \sim \Lambda$ which is appropriate for describing the constituents of hadrons of size $r_\perp \sim 1/\Lambda$. For exclusive processes the theory $\mathrm{SCET_I}$ plays an intermediate role by describing in a local way the fluctuations with $p^2 \sim Q\Lambda$ that are involved in interactions between soft and collinear fields in $\mathrm{SCET_{II}}$. In contrast, $\mathrm{SCET_I}$ suffices for describing factorization in inclusive processes like $B \to X_s\gamma$, as well as some exclusive processes like $B \to \gamma e \nu$ [4,9]. Interactions in $\mathrm{SCET_{II}}$ are discussed in Refs. [4,10] and power corrections in $\mathrm{SCET_I}$ were studied in Refs. [8,11–17]. Quark masses were considered in Ref. [18].

The symmetries of the effective theory provide an important guiding principle for constraining the form of operators, especially at the level of power corrections. The SCET has a rich symmetry structure, reflecting the interplay between the different length scales it describes. The constraints include

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[1] In Ref. [4] a version of SCET was constructed that simultaneously involves collinear, soft, and usoft fields. While it is possible that some physical process may simultaneously require these degrees of freedom, here we restrict ourselves to the degrees of freedom of $\mathrm{SCET_I}$-$\mathrm{SCET_{II}}$ which suffice for most applications.

TABLE I. Gauge transformations for the collinear and usoft fields from Ref. [4], where $i\mathcal{D}^\mu \equiv (n^\mu/2)\overline{\mathcal{P}} + \mathcal{P}_\perp^\mu + (\bar{n}^\mu/2) in\cdot D_{\rm us}$. The collinear fields and transformations are understood to have momentum labels and involve convolutions, but for simplicity these indices are suppressed. The usoft transformations do not change the momentum labels of collinear fields.

Object	Collinear $\mathcal{U}_c$	Usoft U_{us}
ξ_n	$\mathcal{U}_c\ \xi_n$	$U_{us}\xi_n$
gA_n^μ	$\mathcal{U}_c gA_n^\mu\ \mathcal{U}_c^\dagger + \mathcal{U}_c[i\mathcal{D}^\mu, \mathcal{U}_c^\dagger]$	$U_{us} gA_n^\mu U_{us}^\dagger$
W	$\mathcal{U}_c W$	$U_{us} W U_{us}^\dagger$
q_{us}	q_{us}	$U_{us} q_{us}$
gA_{us}^μ	gA_{us}^μ	$U_{us} gA_{us}^\mu U_{us}^\dagger + U_{us}[i\partial^\mu, U_{us}^\dagger]$
Y	Y	$U_{us} Y$

power counting, collinear/soft/ultrasoft gauge invariance, reductions in spin structures, and a reparametrization invariance [1–4,11,13,19] (see Ref. [20] for a brief review of the symmetries). At a given order in λ the most general set of operators for a given process can be constructed using the following.

(i) *Power counting:* Restricts the type of fields and derivatives allowed in the operator.

(ii) *Gauge invariance:* Requires operators to be built out of gauge invariant building blocks.

(iii) *Reparametrization invariance:* Corresponds to the restoration of Lorentz invariance order by order in λ.

(iv) *Locality:* The theory $\mathrm{SCET_I}$ is only nonlocal in $\mathcal{O}(Q)$ momenta. Only inverse powers of the large label momentum are allowed and collinear Wilson lines have to be built out of $\mathcal{O}(1)$ gluons.

Note that $\mathrm{SCET_I}$ is constructed in a local manner, but after doing this it is useful to consider a field redefinition $\xi_n \to Y\xi_n$ which introduces nonlocality at the usoft scale. The locality restriction does not apply to $\mathrm{SCET_{II}}$. Integrating out $p^2 \sim Q\Lambda$ modes immediately results in operators involving the soft Wilson line S [4], and it contains inverse powers of $1/\Lambda$ momenta. In the following we will focus on gauge invariance and discuss subtleties which arise in constructing invariant operators at subleading order.

The gauge transformations for the SCET fields were derived in [4] and are summarized in Tables I and II. Here $\partial_c^\mu \mathcal{U}_c \sim Q(\lambda^2,1,\lambda)$, $\partial_s^\mu U_s \sim Q\lambda$, and $\partial^\mu U_{us} \sim Q\lambda^2$ distinguish the collinear, soft and usoft gauge transformations respectively. Partial derivatives without a subscript are usoft, so $i\partial_\mu \sim Q\lambda^2$. In Table I we have used

$$i\mathcal{D}^\mu \equiv \frac{n^\mu}{2}\overline{\mathcal{P}} + \mathcal{P}_\perp^\mu + \frac{\bar{n}^\mu}{2} in\cdot D_{\rm us} \tag{1}$$

in the fundamental representation. Note that only the $n\cdot A_{us}$ component of the usoft gauge field appears here and that the components of $\mathcal{D}^\mu$ have the same scaling in λ as the collinear gluon field, so all transformations are homogeneous. Thus, power counting strongly constrains the leading usoft-collinear interactions. It also forces us to have a multipole expansion so that only the $n\cdot k$ momenta of collinear particles can be changed by interactions with usoft gluons. In Refs. [1–4] this expansion is done in momentum space while in Refs. [10,14,15] it is done in position space. This leads to formulations of SCET whose operators appear slightly different, but whose final predictions for physical observables have to be the same.

In this paper we discuss how gauge invariance is realized for power suppressed operators in both $\mathrm{SCET_I}$ and $\mathrm{SCET_{II}}$. $\mathrm{SCET_I}$ is studied in Sec. II where we clarify the nature of collinear gauge invariance in power suppressed operators with ultrasoft derivatives. This is done by showing that it is possible to arrange these power suppressed operators such that only the original *leading order* gauge transformations are needed at any order in the power expansion. This was also the goal of a recent study by Beneke and Feldmann [15] and a comparison is given with their results. The form of our transformed fields is different from theirs, reflecting a freedom in choice of viable field redefinitions. We found that it was not necessary to do any gauge fixing in our manipulations.

TABLE II. Gauge transformations for collinear and soft fields in $\mathrm{SCET_{II}}$ from Ref. [4]. Momentum labels are suppressed, and ∂_c^μ and ∂_s^μ are defined to only pick out collinear and soft momenta, respectively. Here $i\partial_c^\mu \neq i\mathcal{D}^\mu$ since usoft fields are not included in $\mathrm{SCET_{II}}$.

Objects	Collinear $\mathcal{U}_c$	Soft U_s
ξ_n	$\mathcal{U}_c\xi_n$	ξ_n
gA_n^μ	$\mathcal{U}_c gA_n^\mu \mathcal{U}_c^\dagger + \mathcal{U}_c[i\partial_c^\mu \mathcal{U}_c^\dagger]$	gA_n^μ
W	$\mathcal{U}_c W$	W
q_s	q_s	$U_s q_s$
gA_s^μ	gA_s^μ	$U_s gA_s^\mu U_s^\dagger + U_s[i\partial_s^\mu, U_s^\dagger]$
S	S	$U_s S$

In $\mathrm{SCET_{II}}$ the soft and collinear gauge invariance alone allow a large number of operators, reflecting the more nonlocal nature of this theory. In particular, gauge invariance does not uniquely fix the path of the Wilson lines. However, since $\mathrm{SCET_{II}}$ is matched on from $\mathrm{SCET_{I}}$ and not from full QCD, one can obtain information about the operators relevant for a given process from the structure of operators in $\mathrm{SCET_{I}}$. We illustrate the $\mathrm{SCET_{I}}\to\mathrm{SCET_{II}}$ matching by several examples in Sec. III.

II. GAUGE INVARIANCE IN $\mathrm{SCET_{I}}$

At leading order the $\mathrm{SCET_{I}}$ Lagrangian for collinear quarks is [2,3]

$$\mathcal{L}_{\xi\xi}^{(0)}=\bar{\xi}_n\left[in\cdot D+i\not{D}_c^{\perp}W\frac{1}{\bar{\mathcal{P}}}W^{\dagger}i\not{D}_c^{\perp}\right]\frac{\bar{\not{n}}}{2}\xi_n, \tag{2}$$

where the collinear covariant derivatives are $iD_c^{\mu}=\mathcal{P}^{\mu}+gA_n^{\mu}$ with label operators $\mathcal{P}^{\mu}$, the full derivative $in\cdot D=in\cdot\partial+gn\cdot A_{us}+gn\cdot A_n$, and the Wilson line W is built out of $\bar{n}\cdot A_n$ fields where $f(i\bar{n}\cdot D_c)=Wf(\bar{\mathcal{P}})W^{\dagger}$,

$$W=\left[\sum_{\text{perms}}\exp\left(-\frac{g}{\bar{\mathcal{P}}}\bar{n}\cdot A_{n,q}(x)\right)\right]. \tag{3}$$

Under the gauge transformations in Table I covariant derivatives acting in the fundamental representation transform under collinear and usoft transformations as

$$\begin{aligned}\mathcal{U}_c:\quad & in\cdot D\to\mathcal{U}_c in\cdot D\mathcal{U}_c^{\dagger},\quad iD_c^{\perp}\to\mathcal{U}_c iD_c^{\perp}\mathcal{U}_c^{\dagger},\\ & i\bar{n}\cdot D_c\to\mathcal{U}_c i\bar{n}\cdot D_c\mathcal{U}_c^{\dagger},\\ U_{us}:\quad & in\cdot D\to U_{us}in\cdot DU_{us}^{\dagger},\quad iD_c^{\perp}\to U_{us}iD_c^{\perp}U_{us}^{\dagger},\\ & i\bar{n}\cdot D_c\to U_{us}i\bar{n}\cdot D_cU_{us}^{\dagger}.\end{aligned} \tag{4}$$

It is straightforward to verify that all factors of $\mathcal{U}_c$ or U_{us} drop out of $\mathcal{L}_{\xi\xi}^{(0)}$, which has been shown to be the most general possible operator consistent with gauge invariance, power counting, and reparametrization invariance [4,19]. The same is true of the leading order collinear gluon action

$$\begin{aligned}\mathcal{L}_{cg}^{(0)}=&\frac{1}{2g^2}\mathrm{tr}\{[i\mathcal{D}^{\mu}+gA_{n,q}^{\mu},i\mathcal{D}^{\nu}+gA_{n,q'}^{\nu}]\}^2\\ &+2\,\mathrm{tr}\{\bar{c}_{n,p'}[i\mathcal{D}_{\mu},[i\mathcal{D}^{\mu}+gA_{n,q}^{\mu},c_{n,p}]]\}\\ &+\frac{1}{\alpha}\mathrm{tr}\{[i\mathcal{D}_{\mu},A_{n,q}^{\mu}]\}^2.\end{aligned} \tag{5}$$

The terms on the second line are the gauge fixing terms for a general covariant gauge, where c_n are adjoint ghost fields.

Beyond leading order the form of the subleading Lagrangians can be determined by matching calculations and use of the SCET symmetries. There is a reparametrization invariance [21] (RPI), which in SCET is due to the freedom in choosing the basis vectors n and $\bar{n}$, and in decomposing the momenta $\bar{n}\cdot(p+k)$ and $(p_{\perp}^{\mu}+k_{\perp}^{\mu})$ into collinear p and usoft k components [11,19]. This RPI connects collinear and usoft derivatives,

$$\bar{\mathcal{P}}+i\bar{n}\cdot\partial,\quad \mathcal{P}_{\perp}^{\mu}+i\partial_{\perp}^{\mu}, \tag{6}$$

and also relates the Wilson coefficients of leading and subleading operators [11,14,16,19].

To turn the derivatives in Eq. (6) into covariant derivatives we make use of gauge symmetry. This forces the label operator to be replaced by the collinear covariant derivative iD_c^{μ}, but as we shall see it allows some freedom in the usoft term [12]. In Refs. [11,19] the usoft derivative was made covariant with the choice iD_{us}^{μ}, so the RPI combinations in Eq. (6) become

$$\begin{aligned}\text{choice (i)}\quad & i\bar{n}\cdot D=i\bar{n}\cdot D_c+i\bar{n}\cdot D_{us},\\ & iD_{\perp}^{\mu}=iD_{c,\perp}^{\mu}+iD_{us,\perp}^{\mu}.\end{aligned} \tag{7}$$

For the purpose of gauge transformations this corresponds to promoting the ultrasoft field to a full background field of a quantum collinear gauge field so that

$$gA_n^{\mu}\to\mathcal{U}_c gA_n^{\mu}\mathcal{U}_c^{\dagger}+\mathcal{U}_c[\mathcal{P}^{\mu}+iD_{\mathrm{us}}^{\mu},\mathcal{U}_c^{\dagger}], \tag{8}$$

and the combined field $A^{\mu}=A_n^{\mu}+A_{\mathrm{us}}^{\mu}$ transforms as

$$gA^{\mu}\to\mathcal{U}_c gA^{\mu}\mathcal{U}_c^{\dagger}+\mathcal{U}_c[\mathcal{P}^{\mu}+i\partial_{\mathrm{us}}^{\mu},\mathcal{U}_c^{\dagger}]. \tag{9}$$

With this choice one still has homogeneous gauge transformations in Table I at leading order, which we will call $G^{(0)}$, however one also induces subleading collinear transformations for $A_n^{\perp}$ and $\bar{n}\cdot A_n$ suppressed by λ and λ^2, respectively

$$\begin{aligned}G^{(1)}:\quad & A_{n,\perp}^{\mu}\to\mathcal{U}_c[iD_{\perp,us}^{\mu},\mathcal{U}_c^{\dagger}],\\ & \bar{n}\cdot A_n\to\mathcal{U}_c[i\bar{n}\cdot D_{us},\mathcal{U}_c^{\dagger}].\end{aligned} \tag{10}$$

Thus, much like the reparametrization invariance, there are gauge transformations that connect the leading and subleading terms. This observation was first made in Ref. [12]. For example, using the gauge completion given in Eq. (7) the $O(\lambda)$ Lagrangian is

$$\mathcal{L}_{\xi\xi}^{(1)}=\bar{\xi}_n\left[i\not{D}_{\mathrm{us}}^{\perp}\frac{1}{\bar{n}\cdot iD_c}i\not{D}_c^{\perp}+i\not{D}_c^{\perp}\frac{1}{\bar{n}\cdot iD_c}i\not{D}_{us}^{\perp}\right]\frac{\bar{\not{n}}}{2}\xi_n. \tag{11}$$

Under a collinear gauge transformation $G^{(0)}$ from Table I one finds

$$\begin{aligned}\mathcal{L}^{(1)}\to\mathcal{L}^{(1)}-\bar{\xi}_n\Bigg[&[i\not{D}_{\mathrm{us}}^{\perp},\mathcal{U}_c^{\dagger}]\mathcal{U}_c\frac{1}{\bar{n}\cdot iD_c}i\not{D}_c^{\perp}\\ &+i\not{D}_c^{\perp}\frac{1}{\bar{n}\cdot iD_c}\mathcal{U}_c^{\dagger}[i\not{D}_{us}^{\perp},\mathcal{U}_c]\Bigg]\frac{\bar{\not{n}}}{2}\xi_n.\end{aligned} \tag{12}$$

The second term cancels against the $G^{(1)}$ variation of the leading order Lagrangian $\mathcal{L}_{\xi\xi}^{(0)}$, implying that the effective Lagrangian is invariant up to this order. The other subleading actions with usoft fields are [8,14,16,22]

$$\mathcal{L}_{\xi q}^{(2a)}=\bar{\xi}_n\frac{1}{i\bar{n}\cdot D_c}ig\left\{ \not{M}_\perp+\frac{\bar{\not{n}}}{2}n\cdot M\right\} Wq_{us}+\text{H.c.},$$

$$\mathcal{L}_{cg}^{(1)}=\frac{2}{g^2}\text{tr}\{[iD_0^\mu,iD_c^{\perp\nu}][iD_{0\mu},iD_{us\nu}^\perp]\},$$

$$\begin{aligned}\mathcal{L}_{cg}^{(2)}=&\frac{1}{g^2}\text{tr}\{[iD_0^\mu,iD_{us}^{\perp\nu}][iD_{0\mu},iD_{us\nu}^\perp]\}\\&+\frac{1}{g^2}\text{tr}\{[iD_{us}^{\perp\mu},iD_{us}^{\perp\nu}][iD_{c\mu}^\perp,iD_{c\nu}^\perp]\}\\&+\frac{1}{g^2}\text{tr}\{[iD_0^\mu,in\cdot D][iD_{0\mu},i\bar{n}\cdot D_{us}]\}\\&+\frac{1}{g^2}\text{tr}\{[iD_{us}^{\perp\mu},iD_c^{\perp\nu}][iD_{c\mu}^\perp,iD_{us\nu}^\perp]\},\end{aligned}\tag{13}$$

where $igM^\mu=[i\bar{n}\cdot D_c,iD_{us}^\mu+\bar{n}^\mu gn\cdot A_n/2]$ and $iD_0^\mu=iD_c^\mu+i\bar{n}^\mu n\cdot D_{us}/2$. [The terms $\mathcal{L}_{\xi q}^{(1)}$ and $\mathcal{L}_{\xi q}^{(2b)}$ do not depend on ultrasoft covariant derivatives and are shown below in Eq. (27).] Similar manipulations show that the results in Eq. (13) are invariant with terms canceled by the $G^{(1)}$ transformation of $\mathcal{L}_{\xi q}^{(1)}$ and $\mathcal{L}_{cg}^{(0,1)}$.

Although operators with usoft fields are gauge invariant, the presence of $G^{(1)}$ requires transformations of operators at different powers in λ to cancel one another. This is unsatisfactory since constraining operators at any particular order requires transforming lower order operators. Furthermore this would mean we would only be able to assign an unambiguous meaning to the sum of leading and subleading matrix elements. Instead, we would like to use fields with no $G^{(1)}$ transformation, so that operators are manifestly invariant under $G^{(0)}$ at each order in λ. In other words the terms at a given order are invariant without needing the transformation of lower order terms. To this end, consider the field redefinitions

$$\begin{aligned}g\bar{n}\cdot\hat{A}_n&=g\bar{n}\cdot A_n-\mathcal{W}[i\bar{n}\cdot D_{us},\mathcal{W}^\dagger],\\ g\hat{A}_n^\perp&=gA_n^\perp-\mathcal{W}[iD_{us}^\perp,\mathcal{W}^\dagger],\end{aligned}\tag{14}$$

where $gn\cdot\hat{A}_n=gn\cdot A_n$, and $\hat{A}_n^\mu$ are new collinear gluon fields. Here $\mathcal{W}$ is the product of Wilson lines defined in Ref. [14] which in position space is

$$\begin{aligned}\mathcal{W}(x)=&P\exp\left(ig\int_{-\infty}^0 ds\bar{n}\cdot(A_n+A_{us})(\bar{n}s+x)\right)\\&\times\left[P\exp\left(ig\int_{-\infty}^0 ds\bar{n}\cdot A_{us}(\bar{n}s+x)\right)\right]^\dagger.\end{aligned}\tag{15}$$

In Eq. (15) the collinear fields $A_n^\mu(X+x)$ are the Fourier transforms of $A_{n,p}^\mu(x)$ with X the conjugate variable to p. Under collinear gauge transformations $\mathcal{W}\to U_c\mathcal{W}$, while under usoft gauge transformations $\mathcal{W}\to U_{us}\mathcal{W}U_{us}^\dagger$. The presence of $\mathcal{W}$ in Eq. (14) causes $\hat{A}_n$ to be defined in terms of a nonlinear function of A_n. Note that our transformation in Eq. (14) differs from that in Ref. [15], as we discuss in more detail below. Under a collinear gauge transformation the $\perp$ component of the new collinear gluon field transforms as (suppressing momentum space labels)

$$\begin{aligned}g\hat{A}_n^\perp\to&\,\mathcal{U}_c gA_n^\perp\mathcal{U}_c^\dagger+\mathcal{U}_c[\mathcal{P}_\perp+iD_{us}^\perp,\mathcal{U}_c^\dagger]\\&-\mathcal{U}_c\mathcal{W}[iD_{us}^\perp,\mathcal{W}^\dagger\mathcal{U}_c^\dagger]\\=&\,\mathcal{U}_c gA_n^\perp\mathcal{U}_c^\dagger+\mathcal{U}_c\mathcal{P}^\perp\mathcal{U}_c^\dagger+\mathcal{U}_c iD_{us}^\perp\mathcal{U}_c^\dagger\\&-\mathcal{U}_c\mathcal{W}iD_{us}^\perp\mathcal{W}^\dagger\mathcal{U}_c^\dagger\\=&\,\mathcal{U}_c g\hat{A}_n^\perp\mathcal{U}_c^\dagger+\mathcal{U}_c\mathcal{P}_\perp\mathcal{U}_c^\dagger.\end{aligned}\tag{16}$$

Only hatted fields appear in the final result. With a similar set of steps we find $g\bar{n}\cdot\hat{A}_n\to\mathcal{U}_c g\bar{n}\cdot\hat{A}_n\mathcal{U}_c^\dagger+\mathcal{U}_c\bar{\mathcal{P}}\mathcal{U}_c^\dagger$. Therefore

$$g\hat{A}_n^\mu\to\mathcal{U}_c g\hat{A}_n^\mu\mathcal{U}_c^\dagger+\mathcal{U}_c[i\mathcal{D}^\mu,\mathcal{U}_c^\dagger],\tag{17}$$

just like in Table I. Thus, in terms of the hatted fields, transformations that involve suppressed terms like $G^{(1)}$ never appear. This is the desired result.

To express the Lagrangians in terms of hatted fields it is useful to have the inverse transformation to Eq. (14). This is complicated by the factors of $\mathcal{W}=\mathcal{W}[\bar{n}\cdot A_n,\bar{n}\cdot A_{us}]$ given in Eq. (14), which depend nonlinearly on the gluon fields. Now, we know that

$$i\bar{n}\cdot D\mathcal{W}=\mathcal{W}g\bar{n}\cdot A_{us},\tag{18}$$

which implies that in terms of the hatted fields $\mathcal{W}=\mathcal{W}[\bar{n}\cdot\hat{A}_n,\bar{n}\cdot A_{us}]$ satisfies the equation

$$\begin{aligned}0&=(i\bar{n}\cdot\hat{D}_c+\mathcal{W}i\bar{n}\cdot D_{us}\mathcal{W}^\dagger)\mathcal{W}-\mathcal{W}g\bar{n}\cdot A_{us}\\&=i\bar{n}\cdot\hat{D}_c\mathcal{W}.\end{aligned}\tag{19}$$

However, this equation has a unique solution $\hat{W}$. Switching to momentum labels and residual coordinates x [3], this $\hat{W}$ is just the standard Wilson line in Eq. (3) expressed in terms of the $\bar{n}\cdot\hat{A}_n$ collinear field (since they are defined by the same equation). This gives the remarkable result that after the field redefinition we have to all orders in λ

$$\mathcal{W}=\hat{W}=\left[\sum_{\text{perms}}\exp\left(-\frac{g}{\bar{\mathcal{P}}}\bar{n}\cdot\hat{A}_{n,q}(x)\right)\right],\tag{20}$$

which is independent of the usoft gauge field. Under the gauge transformations $\hat{W}\to U_c\hat{W}$ and $\hat{W}\to U_{us}\hat{W}U_{us}^\dagger$ just like we had for W. Thus, the inverse transformation to Eq. (14) can be written

$$g\bar{n}\cdot A_n = g\bar{n}\cdot \hat{A}_n + \hat{W}[i\bar{n}\cdot D_{us},\hat{W}^\dagger],$$
$$gA_n^\perp = g\hat{A}_n^\perp + \hat{W}[iD_{us}^\perp,\hat{W}^\dagger]. \tag{21}$$

This corresponds to gauging the RPI combinations in Eq. (6) to

$$\text{choice (ii)}\quad i\bar{n}\cdot\hat{D} = i\bar{n}\cdot\hat{D}_c + \hat{W} i\bar{n}\cdot D_{us}\hat{W}^\dagger,$$
$$i\hat{D}_\perp^\mu = i\hat{D}_{c,\perp}^\mu + \hat{W} iD_{us,\perp}^\mu \hat{W}^\dagger, \tag{22}$$

rather than using choice (i) in Eq. (7). Under collinear and usoft gauge transformations these derivatives transform exactly as in Eq. (4)

$$\mathcal{U}_c:\quad in\cdot\hat{D}\to\mathcal{U}_c in\cdot\hat{D}\mathcal{U}_c^\dagger,\quad i\hat{D}_c^\perp\to\mathcal{U}_c i\hat{D}_c^\perp\mathcal{U}_c^\dagger,$$
$$i\bar{n}\cdot\hat{D}_c\to\mathcal{U}_c i\bar{n}\cdot\hat{D}_c\mathcal{U}_c^\dagger, \tag{23}$$
$$U_{us}:\quad in\cdot\hat{D}\to U_{us} in\cdot\hat{D}U_{us}^\dagger,\quad i\hat{D}_c^\perp\to U_{us} i\hat{D}_c^\perp U_{us}^\dagger,$$
$$i\bar{n}\cdot\hat{D}_c\to U_{us} i\bar{n}\cdot\hat{D}_c U_{us}^\dagger.$$

In Ref. [15] transformations were also made with the aim of determining fields that could be used in power suppressed operators while avoiding gauge transformations that mix different orders in λ. Similar to the construction here their initial fields transform as in Eq. (8) and the desired final collinear transformations are identical to the form in Ref. [4], shown in our Table I. In Ref. [15] the new collinear quark and gluon fields were defined as

$$\xi_n = RW_c^\dagger\hat{\xi}_n,$$
$$gA_{\perp c} = R(W_c^\dagger i\hat{D}_{\perp c}W_c^\dagger - i\partial_c^\perp)R^\dagger, \tag{24}$$
$$gn\cdot A_c = R[W_c^\dagger in\cdot\hat{D}W_c - in\cdot D_{us}(\bar{n}\cdot xn/2)]R^\dagger,$$

where the fields on the left-hand side are understood to be in a light-like axial gauge with $\bar{n}\cdot A_c = 1$. The matrix R is defined as $R(x) = P\exp(ig\int_C dz_\mu A_{\rm us}^\mu(z))$ with the path C a straight line connecting $\frac{1}{2}\bar{n}_\mu n\cdot x$ to x. In Ref. [15] the collinear fields were constructed entirely in position space, and a multipole expansion was performed on the usoft fields $\phi_{us}(x) = \phi_{us}(x_-) + (x_\perp\cdot i\partial_\perp)\phi_{us}(x_-) + \cdots$. The transformation with the matrix R was then necessary to connect collinear and usoft fields which are at different space-time points. After inserting these fields into the effective Lagrangian, operators involving the matrix R were expanded using the Fock-Schwinger gauge for the ultrasoft gluon field.

The results in Eq. (24) differ from our field transformation in Eq. (21) in several respects. First, we did not need to redefine the collinear quark field $\xi_{n,p}(x)$ since our labeled collinear fields carry residual ultrasoft momentum through their x dependence. For the gluons our transformation changes $\bar{n}\cdot A_n$ but not the $n\cdot A_n$ field, whereas Eq. (24) does the exact opposite. For the $A_n^\perp$ field our hatted field is not surrounded by W's, and we have a covariant usoft derivative while Eq. (24) has a normal derivative. The fact that both our usoft and collinear fields are local in the coordinate x representing residual momenta $k^\mu\sim Q\lambda^2$ means that we did not need to consider a matrix like R. Also, note that in our procedure for transforming the fields we did not require any gauge fixing at intermediate steps. Finally, we comment that the form of our field redefinition leads to an interesting result for $\mathcal{W}$ in terms of the new fields, namely $\mathcal{W} = \hat{W}$ with no higher order terms in λ.

The use of position and momentum space makes a more direct comparison difficult. However, any field redefinitions that lead to the desired result are equally valid and both Eq. (24) and Eq. (21) satisfy this criterion. In general one knows that field redefinitions should only affect the form of operators and the result for Green's functions, but should not affect S-matrix elements. Thus, equivalent effective theories are often realized with different fields. We expect that there should be a field redefinition which would relate our fields $\hat{A}_n$ to the fields $\hat{A}_n$ in Ref. [15], although we have not constructed it in closed form.

Lagrangian results

Having established collinear gauge fields whose transformations never mix orders in λ, we now rewrite all subleading Lagrangians to order λ^2 using Eq. (21). For simplicity we omit the hats in the following equations, however all collinear gauge fields should be understood to be the hatted ones. For the collinear quark Lagrangian we find

$$\mathcal{L}_{\xi\xi}^{(1)} = (\bar{\xi}_n W) i\not{D}_{us}^\perp \frac{1}{\bar{\mathcal{P}}}\left(W^\dagger i\not{D}_c^\perp \frac{\bar{\not{n}}}{2}\xi_n\right) + (\bar{\xi}_n i\not{D}_c^\perp W)\frac{1}{\bar{\mathcal{P}}} i\not{D}_{us}^\perp\left(W^\dagger\frac{\bar{\not{n}}}{2}\xi_n\right)$$

$$\mathcal{L}_{\xi\xi}^{(2)} = (\bar{\xi}_n W) i\not{D}_{us}^\perp \frac{1}{\bar{\mathcal{P}}} i\not{D}_{us}^\perp \frac{\bar{\not{n}}}{2}(W^\dagger\xi_n) + (\bar{\xi}_n i\not{D}_c^\perp W)\frac{1}{\bar{\mathcal{P}}^2} i\bar{n}\cdot D_{us}\frac{\bar{\not{n}}}{2}(W^\dagger i\not{D}_c^\perp\xi_n), \tag{25}$$

where we have used the fact that

$$\frac{1}{i\bar{n}\cdot D} = \frac{1}{i\bar{n}\cdot D_c} - W\frac{1}{\bar{\mathcal{P}}^2} i\bar{n}\cdot D_{us}W^\dagger + \cdots. \tag{26}$$

It is easy to see that the results in Eq. (25) are invariant under the transformations in Table I. For the mixed collinear-usoft quark interactions we find the invariant results

$$\mathcal{L}_{\xi q}^{(1)} = \bar{\xi}_n\frac{1}{i\bar{n}\cdot D_c} ig\not{B}_\perp^c W q_{us} + \text{H.c.},$$

$$\mathcal{L}_{\xi q}^{(2a)} = \bar{\xi}_n\frac{\bar{\not{n}}}{2}\frac{1}{i\bar{n}\cdot D_c} ign\cdot M W q_{us} + \text{H.c.},$$

$$\mathcal{L}^{(2b)}_{\xi q} = \bar{\xi}_n \frac{\bar{n}\!\!\!/}{2} i D\!\!\!\!/^{\,c}_{\perp} \frac{1}{(i\bar{n}\cdot D_c)^2} ig B\!\!\!\!/^{\,c}_{\perp} W q_{us} + \text{H.c.}, \tag{27}$$

where $igB\!\!\!\!/^{\,c}_{\perp} = [i\bar{n}\cdot D^c, iD\!\!\!\!/^{\,c}_{\perp}]$ and we have used the fact that the transformation of $\mathcal{L}^{(1)}_{\xi q}$ makes

$$\begin{aligned} igM^{\mu}_{\perp} &= [i\bar{n}\cdot D_c, WiD^{\mu}_{us\perp}W^{\dagger}] \\ &= [W\bar{\mathcal{P}}W^{\dagger}, WiD^{\mu}_{us\perp}W^{\dagger}] \\ &= W[\bar{\mathcal{P}}, iD^{\mu}_{us\perp}]W^{\dagger} \\ &= 0. \end{aligned} \tag{28}$$

Finally, for the subleading terms in the mixed usoft-collinear gluon action we find

$$\begin{aligned} \mathcal{L}^{(1)}_{cg} &= \frac{2}{g^2}\,\mathrm{tr}\{[iD_0^{\mu}, iD_c^{\perp\nu}][iD_{0\mu}, WiD^{\perp}_{us\,\nu}W^{\dagger}]\}, \\ \mathcal{L}^{(2)}_{cg} &= \frac{1}{g^2}\,\mathrm{tr}\{[iD_0^{\mu}, WiD^{\perp\nu}_{us}W^{\dagger}][iD_{0\mu}, WiD^{\perp}_{us\,\nu}W^{\dagger}]\} \\ &\quad + \frac{1}{g^2}\,\mathrm{tr}\{W[iD^{\perp\mu}_{us}, iD^{\perp\nu}_{us}]W^{\dagger}[iD^{\perp}_{c\mu}, iD^{\perp}_{c\nu}]\} + \frac{1}{g^2}\,\mathrm{tr}\{[iD_0^{\mu}, in\cdot D][iD_{0\mu}, Wi\bar{n}\cdot D_{us}W^{\dagger}]\} \\ &\quad + \frac{1}{g^2}\,\mathrm{tr}\{[WiD^{\perp\mu}_{us}W^{\dagger}, iD_c^{\perp\nu}][iD^{\perp}_{c\mu}, WiD^{\perp}_{us\,\nu}W^{\dagger}]\}, \end{aligned} \tag{29}$$

where $iD_0^{\mu} = i\mathcal{D}^{\mu} + gA_n^{\mu}$.

III. POWER SUPPRESSED SOFT-COLLINEAR OPERATORS

In $\mathrm{SCET_{II}}$ the structure of operators with soft and collinear fields is still constrained by properties such as power counting, gauge invariance, and reparametrization invariance. However the nonlocal nature of the theory makes it more difficult to simply write down the most general operators in an arbitrary case. To see this we consider a simple example, namely a heavy-to-light current. In the full theory we have $\bar{q}\Gamma b$ and in the effective theory

$$C(\bar{\mathcal{P}})\bar{\xi}_n W\Gamma S^{\dagger} h_v . \tag{30}$$

The Wilson lines W and S are required to ensure collinear and soft gauge invariance, respectively. However, neither gauge invariance nor power counting determines the exact path of S from x to ∞, since all A_s^{μ} fields scale the same way. Thus, additional input is needed to constrain these operators. From direct matching calculations, which integrate out fluctuations with $p^2 \sim Q\Lambda$, it is straightforward to determine that S is a straight Wilson line along the n direction built out of $n\cdot A_s$ fields [4]. An alternative procedure is as follows [8]:

(i) Match QCD onto $\mathrm{SCET_I}$ at a scale $\mu^2 \sim Q^2$ (with $p_c^2 \sim Q\Lambda$).

(ii) Factorize the usoft-collinear interactions with the field redefinitions,

$$\xi_n = Y\xi_n^{(0)} \text{ and } A_n^{\mu} = YA_n^{(0)\mu}Y^{\dagger}.$$

(iii) Match $\mathrm{SCET_I}$ onto $\mathrm{SCET_{II}}$ at a scale $\mu^2 \sim Q\Lambda$ (with $p_c^2 \sim \Lambda^2$).

For the heavy-to-light case we have (i) $\bar{q}\Gamma b \to C(\bar{\mathcal{P}})\bar{\xi}_n W\Gamma h_v^{us}$, and then (ii) $C(\bar{\mathcal{P}})\bar{\xi}_n W\Gamma h_v^{us} = C(\bar{\mathcal{P}})\bar{\xi}_n^{(0)} W^{(0)}\Gamma Y^{\dagger} h_v^{us}$. For the final step we rename the usoft fields as soft fields $Y^{\dagger}h_v^{us} = S^{\dagger}h_v^{s}$, and then lower the off-shellness of the collinear fields. Since the leading collinear Lagrangians in $\mathrm{SCET_I}$ and $\mathrm{SCET_{II}}$ are the same all possible time-ordered products agree exactly and we can simply replace $C(\bar{\mathcal{P}})\bar{\xi}_n^{(0)} W^{(0)} \to C(\bar{\mathcal{P}})\bar{\xi}_n^{\mathrm{II}} W^{\mathrm{II}}$. The final result is identical to Eq. (30) but the steps are simpler than those carried out in the Appendix of Ref. [4]. From the two-step approach it is also clear why the Wilson coefficient does not pick up any dependence on the soft momentum in this example.

The two-stage matching procedure becomes even more useful in cases where $\mathrm{SCET_I}$ contains time-ordered products, since these can induce nontrivial jet functions involving $p^2 \sim Q\Lambda$ fluctuations. $\mathrm{SCET_I}$ gives a well defined set of Feynman rules for computing these jet functions at tree level and in loops, and does so in a manner independent of the computation of Wilson coefficients at the hard scale $p^2 \sim Q^2$. Since the operator in $\mathrm{SCET_I}$ is a time-ordered product we are guaranteed that the running to the scale $\mu^2 = Q\Lambda$ is determined by that of the product of the hard Wilson coefficients. A final benefit is that power counting in $\mathrm{SCET_I}$ constrains the allowed scaling of operators in $\mathrm{SCET_{II}}$, and in particular, places a limit on the number of factors of $1/\Lambda$ that can be induced from $1/(Q\Lambda)$ terms as we discuss below. This provides a complementary procedure to constraining the powers of $1/\Lambda$ with reparametrization invariance as first described in Ref. [10].

Let us consider a generic matching calculation

$$\mathrm{SCET_I}[p_c^2\sim Q\Lambda, p_{us}^2\sim\Lambda^2] \xrightarrow{\mu^2\sim Q\Lambda} \mathrm{SCET_{II}}[p_c^2\sim\Lambda^2, p_s^2\sim\Lambda^2]. \tag{31}$$

First construct all time-ordered products, $T_{\rm I}^j$, of $\mathrm{SCET_I}$ operators which contribute at a given order in the power counting. To match these onto $\mathrm{SCET_{II}}$ operators we take matrix elements,

$$\langle \phi_I(p_i^2\sim\Lambda^2)|T_I^j|\phi_I'(p_i^2\sim\Lambda^2)\rangle. \tag{32}$$

Here the states have particles with ultrasoft momenta $p_{us}^2\sim\Lambda^2$, but with small collinear momenta $p_c^2\sim\Lambda^2$. These are allowed states in the Hilbert space of $\mathrm{SCET_I}$, since for example $p_\perp^2$ momenta of this size correspond to having zero label $\perp$ momenta, but nonzero residual $\perp$ momenta. These are also obviously states in $\mathrm{SCET_{II}}$. As in any matching calculation, we can use any convenient states, and one usually chooses free particle states. Note that the external collinear particles in Eq. (32) have reduced off-shellness, however this is not in general the case for the internal propagators.

As an additional constraint, the matching in Eq. (31) must be carried out in a manner that accounts for the fact that only certain products of collinear fields have *gauge invariant* label momentum, and that these momentum components are not lowered in matching these products of fields onto collinear fields in $\mathrm{SCET_{II}}$. This means that only gauge invariant products of collinear fields should be integrated out in the matching (guaranteeing that gauge invariant products are also left over). This automatically builds in the fact that the low energy operators in $\mathrm{SCET_{II}}$ must be built out of gauge invariant products $\Phi_1=W^\dagger\xi_n$, $\Phi_2=[W^\dagger D_c^\perp W]$, $\mathcal{S}_1=S^\dagger q_s$, etc. This properly matches the theory $\mathrm{SCET_I}$ onto the subset of phase space that is described by fields in $\mathrm{SCET_{II}}$. This matching will be perturbative as long as the scale $\sqrt{Q\Lambda}\gg\Lambda$.

A useful benefit of the two-stage procedure is that the power counting is transparent. Thus even though we are integrating out an intermediate scale $p^2\sim Q\Lambda$ that involves factors of the hadronic scale Λ, we need not worry about missing operators that would be power suppressed but are enhanced by explicit factors of $1/\Lambda$. The power counting for the matching process is

$$T^I\sim\lambda^{2k}\rightarrow\mathcal{O}^{II}\sim\eta^{k+E}, \tag{33}$$

where the final scaling is independent of how factors of η are partitioned between coefficients and operators in $\mathrm{SCET_{II}}$ (we will choose to make Wilson coefficients in $\mathrm{SCET_{II}}$ dimensionless and order η^0). This equation says that T-products which are order λ^{2k} in $\mathrm{SCET_I}$ will match onto operators in $\mathrm{SCET_{II}}$ that are order η^{k+E} with $E\geqslant 0$. Here the factor η^E is the extra factor obtained by lowering the off-shellness of the external collinear fields and thereby changing their power counting. For example $(\xi_n^{\rm I}\sim\lambda=\sqrt{\eta})\rightarrow(\xi_n^{\rm II}\sim\eta)$, which agrees with the formula having $E=1/2$. In general $E=1/2$ for external ξ_n or $A_n^\perp$, $E=0$ for external $\overline{n}\cdot A_n$ or W, and $E=1$ for external $n\cdot A_n$.

To illustrate these points we consider several examples. First consider the example of factorization in $B\rightarrow D\pi$ [23], but using the two-stage procedure. Matching the two $(\overline{c}b)_{V-A}(\overline{d}u)_{V-A}$ electroweak four quark operators onto operators in $\mathrm{SCET_I}$ gives

$$Q_{\mathbf{0}}^{\rm I}=[\overline{h}_{v'}^{us}\Gamma_h h_v^{us}][\overline{\xi}_{n,p'}WC_{\mathbf{0}}(\overline{\mathcal{P}}_+)\Gamma_l W^\dagger\xi_{n,p}],$$

$$Q_{\mathbf{8}}^{\rm I}=[\overline{h}_{v'}^{us}\Gamma_h T^A h_v^{us}][\overline{\xi}_{n,p'}WC_{\mathbf{8}}(\overline{\mathcal{P}}_+)\Gamma_l T^A W^\dagger\xi_{n,p}], \tag{34}$$

where $\overline{\mathcal{P}}_+=\overline{\mathcal{P}}^\dagger+\overline{\mathcal{P}}$ and the Wilson coefficients $C_{\mathbf{0,8}}$ contain the hard $p^2\sim Q^2$ effects. Next decouple the usoft interactions from the leading collinear Lagrangian with the field redefinitions $\xi_n=Y\xi_n^{(0)}$ and $A_n^\mu=YA_n^{(0)\mu}Y^\dagger$ [4]. This leaves

$$Q_{\mathbf{0}}^{\rm I}=[\overline{h}_{v'}^{us}\Gamma_h h_v^{us}][\overline{\xi}_{n,p'}^{(0)}W^{(0)}C_{\mathbf{0}}(\overline{\mathcal{P}}_+)\Gamma_l W^{(0)\dagger}\xi_{n,p}^{(0)}],$$

$$Q_{\mathbf{8}}^{\rm I}=[\overline{h}_{v'}^{us}\Gamma_h YT^AY^\dagger h_v^{us}]\times[\overline{\xi}_{n,p'}^{(0)}W^{(0)}C_{\mathbf{8}}(\overline{\mathcal{P}}_+)\Gamma_l T^A W^{(0)\dagger}\xi_{n,p}^{(0)}]. \tag{35}$$

In this result the ultrasoft and collinear fields are completely factorized. The collinear fields still have large off-shellness $p^2\sim Q\Lambda$, so we need step (iii). Taken with leading order Lagrangian insertions this example is just like the heavy-to-light current, so we match directly onto the $\mathrm{SCET_{II}}$ operators

$$Q_{\mathbf{0}}^{\rm II}=[\overline{h}_{v'}^{s}\Gamma_h h_v^{s}][\overline{\xi}_{n,p'}WC_{\mathbf{0}}(\overline{\mathcal{P}}_+)\Gamma_l W^\dagger\xi_{n,p}],$$

$$Q_{\mathbf{8}}^{\rm II}=[\overline{h}_{v'}^{s}\Gamma_h ST^AS^\dagger h_v^{s}]\times[\overline{\xi}_{n,p'}WC_{\mathbf{8}}(\overline{\mathcal{P}}_+)\Gamma_l T^A W^\dagger\xi_{n,p}]. \tag{36}$$

This is the same as the result originally derived in Ref. [23]. It is easy to see that no other $\mathrm{SCET_{II}}$ operators are possible at this order.

This algebra was quite simple, however we have not yet seen the full power of the intermediate theory with the above example. The procedure becomes useful once we consider time-ordered products in $\mathrm{SCET_I}$, since then one can obtain nontrivial jet functions J in $\mathrm{SCET_{II}}$ which lead to Wilson coefficients $C(z_i)J(z_i,x_j,y_k)$ for the $\mathrm{SCET_{II}}$ operators. This jet function has convolutions with variables z_i that correspond to the p^- momentum dependence in the hard coefficient C. It also can have dependence on the x_j momentum fractions of collinear fields in the $\mathrm{SCET_{II}}$ operators we match onto. Finally, since collinear fields in $\mathrm{SCET_I}$ are affected by the k^+ usoft momenta (through the $in\cdot\partial$ term in their action) the jet J can depend on the momentum fractions y_k which correspond to the soft $+$-momenta of gauge invariant products of soft fields in $\mathrm{SCET_{II}}$.

An example of a more involved matching calculation was given for the case of heavy-to-light form factors in Refs. [8,16] and we will not repeat this example here. To illustrate this case of matching further consider the toy example of light-light soft-collinear currents. In Ref. [10] these currents were derived by direct matching from QCD, so we contrast this procedure with the matching onto $\mathrm{SCET_{II}}$ operators by using $\mathrm{SCET_I}$. Such operators are matched from contributions in $\mathrm{SCET_I}$ which provide mixing between collinear and usoft quarks. Consider

$$T_0^{(3)}=\int d^4x T[J_{\xi\xi}^{(2)}(0),i\mathcal{L}_{\xi q}^{(1)}(x)]$$

$$J_{\xi q}^{(4)}=\bar{\xi}_n W\Gamma q_{us}, \tag{37}$$

where $J_{\xi\xi}^{(2)}=\bar{\xi}_n W\Gamma W^\dagger \xi_n$ and $\mathcal{L}_{\xi q}^{(1)}(x)$ is given in Eq. (27) (hard coefficients are suppressed since they are not crucial to our discussion). The order in λ is denoted by the exponent in brackets. To match these operators onto $\mathrm{SCET_{II}}$ we use the procedure explained above. For the local operator $J_{\xi q}^{(4)}$ this matching is simple. We first perform the field redefinition $\xi_n=Y\xi_n^{(0)}$ and $A_n^\mu=YA_n^{(0)\mu}Y^\dagger$ to write

$$J_{\xi q}^{(4)}=[\bar{\xi}_n^{(0)}W^{(0)}]\Gamma[Y^\dagger q_{us}] \tag{38}$$

where we have indicated the gauge invariant blocks of fields by the square brackets. The final step is to identify the usoft fields with soft fields and to lower the off-shellness of the collinear fields. At tree level this leads to the operator

$$O_1=[\bar{\xi}_n W]\Gamma[S^\dagger q_s] \tag{39}$$

in $\mathrm{SCET_{II}}$ which is order $\eta^{5/2}$. This follows from Eq. (33) with $k=2$ and $E=1/2$.

For the time-ordered product $T_0^{(3)}$ we follow similar steps. After the field redefinition

$$T_0^{(3)}=\int d^4x T\{[\bar{\xi}_n^{(0)}W^{(0)}]\Gamma[W^{(0)\dagger}\xi_n^{(0)}](0),[\bar{\xi}_n^{(0)}W^{(0)}]$$
$$\times[W^{(0)\dagger}i\not{D}_\perp^c W^{(0)}][Y^\dagger q_{us}](x)\}. \tag{40}$$

Consider the matrix element between a collinear fermion, a $\perp$ collinear gluon and a soft fermion. To match onto $\mathrm{SCET_{II}}$ we contract the $[W^{(0)\dagger}\xi_n^{(0)}][\bar{\xi}_n^{(0)}W^{(0)}]$ product, lower the off-shellness of the remaining $[\bar{\xi}_n^{(0)}W^{(0)}]$ and $[W^{(0)\dagger}i\not{D}_\perp^c W^{(0)}]$ and rename the $[Y^\dagger q_{us}]$ to $[S^\dagger q_s]$. At tree the two collinear fermion fields get contracted giving a propagator as shown in the first diagram of Fig. 1. This gives the operator

$$O_2=[\bar{\xi}_n W]\Gamma\frac{\not{\bar{n}}}{2}[W^\dagger i\not{D}_\perp^c W]\frac{1}{\bar{n}\cdot\mathcal{P}}[S^\dagger q_s] \tag{41}$$

in $\mathrm{SCET_{II}}$ which is the same operator as Ref. [10]. Note that while in $\mathrm{SCET_I}$ $T_0^{(3)}$ was larger by one power of λ than $J_{\xi q}^{(4)}$, the resulting two operators are the same order in η. This is because in lowering the off-shellness of $[W^{(0)\dagger}i\not{D}_\perp^c W^{(0)}]$ the power counting of the $\perp$ gluon is reduced from λ to $\eta=\lambda^2$. This agrees with Eq. (33) with $E=1/2$, so $O_2\sim\eta^{5/2}$ just like O_1.[2]

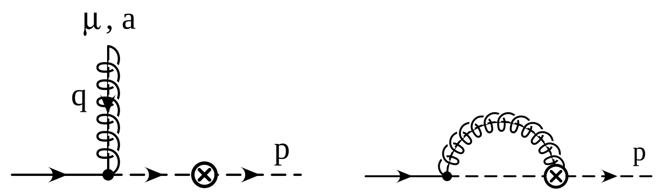

FIG. 1. Examples of graphs contributing to the matching of the $\mathrm{SCET_I}$ T-products onto $\mathrm{SCET_{II}}$ operators in Eq. (43). The dots denote the insertion of a $\mathcal{L}_{\xi q}^{(1)}$ and the circled crosses in the two diagrams are $J_{\xi\xi}^{(2,3)}$ operators, respectively.

There are additional contributions in $\mathrm{SCET_I}$ that one can write down at order λ^4, such as $T[J_{\xi\xi}^{(2)}(0),i\mathcal{L}_{\xi q}^{(2)}(x)]$, $T[J_{\xi\xi}^{(2)}(0),i\mathcal{L}_{\xi q}^{(1)}(x),i\mathcal{L}_{\xi q}^{(1)}(y)]$, and $T[J_{\xi\xi}^{(3)}(0),i\mathcal{L}_{\xi q}^{(1)}(x)]$. At tree level all these contributions contain factors of D^c, which receive an additional suppression factor when matching onto $\mathrm{SCET_{II}}$. However, at higher orders in perturbation theory these operators can contribute since more collinear fields are contracted. For the operators displayed in Eqs. (39), (41) they give rise to nontrivial jet functions. Consider for example the time-ordered product

$$T_0^{(4)}=\int d^4x T[J_{\xi\xi}^{(3)}(0),i\mathcal{L}_{\xi q}^{(1)}(x)] \tag{42}$$

where $J_{\xi\xi}^{(3)}=(\bar{\xi}_n W)\Gamma(1/\bar{\mathcal{P}})(W^\dagger i\not{D}_\perp^c W)(\not{\bar{n}}/2)(W^\dagger\xi_n)$. Operators like $T_0^{(4)}$ appear for example in the matching of QCD onto $\mathrm{SCET_I}$ for the electromagnetic current of light quarks (see the second reference in [9]). Gauge invariant blocks of collinear fields in the time-ordered product are contracted when matching onto $\mathrm{SCET_{II}}$. An example is illustrated in the second diagram in Fig. 1 where the factors of fields containing $D_\perp^c$ derivatives are contracted. Such a graph does not exhibit the additional suppression factor, as there is no collinear covariant perpendicular derivative left over. Thus, this operator can contribute to the operator O_1 and induce a nontrivial Wilson coefficient J. Therefore, the operators $\mathcal{O}_{1,2}$ in $\mathrm{SCET_{II}}$ contributing to light-light soft-collinear current at any order in the matching from $\mathrm{SCET_I}$ have the form

$$O_1=J_1(\omega,y)(\bar{\xi}_n W)_\omega\Gamma(S^\dagger q_s)_y,$$

$$O_2=J_2(\omega_i,y)(\bar{\xi}_n W)_{\omega_1}\Gamma\frac{\not{\bar{n}}}{2}[W^\dagger i\not{D}_\perp^c W]_{\omega_2}\frac{1}{n\cdot\mathcal{P}}(S^\dagger q_s)_y, \tag{43}$$

where $(\bar{\xi}_n W)_\omega=[\bar{\xi}_n W\delta(\omega-\bar{\mathcal{P}}^\dagger)]$ and $(S^\dagger q_s)_y=[\delta(y-n\cdot P)S^\dagger q_s]$.

[2]Note that in matching we always expand the upper theory in a series of terms to match it onto the lower theory. Therefore, it is not unusual that operators in $\mathrm{SCET_I}$ match onto operators of different orders in $\mathrm{SCET_{II}}$.

Finally, this procedure can also be used to match onto the Lagrangian for mixed soft-collinear interactions in $\mathrm{SCET_{II}}$. After making the field redefinition in step (ii) there are no usoft-collinear Lagrangian interactions at order λ^0 in $\mathrm{SCET_I}$. Therefore from Eq. (33) it follows that it is not possible to construct a gauge invariant order η^0 soft-collinear Lagrangian. This is true for both quarks and gluons. This very simple power counting argument clarifies the original argument based on gauge invariance and power counting in Ref. [4] and supplements the direct matching calculations in Ref. [10]. In the language of the power counting formulas in Ref. [13] the power counting for soft-collinear Lagrangian terms in $\mathrm{SCET_{II}}$ corresponds to an index factor $(k-3)V_{SC}^k$ in the equation for δ which gives the power counting for an arbitrary time-ordered product. Here V_{SC}^k counts the number of insertions of soft-collinear Lagrangian operators that are order η^k. The factor of $(k-3)$ agrees with the phase space argument in Ref. [10].

At order λ^2 we have a time-ordered product $\int d^4x T\{\mathcal{L}_{\xi q}^{(1)}(0), i\mathcal{L}_{\xi q}^{(1)}(x)\}$, which can induce suppressed operators in the $\mathrm{SCET_{II}}$ Lagrangian. Contracting the collinear quarks in a $W^\dagger \xi_n(0)\bar{\xi}_n W(x)$ factor this gives an operator whose form agrees with Eq. (17) of Ref. [10]. At tree level in the matching we find

$$\mathcal{L}_{qqBB}^{(1)} = (\bar{q}_s S)\left(W^\dagger ig \mathcal{B}_\perp^c W \frac{1}{\bar{\mathcal{P}}^\dagger}\right) \frac{\not{n}}{2}\left(\frac{1}{\bar{\mathcal{P}}} W^\dagger ig \mathcal{B}_\perp^c W\right) \times \frac{1}{n\cdot\mathcal{P}}(S^\dagger q_s). \tag{44}$$

Here the factor $\not{n}/(2n\cdot\mathcal{P})$ is again from the collinear quark propagator, and from Eq. (33) we count $E=1$ since two $\perp$ gluons are external and have their power counting changed in passing to $\mathrm{SCET_{II}}$. The superscript (1) indicates that this operator contributes at order η in $\mathrm{SCET_{II}}$. The factor of η is derived by noting that the operator in Eq. (44) is $\sim\eta^4$ and so counts as $V_{SC}^4=1$. Thus subtracting three we see that it contributes an η to the δ power counting formula.

IV. CONCLUSION

In this paper we discussed a few issues related to the gauge invariance of the soft-collinear effective theory beyond leading order. Together with power counting and reparametrization invariance, gauge invariance constrains the form of the allowed effective theory operators. However, there is some freedom in splitting the QCD gluon field into collinear and ultrasoft fields in the effective theory. In Sec. II we showed that the choice which gives

$$i\bar{n}\cdot\hat{D} = i\bar{n}\cdot\hat{D}_c + \hat{W} i\bar{n}\cdot D_{us}\hat{W}^\dagger,$$

$$i\hat{D}_\perp^\mu = i\hat{D}_{c,\perp}^\mu + \hat{W} iD_{us,\perp}^\mu \hat{W}^\dagger, \tag{45}$$

corresponds to collinear and usoft fields which transform in a homogeneous way under the gauge transformations at any order in λ. This result uniquely fixes how power suppressed ultrasoft derivatives appear which are related to the collinear derivatives by reparametrization invariance. Using the new fields we then gave results for the subleading collinear and usoft-collinear effective Lagrangians to $O(\lambda^2)$, which by themselves are invariant under the collinear gauge transformations in Table I.

A related construction was presented in Ref. [15] using a position space multipole expansion. The collinear field redefinition adopted here differs from the one there. Our construction has the benefit of avoiding gauge fixing in the derivation. The explicit form of the transformation relating the fields in Ref. [15] to the fields we have here remains an open and interesting question.

For $\mathrm{SCET_{II}}$, power counting, RPI and gauge invariance also give restrictions on allowed operators, which are however not as strict as in $\mathrm{SCET_I}$. The reason is that $\mathrm{SCET_{II}}$ is nonlocal at the scale over which soft particles are propagating, whereas $\mathrm{SCET_I}$ is only nonlocal at the hard scale Q. (This is the case before we decide to induce by hand a nonlocal Y in $\mathrm{SCET_I}$ by making a field redefinition.) Thus, additional input is needed to construct operators in $\mathrm{SCET_{II}}$, and one has to carefully consider which modes are integrated out in arriving at the low energy theory. In Ref. [8] it was proposed that soft-collinear operators in $\mathrm{SCET_{II}}$ could be constructed in an elegant manner by making use of factored ultrasoft-collinear operators in $\mathrm{SCET_I}$. In Sec. III we presented details of this matching calculation in several examples, and showed how the constraints from power counting and gauge invariance on $\mathrm{SCET_I}$ restrict the form of the operators induced in matching onto $\mathrm{SCET_{II}}$.

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