

Positivity and dense matter

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We elaborate on previous results concerning the positivity of the Euclidean path integral measure for low-energy modes in dense fermionic matter. We show that the sign problem usually associated with fermions is absent if one considers only low-energy degrees of freedom. We describe a method for simulating dense QCD on the lattice and give a proof using rigorous inequalities that the color-flavor locked (CFL) phase is the true vacuum of three flavor, massless QCD. We also discuss applications to electronic systems in condensed matter, such as generalized Hubbard models.

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I. INTRODUCTION

Euclidean quantum chromodynamics (QCD) with a non-zero chemical potential has a complex measure, which has made lattice simulation particularly difficult [1]. (For recent lattice investigation of the QCD phase boundary at finite density, see [2,3].) This problem is often referred to as the sign problem, because by appropriately grouping terms quantities such as the partition function can be written as a sum over real, but potentially negative, terms. (That this grouping can be accomplished is in many systems a consequence of a discrete symmetry such as parity or time-reversal invariance.) Indeterminate signs are enough to preclude the use of importance sampling, the main technique for speeding up Monte Carlo integration. It is important to note that while the sign problem often arises in systems of fermions, it is neither inevitable nor inescapable. For example, in QCD at zero chemical potential and in the Hubbard model at half filling one can organize the sum so that terms are real and positive. For recent work on the sign problem, see [4].

Analytical work in color superconductivity [5] has demonstrated a rich phase structure at high density, and stimulated interest in QCD at non-zero baryon density. Several experiments have been proposed to probe matter at density of a few times nuclear matter density [6]. Even rudimentary information about the behavior of dense matter would be useful to the experimental program, as well as to the study of compact astrophysical objects such as neutron stars. In an earlier paper [7], we showed that QCD near a Fermi surface has positive, semi-definite measure. In the limit of low energies, the contribution of the remaining modes far from the Fermi surface can be systematically expanded, using a high density effective theory previously introduced by one of us [8,9]. This effective theory is sufficient to study phenomena like color superconductivity, although quantities like the equation of state may be largely determined by dynamics deep in the Fermi sea.

The expansion about the Fermi surface is in powers of $1/\mu$, where μ is the chemical potential. For this expansion to be controlled, the ultraviolet cutoff of our effective theory must be less than μ , or equivalently the scale of the physics of interest must be small relative to the chemical potential. In QCD at asymptotic density, the superconducting gap is exponentially small, so this condition is satisfied. However, it is also quite possible that at intermediate densities (e.g., those inside a neutron star) the gap is somewhat smaller than μ , providing us with an additional small dimensionless parameter. Even if this is not the case, the power expansion of the effective theory is qualitatively different from the usual perturbation in α_s , and therefore worth exploring. Finally, we note that models (e.g., of electronic systems) which are not asymptotically free may exhibit strongly coupled quasiparticle excitations even at high density. The results described here still apply to such systems and may be of use in their simulation.

This paper is organized as follows. We begin in Sec. II with simple examples in (1+1) and higher dimensions which illustrate how the effective Fermi surface description can have positive measure even if the original model has a sign problem. In Sec. III we review the results from our previous paper, and include some discussion of how they apply to electronic systems that may arise in condensed matter. In Sec. IV we describe how they can be applied to lattice simulations of dense matter. In Sec. V we discuss correlator inequalities (also known as QCD inequalities) that result from positivity, and how they restrict the possible ground states of QCD. In Sec. VI we conclude with a summary and future prospects.

II. EXAMPLE: (1+1) DIMENSIONS AND BEYOND

We begin with an example that illustrates the basic ideas in a simple setting. Consider the Euclidean (1+1) action of non-relativistic fermions interacting with a gauge field A

$$S = \int d\tau dx \psi_\sigma^* [(-\partial_\tau + i\phi + \epsilon_F) - \epsilon(-i\partial_x + A)] \psi_\sigma \quad (1)$$

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where $\epsilon(p)$ is the energy as a function of momentum [e.g. $\epsilon(p) \approx (p^2/2m) + \dots$]. In Eq. (1) and below we may consider it as a function of the operator $(-i\partial_x + A)$. The dispersion relation in the presence of the chemical potential ϵ_F is $E(p) = \epsilon(p) - \epsilon_F$, and a low energy mode must have momentum close to $\pm p_F$, where $\epsilon(\pm p_F) = \epsilon_F$. The Fermi surface in (1+1) dimensions is reduced to the two points $p = \pm p_F$. Near these points we have

$$E(p \pm p_F) \approx \pm v_F p, \quad (2)$$

where $v_F = \partial E / \partial p|_{p_F}$ is the Fermi velocity.

The action (1) is not obviously positive. In fact, the operator in brackets $[\dots]$ clearly has Hermitian as well as anti-Hermitian components, and hence complex eigenvalues.

Let us assume that the gauge field has small amplitude and is slowly varying relative to the scale p_F . We will extract the slowly varying component of the fermion field to construct a low energy effective theory involving quasiparticles and gauge fields. This effective theory will have a positive, semi-definite determinant.

First, we extract the quasiparticle modes (we suppress the spin index in what follows)

$$\psi(x, \tau) = \psi_L e^{+ip_F x} + \psi_R e^{-ip_F x}, \quad (3)$$

where the functions $\psi_{L,R}$ are slowly varying. To simplify the action, we use the identity

$$e^{\pm ip_F x} E(-i\partial_x + A) e^{\mp ip_F x} \psi(x) \approx \pm v_F (-i\partial_x + A) \psi(x), \quad (4)$$

to obtain [10]

$$S_{\text{eff}} = \int d\tau dx [\psi_L^\dagger (-\partial_\tau + i\phi + i\partial_x - A) \psi_L + \psi_R^* (-\partial_\tau + i\phi - i\partial_x + A) \psi_R]. \quad (5)$$

We can write this in a more familiar form by introducing the Euclidean (1+1) gamma matrices $\gamma_{0,1,2}$, which are Hermitian and can be taken as $\gamma_i = \sigma_i$ where σ are the Pauli matrices. Using $\psi_{L,R} = \frac{1}{2}(1 \pm \gamma_2)\psi$ we obtain

$$S_{\text{eff}} = \int d\tau dx \bar{\psi} \gamma^\mu (\partial_\mu + iA_\mu) \psi \equiv \int d\tau dx \bar{\psi} \mathcal{D} \psi. \quad (6)$$

Since the gamma matrices are Hermitian, and the operator $(\partial_\mu + iA_\mu)$ is anti-Hermitian, the operator \mathcal{D} in Eq. (6) has purely imaginary eigenvalues. However, because γ_2 anti-commutes with \mathcal{D} , the eigenvalues come in conjugate pairs: given $\mathcal{D} \phi = \lambda \phi$, we have

$$\mathcal{D} (\gamma_2 \phi) = -\gamma_2 \mathcal{D} \phi = -\gamma_2 \lambda \phi = -\lambda (\gamma_2 \phi).$$

Hence the determinant $\det \mathcal{D} = \prod \lambda^* \lambda$ is real and positive semi-definite.

Thus, by considering only the low-energy modes near the Fermi points of the original model (1), we obtain an effective theory with desirable positivity properties. Note that it is necessary that the interactions (in this case, the background

gauge field A) not couple strongly the low-energy modes to fast modes which are far from the Fermi points. This is a reasonable approximation in many physical situations, where it is the interactions among quasiparticles that are of primary interest. In what follows, we will apply this basic idea to more complex models such as QCD.

It is straightforward to go beyond (1+1) dimensions. Consider an electron system, described by

$$\mathcal{L} = \psi^\dagger [i\partial_t - \epsilon(\vec{p})] \psi + \mu \psi^\dagger \psi, \quad (7)$$

where $\epsilon(\vec{p})$ is the electron energy, a function of momentum \vec{p} . It is interesting to note that the non-relativistic system already has a sign problem even at the zero density, $\mu=0$, though the free case does not suffer this, thanks to the separation of variables. In fact, it is quite unusual to have a system like vacuum QCD which has no sign problem. In Euclidean space the electron determinant is

$$M = -\partial_\tau - \epsilon(\vec{p}) + \mu. \quad (8)$$

The first term in operator (8) is anti-Hermitian, while the rest are Hermitian. Since there is no constant matrix P in the spin space that satisfies $M^\dagger = P M P^{-1}$, it has a sign problem in general.

Let us decompose the fermion momentum as

$$\vec{p} = \vec{p}_F + \vec{l}. \quad (9)$$

Again, the Fermi momentum is defined to be a momentum at which the energy equals to the chemical potential at zero temperature: $\mu = \epsilon(p_F)$, and the Fermi velocity is defined as

$$\vec{v}_F = \left. \frac{\partial \epsilon(p)}{\partial \vec{p}} \right|_{p=p_F}. \quad (10)$$

If we are interested in low energies, $|\vec{l}| \ll p_F$, we may integrate out the fast modes to get an effective operator,

$$M_{\text{EFT}} = -\partial_\tau - \vec{v}_F \cdot \vec{l}, \quad (11)$$

which has complex eigenvalues. However, when we include the $-\vec{v}_F$ sector, we have $M_{\text{EFT}}(\vec{v}_F) M_{\text{EFT}}(-\vec{v}_F) \leq 0$ (i.e., has real negative eigenvalues), assuming $\epsilon(\vec{p}) = \epsilon(-\vec{p})$. We again see that the sign problem in the electron system is alleviated in the low-energy effective theory.

III. QCD

Let us recall why the measure of dense QCD is complex in Euclidean space. We use the following analytic continuation of the Dirac Lagrangian to Euclidean space:

$$x_0 \rightarrow -ix_E^4, \quad x_i \rightarrow x_E^i, \quad \gamma_0 \rightarrow \gamma_E^4, \quad \gamma_i \rightarrow i\gamma_E^i. \quad (12)$$

The Euclidean gamma matrices satisfy

$$\gamma_E^{\mu\dagger} = \gamma_E^\mu, \quad \{\gamma_E^\mu, \gamma_E^\nu\} = 2\delta^{\mu\nu}. \quad (13)$$

The Dirac-conjugated field, $\bar{\psi} = \psi^\dagger \gamma^0$, is mapped into a field, still denoted as $\bar{\psi}$, which is independent of ψ and transforms as ψ^\dagger under $SO(4)$. Then, the grand canonical partition function for QCD is

$$Z(\mu) = \int dA_\mu \det(M) e^{-S(A_\mu)}, \quad (14)$$

where $S(A_\mu)$ is the positive semi-definite gauge action, and the Dirac operator

$$M = \gamma_E^\mu D_E^\mu + \mu \gamma_E^4, \quad (15)$$

where $D_E = \partial_E + iA_E$ is the analytic continuation of the covariant derivative. The Hermitian conjugate of the Dirac operator is

$$M^\dagger = -\gamma_E^\mu D_E^\mu + \mu \gamma_E^4. \quad (16)$$

The first term in Eq. (15) is anti-Hermitian, while the second is Hermitian, hence the generally complex eigenvalues. When $\mu=0$, the eigenvalues are purely imaginary, but come in conjugate pairs (λ, λ^*) [11], so the resulting determinant is real and positive semi-definite:

$$\det M = \prod \lambda^* \lambda \geq 0. \quad (17)$$

In what follows we investigate the positivity properties of an effective theory describing only modes near the Fermi surface. A system of degenerate quarks with a net baryon number asymmetry is described by the QCD Lagrangian density with a chemical potential μ ,

$$\mathcal{L}_{\text{QCD}} = \bar{\psi} i \not{D} \psi - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \mu \bar{\psi} \gamma_0 \psi, \quad (18)$$

where the covariant derivative $D_\mu = \partial_\mu + iA_\mu$ and we neglect the quark mass for simplicity.

At high density ($\mu \gg \Lambda_{\text{QCD}}$), due to asymptotic freedom the energy spectrum of quarks near the Fermi surface is approximately given by a free Dirac eigenvalue equation,

$$(\vec{\alpha} \cdot \vec{p} - \mu) \psi_\pm = E_\pm \psi_\pm, \quad (19)$$

where $\vec{\alpha} = \gamma_0 \vec{\gamma}$ and ψ_\pm denote the energy eigenfunctions with eigenvalues $E_\pm = -\mu \pm |\vec{p}|$, respectively. At low energy $E < \mu$, the states ψ_+ near the Fermi surface, $|\vec{p}| \sim \mu$, are easily excited but ψ_- , which correspond to the states in the Dirac sea, are completely decoupled due to the presence of the energy gap μ provided by the Fermi sea. Therefore the appropriate degrees of freedom at low energy consist of gluons and ψ_+ only.

Now, we wish to construct an effective theory describing the dynamics of ψ_+ by integrating out modes whose energy is greater than μ . Consider a quark near the Fermi surface, whose momentum is close to $\mu \vec{v}_F$. Without loss of generality, we may decompose the momentum of a quark into a Fermi momentum and a residual momentum as

$$p_\mu = \mu v_\mu + l_\mu, \quad (20)$$

where $v^\mu = (0, \vec{v}_F)$. Since the quark energy is given as

$$E = -\mu + \sqrt{(l_\parallel + \mu)^2 + l_\perp^2}, \quad (21)$$

the residual momentum should satisfy $(l_\parallel + \mu)^2 + l_\perp^2 \leq 4\mu^2$ with $\vec{l}_\parallel = \vec{v}_F \vec{l} \cdot \vec{v}_F$ and $\vec{l}_\perp = \vec{l} - \vec{l}_\parallel$.

To describe the small excitations of the quark with Fermi momentum, $\mu \vec{v}_F$, we decompose the quark fields as

$$\psi(x) = e^{i\mu \vec{v}_F \cdot \vec{x}} [\psi_+(\vec{v}_F, x) + \psi_-(\vec{v}_F, x)], \quad (22)$$

where

$$\begin{aligned} \psi_\pm(\vec{v}_F, x) &= P_\pm(\vec{v}_F) e^{-i\mu \vec{v}_F \cdot \vec{x}} \psi(x) \\ \text{with } P_\pm(\vec{v}_F) &\equiv \frac{1 \pm \vec{\alpha} \cdot \vec{v}_F}{2}. \end{aligned} \quad (23)$$

The quark Lagrangian in Eq. (18) then becomes

$$\begin{aligned} \bar{\psi}(i\not{D} + \mu \gamma^0) \psi &= [\bar{\psi}_+(\vec{v}_F, x) i \gamma_\parallel^\mu D_\mu \psi_+(\vec{v}_F, x) \\ &\quad + \bar{\psi}_-(\vec{v}_F, x) \gamma^0 (2\mu + i\bar{D}_\parallel) \psi_-(\vec{v}_F, x)] \\ &\quad + [\bar{\psi}_-(\vec{v}_F, x) i \not{D}_\perp \psi_+(\vec{v}_F, x) + \text{H.c.}] \end{aligned} \quad (24)$$

where $\gamma_\parallel^\mu \equiv (\gamma^0, \vec{v}_F \vec{v}_F \cdot \vec{\gamma})$, $\gamma_\perp^\mu = \gamma^\mu - \gamma_\parallel^\mu$, $\bar{D}_\parallel = \bar{V}^\mu D_\mu$ with $V^\mu = (1, \vec{v}_F)$, $\bar{V}^\mu = (1, -\vec{v}_F)$, and $\not{D}_\perp = \gamma_\perp^\mu D_\mu$.

At low energy, we integrate out all the ‘‘fast’’ modes ψ_- and derive the low energy effective Lagrangian by matching all the one-light-particle irreducible amplitudes containing gluons and ψ_+ in loop expansion. The effects of fast modes will appear in the quantum corrections to the couplings of low energy interactions. At tree level, the matching is equivalent to eliminating ψ_- in terms of equations of motion:

$$\begin{aligned} \psi_-(\vec{v}_F, x) &= -\frac{i\gamma^0}{2\mu + iD_\parallel} \not{D}_\perp \psi_+(\vec{v}_F, x) \\ &= -\frac{i\gamma^0}{2\mu} \sum_{n=0}^{\infty} \left(-\frac{iD_\parallel}{2\mu} \right)^n \not{D}_\perp \psi_+(\vec{v}_F, x). \end{aligned} \quad (25)$$

Therefore, the tree-level Lagrangian for ψ_+ becomes

$$\mathcal{L}_{\text{eff}}^0 = \bar{\psi}_+ i \gamma_\parallel^\mu D_\mu \psi_+ - \frac{1}{2\mu} \bar{\psi}_+ \gamma^0 (\not{D}_\perp)^2 \psi_+ + \dots, \quad (26)$$

where the ellipsis denotes terms with higher derivatives.

Consider the first term in our effective Lagrangian, which when continued to Euclidean space yields the operator

$$M_{\text{eff}} = \gamma_\parallel^E \cdot D(A). \quad (27)$$

M_{eff} is anti-Hermitian and it anti-commutes with γ_5 , so it leads to a positive semi-definite determinant. However, note

that the Dirac operator is not well defined in the space of $\psi_+(\vec{v}_F, x)$ (for fixed v_F), since it maps $\psi_+(\vec{v}_F, x)$ into $\psi_+(-\vec{v}_F, x)$:

$$i\mathcal{D}_{\parallel} P_+ \psi = P_- i\mathcal{D}_{\parallel} \psi. \quad (28)$$

Since $P_-(\vec{v}_F) = P_+(-\vec{v}_F)$, $i\mathcal{D}\psi_+(\vec{v}_F, x)$ are $\psi_+(-\vec{v}_F, x)$ modes, or fluctuations of a quark with momentum $-\mu\vec{v}_F$.

We can demonstrate the necessity of including both $\psi_+(\vec{v}_F, x)$ and $\psi_+(-\vec{v}_F, x)$ modes in our effective theory by considering charge conservation in a world with only $+\vec{v}_F$ quarks. The divergence of the quark current at one loop is at the leading order in the $1/\mu$ expansion

$$\langle \partial_{\mu} J^{a\mu}(\vec{v}_F, x) \rangle = g_s \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} p^{\mu} \Pi_{\mu\nu}^{ab}(p) A_{\parallel}^{b\nu}(-p), \quad (29)$$

where $A_{\parallel} = (A_0, \vec{v}_F \vec{v}_F \cdot \vec{A})$ and $\Pi_{\mu\nu}^{ab}$ is the vacuum polarization tensor in the effective theory given as [8]

$$\Pi_{ab}^{\mu\nu}(p) = -i \frac{\mu^2}{2} \delta_{ab} \left(\frac{-2\vec{p} \cdot \vec{v}_F V^{\mu} V^{\nu}}{p \cdot V + i\epsilon \vec{p} \cdot \vec{v}_F} + g_{\perp}^{\mu\nu} \right), \quad (30)$$

where $g_{\perp}^{\mu\nu} = g^{\mu\nu} - (V^{\mu} \bar{V}^{\nu} + \bar{V}^{\mu} V^{\nu})/2$ is the metric tensor perpendicular to V^{μ} and \bar{V}^{μ} . The polarization tensor has to be transverse to maintain gauge invariance. We find that if we have both fields $\psi_+(\vec{v}_F, x)$ and $\psi_+(-\vec{v}_F, x)$ the current is conserved and the gauge symmetry is not anomalous:

$$\langle \partial_{\mu} J_a^{\mu}(\vec{v}_F, x) + \partial_{\mu} J_a^{\mu}(-\vec{v}_F, x) \rangle = 0. \quad (31)$$

Therefore, we need to introduce quark fields with opposite momenta. The Dirac operator is well defined on this larger space.

This anomaly can be understood in terms of spectral flow, since the Fermi surface is (in a certain sense) not gauge invariant. Under a gauge transformation, $U(x) = e^{iq \cdot x}$, the Hamiltonian changes and the energy spectrum of free modes of residual momentum \vec{l} shifts to $E = \vec{l} \cdot \vec{v}_F + \vec{q} \cdot \vec{v}_F$. Quarks near the Fermi surface with $\vec{v}_F \cdot \vec{q} > 0$ flow out of the Fermi sea, creating charge. This charge creation is compensated by quarks with opposite \vec{v}_F ; their energy decreases and they flow into the Fermi sea. However, unless modes with opposite velocities (i.e. both sides of the Fermi sphere) are included, charge is not conserved.

Thus far we have considered the quark velocity as a parameter labelling different sectors of the quark field. This is similar to the approach of heavy quark effective theory (HQET) [12], in which the velocity of the heavy charm or bottom quark is almost conserved due to the hierarchy of scales between the heavy quark mass and the QCD scale. However, this approach contains an ambiguity often referred to as ‘‘reparametrization invariance,’’ related to the non-uniqueness of the decomposition (20) of quark momenta into a large and residual component. In the dense QCD case, two

$\psi(v_F, x)$ modes whose values of v_F are not very different may actually represent the same degrees of freedom of the original quark field. In what follows we give a different formulation which describes *all* velocity modes of the quark field, and is suitable for defining the quasiparticle determinant.

First, a more precise definition of the breakup of the quark field into Fermi surface modes. Using the momentum operator in a position eigenstate basis: $\vec{p} = -i\vec{\partial}$, we construct the Fermi velocity operator:

$$\vec{v} = \frac{-i}{\sqrt{-\nabla^2}} \frac{\partial}{\partial x}, \quad (32)$$

which is Hermitian, and a unit vector.

Using the velocity operator, we define the projection operators P_{\pm} as before and break up the quark field as $\psi(x) = \psi_+(x) + \psi_-(x)$, with $\psi_{\pm} = P_{\pm} \psi$. By leaving \vec{v} as an operator we can work in coordinate space without introducing the HQET-inspired velocity Fourier transform which introduces v_F as a parameter. If we expand the quark field in the eigenstates of the velocity operators, we recover the previous formalism with all Fermi velocities summed up.

The leading low-energy part of the quark action is given by

$$\mathcal{L}_+ = \bar{\psi} P_-(v) (i\partial - \not{A} + \mu \gamma_0) P_+(v) \psi. \quad (33)$$

As before, we define the fields ψ_+ to absorb the large Fermi momentum:

$$\psi_+(x) = e^{-i\mu \vec{x} \cdot \vec{v}} P_+(v) \psi(x). \quad (34)$$

Let us denote the eigenvalue v obtained by acting on the field ψ (which has momentum of order μ) as v_l (or v ‘‘large’’), whereas eigenvalues obtained by acting on the effective field theory modes ψ_+ are denoted v_r (or v ‘‘residual’’). If the original quark mode had momentum p with $|p| > \mu$ (i.e. was a particle), then v_l and v_r are parallel, whereas if $|p| < \mu$ (as for a hole) then v_r and v_l are antiparallel. In the first case, we have $P_+(v_l) = P_+(v_r)$ whereas in the second case $P_+(v_l) = P_-(v_r)$. Thus, the residual modes ψ_+ can satisfy either of $P_{\pm}(v_r) \psi_+ = \psi_+$, depending on whether the original ψ mode from which it was derived was a particle or a hole. In fact, ψ_+ modes can also satisfy either of $P_{\pm}(v_l) \psi_+ = \psi_+$ since they can originate from ψ modes with momentum $\sim +\mu v$ as well as $-\mu v$ (both are present in the original measure: $D\bar{\psi} D\psi$). So, the functional measure for ψ_+ modes contains all possible spinor functions—the only restriction is on the momenta: $|l_0|, |\vec{l}| < \Lambda$, where Λ is the cutoff.

In light of the ambiguity between v_l and v_r , the equation $\psi = e^{+i\mu \vec{x} \cdot v} \psi_+$ must be modified to

$$\psi = \exp(+i\mu \vec{x} \cdot v \alpha \cdot v) \psi_+ = \exp(+i\mu \vec{x} \cdot v_r \alpha \cdot v_r) \psi_+, \quad (35)$$

where the factor of $\alpha \cdot v_r$ corrects the sign in the momentum shift if v_r and v_l are antiparallel. In general, any expression

with two powers of v is unaffected by this ambiguity. For notational simplicity we define a local operator

$$X \equiv \mu x \cdot v \alpha \cdot v = \mu \frac{\alpha^i x^j}{\nabla^2} \frac{\partial^2}{\partial x^i \partial x^j}. \quad (36)$$

Taking this into account, we obtain the following action:

$$\mathcal{L}_+ = \bar{\psi}_+ e^{-iX} (i \not{\partial} - \not{A} + \mu \gamma_0) e^{+iX} \psi_+. \quad (37)$$

We treat the \not{A} term separately from $i \not{\partial} + \mu \gamma_0$ since the former does not commute with X , while the latter does. Continuing to Euclidean space, and using the identity $P_- \gamma_\mu P_+ = \gamma_\mu^\parallel P_+$, we obtain

$$\mathcal{L}_+ = \bar{\psi}_+ \gamma_\mu^\parallel (\partial^\mu + i A_+^\mu) \psi_+, \quad (38)$$

where

$$A_+^\mu = e^{-iX} A^\mu e^{+iX} \quad (39)$$

and all γ matrices are Euclidean. The term containing A cannot be fully simplified because $[v, A] \neq 0$. Physically, this is because the gauge field carries momentum and can deflect the quark velocity. The redefined ψ_+ modes are functions only of the residual momenta l , and the exponential factors in the A term reflect the fact that the gluon originally couples to the quark field ψ , not the residual mode ψ_+ .

The kinetic term in Eq. (38) can be simplified to

$$\gamma_\mu^\parallel \partial^\mu = \gamma^\mu \partial^\mu \quad (40)$$

since $v \cdot \partial v \cdot \gamma = \partial \cdot \gamma$. The action (38) is the most general dimension 4 term with the rotational, gauge invariance [13] and projection properties appropriate to quark quasiparticles. Therefore, it is a general consequence of any Fermi liquid description of quark-like excitations.

The operator in Eq. (38) is anti-Hermitian and leads to a positive, semi-definite determinant since it anti-commutes with γ_5 . The corrections given in Eq. (26) are all Hermitian, so higher orders in the $1/\mu$ expansion may reintroduce complexity. The structure of the leading term plus corrections is anti-Hermitian plus Hermitian, just as in the original QCD Dirac Lagrangian with chemical potential.

By integrating out the fast modes, the Euclidean QCD partition function can be rewritten as

$$Z(\mu) = \int dA_+ \det M_{\text{eff}}(A_+) e^{-S_{\text{eff}}(A_+)}. \quad (41)$$

The leading terms in the effective action for gluons (these terms are generated when we match our effective theory, with energy cutoff Λ , to QCD) also contribute only real, positive terms to the partition function:

$$S_{\text{eff}}(A) = \int d^4 x_E \left(\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{M^2}{16\pi} \sum_{v_F} A_{\perp\mu}^a A_{\perp\mu}^a \right) \geq 0, \quad (42)$$

where $A_\perp = A - A_\parallel$ and the Debye screening mass is $M = \sqrt{N_f/(2\pi^2)} g_s \mu$. Note that Landau damping is due to softer quark modes which have not been integrated out, and therefore do not contribute to matching.

Although the HDET only describes low-energy modes, it still contains Cooper pairing interactions. This is because Cooper pairing, in which the quasiparticles have nearly equal and opposite momenta, is induced by gluonic interactions with small energy and momentum transfer. That is, although a gluon exchange (or other interaction) which causes a large angular deflection of a quasiparticle

$$|\vec{p}\rangle \rightarrow |\vec{p}'\rangle$$

must involve a large momentum transfer, and hence is not part of the effective theory, a Cooper pairing interaction

$$|\vec{p}, -\vec{p}\rangle \rightarrow |\vec{p}', -\vec{p}'\rangle$$

only involves a small energy and momentum transfer, even if the angle between \vec{p} and \vec{p}' is large. Hence, it is described by the leading order interaction between soft gluons and quarks in the effective theory (38).

Matching of hard gluon effects also leads to four-quark operators in the effective theory. The addition of these four-quark operators still leads to a positive action for attractive channels, since they arise from quasiparticle-gluon interactions which are originally positive. A simple way to study the positivity of four-quark operators is to replace them by a vector field with the trivial quadratic term V_μ^2 which couples to quarks like the original gluon: $V_\mu \bar{\psi} \gamma^\mu \psi$. Completing the square, we see that the resulting path integral is positive. Note that this argument does not apply to interactions involving six or more quarks, or interactions involving virtual anti-quarks. However, these are always suppressed by additional powers of μ .

Finally, some comments on more general models than QCD—in particular electronic systems in condensed matter—which are also affected by a sign problem. We note that as long as the important dynamics of the system involve energies which are small compared to the Fermi energy (or are exhibited at a temperature small compared to the Fermi temperature), the Euclidean description of the system is likely to have a positive effective action after modes far from the Fermi surface are integrated out. Models involving, e.g., attractive four-fermion interactions or long range gauge fields fall into this category. As long as the important interactions are soft interactions between quasiparticle degrees of freedom not too far from the Fermi surface, the model can be simulated without a sign problem, even if it is strongly coupled.

As a specific example, consider the Hamiltonian

$$H = \sum_{ab} t_{ab} c_a^\dagger c_b + \sum_{abcd} v_{abcd} c_a^\dagger c_b^\dagger c_c c_d, \quad (43)$$

where $abcd$ denote lattice site and spin indices. This Hamiltonian includes the Hubbard model as a special case. A constant diagonal part of the kinetic term acts as a chemical

potential $\mu = t_{aa}$, but t may also contain hopping terms such as a nearest-neighbor off-diagonal term in the tight binding approximation. The usual kinetic term, taken by itself, produces a band structure which in momentum space has the form

$$\sum_k E(k) c_k^\dagger c_k, \quad (44)$$

for some function $E(k)$. A non-zero chemical potential leads to filling of levels up to some momentum k_F . There is an effective theory description of modes near this surface if the Fourier transform of the interaction term v_{abcd} only contains soft interactions (i.e. support for typical momentum transfers less than k_F).

To obtain the partition function Z we must extend the fields from the d -dimensional lattice to $(d+1)$ -dimensional Euclidean space, with finite extent β in the time direction. The action density is given by

$$-\frac{\partial}{\partial \tau} - \mathcal{H} \quad (45)$$

where \mathcal{H} is the Hamiltonian density. Using the Hubbard-Stratonovich transformation, we can rewrite the interaction term in terms of a functional integral over a background field U ,

$$Z = \frac{1}{N} \int \prod_{ab} dU_{ab} e^{\sum_{cdef} U_{cd} v_{cdef}^{-1} U_{ef}} \det[T - U], \quad (46)$$

where $T = (\partial/\partial \tau) - t$ contains the time derivative as well as the kinetic term t from the Hamiltonian (43). We assume that the interaction v is real and negative definite so that the exponential is real and the integral converges.

The determinant can be expanded about our effective theory. The leading term is positive and has no sign problem. The correction terms are suppressed as long as the background field U has support on momentum scales small compared to k_F . [See the discussion in the next section, especially Eq. (52) with U playing the role of the gauge field.] Typical fields U in the integral are determined by the exponential term—if the Fourier transform of v has little support at large momenta, then the exponent $\sum_k U(k) v^{-1}(k) U(k)$ will be large and negative, and the corresponding modes of U suppressed.

Unfortunately, the Hubbard model itself contains the interaction term

$$V \sum_i n_{i\uparrow} n_{i\downarrow} \quad (47)$$

where $n_i = c_i^\dagger c_i$ is the number operator. This interaction allows all momentum transfers up to π/a , where a is the lattice spacing. Because it couples low-energy modes to fast modes far from the Fermi surface, the resulting partition function is not well approximated by the positive EFT part.

However, a model with modified (softer) interactions would be approximately positive and might in fact be more physically realistic.

IV. LATTICE SIMULATION

The goal of this section is to give a method for simulating QCD at finite density. We will consider a chemical potential μ much larger than Λ_{QCD} throughout, and divide the functional integral over quark excitations into two parts: (I) modes within a shell of width Λ of the Fermi surface, and (II) modes which are further than Λ from the Fermi surface. We will assume the hierarchy

$$\mu \gg \Lambda \gg \Lambda_{\text{QCD}}. \quad (48)$$

The quark determinant in region (I) is well approximated by the determinant of the leading operator in high density effective theory (HDET) as long as the first inequality in (48) is satisfied. As discussed in the previous section, it is positive and real.

Here we will show that the contributions to the effective action for the gauge field from quark modes in region (II) are small and vanish as the Λ grows large compared to Λ_{QCD} .

First consider the theory in Minkowski space. The Dirac operator is

$$M = i \not{D} + \mu \gamma_0 \quad (49)$$

and the Dirac equation can be written as

$$i \partial_0 \psi = H \psi \quad (50)$$

with

$$H = i \alpha \cdot \partial - \mu \quad (51)$$

a Hermitian operator. The break up into regions (I) and (II) proceeds naturally in terms of energy eigenvalues of H (or l_0 in the HDET notation). The low-lying modes in region (I) are particle states with spatial momenta satisfying $|\vec{p}| \approx \mu$.

The analytic continuation of region (I) to Euclidean space leads to the HDET determinant considered previously.

Modes in region (II) all have large energy eigenvalues, at least as large as Λ . In considering their effect on physics at the scale Λ_{QCD} , we can integrate them out in favor of local operators suppressed by powers of $\Lambda_{\text{QCD}}/\Lambda$.

To make this concrete, consider the effective action for gauge fields with field strengths $F_{\mu\nu}$ of order Λ_{QCD} . The quark contribution to this effective action is simply the logarithm of the determinant we wish to compute. It can be expanded diagrammatically in graphs with external gauge field lines connected to a single quark loop. Restricting to region (II), we require that the quark modes in the loop have large H eigenvalues. Evaluating such graphs leads only to operators which are local in the external fields $A_\mu(x)$.

The resulting renormalizable (dimension 4) operator is the finite density equivalent of $F_{\mu\nu}^2$, except that due to the breaking of Lorentz invariance it contains separate time- and space-like components which represent the contribution of high-energy modes to the renormalization of the coupling

constant and Debye screening. These effects do not introduce a complex component when continued to Euclidean space.

Higher dimension operators, which involve additional powers of $F_{\mu\nu}$ or covariant derivatives D_μ are suppressed by the scale Λ . These are presumably the source of complex terms introduced to the effective action. However, due to the $1/\Lambda$ suppression they are dominated by the contribution from the low-lying modes in region (I), which is necessarily non-local, but real.

The logarithm of the Euclidean quark determinant will have the form

$$\begin{aligned} \ln \det M \sim & \mathcal{O}(\mu^4) + (\text{non-local, real}) \\ & + \mathcal{O}\left(\frac{1}{\Lambda}\right) (\text{local, complex}), \end{aligned} \quad (52)$$

where the first term is the (real, constant) free energy of non-interacting quarks, the second term is from the positive determinant in region (I) and the last term is the suppressed, complex contribution from region (II). Only the last two terms depend on the gauge field $A_\mu(x)$.

On the lattice, one can use the dominant dependence of $\det M$ on the first and second terms to do importance sampling. In order to keep the complex higher dimension operators [last term in Eq. (52)] small, it is important that the gauge field strengths are kept smaller than Λ^2 . One can impose this condition by using two different lattice spacings a_g for the gluons and a_{\det} for the quarks, with $a_g > a_{\det}$. The determinant is calculated on the finer a_{\det} lattice, and is a function of plaquettes which are obtained by interpolation from the plaquettes on the coarser a_g lattice. Interpolation can be defined in a natural way, since each lattice link variable $U_{x\mu}$ is an element of the gauge group, and one can connect any two points g_1, g_2 on the group manifold in a linear fashion: $g(t) = g_1 + t(g_2 - g_1), 0 \leq t \leq 1$.

More explicitly, let x and $x + a_g \mu$ be two neighboring points on the coarse lattice, and

$$z_n = x + n a_{\det} \mu, \quad n = 0, 1, \dots, N \quad (53)$$

be the corresponding points on the fine lattice: $z_0 = x, z_N = x + a_g \mu$, where $N = a_g / a_{\det}$. Then a link

$$U_{x,\mu} = \exp[i a_g G^m t^m]$$

is interpolated to a set of links as

$$\bar{U}_{z_n, z_{n+1}} = \exp[i a_{\det} G^m t^m], \quad (54)$$

where the t^m are $SU(N)$ generators, and the bar denotes the finer lattice. Equation (54) allows us to compute \bar{U} links which are sub-links of original a_g link variables, and lie on the outer perimeter of a plaquette. The remaining \bar{U} links, which are within an a_g plaquette, can be obtained through a similar interpolation starting from opposite sides of the perimeter, yielding an entire set of plaquettes on the a_{\det} lattice. The field strengths resulting from this interpolation are always of order a_g^{-2} and can be kept small compared to the cutoff Λ^2 . To properly include the quasiparticle modes, the

spacing of the fermion lattice must be $a_{\det} \sim 1/\mu$, while $a_g \sim 1/\Lambda_{\text{QCD}}$ is probably sufficient to capture the effects of non-perturbative gauge configurations.

The fermion determinant is to be computed as a function of the finer plaquettes $\{\bar{U}_{x\mu}\}$. The result is (approximately) real and positive and can be used for importance sampling. Further, there is a physical understanding of the complex part of the determinant: it originates in the modes far from the Fermi surface which have been integrated out.

V. INEQUALITIES AND ANOMALY MATCHING

Positivity of the measure allows for rigorous QCD inequalities at asymptotic density. For example, inequalities among masses of bound states can be obtained using bounds on bare quasiparticle propagators. One subtlety that arises is that a quark mass term does not lead to a quasiparticle gap (the mass term just shifts the Fermi surface). Hence, for technical reasons the proof of non-breaking of vector symmetries [14] must be modified. [Naive application of the Vafa-Witten theorem would preclude the breaking of the baryon number that is observed in the color-flavor-locked (CFL) phase [15].] A quasiparticle gap can be inserted by hand to regulate the bare propagator, but it will explicitly violate baryon number. However, following the logic of the Vafa-Witten proof, any symmetries which are preserved by the regulator gap cannot be broken spontaneously. One can, for example, still conclude that isospin symmetry is never spontaneously broken (although see below for a related subtlety). In the case of three flavors, one can introduce a regulator d with the color and flavor structure of the CFL gap to show rigorously that none of the symmetries of the CFL phase are broken at asymptotic density. On the other hand, by applying anomaly matching conditions [16], we can prove that the $SU(3)_A$ symmetries *are* broken. We therefore conclude that the CFL phase is the true ground state for three light flavors at asymptotic density, a result that was first established by explicit calculation [9,17,18].

To examine the long-distance behavior of the vector current, we note that the correlator of the vector current for a given gauge field A can be written as

$$\begin{aligned} & \langle J_\mu^a(\vec{v}_F, x) J_\nu^b(\vec{v}_F, y) \rangle^A \\ & = -\text{Tr} \gamma_\mu T^a S^A(x, y; d) \gamma_\nu T^b S^A(y, x; d), \end{aligned}$$

where the $SU(N_f)$ flavor current $J_\mu^a(\vec{v}_F, x) = \bar{\psi}_+(\vec{v}_F, x) \gamma_\mu T^a \psi_+(\vec{v}_F, x)$. The propagator with $SU(3)_V$ -invariant IR regulator d is given as

$$S^A(x, y; d) = \langle x | \frac{1}{M} | y \rangle = \int_0^\infty d\tau \langle x | e^{-i\tau(-iM)} | y \rangle$$

where with $D = \partial + iA$

$$M = \gamma_0 \begin{pmatrix} D \cdot V & d \\ d^\dagger & D \cdot \bar{V} \end{pmatrix}.$$

Since the eigenvalues of M are bounded from below by d , we have

$$\left| \langle x | \frac{1}{M} | y \rangle \right| \leq \int_R^\infty d\tau e^{-d\tau} \sqrt{\langle x|x \rangle} \sqrt{\langle y|y \rangle} = \frac{e^{-dR}}{d} \sqrt{\langle x|x \rangle} \sqrt{\langle y|y \rangle}, \quad (55)$$

where $R \equiv |x-y|$. The current correlators fall off rapidly as $R \rightarrow \infty$;

$$\begin{aligned} & \left| \int dA_+ \det M_{\text{eff}}(A) e^{-S_{\text{eff}} \langle J_\mu^A(\vec{v}_F, x) J_\nu^B(\vec{v}_F, y) \rangle^{A_+}} \right| \\ & \leq \int_{A_+} |\langle J_\mu^A(\vec{v}_F, x) J_\nu^B(\vec{v}_F, y) \rangle^{A_+}| \\ & \leq \frac{e^{-2dR}}{d^2} \int_{A_+} |\langle x|x \rangle| |\langle y|y \rangle|, \end{aligned} \quad (56)$$

where we used the Schwartz inequality in the first inequality, since the measure of the effective theory is now positive, and Eq. (55) in the second inequality. The IR regulated vector currents do not create massless modes out of the vacuum or Fermi sea, which implies that there is no Nambu-Goldstone mode in the $SU(3)_V$ channel. Therefore, for three massless flavors $SU(3)_V$ has to be unbroken as in CFL. The rigorous result provides a non-trivial check on explicit calculations, and applies to any system in which the quasiparticle dynamics have positive measure.

It is important to note the order of limits necessary to obtain the above results. Because there are higher-order corrections to the HDET, suppressed by powers of Λ/μ that spoil its positivity, there may be contributions on the right-hand side (RHS) of Eq. (56) of the form

$$\mathcal{O}\left(\frac{\Lambda}{\mu}\right) f(R), \quad (57)$$

where $f(R)$ falls off more slowly than the exponential in Eq. (56). To obtain the desired result, we must first take the limit $\mu \rightarrow \infty$ at fixed Λ before taking $R \rightarrow \infty$. Therefore, our results only apply in the limit of asymptotic density.

Although our result precludes breaking of vector symmetries at asymptotic density in the case of three *exactly* massless quarks [21], it does not necessarily apply to the case when the quark masses are allowed to be slightly non-zero. In that case the results depend on precisely how the limits of zero quark masses and asymptotic density are taken, as we discuss below.

In [19] the authors investigate the effect of quark masses on the CFL phase. These calculations are done in the asymptotic limit, and are reliable for sufficiently small quark masses. When $m_u = m_d \equiv m \ll m_s$ [unbroken $SU(2)$ isospin, but explicitly broken $SU(3)$], one finds a kaon condensate. The critical value of m_s at which the condensate forms is $m_s^* \sim m^{1/3} \Delta_0^{2/3}$, where Δ_0 is the CFL gap [see, in particular, Eq. (8) of the first paper]. As kaons transform as a doublet under isospin, the vector $SU(2)$ symmetry is broken in seeming contradiction with our result.

However, a subtle order of limits is at work here. For simplicity, let us set $m=0$. Note that the CFL regulator d , which was inserted by hand, explicitly breaks $SU(3)_A$ through color-flavor locking, leading to small positive mass squared for the pions and kaons, given as

$$m_{\pi, K}^2 \sim \alpha_s d^2 \ln\left(\frac{\mu}{d}\right). \quad (58)$$

The meson mass is not suppressed by $1/\mu$, since, unlike the Dirac mass term, the regulator, being a Majorana mass, does not involve antiquarks [20].

Therefore, even when the light quarks are massless, there is a critical value of m_s necessary to drive negative the mass-squared of kaons and cause condensation:

$$m_s^* \sim \left[g_s d \mu \ln\left(\frac{\mu}{d}\right) \right]^{1/2} > (d\mu)^{1/2}, \quad (59)$$

where g_s is the strong coupling constant. Note the product of g_s with the logarithm grows as μ gets large. To obtain our inequality we must keep the regulator d non-zero until the end of the calculation in order to see the exponential fall off. To find the phase with kaon condensation identified in [19] we must keep m_s larger than m_s^* . (Note $\mu \rightarrow \infty$, so to have any chance of finding this phase we must take $d \rightarrow 0$ keeping dR large and $d\mu$ small.)

Since the UV cutoff of the HDET must be larger than m_s , we have

$$1 > \left(\frac{m_s^*}{\Lambda}\right)^2 > \frac{d}{\Lambda} \frac{\mu}{\Lambda}, \quad (60)$$

which implies

$$\frac{\Lambda}{\mu} f(R) > \frac{d}{\Lambda} f(R). \quad (61)$$

Note the right-hand side of this inequality does not necessarily fall off at large R , and also does not go to zero for $\mu \rightarrow \infty$ at fixed Λ and d . This is a problem since to apply our inequality the exponential falloff from Eq. (56) must dominate the correction term (57), which is just the left-hand side of Eq. (61). Combining these equations, we see that the exponential falloff of the correlator is bounded below,

$$\frac{e^{-2dR}}{d^2} > \frac{d}{\Lambda} f(R), \quad (62)$$

in the scaling region with a kaon condensate, $m_s > m_s^*$.

Alternatively, if we had taken m_s to be finite for fixed regulator d (so that, as $\mu \rightarrow \infty$, eventually $m_s < m_s^*$), the inequality in Eq. (56) could be applied to exclude a Nambu-Goldstone boson, but we would find ourselves in the phase without a kaon condensate.

VI. CONCLUSION

The low-energy physics of dense fermionic matter, ranging from quark matter to electronic systems, is controlled by modes near the Fermi surface. An effective Lagrangian describing the low-energy modes can be given in a systematic expansion in powers of the energy scale over the chemical potential. The leading term in this expansion has a simple form, and we have shown that it leads to a real, positive Euclidean path integral measure.

This observation opens the door to importance sampling in Monte Carlo simulations of dense matter systems. The key requirement is that the interactions do not strongly couple the low-energy modes to modes far from the Fermi surface. QCD at high density satisfies this requirement, as do all asymptotically free models. Electronic systems in which the important interactions involve momentum transfer less than the Fermi energy are in this category, although some idealized models such as the Hubbard model are not. We have given some proposals for how the positive effective theory

might be simulated numerically. Ultimately, we hope that actual practitioners will develop even more practical methods.

Finally, positivity has analytical applications as well, since it allows the use of rigorous inequalities. In QCD we obtain restrictions on symmetry breaking patterns at high density. Similar restrictions can probably be obtained for electronic systems with suitable interactions.

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- [21] To investigate spontaneous symmetry breaking, one ordinarily has to start at finite volume and insert a source which explicitly breaks the symmetry. The source is removed only after the infinite volume limit is taken. We stress that the source does not have to be a quark mass (it could be a higher dimension operator), so one can investigate symmetry breaking even when the quark mass is exactly zero throughout the calculation. (To be precise, a quark mass does not explicitly violate vector symmetries, so it cannot play the role of the source in the thermodynamic limit needed here.)