Tribimaximal mixing, discrete family symmetries, and a conjecture connecting the quark and lepton mixing matrices

Catherine I. Low* and Raymond R. Volkas†
School of Physics, Research Centre for High Energy Physics, The University of Melbourne, Victoria 3010, Australia
(Received 22 May 2003; published 8 August 2003)

Neutrino oscillation experiments (excluding the Liquid Scintillator Neutrino Detector experiment) suggest a tribimaximal form for the lepton mixing matrix. This form indicates that the mixing matrix is probably independent of the lepton masses, and suggests the action of an underlying discrete family symmetry. Using these hints, we conjecture that the contrasting forms of the quark and lepton mixing matrices may both be generated by such a discrete family symmetry. This idea is that the diagonalization matrices out of which the physical mixing matrices are composed have large mixing angles, which cancel out due to a symmetry when the CKM matrix is computed, but do not do so in the MNS case. However, in the cases where the Higgs bosons are singlets under the symmetry, and the family symmetry commutes with $SU(2)_L$, we prove a no-go theorem: no discrete unbroken family symmetry can produce the required mixing patterns. We then suggest avenues for future research.

DOI: 10.1103/PhysRevD.68.033007 PACS number(s): 12.15.Ff, 14.60.Pq

I. INTRODUCTION

Experimental observations of neutrino oscillations[1] point to a mixing matrix of the form

$$U_{\rm MNS}=\begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \tag{1}$$

where the flavor eigenstates are related to the mass eigenstates via $(\nu_e,\nu_\mu,\nu_\tau)^T=U_{\rm MNS}(\nu_1,\nu_2,\nu_3)^T$. Such a mixing pattern has been termed "tribimaximal mixing" [2]. (Majorana phases have not been included in the above mixing matrix as they do not lead to observable effects in oscillations.) A mixing matrix of this form was first investigated by Wolfenstein in 1978 [3] (with degenerate mass eigenstates ν_1 and ν_3), and proposed more recently in the light of the new experimental observations by Harrison, Perkins and Scott [2,4,5] and He and Zee [6,7]. The generation of small deviations from tribimaximal mixing has been investigated by Xing [8].

The tribimaximal form is a very special case of the general mixing matrix parametrized in the usual way by the Euler angles θ_{ij} where $i,j=1,2,3$. The angle θ_{23}, extracted from atmospheric neutrino experiments [9–12], takes the best fit value of $\sin^2\theta_{23}=\frac{1}{2}$ [13]. Solar neutrino results [14–20] are accommodated in Eq. (1) through the choice $\sin^2\theta_{12}=\frac{1}{3}$, which is in the middle of the allowed "large mixing angle (LMA) regions" denoted LMA-I and LMA-II [21]. The third mixing angle, measured by the nonobservation of ν_e disappearance [22], is taken as the current best fit $\theta_{13}=0$ [13]. Note that θ_{23} takes the maximum possible value, while θ_{13} takes the minimum possible value.

A. Mathematics suggested by tribimaximal mixing

If these special mixing angle values are indeed the correct ones, then it is unlikely that they arise from a random choice of parameters [23]. This encourages one to look for exact or approximate symmetries of nature, operative even at low energy scales, that enforce the special tribimaximal form (or something close to it).

1. Mixing angles independent of masses

The elements of the tribimaximal mixing matrix are square roots of fractions, whereas the charged lepton masses appear to have no precise fractional relationships, and neither do the preferred neutrino Δm^2 parameters. This motivates the construction of models where the mixing angles, though precisely defined, are independent of the mass eigenvalues. Such an approach is to be contrasted with the often considered alternative proposal that relates mixing angles to mass ratios [24–27].

2. Abelian symmetries

Harrison, Perkins and Scott [2] proposed weak basis mass matrices for charged leptons and neutrinos that generate tribimaximal mixing. An attractive feature of the proposed mass matrices is that they can be generated by discrete Abelian symmetries acting on the three generations of charged leptons and neutrinos. These symmetries dictate the form of the mixing matrix, but leave the masses as free parameters (see above discussion). The utility of these mass matrices

---

*Electronic address: c.low@physics.unimelb.edu.au
†Electronic address: r.volkas@physics.unimelb.edu.au

[1]For the purpose of this study, we have assumed the Liquid Scintillator Neutrino Detector (LSND) results [1] have a nonoscillation explanation. The reader should be aware, however, that this assumption may be false.

suggests that Abelian generation symmetries are interesting candidates for the new symmetries that might explain the neutrino mixing pattern.

B. Aims of this paper

1. Quark and lepton mixing matrices derived from a symmetry

In Sec. II the Harrison, Perkins and Scott proposal [2] will be reviewed. We will then extend their ideas by conjecturing that the underlying symmetries might simultaneously produce a quark mixing matrix that is almost the identity matrix and a leptonic analogue that has the very different tribimaximal form. While we find this an attractive hypothesis, it is not so easy to implement in a completely well-defined extension of the standard model. As we shall see, this proposal requires that left-handed charged leptons and left-handed neutrinos transform differently. But to have the symmetry group $G_{\rm SM}$ of the standard model extended to $G_{\rm SM}\otimes G_H$, where G_H is a discrete horizontal or generation symmetry, the left-handed charged leptons and left-handed neutrinos must transform in the same way under the symmetry, as they are members of the same $SU(2)_L$ doublet.

2. Form-diagonalizable matrices

Section III will define a class of matrices that are invariant under a symmetry and where the unitary matrices that diagonalize them are independent of the eigenvalues. We dub matrices such as these ''form-diagonalizable'' and propose them as good candidates for lepton mass matrices because they generate mixing angles that are independent of the eigenvalues. This section will look at some interesting mathematics that relates the symmetry group to the diagonalization matrices.

3. No-go theorem

Motivated by the symmetries proposed by Harrison, Perkins, and Scott [2], Sec. IV will investigate the possibility of using such symmetries to extend the standard model. We assume left-handed neutrinos transform under the symmetry in the same way as left-handed charged leptons, and that the Higgs bosons are singlets. Given these assumptions, we find that tribimaximal mixing, or any other form that is both phenomenologically acceptable and predictive, cannot be generated by an unbroken family symmetry.

4. Further symmetries to investigate

Ways around the no-go theorem will be briefly discussed in Sec. V. Either or both of the assumptions of the theorem — that the Higgs fields are singlets and that the symmetry is unbroken — must be relaxed. The generation symmetry can be extended to the Higgs sector by introducing a number of generations of Higgs fields that transform under the symmetry. Majorana neutrinos have different couplings to the Higgs fields from the Dirac charged leptons. As a result a symmetry that transforms Higgs fields could potentially explain the differences between the mixing matrices of the leptons and the quarks. Vacuum expectation values of the Higgs fields can break the symmetry and result in different mixing matrices from those of the exact symmetry cases. Work along these lines is in progress. For some recent efforts, see for instance [28–32].

II. DISCRETE SYMMETRIES CONSTRAIN MIXING MATRICES

Many theories have been constructed using symmetries to generate preferred mass patterns and mixing angles. For example, democratic mass matrices can be generated from an $S_{3L}\times S_{3R}$ generation symmetry [33–35], L_e-L_μ-L_τ symmetry leads to bimaximal mixing (disfavored by the current data) [36–38], and S_2 permutation symmetry acting on ν_μ and ν_τ results in maximal atmospheric mixing [39–41].

A. How symmetries constrain mixing matrices

The mixing matrix is related to the charged lepton mass matrix M_ℓ and the neutrino mass matrix M_ν in any weak basis by the unitary diagonalization matrices U_{ℓ_L} and U_ν. We use

$$\mathrm{diag}(m_e,m_\mu,m_\tau)=U^\dagger_{\ell_L}M_\ell U_{l_R},$$

$$\mathrm{diag}(m_1,m_2,m_3)=U^\dagger_\nu M_\nu U^*_\nu, \tag{2}$$

to extract the lepton mixing matrix via

$$U_{\rm MNS}=U^\dagger_{\ell_L}U_\nu. \tag{3}$$

The symmetries of the standard model do not dictate the form of the mass matrices. The charged lepton mass matrix M_ℓ can be any 3×3 matrix, and if neutrinos are Majorana, then M_ν must be symmetric, but is otherwise unconstrained. As a result the mixing matrix can be of any unitary form, and the masses are unrestricted by the standard model symmetries. However, if a generation symmetry holds, the form of the mass matrices — and hence the mixing matrix — is constrained. For the Lagrangian to be invariant under transformations of the three generations of Majorana neutrinos, the left-handed charged leptons and the right-handed charged leptons,

$$\nu\to X_\nu\nu,\quad \ell_L\to X_L\ell_L,\quad \ell_R\to X_R\ell_R, \tag{4}$$

the mass matrices must obey the restrictions,

$$M_\nu=X^\dagger_\nu M_\nu X^*_\nu,\quad M_\ell=X^\dagger_L M_\ell X_R, \tag{5}$$

where X_ν, X_L and X_R are 3×3 unitary matrices. The special case of the vector-like symmetry would have left and right-handed fields transforming identically, with $X_L=X_R$.

B. Harrison, Perkins and Scott's proposed symmetries

Harrison, Perkins and Scott [2] suggested mass matrices of the form

$$M_\ell=\begin{pmatrix} a & b & c\\ c & a & b\\ b & c & a\end{pmatrix},\quad M_\nu=\begin{pmatrix} x & 0 & y\\ 0 & z & 0\\ y & 0 & x\end{pmatrix},\qquad (6)$$

where the parameters a,b,c are related to the three charged lepton masses, and x,y,z provide three independent neutrino masses. The charged lepton mass matrix is of circulant form and can be generated by a cyclic permutation (C_3) symmetry. An $S_2\times S_2$ symmetry generates the neutrino mass matrix.

The unitary transformation matrices are

$$X_{L1}=X_{R1}=\begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix};\quad X_{L2}=X_{R2}=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix};$$

$$X_{\nu1}=\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix};\quad X_{\nu2}=\begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}.\qquad (7)$$

The proposed mass matrices are diagonalized by

$$U_{\ell_L}=U_{\ell_R}=\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1\\ 1 & \omega & \omega^*\\ 1 & \omega^* & \omega\end{pmatrix},$$

$$U_\nu=\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix},\qquad (8)$$

where $\omega\equiv e^{2\pi i/3}$, which combine to give tribimaximal mixing.

C. Using the symmetry to constrain quark mixing to small angles: A conjecture

Harrison, Perkins and Scott's idea can be extended to include the quarks, and produce small quark mixing. We conjecture that the up-type quarks and the down-type quarks transform under the C_3 generation symmetry in the same way as the charged leptons transform above. This will force both quark mass matrices into circulant form

$$M_u=\begin{pmatrix} a_u & b_u & c_u\\ c_u & a_u & b_u\\ b_u & c_u & a_u\end{pmatrix},\quad M_d=\begin{pmatrix} a_d & b_d & c_d\\ c_d & a_d & b_d\\ b_d & c_d & a_d\end{pmatrix}.\qquad (9)$$

These mass matrices are diagonalized by the same matrix $U_u=U_d$, resulting in $U_{\rm CKM}=U_u^\dagger U_d=I$, corresponding to no quark mixing. As with the leptons, all quark masses are unrestricted by the symmetry.

The unbroken symmetry produces $U_{\rm CKM}=I$, and $U_{\rm MNS}$ to be of tribimaximal form. Small symmetry breaking can be introduced to generate off-diagonal terms in the quark mixing matrix. This breaking may also deviate the lepton mixing matrix away from tribimaximal form.

The quarks transform together, whereas the neutrinos transform independently of the charged leptons. This accounts for the differences between the quark and the lepton mixing matrices.

Under the symmetry the neutrinos transform in a different way from all the other fermions. This may be associated with other special characteristics of the neutrinos, for example, the Majorana nature of the neutrino, or the lack of electric charge.

D. $SU(2)_L$ constraint on standard model extensions

The conjecture outlined above shows that discrete generation symmetries can produce tribimaximal lepton mixing and small quark mixing. However, these symmetries cannot be incorporated into an extension of the standard model with the structure $SU(2)_L\otimes G_H$, where the G_H is the discrete horizontal or family symmetry. The symmetries of Eq. (7) do not commute with $SU(2)_L$, as the left-handed neutrinos transform under the symmetry in a different way from the left-handed charged leptons, whereas a symmetry that is an extension to the standard model should preserve the standard model symmetry $SU(2)_L$, by having members of the same $SU(2)_L$ doublet transform together [42]. $SU(2)_L$ is not violated by the quark transformations as the up and down-type quarks transform in the same way.

This constraint makes it difficult to find any symmetry that gives rise to tribimaximal mixing. Section IV investigates whether it is possible for any discrete family symmetry to predict tribimaximal mixing when the $SU(2)_L$ constraint is included.

III. FORM-DIAGONALIZABLE MATRICES

A. Definition

A form-diagonalizable matrix is a matrix that is invariant under a symmetry, and with diagonalization matrices whose elements depend on the form of the original matrix only. As a result the diagonalization matrices are independent of the matrices' eigenvalues.

An $n\times n$ form-diagonalizable matrix is defined by

$$F=\sum_i^k \alpha_i\lambda_i\qquad (10)$$

where

(1) λ_i are $n\times n$ matrices of pure numbers, and α_i are n complex parameters;

(2) λ_i are simultaneously diagonalizable by two unitary matrices U_L and U_R, where $U_L^\dagger\lambda_i U_R$ is diagonal for all i;

(3) λ_i are invariant under a group transformation: $\lambda_i=X_L^\dagger\lambda_i X_R$;

(4) $k\leqslant n$.

Note that for $k<n$, only k eigenvalues are independent.

These conditions result in the masses being linear combinations of α_i, and the diagonalization matrices, U_L and U_R, being independent of these masses.

B. Examples of form-diagonalizable matrices with Abelian symmetries

Equation (6) has two examples of form diagonalizable mass matrices, with the symmetries being the Abelian groups C_3 and $S_2 \times S_2$.

The form of the mass matrices is dependent not only on the symmetry group, but also on the representation of the group that the transformation matrices X_L and X_R take.

1. Regular representation of Abelian groups

An interesting relationship occurs between the symmetry group and the diagonalization matrix when the symmetry is an Abelian group in the regular representation. The regular representation of a group of order *n*, is a set of *n* matrices X_i. The matrices are unitary, have size $n \times n$, and their elements are 0 or 1. A matrix *M* is considered to be invariant under the regular representation of a group when $M = X_i^T M X_i$ for all *i*.

For Abelian symmetries the mass matrix that is invariant under the regular representation is a linear combination of all the representation matrices themselves, i.e. λ_i of Eq. (10) are the X_i. This is shown in Appendix A.

The matrix *U* that diagonalizes the mass matrix *M* can be simply derived from the *n* one-dimensional representations of the group *G*: Each column of the diagonalization matrix is made up of a normalized list of the elements of the one-dimensional representations, and each column corresponds to a different one-dimensional representation. As all the irreducible representations of Abelian groups are one dimensional, the character table lists these representations, and the diagonalization matrix can be read directly off the table.

2. C_3 example

This relationship between the regular representation and the diagonalization matrices is illustrated by the C_3 symmetry of the charged leptons outlined in Sec. II B.

The charged lepton mass matrix of Eq. (6) is invariant under the regular representation of C_3 which is given by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}. \tag{11}$$

The mass matrix is made up of a linear combination of invariant matrices λ_i. In this case the λ_i are the representation matrices themselves, forming the mass matrix M_ℓ of Eq. (6). The diagonalization matrix is

$$U_\ell = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^* \\ 1 & \omega^* & \omega \end{pmatrix} \tag{12}$$

where $\omega = e^{2\pi i/3}$, ω^*, 1 are the cube roots of unity. This diagonalization matrix can be constructed using the one-dimensional representations of C_3 which are $\{1,1,1\}$, $\{1,\omega,\omega^*\}$, $\{1,\omega^*,\omega\}$. Each column of the diagonalization matrix is made up of a one-dimensional representation, and the matrix is normalized.

Representations other than the regular representations can also produce form-diagonalizable mass matrices. An example of this is the $S_2 \times S_2$ symmetry which generates the mass matrix M_ν of Eq. (6). In cases other than the regular representation, the relationship between the representation of the symmetry and the diagonalization matrix is not clear.

IV. NO-GO THEOREM FOR DISCRETE FAMILY SYMMETRIES

Individual lepton number symmetry $U(1)_{L_e} \otimes U(1)_{L_\mu} \otimes U(1)_{L_\tau}$ is a symmetry of the standard model with massless neutrinos, and is known to be broken by neutrino oscillations. However, if a discrete subgroup of this symmetry is unbroken by the neutrino mass term, this will constrain the form of the mixing matrix.

The success of the symmetries in Eq. (7) in generating tribimaximal mixing, and the idea that a subgroup of $U(1)_{L_e} \otimes U(1)_{L_\mu} \otimes U(1)_{L_\tau}$ may still remain unbroken with massive neutrinos motivates the systematic study of discrete Abelian group symmetries, with the added constraint of having the left-handed charged leptons transform in the same way as the left-handed neutrinos.

This section shows that discrete unbroken generation symmetries (Abelian and non-Abelian) with the $SU(2)_L$ constraint and the other assumptions stated below cannot generate tribimaximal mixing. In fact, the only mixing matrix that falls within experimental bounds and is generated by a symmetry, is the mixing matrix that is completely unrestricted by the symmetry. In this section we assume that the Higgs bosons are singlets of the symmetry.

Section IV B shows that discrete non-Abelian generation symmetries give rise to degenerate charged leptons, proving that non-Abelian symmetries cannot produce mass and mixing schemes that agree with experiment.

Section IV C considers how Abelian groups can constrain the charged lepton Dirac mass matrix. Exactly how the transformations alter the neutrino mass matrix depends on the type of mass term, because Majorana mass terms are constrained by the symmetry in a different way from Dirac mass terms. Because of this the no-go theorem for Abelian groups is segmented into three cases; Majorana neutrinos (Sec. IV D), Dirac neutrinos (Sec. IV E), and Majorana neutrinos when the mass term is generated by the seesaw mechanism (Sec. IV F). In the seesaw case we assume that the right-handed Majorana mass matrix is invertible.

We show that in all three cases all mixing schemes that can be produced by Abelian symmetries are not allowed by experiment, except for the case where the mixing is not constrained by the symmetry at all.

A. Equivalent representations yield identical mixing

The matrices X_{Li} and X_{Ri} of Eq. (5) that transform the leptons are representations of the symmetry group. Different representations of the same symmetry group provide different restrictions on the mass matrices. As there are three generations of leptons we are interested in three dimensional representations only. A given symmetry group has an infinite number of three dimensional representations, but only a finite number of inequivalent representations.

Two different representations X_i and Y_i, are considered to be equivalent if they are related by a similarity transformation

$$Y_i = V^\dagger X_i V, \tag{13}$$

where V is any unitary matrix.

Appendix B shows that two equivalent transformation matrices restrict the mixing matrix in an identical way. This is because the weak basis leptons $(\nu_{L_X}, l_{L_X})^T$ in the case where the representation X_i is chosen, are related to the weak basis leptons $(\nu_{L_Y}, l_{L_Y})^T$ in the Y_i case by a basis change, as per

$$\begin{pmatrix} \nu_{L_X} \\ l_{L_X} \end{pmatrix} \rightarrow \begin{pmatrix} \nu_{L_Y} \\ l_{L_Y} \end{pmatrix} = V^\dagger \begin{pmatrix} \nu_{L_X} \\ l_{L_X} \end{pmatrix}. \tag{14}$$

Since the mixing matrix is associated with the mass basis of the leptons, not the weak basis, the two equivalent representations will restrict the mixing matrix in an identical way. As there are only a finite number of inequivalent representations of any discrete group it is possible to find all mixing matrices that can be generated by a given group.

All Abelian representations are equivalent to a diagonal representation—a representation where all matrices are diagonal. The converse is also true; no non-Abelian representation has matrices that are all diagonal (as diagonal matrices commute). This provides a convenient way of analyzing many groups at once. First we will consider non-Abelian groups by examining how nondiagonal transformations affect mass matrices and mixing, and then we consider Abelian representations by looking at diagonal representations.

B. Non-Abelian groups

Non-Abelian groups have Abelian (for example the trivial representation) and non-Abelian representations. Abelian representations of non-Abelian groups are not faithful, and are also representations of Abelian groups. This section shows that non-Abelian representations constrain some charged leptons to be degenerate. Abelian representations are covered by Secs. IV D, IV E and IV F.

As explained in Sec. IV A, two equivalent representations correspond to two different bases. So if the mass matrices are invariant under some non-Abelian transformation, there exists a non-Abelian representation of the group that corresponds to the charged lepton mass basis $M_\ell = \mathrm{diag}(m_e, m_\mu, m_\tau)$. As this representation is non-Abelian, there is at least one matrix that is not diagonal.

Mass degeneracy can be concluded by considering just one nondiagonal transformation matrix. For example a block diagonal unitary matrix

$$X_L = \begin{pmatrix} x & 0 & 0 \\ 0 & y & w \\ 0 & z & v \end{pmatrix}, \tag{15}$$

constrains $M_\ell M_\ell^\dagger$ by

$$M_\ell M_\ell^\dagger = X_L^\dagger M_\ell M_\ell^\dagger X_L = \begin{pmatrix} x^* & 0 & 0 \\ 0 & y^* & z^* \\ 0 & w^* & v^* \end{pmatrix} \begin{pmatrix} m_e^2 & 0 & 0 \\ 0 & m_\mu^2 & 0 \\ 0 & 0 & m_\tau^2 \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ 0 & y & w \\ 0 & z & v \end{pmatrix} \tag{16}$$

$$= \begin{pmatrix} m_e^2|x|^2 & 0 & 0 \\ 0 & m_\mu^2|y|^2 + m_\tau^2|z|^2 & m_\mu^2 y^* w + m_\tau^2 z^* v \\ 0 & m_\mu^2 y w^* + m_\tau^2 z v^* & m_\tau^2|v|^2 + m_\mu^2|w|^2 \end{pmatrix}. \tag{17}$$

The 2×2 block in X_L rotates m_μ^2 and m_τ^2, so the diagonal mass matrix will only be invariant under this transformation if $m_\mu^2 = m_\tau^2$. An X_L that is not in block diagonal form will result in three degenerate charged leptons.

The same argument also applies when the X_R transformation is non-Abelian. In this case the X_R transformation constrains $M_\ell^\dagger M_\ell = \mathrm{diag}(m_e^2, m_\mu^2, m_\tau^2)$ by $M_\ell^\dagger M_\ell = X_R^\dagger M_\ell^\dagger M_\ell X_R$, also resulting in degenerate masses.

C. Abelian representations and charged lepton mass matrices

In the case of Abelian groups, every representation is equivalent to a diagonal matrix representation, so to find out all the mixing matrices that can be produced by an Abelian group, we can restrict the study to how mass matrices can be constrained by diagonal representations.

The diagonal representations

$$X_L = \mathrm{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}), \quad X_R = \mathrm{diag}(e^{i\sigma_1}, e^{i\sigma_2}, e^{i\sigma_3}), \tag{18}$$

constrain the charged lepton mass matrix M_ℓ by $M_\ell = X_L^\dagger M_\ell X_R$, or, more explicitly,

$$M_\ell=\begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix}=\begin{pmatrix} re^{-i(\phi_1-\sigma_1)} & se^{-i(\phi_1-\sigma_2)} & te^{-i(\phi_1-\sigma_3)} \\ ue^{-i(\phi_2-\sigma_1)} & ve^{-i(\phi_2-\sigma_2)} & we^{-i(\phi_2-\sigma_3)} \\ xe^{-i(\phi_3-\sigma_1)} & ye^{-i(\phi_3-\sigma_2)} & ze^{-i(\phi_3-\sigma_3)} \end{pmatrix}. \tag{19}$$

Not all of the information contained in the mass matrix is required in order to find the masses and the mixing matrix. One may simply compute the Hermitian squared mass matrix $M_\ell M_\ell^\dagger$ and then diagonalize it via the left-handed matrix U_{ℓ_L} only, as per $U_{\ell_L}^\dagger M_\ell M_\ell^\dagger U_{\ell_L}=\mathrm{diag}(m_e^2,m_\mu^2,m_\tau^2)$. Now, $M_\ell M_\ell^\dagger$ is restricted by the X_L transformation by

$$M_\ell M_\ell^\dagger=\begin{pmatrix} a & b & c \\ b^* & d & f \\ c^* & f^* & g \end{pmatrix}=X_L^\dagger M_\ell M_\ell^\dagger X_L=\begin{pmatrix} a & be^{-i(\phi_1-\phi_2)} & ce^{-i(\phi_1-\phi_3)} \\ b^*e^{i(\phi_1-\phi_2)} & d & fe^{-i(\phi_2-\phi_3)} \\ c^*e^{i(\phi_1-\phi_3)} & f^*e^{i(\phi_2-\phi_3)} & g \end{pmatrix}. \tag{20}$$

The X_L transformation constrains the Hermitian squared mass matrix in the following way: The diagonal elements of $M_\ell M_\ell^\dagger$ are unrestricted by the symmetry; when $\phi_i=\phi_j$ the ijth term in $M_\ell M_\ell^\dagger$ is unrestricted by the symmetry; otherwise the ijth element will be zero.

Note that $M_\ell M_\ell^\dagger$ can also be constrained by the X_R matrix. For example, if $X_L=I$ and $X_R=-I$ then $M_\ell=M_\ell M_\ell^\dagger=0$, even though the X_L transformation does not constrain the mass matrix.

To make the no-go theorem simpler, we look first at how U_{MNS} can be constrained by the X_L transformation, before analyzing how the X_R transformation alters the situation. For nearly all choices of X_L, the X_L tranformation constrains $M_\ell M_\ell^\dagger$ and M_ν in such a way to force the mixing matrix U_{MNS} into a form that has been ruled out experimentally. In these cases the X_R transformations are irrelevant, the symmetry having been ruled out for all possible choices of X_R.

D. Abelian representations and Majorana neutrinos

The left-handed transformation X_L restricts the Majorana neutrino mass matrix by

$$M_\nu=\begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix}=\begin{pmatrix} Ae^{-2i\phi_1} & Be^{-i(\phi_1+\phi_2)} & Ce^{-i(\phi_1+\phi_3)} \\ Be^{-i(\phi_1+\phi_2)} & De^{-2i\phi_2} & Ee^{-i(\phi_2+\phi_3)} \\ Ce^{-i(\phi_1+\phi_3)} & Ee^{-i(\phi_2+\phi_3} & Fe^{-2i\phi_3} \end{pmatrix}. \tag{21}$$

The X_L transformation multiplies each element of the mass matrix by a phase. If the phase equals 1, then the element is unconstrained by the symmetry. If the phase is not equal to 1, then the matrix element is forced to be zero. If $e^{i\phi_i}=\pm 1$, then the iith element of the matrix will be unrestricted by the symmetry. If $e^{i\phi_i}=e^{-i\phi_j}$ then the ijth element will be unrestricted. Otherwise the elements will be zero.

We have performed an exhaustive analysis of all possible forms of lepton mixing matrices that can be produced by an Abelian generation symmetry. The mixing matrices are listed below. Interchanging columns corresponds to relabeling neutrino mass eigenstates.

In the following matrices $s\equiv\sin\theta$ and $c\equiv\cos\theta$, where θ is unconstrained by the symmetry. The phases $e^{i\delta_i}$ are not neccesarily physical.

Mixing matrix **Form of X_L required for all X_L**

$$U_{\mathrm{MNS}_1}=\begin{pmatrix} ce^{i\delta_1} & se^{i\delta_2} & 0 \\ -se^{i\delta_3} & ce^{i\delta_4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{matrix} X_L=\mathrm{diag}(e^{i\phi_1},e^{i\phi_1},\pm 1) \\ X_L=\mathrm{diag}(\pm 1,\pm 1,e^{i\phi_3}) \\ X_L=\mathrm{diag}(\pm 1,\pm 1,\mp 1) \end{matrix} \tag{22}$$

$$U_{\mathrm{MNS}_2}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & ce^{i\delta_1} & se^{i\delta_2} \\ 0 & -se^{i\delta_3} & ce^{i\delta_4} \end{pmatrix} \qquad \begin{matrix} X_L=\mathrm{diag}(\pm 1,e^{i\phi_2},e^{i\phi_2}) \\ X_L=\mathrm{diag}(e^{i\phi_1},\pm 1,\pm 1) \\ X_L=\mathrm{diag}(\pm 1,\mp 1,\mp 1) \end{matrix} \tag{23}$$

$$U_{\mathrm{MNS}_3}=\begin{pmatrix} ce^{i\delta_1} & 0 & se^{i\delta_2} \\ 0 & 1 & 0 \\ -se^{i\delta_3} & 0 & ce^{i\delta_4} \end{pmatrix} \qquad \begin{matrix} X_L=\mathrm{diag}(e^{i\phi_1},\pm 1,e^{i\phi_1}) \\ X_L=\mathrm{diag}(\pm 1,e^{i\phi_1},\pm 1) \\ X_L=\mathrm{diag}(\mp 1,\pm 1,\mp 1) \end{matrix} \tag{24}$$

$$U_{\mathrm{MNS}_4}=\begin{pmatrix} -\frac{1}{\sqrt{2}}e^{i\delta_5} & \frac{1}{\sqrt{2}}e^{i\delta_5} & 0 \\ \frac{s}{\sqrt{2}}e^{i\delta_1} & \frac{s}{\sqrt{2}}e^{i\delta_1} & ce^{i\delta_2} \\ \frac{c}{\sqrt{2}}e^{i\delta_3} & \frac{c}{\sqrt{2}}e^{i\delta_3} & -se^{i\delta_4} \end{pmatrix} \qquad X_L=\mathrm{diag}(e^{i\phi_1},e^{-i\phi_1},e^{-i\phi_1}) \tag{25}$$

$$U_{\mathrm{MNS}_5}=\begin{pmatrix} \frac{s}{\sqrt{2}}e^{i\delta_1} & \frac{s}{\sqrt{2}}e^{i\delta_1} & ce^{i\delta_2} \\ \frac{c}{\sqrt{2}}e^{i\delta_3} & \frac{c}{\sqrt{2}}e^{i\delta_3} & -se^{i\delta_4} \\ -\frac{1}{\sqrt{2}}e^{i\delta_5} & \frac{1}{\sqrt{2}}e^{i\delta_5} & 0 \end{pmatrix} \qquad X_L=\mathrm{diag}(e^{i\phi_1},e^{i\phi_1},e^{-i\phi_1}) \tag{26}$$

$$U_{\mathrm{MNS}_6}=\begin{pmatrix} \frac{s}{\sqrt{2}}e^{i\delta_1} & \frac{s}{\sqrt{2}}e^{i\delta_1} & ce^{i\delta_2} \\ -\frac{1}{\sqrt{2}}e^{i\delta_5} & \frac{1}{\sqrt{2}}e^{i\delta_5} & 0 \\ \frac{c}{\sqrt{2}}e^{i\delta_3} & \frac{c}{\sqrt{2}}e^{i\delta_3} & -se^{i\delta_4} \end{pmatrix} \qquad X_L=\mathrm{diag}(e^{i\phi_1},e^{-i\phi_1},e^{i\phi_1}) \tag{27}$$

$$U_{\mathrm{MNS}_7}=\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{matrix} X_L=\mathrm{diag}(e^{i\phi_1},e^{-i\phi_1},e^{-i\phi_3}) \\ X_L=\mathrm{diag}(e^{i\phi_1},e^{-i\phi_1},\pm 1) \end{matrix} \tag{28}$$

$$U_{\mathrm{MNS}_8}=\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \qquad \begin{matrix} X_L=\mathrm{diag}(e^{i\phi_1},e^{i\phi_2},e^{-i\phi_1}) \\ X_L=\mathrm{diag}(e^{i\phi_1},\pm 1,e^{-i\phi_1}) \end{matrix} \tag{29}$$

$$U_{\mathrm{MNS}_9}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \qquad \begin{matrix} X_L=\mathrm{diag}(e^{i\phi_1},e^{i\phi_2},e^{-i\phi_2}) \\ X_L=\mathrm{diag}(\pm 1,e^{i\phi_2},e^{-i\phi_2}) \end{matrix} \tag{30}$$

$$U_{\mathrm{MNS}_{10}}=\text{Trivial}-\text{massless neutrinos} \qquad \begin{matrix} X_L=\mathrm{diag}(e^{i\phi_1},e^{i\phi_2},e^{i\phi_3}) \\ \phi_i\neq\pm 1 \quad \text{for at least one} \quad X_L, \quad \text{for all} \quad i, \\ \phi_j\neq\phi_i \quad \text{for at least one} \quad X_L, \quad \text{for all} \quad i, \quad j. \end{matrix} \tag{31}$$

$$U_{\mathrm{MNS}_{11}}=\text{Unrestricted by the symmetry} \qquad X_L=\pm I. \tag{32}$$

In cases $U_{\mathrm{MNS}_{4,5,6}}$, $m_1=-m_2$ and $m_3=0$. In cases $U_{\mathrm{MNS}_{7,8,9}}$, the two mixed neutrinos have $m_i=-m_j$.

Except for the case where the mixing is unrestricted by the symmetry, none of the above mixing matrices fall within experimental bounds. In the unrestricted case U_ν is unrestricted, so although right-handed charged lepton transformations can alter U_{ℓ_L}, the mixing matrix $U_{\mathrm{MNS}}=U_{\ell_L}^\dagger U_\nu$ will remain unconstrained by the symmetry.

E. Abelian representations and Dirac neutrinos

An Abelian symmetry constrains the neutrino Dirac mass matrix in the same way as the charged lepton Dirac mass matrix, Eq. (19), except that the right-handed neutrino may transform in a different way to the right-handed charged leptons.

Dirac neutrino mass matrices are diagonalized by $\mathrm{diag}(m_1,m_2,m_3)=U^\dagger_{\nu_L}M_\nu U_{\nu_R}$, and the mixing matrix incorporates only the left diagonalization matrices. U_{ν_L} can be obtained from $M_\nu M_\nu^\dagger$ which is restricted by the X_L transformation by $M_\nu M_\nu^\dagger = X_L^\dagger M_\nu M_\nu^\dagger X_L$.

The possible U_{MNS} matrices obtainable by the left-handed transformation are listed below. It is possible that the right-handed transformations will be able to further restrict the mixing matrices.

Mixing matrix	**Form of X_L required for all X_L**	
$U_{\mathrm{MNS}_1}=\begin{pmatrix} c_l & s_l & 0 \\ -s_l e^{i\delta_l} & c_l e^{i\delta_l} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$X_L=\mathrm{diag}(e^{i\phi_1},e^{i\phi_1},e^{i\phi_3})$	(33)
$U_{\mathrm{MNS}_2}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_l & s_l \\ 0 & -s_l e^{i\delta_l} & c_l e^{i\delta_l} \end{pmatrix}$	$X_L=\mathrm{diag}(e^{i\phi_1},e^{i\phi_2},e^{i\phi_2})$	(34)
$U_{\mathrm{MNS}_3}=\begin{pmatrix} c_l & 0 & s_l \\ 0 & 1 & 0 \\ -s_l e^{i\delta_l} & 0 & c_l e^{i\delta_l} \end{pmatrix}$	$X_L=\mathrm{diag}(e^{i\phi_1},e^{i\phi_2},e^{i\phi_1})$	(35)
$U_{\mathrm{MNS}_4}=$ unrestricted by X_L	$X_L=e^{i\phi_1}I$	(36)
$U_{\mathrm{MNS}}=I$	$e^{i\phi_1}\neq e^{i\phi_2}$ for some X_L, $e^{i\phi_1}\neq e^{i\phi_3}$ for some X_L, and $e^{i\phi_2}\neq e^{i\phi_3}$ for some X_L.	(37)

The only U_{MNS} that fits in with experiment is the one that is unrestricted by X_L, which occurs when $X_L=e^{i\phi}I$. In this case both U_{ℓ_L} and U_{ν_L} are unconstrained by the X_L transformation. However, U_{ℓ_L} and U_{ν_L} can be restricted by the right-handed transformations X_{ℓ_R} and X_{ν_R}. If one or both of the two diagonalization matrices remains unrestricted under the right-handed transformations, then $U_{\mathrm{MNS}}=U^\dagger_{\ell_L}U_{\nu_L}$ will be unrestricted, independent of how the second diagonalization matrix is restricted by the symmetry.

The transformation

$$X_{\ell_R}=\mathrm{diag}(e^{i\sigma_1},e^{i\sigma_2},e^{i\sigma_3}) \tag{38}$$

restricts the charged lepton mass matrix by

$$M_\ell=\begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix} \tag{39}$$

$$=X_L^\dagger M_\ell X_{\ell_R}$$

$$=\begin{pmatrix} e^{-i(\phi-\sigma_1)}r & e^{-i(\phi-\sigma_2)}s & e^{-i(\phi-\sigma_3)}t \\ e^{-i(\phi-\sigma_1)}u & e^{-i(\phi-\sigma_2)}v & e^{-i(\phi-\sigma_3)}w \\ e^{-i(\phi-\sigma_1)}x & e^{-i(\phi-\sigma_2)}y & e^{-i(\phi-\sigma_3)}z \end{pmatrix}. \tag{40}$$

Either the ith column is unrestricted by the symmetry ($\phi=\sigma_i$), or the symmetry constrains column i to be a column of zeros ($\phi\neq\sigma_i$). A matrix that has one column of zeros has one massless charged lepton. A matrix that has no columns of zeros is completely unconstrained by the symmetry, and will give an unrestricted U_{ℓ_L}.

Therefore, in the case where $X_L = e^{i\phi}I$, $U_{\rm MNS}$ is unrestricted unless one or more of the charged leptons are massless. As there are no massless charged leptons, we can conclude that for Dirac neutrinos no mixing matrix is compatible with experiment, except for when $U_{\rm MNS}$ is completely unconstrained by the symmetry.

In fact, if the electron is taken to be massless (corresponding to a single column of zeros), we are convinced that U_{ℓ_L} is also completely general, and hence, the mixing matrix is unrestricted by the symmetry. In this case U_{ℓ_L} has the same number of free parameters as a completely unconstrained diagonalization matrix. This has been backed up by numerical calculations. The right-handed diagonalization matrix U_{ℓ_R}, however, is restricted by the right-handed transformation.

F. Abelian representations and seesaw neutrinos

Majorana neutrino mass matrices that are generated by the seesaw mechanism can be expressed as

$$M_\nu = M_d^T M_M^{-1} M_d, \tag{41}$$

where M_d is the Dirac mass matrix, and M_M is the right-handed Majorana mass matrix. This equation is valid when M_M is invertible. In this section we assume that M_M is invertible. (If the Majorana mass matrix was not invertible, and had rank $n > 3$, the physical particles would be n ultralight neutrinos, n heavy neutrinos and $2n-6$ neutrinos whose masses are naturally the same size as the other fermions [43,44].)

Under the X_L transformations M_ν is restricted by

$$M_\nu = X_L^\dagger M_\nu X_L^*, \tag{42}$$

the same as when the neutrinos are Majorana but do not have mass terms generated by the seesaw mechanism. Section IV D lists all the ways that X_L can restrict the mixing matrix. Again, the only mixing matrix that fits with experiment is the mixing matrix that is unrestricted by the symmetry, which occurs when $X_L = \pm I$. In this case the diagonalization matrices U_{ℓ_L} and U_ν are both unrestricted by the X_L transformation, but can be further restricted by right-handed transformations.

The right-handed charged lepton transformation restricts the mass matrix by

$$M_\ell = \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix} \tag{43}$$

$$= X_L^\dagger M_\ell X_{\ell_R} = \pm \begin{pmatrix} e^{i\sigma_1}r & e^{i\sigma_2}s & e^{i\sigma_3}t \\ e^{i\sigma_1}u & e^{i\sigma_2}v & e^{i\sigma_3}w \\ e^{i\sigma_1}x & e^{i\sigma_2}y & e^{i\sigma_3}z \end{pmatrix}. \tag{44}$$

The argument in the Dirac neutrino section is applicable here also. Either a column of the mass matrix is unrestricted by the symmetry, or it is zero. If all columns are unrestricted, U_{ℓ_L} is unrestricted by the symmetry, giving a mixing matrix that is unconstrained by the symmetry. For each column that is constrained to be zero, there is a corresponding massless charged lepton which is not seen in nature. If one charged lepton is taken to be massless, the mixing is still unconstrained by the symmetry. Therefore, the only mixing matrix that can be generated by a discrete unbroken symmetry, and is consistent with experiments is the mixing matrix that is completely unconstrained by the symmetry.

V. CONCLUSIONS AND FUTURE WORK

It is tantalizing to suppose that a family symmetry could simultaneously explain both the lepton and the quark mixing matrices. We have shown however, that given certain assumptions, unbroken symmetries acting on the generations of the fermions cannot produce a lepton mixing matrix of tribimaximal form, or anything approaching this form. Relaxing the assumptions of this no-go theorem may make it possible for a symmetry to generate an experimentally allowed mixing matrix.

An option for trying to generate nontrivial mixing in the lepton sector, while still including the $SU(2)_L$ restriction, is to utilize the different mass generation mechanisms for the neutrinos and charged leptons. Charged lepton masses come from Yukawa couplings with the standard model Higgs doublet. Majorana neutrinos will gain masses from another mechanism, possibly using the same Higgs doublets in the seesaw mechanism, or by interaction with a Higgs triplet, or by a different mechanism.

If the Higgs sector is extended by introducing a number of generations of Higgs fields, these Higgs fields can also transform under the symmetry. Since the action of the Higgs fields in creating mass matrices is different for neutrinos compared to charged leptons, different restrictions for the two mass matrices will in general result. This in turn will lead to the diagonalization matrices for neutrinos being different from that of the charged leptons, possibly resulting in phenomenologically acceptable lepton mixing.

Since both up-like and down-like quarks are Dirac particles, the action of the Higgs fields in creating their mass matrices is similar for both sectors. It might be possible, then, to construct a model whereby these mass matrices are sufficiently similar so as to yield very similar left-diagonalization matrices. The resulting $U_{\rm CKM}$ may then be approximately diagonal, in agreement with the observed form of this matrix. This kind of setting — models with a nonminimal Higgs sector — may be the appropriate one in which to realize our conjecture (see Sec. II C) within a complete and consistent standard model extension, despite its original inspiration coming from the rather different Harrison, Perkins and Scott proposal.

ACKNOWLEDGMENTS

R.R.V. would like to thank Tony Zee and Lincoln Wolfenstein for useful discussions during the *Neutrinos: Data, Cosmos, and Planck Scale* workshop held at the Kavli Institute for Theoretical Physics at the University of California–

Santa Barbara. This work was supported in part by the Australian Research Council, and in part by The University of Melbourne.

APPENDIX A: REGULAR REPRESENTATIONS OF ABELIAN GROUPS

For a group of rank *n*, the regular representation involves *n*, $n\times n$ matrices, with elements 0 and 1. Each row or column contains one 1. The ijth term equals 1 for one and only one matrix in the representation. One of the matrices is the identity.

M is invariant under the regular representation of an Abelian group if *M* commutes with all *X*:

$$M = X_a^T M X_a \quad \text{for all a,} \tag{A1}$$

where *a* labels the *X* matrices, or for each element

$$M_{ij} = \sum_{kl} (X^{aT})_{ik} M_{kl} X^a_{lj} \quad \text{for all a.} \tag{A2}$$

As the group is Abelian, all the *X* matrices commute with each other, so an arbitrary linear combination of the *X* matrices will also commute with all *X*. The following argument shows that if *M* commutes with *X*, *the most general M* must be a linear combination of the *X* matrices. The restriction forces the diagonal elements of *M* to be equal:

$$\begin{aligned} M_{11} &= \sum_{kl} (X^{aT})_{1k} M_{kl} X^a_{l1} \\ &= (X^T)_{1j} M_{jj} X_{j1} = M_{jj} = M_{jj} \\ &\quad \text{choosing the } X \text{ to be the one that has } X_{j1}=1. \end{aligned} \tag{A3}$$

Since there exists a matrix *X* such that $X_{j1}=1$ for all *j*, all the diagonal elements are equal. The diagonal elements of *M* can be written as $M_{11}I$.

By looking just at the restrictions placed on the mass matrix by an *X* that has $X_{ij}=1$, we show that if X_{kl} also equals 1, then the klth element of the mass matrix must be equal to the ijth element, $M_{ij}=M_{kl}$. Let us take the *X* that has $X_{12}=1$.

$$M_{12} = \sum_{kl} = (X^T)_{1k} M_{kl} X_{l2} \quad \text{choose the } X \text{ that has } X_{12}=1. \tag{A4}$$

$$\begin{aligned} &= \sum_k (X^T)_{1k} M_{k1} X_{12} \quad \text{choose a } k \text{ such that } X_{k1}=1 \\ &= M_{k1} \end{aligned}$$

$$\begin{aligned} M_{k1} &= \sum_j (X^T)_{kj} M_{jk} X_{k1} \quad \text{choose } j \text{ such that } X_{jk}=1 \\ &= M_{jk}. \end{aligned}$$

Repeating this will show that the restrictions from the *X* that has $X_{12}=1$, ensure that $M_{12}=M_{ij}$ if $X_{ij}=1$. $M_{12}X$ describes the ij terms of the mass matrix, where $X_{ij}=1$. The same argument can be made for any *M* element. If $X_{ij}=X_{kl}$ for a given *X*, then $M_{ij}=M_{kl}$, showing that the klth elements of *M* can be expressed as $M_{ij}X$. Therefore *M* is a linear combination of the *X* matrices.

APPENDIX B: PROOF THAT TWO EQUIVALENT REPRESENTATIONS CONSTRAIN THE MIXING MATRIX IN AN IDENTICAL WAY

This proof assumes that Higgs bosons are singlets of the generation symmetry, and that the generation symmetry commutes with $SU(2)_L$ meaning ν_L transforms in the same way as ℓ_L. The seesaw section assumes that the right-handed Majorana mass matrix is invertible.

1. Charged leptons

A_{Li} and B_{Li} are equivalent representations which will transform the left-handed leptons. Each matrix is labelled by an index *i*. $A_{\ell_R i}$and $B_{\ell_R i}$ are also equivalent representations which transform the right-handed charged leptons:

$$U_1^\dagger A_{Li} U_1 = B_{Li}, \quad U_2^\dagger A_{\ell_R i} U_2 = B_{\ell_R i}. \tag{B1}$$

The two different representations restrict the charged lepton mass matrix by

$$\begin{aligned} M_{\ell A} &= A_{Li}^\dagger M_{\ell A} A_{\ell_R i} \quad \text{for all } i, \\ M_{\ell B} &= B_{Li}^\dagger M_{\ell B} B_{\ell_R i} \quad \text{for all } i \\ &= U_1^\dagger A_{Li}^\dagger U_1 M_{\ell B} U_2^\dagger A_{\ell_R i} U_2. \end{aligned} \tag{B2}$$

$U_1 M_{\ell B} U_2^\dagger$ has the same restrictions as $M_{\ell A}$. As we assume that the mass matrices are completely unconstrained apart from the generation symmetry constraints, we can set

$$U_1 M_{\ell B} U_2^\dagger = M_{\ell A}. \tag{B3}$$

M_ℓ is diagonalized by U_{ℓ_L} and U_{ℓ_R} via

$$\text{diag}(m_e, m_\mu, m_\tau) = U_{l_L A}^\dagger M_{lA} U_{l_R A} = U_{l_L B}^\dagger M_{lB} U_{l_R B}, \tag{B4}$$

so $U_{\ell_L B} = U_1^\dagger U_{\ell_L A}$ and $U_{\ell_R B} = U_2^\dagger U_{\ell_R A}$.

2. Majorana neutrinos

The two representations restrict the neutrino mass matrix by

$$\begin{aligned} M_{\nu A} &= A_{Li}^\dagger M_{\nu A} A_{Li}^* \quad \text{for all } i, \\ M_{\nu B} &= B_{Li}^\dagger M_{\nu B} B_{Li}^* \quad \text{for all } i, \\ &= U_1^\dagger A_{Li}^\dagger U_1 M_{\nu B} U_1^T A_{Li}^* U_1^*. \end{aligned} \tag{B5}$$

$U_1 M_{\nu B} U_1^T$ has the same restrictions as $M_{\nu A}$, and we can equate $U_1 M_{\nu B} U_1^T = M_{\nu A}$. M_ν is diagonalized by U_ν via

$$\mathrm{diag}(m_1, m_2, m_3) = U_{\nu A}^\dagger M_{\nu A} U_{\nu A}^* = U_{\nu B}^\dagger M_{\nu B} U_{\nu B}^* . \quad \text{(B6)}$$

So $U_{\nu B} = U_1^\dagger U_{\nu A}$. Combining this result with the charged lepton results we see

$$U_{\mathrm{MNS}B} = U_{\ell_L B}^\dagger U_{\nu B} = U_{\ell_L A}^\dagger U_1 U_1^\dagger U_{\nu A} = U_{\ell_L A}^\dagger U_{\nu A} = U_{\mathrm{MNS\,A}} \quad \text{(B7)}$$

showing that representation A gives the same mixing matrix restrictions as representation B.

3. Dirac neutrinos

The right-handed neutrinos transform by the representations $A_{\nu_R i}$ and $B_{\nu_R i}$ which are related by

$$U_3^\dagger A_{\nu_R i} U_3 = B_{\nu_R i} . \quad \text{(B8)}$$

An identical argument to Appendix B 1 shows $U_1 M_{\nu B} U_3^\dagger$ has the same restrictions as $M_{\nu A}$, enabling us to set $U_1 M_{\nu B} U_3^\dagger = M_{\nu A}$. So $U_{\nu_L B} = U_1^\dagger U_{\nu_L A}$, $U_{\nu_R B} = U_3^\dagger U_{\nu_R A}$. Combining this with the charged lepton result we see that the mixing matrix for A is the same as the mixing matrix for B:

$$U_{\mathrm{MNSB}} = U_{l_L B}^\dagger U_{\nu B} = U_{l_L A}^\dagger U_1 U_1^\dagger U_{\nu A} = U_{l_L A}^\dagger U_{\nu A} = U_{\mathrm{MNSA}} , \quad \text{(B9)}$$

showing that the two equivalent representations restrict the mixing in the same way.

4. Seesaw neutrinos

This section assumes that the Majorana mass matrix is invertible, so the resultant light neutrino mass matrix is given by $M_\nu = M_d^T M_M^{-1} M_d$.

From Appendix B 3, $(U_1 M_{dB} U_3^\dagger)$ has the same restrictions as M_{dA}, so set them to be equal.

From Appendix B 2, the right-handed Majorana mass term constraints show $(U_3^* M_{MB} U_3^\dagger)$ has the same restrictions as M_{MA}, so they can be set equal.

The resultant light neutrino mass term has the restrictions

$$\begin{aligned} M_{\nu A} &= M_{dA} M_{MA}^{-1} M_{dA}^T \\ &= (U_1 M_{dB} U_3^\dagger)(U_3 M_{MB}^{-1} U_3^T)(U_3^* M_{dB}^T U_1^T) \\ &= U_1 M_{dB} M_{MB}^{-1} M_{dB}^T U_1^T \\ &= U_1 M_{\nu B} U_1^T . \end{aligned} \quad \text{(B10)}$$

So $M_{\nu A}$ and $M_{\nu B}$ are related by a basis change — the same as the case with nonseesaw Majorana neutrinos. Diagonalizing,

$$\begin{aligned} \mathrm{diag}(m_1, m_2, m_3) &= U_{\nu A}^\dagger M_{\nu A} U_{\nu A}^* \\ &= U_{\nu A}^\dagger U_1 M_{\nu B} U_1^T U_{\nu A}^* \\ &= U_{\nu B}^\dagger M_{\nu B} U_{\nu B}^* . \end{aligned} \quad \text{(B11)}$$

So $U_{\nu B} = U_1^\dagger U_{\nu A}$.

So the mixing matrices for the two representations are

$$U_{\mathrm{MNSB}} = U_{l_L B}^\dagger U_{\nu B} = U_{l_L A}^\dagger U_1 U_1^\dagger U_{\nu A} = U_{l_L A}^\dagger U_{\nu A} = U_{\mathrm{MNSA}} . \quad \text{(B12)}$$

Therefore, two different, but equivalent, representations restrict the mixing matrix in the same way.

[1] LSND Collaboration, L. B. Auerbach *et al.*. Phys. Rev. C **64**, 065501 (2001).
[2] P. F. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. B **530**, 167 (2002).
[3] L. Wolfenstein, Phys. Rev. D **18**, 958 (1978).
[4] P.F. Harrison and W. G. Scott, Phys. Lett. B **535**, 163 (2002).
[5] P. F. Harrison and W. G. Scott, Phys. Lett. B **557**, 76 (2003).
[6] X.-G. He and A. Zee, Phys. Lett. B **560**, 87 (2003).
[7] X.-G. He and A. Zee, Phys. Rev. D **68**, 037302 (2003).
[8] Z.-Z. Xing, Phys. Lett. B **533**, 85 (2002).
[9] Kamiokande Collaboration, Y. Fukuda *et al.*, Phys. Lett. B **335**, 237 (1994).
[10] Soudan-2 Collaboration, W. W. M. Allison *et al.*, Phys. Lett. B **449**, 137 (1999).
[11] MACRO Collaboration, M. Ambrosio *et al.*, Phys. Lett. B **434**, 451 (1998).
[12] R. Becker-Szendy *et al.*, Nucl. Phys. B (Proc. Suppl.) **38**, 331 (1995).
[13] G. L. Fogli *et al.*, Phys. Rev. D **67**, 073002 (2003).
[14] KamLAND Collaboration, K. Eguchi *et al.*, Phys. Rev. Lett. **90**, 021802 (2003).
[15] SNO Collaboration, Q. R. Ahmad *et al.*, Phys. Rev. Lett. **89**, 011301 (2002).
[16] K2K Collaboration, M. H. Ahn *et al.*, Phys. Rev. Lett. **90**, 041801 (2003).
[17] Super-Kamiokande Collaboration, S. Fukuda *et al.*, Phys. Lett. B **539**, 179 (2002).
[18] SAGE Collaboration, J. N. Abdurashitov *et al.*, J. Exp. Theor. Phys. **95**, 181 (2002).
[19] B. T. Cleveland *et al.*, Astrophys. J. **496**, 505 (1998).
[20] GNO Collaboration, C. M. Cattadori, Nucl. Phys. B (Proc. Suppl.) **110**, 311 (2002).

[21] G. L. Fogli *et al.*, Phys. Rev. D **67**, 073002 (2003).
[22] CHOOZ Collaboration, M. Apollonio *et al.*, Phys. Lett. B **466**, 415 (1999).
[23] A. de Gouvea and H. Murayama, hep-ph/0301050.
[24] S. Weinberg, Trans. N. Y. Acad. Sci. **38**, 185 (1977).
[25] H. Fritzsch, Phys. Lett. **70B**, 436 (1977).
[26] H. Harari and Y. Nir, Phys. Lett. B **195**, 586 (1987).
[27] C. D. Froggatt and H. B. Nielsen, Nucl. Phys. **B147**, 277 (1979).
[28] J. Kubo, A. Mondragon, M. Mondragon, and E. Rodriguez-Jauregui, Prog. Theor. Phys. **109**, 795 (2002).
[29] W. Grimus and L. Lavoura, hep-ph/0305046.
[30] K. S. Babu, E. Ma, and J. W. F. Valle, Phys. Lett. B **552**, 207 (2003).
[31] E. Ma and G. Rajasekaran, Phys. Rev. D **64**, 113012 (2001).
[32] E. Ma, Mod. Phys. Lett. A **17**, 2361 (2002).
[33] H. Fritzsch and Z.-Z. Xing, Phys. Lett. B **372**, 265 (1996).
[34] H. Fritzsch and Z.-Z. Xing, Phys. Lett. B **440**, 313 (1998).
[35] M. Fukugita, M. Tanimoto, and T. Yanagida, Phys. Rev. D **57**, 4429 (1998).
[36] K. S. Babu and R. N. Mohapatra, Phys. Lett. B **532**, 77 (2002).
[37] L. Lavoura, Phys. Rev. D **62**, 093011 (2000).
[38] L. Lavoura and W. Grimus, J. High Energy Phys. **09**, 007 (2000).
[39] E. Ma, Phys. Rev. Lett. **90**, 221802 (2003).
[40] P. F. Harrison and W. G. Scott, Phys. Lett. B **547**, 219 (2002).
[41] W. Grimus and L. Lavoura, J. High Energy Phys. **07**, 045 (2001).
[42] R. N. Mohapatra and S. Nussinov, Phys. Lett. B **441**, 299 (1998).
[43] S. L. Glashow, Phys. Lett. B **256**, 255 (1991).
[44] M. Fukugita and T. Yanagida, Phys. Rev. Lett. **66**, 2705 (1991).