Differential bremsstrahlung and pair production cross sections at high energies

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Detailed differential cross sections for high energy bremsstrahlung and pair production are derived with specific attention to the differences between the two processes, which are considerable. For the integrated cross sections, which are the only cross sections specifically known until now, the final state integration theorem guarantees that the exact cross section formulas can be exchanged between bremsstrahlung and pair production by the same substitution rules as for the Born-approximation Bethe-Heitler cross sections, for any amount of atomic screening. In fact the theorem states that the Coulomb corrections to the integrated bremsstrahlung and pair production cross sections are identical for any amount of screening. The analysis of the basic differential cross sections leads to fundamental physical differences between bremsstrahlung and pair production. Coulomb corrections occur for pair production in the strong electric field of the atom for "large" momentum transfer of the order of mc. For bremsstrahlung, on the other hand, the Coulomb corrections take place at a "large" distance from the atom of the order of $(\hbar/mc)\epsilon$, with a "small" momentum transfer mc/ϵ, where ϵ is the initial electron energy in units of mc^2. And the Coulomb corrections can be large, of the order of larger than $(Z/137)^2$, which is considerably larger than the integrated cross section corrections.

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I. INTRODUCTION

The high energy integrated cross section for bremsstrahlung and pair production, exact to all orders in αZ including screening effects, was obtained [1–6] at a time when there was a need for these quantities for comparison with experimental results, in particular, for total pair production cross sections. The results of these publications are well recorded and described in review publications [7–9], where also works on pair production for lower energies [10] are included.

In the present article we obtain the differential cross section for high energy bremsstrahlung and pair production, thereby completing our knowledge of two of our most important electron-positron-photon interaction cross sections at high energies. The calculations reveal new aspects of the "Coulomb corrections." At regions of angles and energies the Coulomb corrections may cut the Born-approximation Bethe-Heitler pair cross section by as much as a factor of one-half for a lead target—a situation which may be characterized as a dramatic higher order αZ effect, rather than a correction. The calculation further demonstrates the dramatic differences between the differential cross section of bremsstrahlung and pair production which still must give (and does so) the same Coulomb corrections for the *integrated* cross section—due to the theorem relating the final state integrated cross sections of bremsstrahlung and pair production [3].

II. BREMSSTRAHLUNG

The high energy, small angle bremsstrahlung differential cross section is obtained from a previous article [6]. It can be written in the convenient form [11]

$$d^4\sigma_{brems}=d^4\sigma_{Born}A_B(y_B), \tag{2.1}$$

with the well-known Bethe-Heitler, Born cross section in the small angle approximation:

$$d^4\sigma_{Born}=\sigma_0\frac{\epsilon_2^2}{kq^4}dk\Theta_k d\Theta_k \Theta_2 d\Theta_2 d\Phi[-2\epsilon_1\epsilon_2(\xi-\eta)^2 + ({\epsilon_1}^2+{\epsilon_2}^2)(\vec{u}-\vec{v})^2\xi\eta], \tag{2.2}$$

with

$$\sigma_0=8\frac{a^2}{2\pi}\left(\frac{\hbar}{mc}\right)^2\frac{e^2}{\hbar c}.$$

The Coulomb correction factor to the Bethe-Heitler cross section [6],

$$A_B(y_B)=[V^2(x_B)+a^2y_B^2W^2(x_B)]/V^2(1), \tag{2.3}$$

contains the higher order interactions of the electron in the Coulomb potential $V(r)=Ze^2/r$. The functions $V(x_B)$ and $W(x_B)$ are hypergeometric functions,

$$V(x_B)=F(ia,-ia;1;x_B), \tag{2.4}$$

$$W(x_B)=F(1+ia,1-ia;2;x_B)=\frac{1}{a^2}\frac{\partial}{\partial x_B}V(x_B), \tag{2.5}$$

where $a=\alpha Z$, and the index B stands for bremsstrahlung: $x_B=1-y_B$, with

$$y_B=\frac{(q_{min})^2}{q^2\xi\eta} \quad \text{with} \quad q_{min}=\frac{k}{2\epsilon_1\epsilon_2}, \tag{2.6}$$

the minimum momentum transfer to the nucleus, q_{min}, the minimum of

$$\vec{q}=\vec{p_1}-\vec{p_2}-\vec{k}, \tag{2.7}$$

with $(\vec{p_1},\epsilon_1)$, $(\vec{p_2},\epsilon_2)$ the momenta and energies of the incoming and outgoing electron, respectively, and $(\vec{k},\ k)$ the corresponding quantities of the emitted photon. All momenta and energies are measured in units of mc and mc^2, respectively, with m the electron mass. The energy balance is $\epsilon_1 = \epsilon_2 + k$. As usual the z axis is taken along the photon momentum with the polar angles of the initial and final electron momenta Θ_1 and Θ_2, respectively. The corresponding azimuth angle is Φ. $\vec{u}$ and $\vec{v}$ are the components of $\vec{p_1}$ and $\vec{p_2}$, perpendicular to $\vec{k}$, respectively; in the small angle approximation they are given by $u = p_1\Theta_1$ and $v = p_2\Theta_2$; ξ and η are electron propagators in this system $\xi = (u^2+1)^{-1}$ and $\eta = (v^2+1)^{-1}$.

The Coulomb correction factor $A_B(y_B)$ is a function of the two variables y_B and a only. From Eq. (2.3) the dependence on the other combinations of variables can be obtained directly, as discussed in Sec. IV.

III. PAIR PRODUCTION

The high energy, small angle differential cross section is obtained from Ref. [6]. It may be rewritten [12] in a form close to the bremsstrahlung cross section, Eq. (2.1),

$$d^4\sigma_{pair} = d^4\sigma_{Born}[V^2(x_p) + a^2 y_p^2 W^2(x_p)]/V^2(1) + d^4\sigma' a^2 y_p^2 W^2(x_p)/V^2(1), \tag{3.1}$$

with the well-known Bethe-Heitler cross section

$$d^4\sigma_{Born} = \sigma_0 \frac{\epsilon_1\epsilon_2}{k^3} d\epsilon_1 \Theta_1 d\Theta_1 \Theta_2 d\Theta_2 d\Phi \frac{1}{q^4}[2\epsilon_1\epsilon_2(\xi-\eta)^2 + (\epsilon_1^2+\epsilon_2^2)(\vec{u}+\vec{v})^2 \xi\eta] \tag{3.2}$$

and the additional cross section factor

$$d^4\sigma' = \sigma_0 \frac{\epsilon_1^2\epsilon_2^2}{k^3 q^4} d\epsilon_1 \Theta_1 d\Theta_1 \Theta_2 d\Theta_2 d\Phi [k^2 - 4\epsilon_1\epsilon_2 \xi\eta \times (u^2+v^2) - 2(\epsilon_1^2+\epsilon_2^2)(\vec{u}+\vec{v})^2\xi\eta]. \tag{3.3}$$

$V(x_p)$ and $W(x_p)$ in Eq. (3.1) are the hypergeometric functions defined in Eqs. (2.4) and (2.5), with the variable x_p for pair production in terms of physical variables given by

$$y_p = q^2 \xi\eta, \tag{3.4}$$

rather than x_B in Eq. (2.6) for bremsstrahlung.

The Coulomb correction factor for pair production we define corresponding to $A_B(y_B)$ by

$$d^4\sigma_{pair} = d^4\sigma_{Born} A_p(y_p, \vec{p_1}, \vec{p_2}), \tag{3.5}$$

with

$$A_p(y_p, \vec{p_1}, \vec{p_2}) = [V^2(x_p) + (1+R)a^2 y_p^2 W^2(x_p)]/V^2(1), \tag{3.6}$$

with R given by $R = d^4\sigma'/d^4\sigma_{Born}$, which introduces variables other than y_p in A_p, denoted by $\vec{p_1}$, $\vec{p_2}$ in Eq. (3.6),

$$R(y_{p_\perp}, \xi, \eta) = \frac{k^2 - 4\epsilon_1\epsilon_2(\xi+\eta-2\xi\eta) - 2(\epsilon_1^2+\epsilon_2^2) y_{p_\perp}}{2\epsilon_1\epsilon_2(\xi-\eta)^2 + (\epsilon_1^2+\epsilon_2^2) y_{p_\perp}}, \tag{3.7}$$

where $y_{p_\perp}$ contains the transverse component of q^2:

$$y_{p_\perp} = q_\perp^2 \xi\eta = (\vec{u}+\vec{v})^2 \xi\eta. \tag{3.8}$$

The two y variables are clearly related by

$$y_B \cdot y_p = q_{min}^2. \tag{3.9}$$

TABLE I. Values of y_B and y_p related to values of the momentum transfer q. Here ξ and η are of order 1, since the angles Θ_1 and Θ_2 are assumed to be of order $1/\epsilon$.

	$q^2 \sim q_{min}^2$	$q^2 \sim 1$
$y_B = \dfrac{q_{min}^2}{q^2\xi\eta}$	~ 1	$\sim q_{min}^2$
$y_p = q^2\xi\eta$	$\sim q_{min}^2$	~ 1

The appearance of y_B and y_p in the equations for bremsstrahlung and pair production, Eqs. (2.1) and (3.1), can be related to the fact that in the Born-approximation bremsstrahlung and pair production processes there are two regions of momentum transfer, q, which give equally important contributions to the cross section: $q \sim q_{min} \sim 1/\epsilon$ and $q \sim 1$. In Table I we show that for bremsstrahlung, y_B of order 1 corresponds to $q^2 \sim q_{min}^2$—i.e., very small—while for y_B of order q_{min}^2, q^2 is of order 1. For pair production the situation is exactly the opposite: y_p is of the same order of magnitude as q^2, always. As we shall see this difference leads to that, while for pair production the largest effects of the Coulomb corrections are for large momentum transfers—i.e., for close collisions; for bremsstrahlung, one finds that the Coulomb corrections are most important for the most distant collisions.

IV. DIFFERENTIAL CROSS SECTION COULOMB CORRECTIONS

We define the Coulomb corrections to the bremsstrahlung and pair production C_B and C_p, respectively, by

$$d^4\sigma_{B,p} = d^4\sigma_{B,p,Born}(1 - C_{B,p}), \tag{4.1}$$

related to Eqs. (2.1) and (3.5) by $A_{B,p} = (1 - C_{B,p})$.

In Table II the values for the Coulomb corrections C_B are given for lead for values of the parameter y_B and for the physical parameter $q^2\xi\eta/(10q_{min}^2)$. We use here the extra factor 10 in order to obtain a parameter varying between 0.1 and 1.0 as in the table. As mentioned in Sec. II, the value of the differential cross section as a function of any physical variable, Eq. (4.1), can be written down from Table II, here for lead, by choosing the appropriate variable. The approxi-

TABLE II. Coulomb corrections C_B and C_p for lead for bremsstrahlung and pair production, respectively, as functions of y_B and y_p, and of the physical variable $q^2\xi\eta/(10q^2_{min})$ for bremsstrahlung, and $q^2\xi\eta$ for pair production. Values of u, v, ϵ_1, and ϵ_2, in C_p are chosen as explained in the text.

y_B	$\frac{q^2\xi\eta}{10q^2_{min}}=\frac{0.1}{y_B}$	C_B	$y_p=q^2\xi\eta$	C_p
0.1	1.000	0.1632	0.1	0.0606
0.2	0.500	0.2694	0.2	0.1801
0.3	0.333	0.3106	0.3	0.2492
0.4	0.250	0.3594	0.4	0.3284
0.5	0.200	0.3966	0.5	0.3966
0.6	0.167	0.4279	0.6	0.4591
0.7	0.143	0.4551	0.7	0.5120
0.8	0.125	0.4787	0.8	0.5612
0.9	0.111	0.4937	0.9	0.5997
1.0	0.100	0.5189	1.0	0.6457

mations of $V(x)$ and $W(x)$ in terms of the Euler dilogarithms obtained in Appendix A have been useful in obtaining the results in this section.

While Table II is universal in the variable $q^2\xi\eta/(10q^2_{min})$ for bremsstrahlung, the function R in the pair production cross section, Eq. (3.6), breaks this universality, and specific values of, for instance, u, v, ϵ_1, and ϵ_2 have to be chosen in order to present values for C_p. We choose $u=v=1$ and $\epsilon_1=\epsilon_2$, and with these values u and v are large enough, and we assume the angle Φ such that $q_\perp \gg q_z$—i.e., $q_\perp \approx q$. These cases of "equal energies and angles pairs" are also independent of the photon energy, so Table II covers a fairly sizable region of physical parameters. With these parameters $R=[1-2\xi(1-\xi)-y_p]y_p^{-1}=1/y_p-2$ and

$$A_p=[V^2(x_p)+a^2y_p^2W^2(x_p)+a^2y_pW^2(x_p)\times(1-2y_p)]/V^2(1). \tag{4.2}$$

Table II demonstrates that the Coulomb corrections C_B and C_p may be very large, to an extent that they may be sizably larger than a^2 which for lead is equal to 0.359. In the physical variable $q^2\xi\eta$ they only barely overlap as shown in Fig. 1, again for lead with $q_{min}=10^{-3}$ for bremsstrahlung.

There are in the differential cross sections no substitution rules between bremsstrahlung and pair production as in the Bethe-Heitler Born-approximation cross sections.

Still there is the substitution rule between the *integrated* cross sections, due to the final state integrated cross section theorem [3]: the bremsstrahlung cross section integrated over final state angles,

$$d\sigma_B/dk=(d\sigma_B/dk)_{BH}-4a^2\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2\frac{1}{kp_1^2}\times\left(\epsilon_1^2+\epsilon_2^2-\frac{2}{3}\epsilon_1\epsilon_2\right)f(Z), \tag{4.3}$$

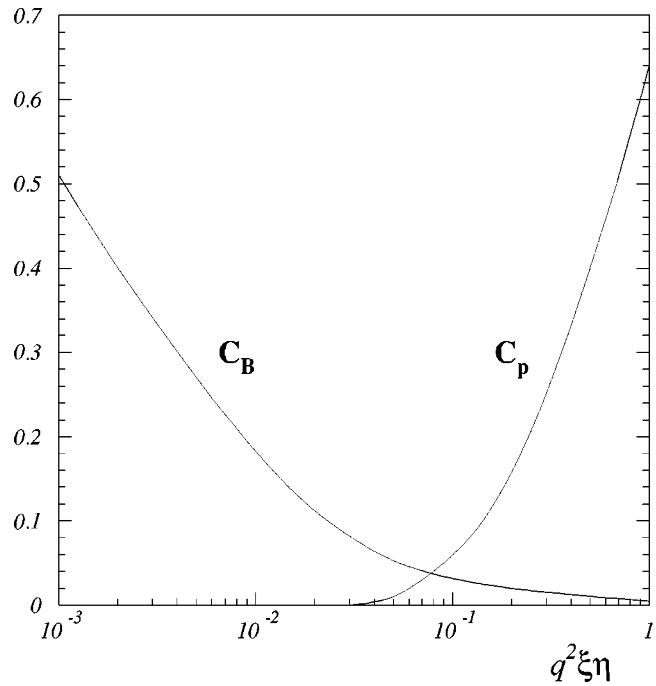

FIG. 1. C_B and C_p curves for the physical variable $q^2\xi\eta$ between 10^{-3} and 1, for $q^2_{min}=10^{-3}$.

is obtained from the corresponding pair production cross section by the substitution $\epsilon_1 \to -\epsilon_1$ or vice versa. This is true for any amount of screening. Here [2]

$$f(Z)=a^2\sum_{n=1}^{\infty}\frac{1}{n(n^2+a^2)}. \tag{4.4}$$

Relations to our differential cross sections for bremsstrahlung, Eqs. (2.1) and (2.3), and pair production, Eq. (3.1), are the following: (1) It can be shown that the angular integral of $d^4\sigma' a^2y_p^2W^2(x_p)$, Eq. (3.1) vanishes. (2) Then the integral of $d^4\sigma_B$ can be performed as in Ref. [6], including any amount of screening, leading to the result of Eq. (4.3), from which the pair production integrated cross section is obtained as described above.

To the lowest order in a^2,

$$f(Z)=a^2L_3(1)=1.2021a^2, \tag{4.5}$$

where $L_3(x)$ is the Euler trilogarithm.

Further $f(Z)$ may be written as

$$f(Z)=\sum_{k=0}^{\infty}a^{2+2k}(-1)^kL_{3+2k}(1)\quad\text{(Appendix B)}.$$

V. SCREENING

The inclusion of the effect of atomic screening is obtained by the method as in Ref. [6] by adding the screening *correction* to the cross section. Following Ref. [6] we obtain, for the differential cross sections,

$$d^4\sigma_{B,p} = d^4\sigma_{Born,B,p}(A_{B,p}(y_{B,p}) + \{[1-F(q)]^2 - 1\})$$
$$= d^4\sigma_{Born,B,p}[1-F(q)]^2$$
$$- d^4\sigma_{Born,B,p}C_{B,p}(y_{B,p}), \quad (5.1)$$

where now the Born cross section is represented by the Bethe-Heitler screened cross section, and the Coulomb correction is added. The integrated cross sections then follow as in Eq. (4.3) in accordance with the final state integration theorem. The Table II giving C_B and C_p is therefore valid irrespective of the amount of screening. For the atom form factor $F(q)$ it is convenient to use, as in Ref. [6], the Molière representation [13] of the Thomas-Fermi model of screening,

$$\frac{1-F(q)}{q^2} = \sum_{i=1}^{3} \frac{\alpha_i}{\beta_i^2 + q^2}, \quad (5.2)$$

with

$$\alpha_1 = 0.10, \quad \alpha_2 = 0.55, \quad \alpha_3 = 0.35,$$

$$\beta_i = (Z^{1/3}/121)b_i, \quad b_1 = 6.0, \quad b_2 = 1.20, \quad b_3 = 0.30. \quad (5.3)$$

VI. MEASUREMENTS OF BREMSSTRAHLUNG AND PAIR PRODUCTION CROSS SECTIONS

Measurements until now of high energy bremsstrahlung and pair production cross sections have been performed on cross sections integrated over the final state electron in bremsstrahlung and over the final state electron or positron in pair production. In fact most of the measurements are on total cross sections. The results are compared with theory in Ref. [2] for pair production. In Ref. [1] it was conjectured that for bremsstrahlung the screening effect would reduce the Coulomb correction to the extent that for complete screening, the Coulomb correction would disappear and the cross section would be given by the Born-approximation Bethe-Heitler cross section. This conjecture was later proved to be incorrect by the final state integration theorem [3]. Brown and others initiated an independent experimental research on bremsstrahlung [5] for strong and complete screening which led to an experimental confirmation of the final state integration theorem: Screening has in bremsstrahlung the same effect as in pair production, as explicitly demonstrated in Eq. (5.1).

As announced in this article, we extend the theory of exact bremsstrahlung and pair production to differential cross sections in explicit presentations to an extent so that also this part of radiation QED theory may be explored experimentally. Central here is the *difference* between bremsstrahlung and pair production at the differential level as so clearly demonstrated in their behavior of the Coulomb correction factors C_B and C_p in Fig. 1. A drawing in configuration space, Fig. 2, demonstrates the extreme differences in physical interactions in the bremsstrahlung and pair production processes: Pair production Coulomb correction interactions occur in the strong Coulomb field, close to the nucleus, which is the usual, normal elementary QED interaction. On the other hand, the interaction of more than one virtual photon between the nucleus and initial electron, leading to radiative corrections to bremsstrahlung, occurs in the weakest part of the field, where the momentum transfer is $\sim q_{min}$, far away from the nucleus. This is an unexpected phenomenon in QED, opposite to the experience of strong QED reactions seeking strong fields. The demonstration of this effect on the differential cross section level has stimulated interest in seeking an experimental finding of the effect.

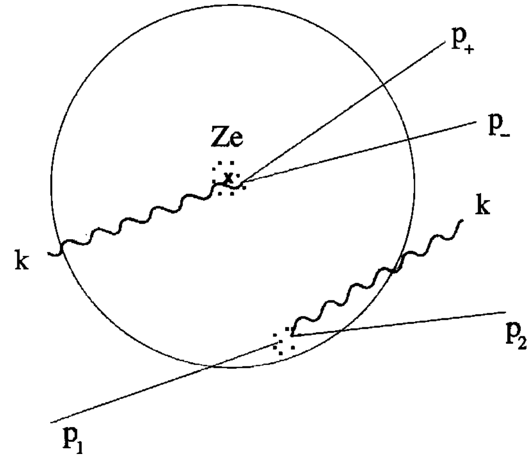

FIG. 2. Picture of Coulomb corrections to pair production and bremsstrahlung as seen in space.

Differential theoretical cross sections for *pair production* for lead are given in Figs. 3 and 4 for the Born approximation I_{Born} and for $I_p = I_{Born}A_p(y_p)$, where I_{Born} is defined from Eq. (3.2) by

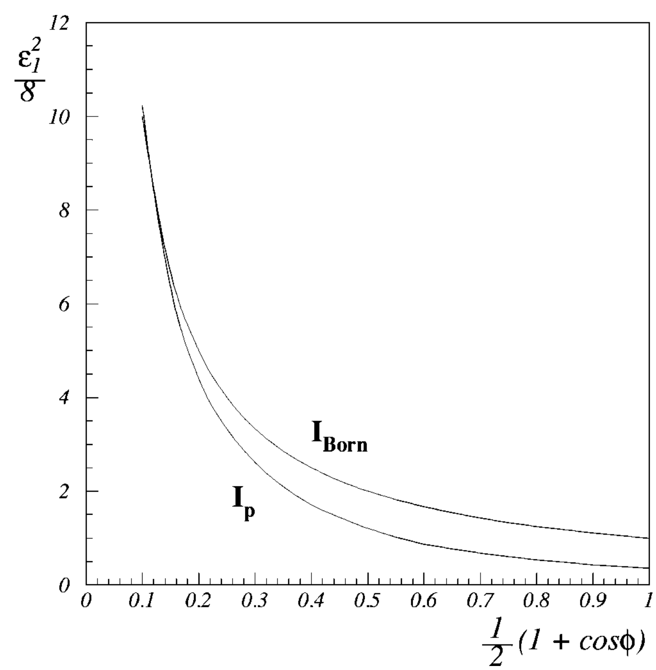

FIG. 3. Differential cross section for pair production in lead, for equal transverse pair momenta, $u = v$. Coulomb corrections are considerably larger than in total cross sections.

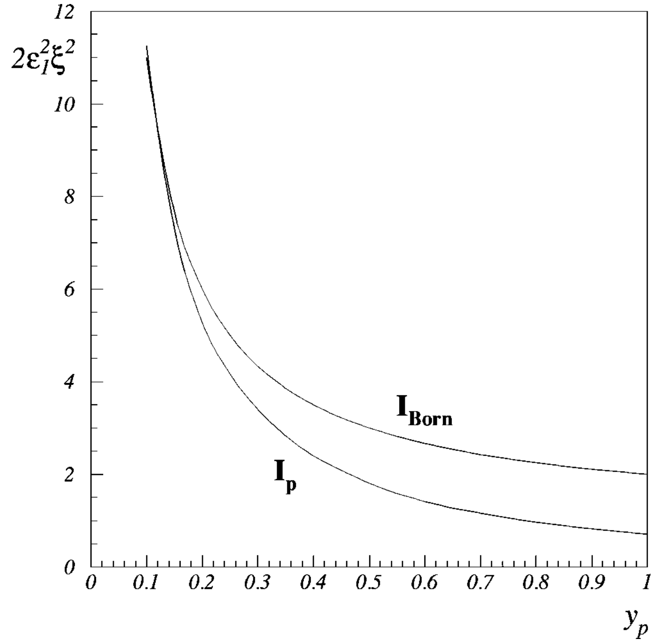

FIG. 4. As in Fig. 3 except that the angular geometry is such that the positron is emitted in the direction of the incoming photon, $v=0$.

$$d^4\sigma_{Born}=\sigma_0\frac{\epsilon_1^2\epsilon_2^2}{k^3}I_{Born}d\epsilon_1\Theta_1 d\Theta_1\Theta_2 d\Theta_2 d\Phi. \quad (6.1)$$

In Fig. 3 the pair cross section is given for Θ_1 and Θ_2 values and energies $\epsilon_1=\epsilon_2$ such that $u=v=1$, for values of the azimuthal angle Φ, from 0 to 2π as shown in Fig. 5. The opening angle of the equal energy pair particles ω is as used by Khubeis *et al.* [14] at lower energies, in the first measurements of opening angles using a comparison with correct theoretical predictions [15]. The methods used by Khubeis *et al.* may be of interest for measurements also at these very high photon energies.

In Fig. 4 the geometry is such that the positron is produced in the direction of the incoming photon, $v=0$, with

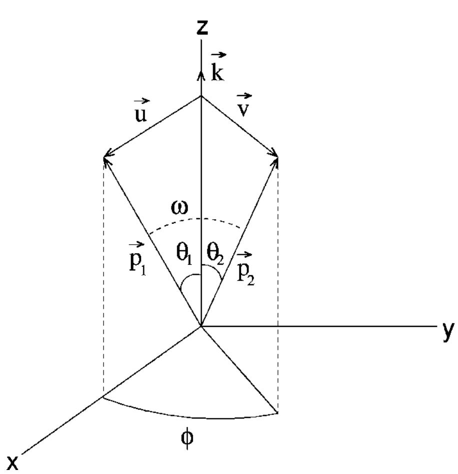

FIG. 5. Geometry of pair production, giving the angles Θ_1, Θ_2, and ϕ in relation to $\vec{u}$ and $\vec{v}$. Here ω is the usual opening angles of the pair.

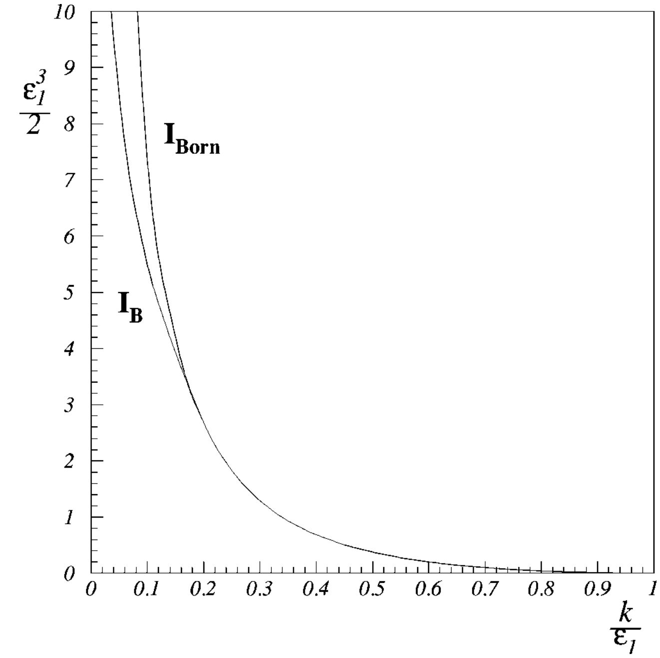

FIG. 6. The photon spectrum for the differential bremsstrahlung cross section in lead. Note the difference from pair production.

energy $\epsilon_1=\epsilon_2$. The observation on u is then the observation of the angle Θ_1. The observations up to $u^2\xi=0.9$ are valid for a not too big value of u: namely, $u=3$.

In both experiments the Coulomb corrections are large, even larger than of the order of a^2, over sizable regions of the angular distributions, and therefore considerably larger than the correction seen in total cross section measurements, which is of the order of 10%.

For *bremsstrahlung* the purely differential cross section is characterized by I_{Born} defined from Eq. (2.2) by

$$d^4\sigma_{Born}=\sigma_0 I_{Born}dk\Theta_1 d\Theta_1\Theta_2 d\Theta_2 d\Phi, \quad (6.2)$$

and I_B from Eq. (2.1) by $I_B=I_{Born}A_B$. Here one finds, in contrast to the case of pair production, that since the Coulomb correction occurs for bremsstrahlung only for q values of the order of $1/\epsilon$, the Coulomb corrections in Fig. 6 are confined to the lower part of the photon spectrum. This could make the Coulomb corrections more difficult to observe than for pair production.

VII. POSTSCRIPT: POSITRONIUM PRODUCTION

In a series of papers [16] I have studied photoproduction and electroproduction of positronium, adding also Coulomb corrections, which are rather substantial and larger than corrections to free pair production. From these studies one can learn the following: In the Coulomb field wave function

$$aF(x)=a[F(ia,-ia;1;x)+iayF(1+ia,1-ia;2;x)] \quad (7.1)$$

related to $|F(x)/V(1)|^2=A(y)$, Eq. (2.3), $aV(x)$ describes the exchange of an odd number of Coulomb field photons, while $a^2yW(x)$ describes the exchange of an even number of photons. This observation is very useful for any process where states, like angular momentum states or polarizations, are of interest.

ACKNOWLEDGMENTS

I acknowledge cooperation over many years with L.C. Maximon on quantum electrodynamic processes. I am grateful for cooperation with experimentalists, in particular with Herwig Schopper for his very accurate measurements of the opening angles of electron-positron pairs. I acknowledge the work of the late Karl L. Brown for his measurements of Coulomb corrections to bremsstrahlung, which I see as experimental verification of the final state integration theorem.

APPENDIX A

We develop here a convenient approximation for the hypergeometric function given in Eq. (2.4):

$$V(x)=F(ia,-ia;1;x)=1+a^2x+\frac{a^2(a^2+1)}{(2!)^2}x^2+\cdots. \tag{A1}$$

Changing the $(n!)^2$ in the denominator to n^2 by the factors $[(n-1)!]^2$ in the numerator successively, the series may be written in the form

$$\begin{aligned}V(x)&=1+a^2L_2(x)+a^4[L_2(x)-x]+\frac{a^6}{4}\left[L_2(x)-\frac{x^2}{4}-x\right]\\&\quad+\frac{a^8}{4\cdot 9}\left[L_2(x)-\frac{x^3}{9}-\frac{x^2}{4}-x\right]+\text{small terms}\\&=1+\left[a^2+a^4+\frac{a^6}{4}+\frac{a^8}{36}\right]L_2(x)-\left[a^4+\frac{a^6}{4}+\frac{a^8}{36}\right]x\\&\quad-\left[a^6+\frac{a^6}{9}\right]\frac{x^2}{16}-\frac{a^8}{9\times 36}x^3+\text{small terms},\end{aligned} \tag{A2}$$

where

$$L_2(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}=-\int_0^x\frac{\ln(1-x)}{x}dx$$

is the Euler dilogarithm.

It can be shown by explicit calculation or by comparison with the known value of $F(ia,-ia;1;1)=(\sinh \pi a/\pi a)$ that for all elements up to uranium included, $a=0.672$, and for $x\leqslant 1$ the errors of the "small terms" are less than 0.6%.

By derivation $W(x)$ is obtained:

$$\begin{aligned}W(x)&=\frac{1}{a^2}\frac{d}{dx}V(x)=\left[1+a^2+\frac{a^4}{4}+\frac{a^6}{36}\right]\frac{\ln(1-x)}{x}-\left[a^2\right.\\&\quad\left.+\frac{a^4}{4}+\frac{a^6}{36}\right]\left[a^4+\frac{a^6}{9}\right]\frac{x}{8}-\frac{a^6}{36}\frac{x^2}{3}+\text{small terms}.\end{aligned} \tag{A3}$$

These results are used in deriving the numerical results in Sec. IV. To lowest order in a^2, to all orders in x^n, one finds

$$V(x)=1+a^2L_2(x), \tag{A4}$$

$$W(x)=-(1+a^2)\frac{\ln(1-x)}{x}-a^2. \tag{A5}$$

APPENDIX B

The higher order Euler logarithms may be useful in the calculations. The $f(Z)$ function, Eq. (4.4),

$$f(Z)=a^2\sum_{n=1}^{\infty}\frac{1}{n(n^2+a^2)},$$

may be written as

$$\begin{aligned}f(Z)&=a^2L_3(1)-a^4L_5(1)+a^6L_7(1)+\cdots\\&=\sum_{k=0}^{\infty}a^{2+2k}(-1)^kL_{3+2k}(1),\end{aligned} \tag{B1}$$

where the multilogarithm is

$$L_n(x)=\sum_m\frac{x^m}{m^n}, \tag{B2}$$

also given by

$$L_n(x)=\int_0^x L_{n-1}(y)\frac{dy}{y}. \tag{B3}$$

From these formulas the results for any element may be obtained without any assisting tables.

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