Gauge invariant reduction to the light front

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The problem of constructing gauge invariant currents in terms of light-front bound-state wave functions is solved by utilizing the gauging of equations method. In particular, we show how to construct perturbative expansions of the electromagnetic current in the light-front formalism such that current conservation is satisfied at each order of the perturbation theory.

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I. INTRODUCTION

Equal light-front (LF) ''time'' wave functions possess the important property of having boost transformations which are kinematical. This feature makes the LF formalism a powerful tool in the investigation of relativistic processes. Lately the LF approach has often been mentioned [1] in relation to recent measurements of proton electromagnetic form factors [2,3]. The LF formalism allows one to maintain Poincaré invariance in a simple way, and this can be of great benefit in analyzing the physics behind any particular form-factor behavior [1,4,5]. In this respect it would be extremely desirable to develop an approach to the problem of electromagnetic currents which combines the three-dimensional nature of the boost invariant LF wave functions with gauge invariance. The theory of gauge invariant currents has recently been developed for the usual four-dimensional Bethe-Salpeter (BS) approach [6] and for its three-dimensional spectator reduction [7,8]. Here we shall extend this theory to obtain the gauge invariant three-dimensional reduction to the LF.

So far, what has been known [9,10] is that, for any given two-body BS Green function G and two-body vertex function Γ^μ [11], one can derive a LF reduced vertex function Λ^μ such that, when sandwiched between LF wave functions [see Eq. (9)], it gives the matrix element of Γ^μ between BS wave functions [see Eq. (5)]. The latter is the initial expression for the transition current, and if it is gauge invariant, then the goal of constructing a gauge invariant current in terms of the LF wave functions is achieved. Unfortunately this is not satisfactory from a practical point of view because Λ^μ represents an infinite series even in the simplest case of the one-body Mandelstam current $\Gamma^\mu=\Gamma_0^\mu$. In addition, the potential V defining the LF wave function [see Eq. (4)] is also only expressible as an infinite series.

The goal of this paper is to derive a conserved current in terms of LF bound-state wave functions corresponding to any LF potential given by equal-time Feynman diagrams. This enables us to go further: namely, to derive a gauge invariant expansion of the current when only a part of the potential is taken into account nonperturbatively. Current conservation is satisfied at each order of the perturbation theory.

II. LF REDUCTION OF THE TWO-BODY EQUATION

Consider the Green function BS equation for the case of two scalar particles:[1]

$$G=G_0+G_0KG. \tag{1}$$

Define the LF two-''time'' Green function $\tilde{G}(P,\underline{k},\underline{p})$ as [9]

$$\tilde{G}(P,\underline{k},\underline{p})=\frac{1}{(2\pi)^2}\int dk^- dp^- G(P,k,p), \tag{2}$$

where the underlined momenta $\underline{p}=(p^+,p_\perp)$ denote the LF three-dimensional part of the four-vector $p=(p^-,p^+,p_\perp)$, where $p^\pm=(p^0\pm p^3)/\sqrt{2}$ and $p_\perp=(p^1,p^2)$. There is no necessity here to specify the precise form for the relative momenta p and k; they could be chosen, for example, as the initial and final momenta of the second particle. The equation on the LF corresponding to the BS of Eq. (1) is

$$\tilde{G}=\tilde{G}_0+\tilde{G}_0V\tilde{G}, \tag{3}$$

where the LF potential is [9,12,13]

$$\begin{aligned} V&=\tilde{G}_0^{-1}-\tilde{G}^{-1}\\ &=\tilde{G}_0^{-1}[\langle G_0KG_0\rangle+\langle G_0KG_0KG_0\rangle\\ &\quad-\langle G_0KG_0\rangle\tilde{G}_0^{-1}\langle G_0KG_0\rangle+\cdots]\tilde{G}_0^{-1}. \end{aligned} \tag{4}$$

Here the angular brackets $\langle$ and $\rangle$ stand for equating LF ''times'' (corresponding to the integration over relative LF energies) in the final and initial states, respectively, as in Eq. (2). Note that products of LF operators (quantities labeled with a tilde or enclosed by angular brackets) have implied three-dimensional integrations over $d^3\underline{p}=dp^+dp_\perp=P^+dx\,dp_\perp$ in contrast to the four-dimensional integrations implied by products of BS quantities.

A similar expansion was first derived in [14] (and later rederived in many papers) for the projection onto the hyper-

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[1]The case of spinor particles will be considered elsewhere.

plane where particles have the same usual time. The infinite series of Eq. (4) for the LF potential V suggests that the LF wave function be expanded in orders of the strength of the BS potential K. Our task is to construct gauge invariant currents in terms of these LF wave functions.

III. GAUGE INVARIANT CURRENTS

In order to calculate electromagnetic or weak properties of bound states, we need to construct the corresponding currents. We first start with the BS approach where the electromagnetic current can be obtained diagramatically by attaching a photon everywhere in Eq. (1) [6]. The resulting expression consists of the matrix element of the vertex function Γ^μ taken between initial $\Psi \equiv \Psi(P,p)$ and final $\bar{\Psi} \equiv \bar{\Psi}(K,k)$ BS bound-state wave functions:

$$J^\mu(K,P) = \bar{\Psi}\Gamma^\mu\Psi, \tag{5}$$

where

$$\Gamma^\mu = \Gamma_0^\mu + K^\mu. \tag{6}$$

Here Γ_0^μ denotes the sum of single-particle currents and K^μ is the interaction current. The vertex function Γ^μ is related to the gauged Green function G^μ (five-point function) by

$$G^\mu = G\Gamma^\mu G. \tag{7}$$

Equation (5) is obtained from Eq. (7) by taking residues at the initial and final bound-state poles [11]. The corresponding axial current can be found in the same way by making an axial-vector insertion instead of attaching a photon. Define the LF two-time five-point Green function $\tilde{G}^\mu$ and the corresponding vertex function Λ^μ by [9]

$$\tilde{G}^\mu(K,\underline{k};P,\underline{p}) = \frac{1}{(2\pi)^2}\int dk^- dp^- G^\mu(K,k;P,p) = \tilde{G}\Lambda^\mu\tilde{G}. \tag{8}$$

Then it is easy to see that the current of Eq. (5) is also given by a corresponding matrix element of LF quantities:

$$J^\mu(K,P) = \bar{\tilde{\Psi}}\Lambda^\mu\tilde{\Psi}, \tag{9}$$

where $\tilde{\Psi}$ is the LF bound-state wave function given by

$$\tilde{\Psi}(P,\underline{p}) = \frac{1}{2\pi}\int dp^- \Psi(P,p). \tag{10}$$

The price paid for the relative simplicity of Eq. (9) (involving the LF wave function $\tilde{\Psi}$ which depends only on physical three-dimensional momenta) is the complexity of the LF vertex function Λ^μ which involves an infinite series in powers of K (even for the case of one-body Mandelstam BS currents, $\Gamma^\mu = \Gamma_0^\mu$) and the potential V which is also given by an infinite series, Eq. (4).

Clearly, if Λ^μ is given (in all its complexity) by Eq. (8) and $\tilde{\Psi}$ is the bound-state solution of the LF bound-state equation defined by the homogeneous part of Eq. (3),

$$\tilde{\Psi} = \tilde{G}_0 V\tilde{\Psi}, \tag{11}$$

then the current expressed as the matrix element of Eq. (9) is conserved, as it is equal to the matrix element of Eq. (5). However, for practical applications, it is useful to develop a gauge invariant perturbation theory based on the expansion given by Eq. (4). In this paper our task is to develop such a theory where gauge invariance is achieved at each order of the perturbation. By contrast, in a recent series of papers [15] this problem has been approached with the strategy of improving gauge invariance by increasing the order of perturbation.

Our approach is founded on the fact that equating LF times in the initial state ($x_1^+ - x_2^+ = 0$) and similarly in the final state, which implies integration over relative "energies" in momentum space, as in Eq. (8), does not change either the Ward-Takahashi identity (WTI) or the Ward identity (WI): i.e.,

$$q_\mu\tilde{G}^\mu = \hat{e}\tilde{G} - \tilde{G}\hat{e}, \tag{12}$$

where $q = K - P$ is the momentum transferred by the current to the initial bound state and where the operator $\hat{e}$ shifts the momenta and picks up the charges of the constituents as required. Its four-dimensional form can be found in [6], while in the present LF version $\hat{e}$ is defined by

$$\begin{aligned}\hat{e}(K,\underline{k};P,\underline{p}) &= i(2\pi)^7\delta^4(K-P-q)[e_1\delta^3(\underline{k}_1-\underline{p}_1-\underline{q})\\ &\quad + e_2\delta^3(\underline{k}_2-\underline{p}_2-\underline{q})]\\ &= i(2\pi)^7\delta^4(K-P-q)[e_1\delta^3(\underline{k}_2-\underline{p}_2)\\ &\quad + e_2\delta^3(\underline{k}_1-\underline{p}_1)].\end{aligned} \tag{13}$$

Here e_i (without a caret) is the ith-particle charge operator. We then define the gauging of a two-time quantity as, first, the gauging of the corresponding four-dimensional quantity and, then, the equating of times in the initial and final states; e.g., for the Green function we have

$$\langle G\rangle^\mu = (\tilde{G})^\mu = \langle G^\mu\rangle. \tag{14}$$

It can be argued that Eq. (14) is not even a matter of definition, if one recalls that "gauging" is equivalent to taking a functional derivative over an auxilary field associated with a given current [6], and as such, does not depend on whether it is taken before or after the times of the particles are equated.

Using this definition, one can gauge Eq. (4), in this way obtaining V^μ expressed as a perturbation series with respect to powers of the strength of the BS interaction K. A similar perturbation series can be written for Λ^μ simply from its definition in Eq. (8). It is then easy to see that

$$\Lambda^\mu = \Lambda_0^\mu + V^\mu, \tag{15}$$

where Λ_0^μ is defined in the same way as Λ^μ: namely,

$$\Lambda_0^\mu = \tilde{G}_0^{-1}\tilde{G}_0^\mu\tilde{G}_0^{-1}. \qquad (16)$$

Note that to obtain Eq. (15) one needs to use the fact that

$$[\tilde{G}_0^{-1}]^\mu = -\Lambda_0^\mu, \qquad (17)$$

which follows from formally gauging the identity operator as

$$[\tilde{G}_0^{-1}\tilde{G}_0]^\mu = \tilde{G}_0^{-1}\tilde{G}_0^\mu + [\tilde{G}_0^{-1}]^\mu\tilde{G}_0 = 0. \qquad (18)$$

The result of Eq. (15) is central to this paper and allows us to develop the sought-after gauge invariant perturbation theory.

It is evident that no matter how one defines the perturbation expansion of the LF potential V, each nth-order term V_n of the expansion of Eq. (4) and the corresponding term V_n^μ in the expansion of Eq. (15) are related to each other via the WTI's

$$q_\mu V_n^\mu = \hat{e}V_n - V_n\hat{e}. \qquad (19)$$

The current calculated up to nth order is given by

$$J_n^\mu = \bar{\tilde{\Psi}}_n\left(\Lambda_0^\mu + \sum_{i=1}^{n} V_i^\mu\right)\tilde{\Psi}_n, \qquad (20)$$

where $\tilde{\Psi}_n$ is the corresponding LF bound-state wave function satisfying

$$\left(\tilde{G}_0^{-1} - \sum_{i=1}^{n} V_i\right)\tilde{\Psi}_n = 0. \qquad (21)$$

Then with the help of Eq. (19) and the WTI for Λ_0^μ it is easy to see that J_n^μ is conserved.

To give a concrete example of a possible choice for V_n, let us define it to be the n-particle exchange contribution in a particle exchange model for K. In particular, if we write $K = K_1 + K_2 + \cdots$ where K_1 is the BS one-particle exchange term, K_2 is the BS crossed two-particle exchange, etc., then, from Eq. (4), the leading order (LO) contribution to the LF potential would be given by

$$V_1 = \tilde{G}_0^{-1}\langle G_0K_1G_0\rangle\tilde{G}_0^{-1}, \qquad (22)$$

the next-to leading order (NLO) contribution by

$$V_2 = \tilde{G}_0^{-1}[\langle G_0K_2G_0\rangle + \langle G_0K_1G_0K_1G_0\rangle - \langle G_0K_1G_0\rangle\tilde{G}_0^{-1}\langle G_0K_1G_0\rangle]\tilde{G}_0^{-1}, \qquad (23)$$

and so on. To obtain V_1^μ we simply gauge Eq. (22):

$$V_1^\mu = \tilde{G}_0^{-1}\langle G_0K_1^\mu G_0\rangle\tilde{G}_0^{-1} - \Lambda_0^\mu\langle G_0K_1G_0\rangle\tilde{G}_0^{-1} - \tilde{G}_0^{-1}\langle G_0K_1G_0\rangle\Lambda_0^\mu + \tilde{G}_0^{-1}\langle G_0^\mu K_1G_0\rangle\tilde{G}_0^{-1} + \tilde{G}_0^{-1}\langle G_0K_1G_0^\mu\rangle\tilde{G}_0^{-1}. \qquad (24)$$

The terms contributing to $\tilde{G}_0V_1^\mu\tilde{G}_0$ are illustrated in Fig. 1.

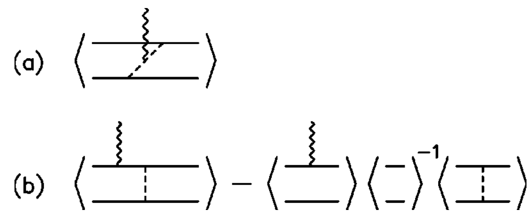

FIG. 1. Contributions to $\tilde{G}_0V_1^\mu\tilde{G}_0$, where V_1^μ is the gauged one-particle-exchange light-front two-body quasipotential, as given in Eq. (24): (a) gauged Bethe-Salpeter kernel contribution, (b) gauged constituent contribution. Left (right) angular brackets indicate that the light-front ''times'' of the left (right) constituent legs of the enclosed Feynman diagrams are set equal. The subtracted second term in (b) removes the equal light-front ''time'' contribution coming from the intermediate state of the first term.

Apart from the term involving K_1^μ, which corresponds to attachments of a photon inside the kernel [Fig. 1(a)], V_1^μ contains attachments to the constituents—e.g., $\tilde{G}_0^{-1}\langle G_0^\mu K_1G_0\rangle\tilde{G}_0^{-1}$. The two terms with a negative sign can be thought of as subtractions to the last two terms [Fig. 1(b)]. These subtract the contributions of the intermediate states of the constituents whose times are equal to each other, as the latter are exposed in the two-time free vertex function Λ_0^μ of Eq. (26). This can be seen by using Eq. (16) and noting that, with the help of Eq. (27), the subtraction terms can be replaced in Eq. (26) by one-body currents:

$$-\Lambda_0^\mu\langle G_0K_1G_0\rangle\tilde{G}_0^{-1} - \tilde{G}_0^{-1}\langle G_0K_1G_0\rangle\Lambda_0^\mu = -\tilde{G}_0^{-1}\tilde{G}_0^\mu V_1 - V_1\tilde{G}_0^\mu\tilde{G}_0^{-1} \to -2\Lambda_0^\mu. \qquad (25)$$

The current in LO is then

$$J_1^\mu = \bar{\Phi}(\Lambda_0^\mu + V_1^\mu)\Phi, \qquad (26)$$

where Φ is the solution of the LF bound-state equation in LO:

$$(\tilde{G}_0^{-1} - V_1)\Phi = 0. \qquad (27)$$

Conservation of the LO current of Eq. (26) follows from the WTI's for V_1^μ [Eq. (19)] and the one-body vertex function Λ_0^μ and the equation for the bound state, Eq. (27).

It is interesting to compare our prescription for constructing the LO current by gauging the LO LF potential [Eqs. (15), (19), and (24)] with the related results of Refs. [16,17] for the case of the usual equal-time quasipotential approach. Using the gauging of equations method, it was shown in Refs. [16,17] that in order to obtain a gauge invariant transition current, both the quasipotential and electromagnetic current operator should be truncated at the same order of the coupling constant. Thus, in Refs. [16,17], the construction of the gauge invariant approximate current involves expansions of both the four-dimensional five-point Green function and the quasipotential, whereas we only need the LF potential given as part of a series expansion [Eq. (4)]. Gauging just this part (viz., the LF potential), we derive the gauge invariant approximate current. This is a nice but formal feature of our approach. A more important difference lies in the fact that the boost transformation of the usual equal-time wave functions is dynamical; i.e., it depends on the interaction [16,17]. Our gauge invariant LF reduction offers wave functions which depend only on three-dimensional momenta; they have kinematical boost transformations and provide gauge invariant currents, all at the same time. It is a difficult task to construct the approximate gauge invariant currents in terms of the covariant wave functions projected onto the hyperplane $P(x_1-x_2)=0$ [9,16,17]. One of the ways for this to be achieved would be in a modification of our gauging prescription for such projected Green functions.

Finally, it is worth noting that we have not addressed the problem of dynamical three-dimensional rotation transformations inherent in the LF quasipotential approach. This problem leads to wave functions that are not invariant under rotations. For instance, in the case of two scalar particles, LF wave functions depend on two variables—say, $x=p^+/P^+$ and $(p_\perp - xP_\perp)^2$—rather than on one rotationally invariant modulus of the three-dimensional relative momentum. Despite this difficulty, the equal LF "time" approach still has an advantage over the usual equal-time approach where the lack of boost invariance leads to functions of *three* rotationally invariant scalar combinations of the total and relative momenta.

A. Currents at NLO

Above, we have formally solved the problem of constructing conserved LF equal-time currents up to any order in the interaction. In this subsection we would like to apply our formalism to the case where only the LO term of V is taken into account exactly, with all higher order contributions being included as a perturbation.

For this purpose, denote the LO contribution to V by V_1 (it can be the single-particle exchange potential discussed above, or it can be defined some other way) and the contributions making up the NLO term by Δ:

$$V=V_1+\Delta+\cdots. \tag{28}$$

Denoting the correction to the wave function Φ due to Δ by $\delta\Phi$, the following LF equation should be satisfied:

$$(\Phi+\delta\Phi)=\tilde{G}_0(V_1+\Delta)(\Phi+\delta\Phi). \tag{29}$$

Treating Δ as a perturbation and keeping terms that are at most linear in Δ, the wave function correction $\delta\Phi$ can be expressed as [18,19]

$$\delta\Phi=\left[\tilde{G}_1^b\Delta+\frac{iP^\mu}{4M^2}\left(\bar{\Phi}\frac{\partial\Delta}{\partial P^\mu}\Phi\right)\right]\Phi, \tag{30}$$

where

$$\tilde{G}_1^b=\tilde{G}_1-\frac{i\Phi\bar{\Phi}}{P^2-M^2} \tag{31}$$

is the LO Green function $\tilde{G}_1$ with the (unperturbed) bound-state pole subtracted off. The second term in Eq. (30) is just a wave-function renormalization due to the dependence of Δ on the total momentum P^μ. Here $\tilde{G}_1$ satisfies the inhomogeneous LF equation

$$\tilde{G}_1=\tilde{G}_0+\tilde{G}_0V_1\tilde{G}_1. \tag{32}$$

The full linear in Δ correction to the current matrix element is [19]

$$\delta J^\mu=\bar{\Phi}\,\delta\Lambda_1^\mu\Phi+\bar{\Phi}\Delta^\mu\Phi+\bar{\Phi}\Lambda_1^\mu\delta\Phi+\delta\bar{\Phi}\Lambda_1^\mu\Phi, \tag{33}$$

where

$$\Lambda_1^\mu=\Lambda_0^\mu+V_1^\mu. \tag{34}$$

The first term stems from the bound-state mass correction to the LO vertex function: that is,

$$\delta\Lambda_1^\mu=\delta M^2\frac{\partial\Lambda_1^\mu}{\partial M^2}, \tag{35}$$

where [18]

$$\delta M^2=i\bar{\Phi}\Delta\Phi. \tag{36}$$

The correction to the current, given by Eq. (33), is conserved by construction, since the exact current corresponding to the potential $V_1+\Delta$ is conserved and so therefore should be the part that is linear in Δ. Nevertheless, it will be instructive to show this current conservation explicitly. In this way we will see that the first term in Eq. (33) is essential for current conservation.

B. Current conservation at NLO

Using the WTI for V_1^μ given in Eq. (19) and the corresponding WTI's for the one-body current Λ_0^μ and for Δ^μ, one obtains

$$q_\mu\delta J^\mu=-\delta M^2\bar{\Phi}\frac{\partial(\hat{e}\tilde{G}_1^{-1}-\tilde{G}_1^{-1}\hat{e})}{\partial M^2}\Phi+\bar{\Phi}(\hat{e}\Delta-\Delta\hat{e})\Phi$$
$$-\bar{\Phi}(\hat{e}\tilde{G}_1^{-1}-\tilde{G}_1^{-1}\hat{e})\delta\Phi-\delta\bar{\Phi}(\hat{e}\tilde{G}_1^{-1}-\tilde{G}_1^{-1}\hat{e})\Phi. \tag{37}$$

In the above expression $q=P'-P$ where P and P' are the total initial and final momenta, respectively; in this respect, it should be noted that in each of the summed terms above, all quantities standing to the right of operator $\hat{e}$ have total momentum P, while those standing to the left of $\hat{e}$ have total momentum P'. Exploiting the bound-state equations $\tilde{G}_1^{-1}\Phi=\bar{\Phi}\tilde{G}_1^{-1}=0$ and making use of Eq. (30), the previous equation can be written as

$$q_\mu \delta J^\mu = -\delta M^2 \bar{\Phi} \frac{\partial(\hat{e}\tilde{G}_1^{-1} - \tilde{G}_1^{-1}\hat{e})}{\partial M^2}\Phi + \bar{\Phi}(\hat{e}\Delta - \Delta\hat{e})\Phi$$
$$- \bar{\Phi}\hat{e}\tilde{G}_1^{-1}\left[\tilde{G}_1^b\Delta + \frac{iP_\mu}{4M^2}\left(\bar{\Phi}\frac{\partial\Delta}{\partial P_\mu}\Phi\right)\right]\Phi$$
$$+ \bar{\Phi}\left[\Delta\tilde{G}_1^b + \frac{iP'_\mu}{4M^2}\left(\bar{\Phi}\frac{\partial\Delta}{\partial P'_\mu}\Phi\right)\right]\tilde{G}_1^{-1}\hat{e}\Phi. \qquad (38)$$

A further application of the bound-state equations gives

$$q_\mu \delta J^\mu = -\delta M^2 \bar{\Phi} \frac{\partial(\hat{e}\tilde{G}_1^{-1} - \tilde{G}_1^{-1}\hat{e})}{\partial M^2}\Phi + \bar{\Phi}(\hat{e}\Delta - \Delta\hat{e})\Phi$$
$$- \bar{\Phi}\hat{e}\tilde{G}_1^{-1}\tilde{G}_1^b\Delta\Phi + \bar{\Phi}\Delta\tilde{G}_1^b\tilde{G}_1^{-1}\hat{e}\Phi. \qquad (39)$$

One can see that the terms responsible for bound-state wave-function renormalization drop out by themselves, whereas other terms contribute zero to $q_\mu \delta J^\mu$ only as a result of partial cancellation between each other. We will see below that the renormalization terms are important in the charge conservation relation as they account for the charge flowing in the intermediate states which are accounted for in Δ. Using Eq. (31), it is easy to show that for $P^2 = M^2$ (see the Appendix)

$$\tilde{G}_1^{-1}\tilde{G}_1^b = 1 - i\frac{\partial\tilde{G}_1^{-1}}{\partial M^2}\Phi\bar{\Phi}, \quad \tilde{G}_1^b\tilde{G}_1^{-1} = 1 - i\Phi\bar{\Phi}\frac{\partial\tilde{G}_1^{-1}}{\partial M^2}, \qquad (40)$$

where the derivative of the inverse Green function, $N = i\partial\tilde{G}_1^{-1}/\partial M^2$, also appears in the normalization condition for the bound-state wave function:

$$\bar{\Phi}N\Phi = 1. \qquad (41)$$

Using these results in Eq. (40) one obtains

$$q_\mu \delta J^\mu = -\delta M^2 \bar{\Phi}' \frac{\partial(\hat{e}\tilde{G}_1^{-1} - \tilde{G}'^{-1}_1\hat{e})}{\partial M^2}\Phi + (\bar{\Phi}'\hat{e}N\Phi)$$
$$\times(\bar{\Phi}\Delta\Phi) - (\bar{\Phi}'\Delta'\Phi')(\bar{\Phi}'N'\hat{e}\Phi), \qquad (42)$$

where we have explicitly indicated with a prime those quantities for which the total momentum is P' and left unprimed those for which the total momentum is P. The Lorentz invariance of the mass correction, Eq. (36), then leads to current conservation

$$q_\mu \delta J^\mu = 0. \qquad (43)$$

C. Charge conservation at NLO

For a two-particle bound state, the requirement of "charge conservation" is given by the condition

$$J^\mu(P,P) = 2(e_1 + e_2)P^\mu. \qquad (44)$$

It is straightforward to show that the LO current J_1^μ, given by Eq. (26), satisfies this condition [as indeed does J_n^μ of Eq. (20) for any n]. Here we show that exact charge conservation holds also in the case where J^μ is calculated to NLO in perturbation theory. For this purpose, it will be sufficient to show that $n_\mu \delta J^\mu(P,P) = 0$ where $n = P/\sqrt{P^2}$ is the unit four-vector along P.

We start by using the WI's for Λ_1^μ and Δ^μ in Eq. (33). The WI for Λ_1^μ can be written as

$$\Lambda_1^\mu(P,\underline{k};P,\underline{p}) = -[\tilde{G}_1^{-1}]^\mu(P,\underline{k};P,\underline{p})$$
$$= i\left[e_2 \frac{\partial\tilde{G}_1^{-1}(P,\underline{k},\underline{p})}{\partial k_\mu} + \frac{\partial\tilde{G}_1^{-1}(P,\underline{k},\underline{p})}{\partial p_\mu}e_2 + (e_1+e_2) \times \frac{\partial\tilde{G}_1^{-1}(P,\underline{k},\underline{p})}{\partial P_\mu}\right], \qquad (45)$$

where we have taken the particular choice $\underline{p} = \underline{p}_2$ and $\underline{k} = \underline{k}_2$ for the relative variables. The derivation of $n_\mu \delta J^\mu(P,P) = 0$ for the two first terms involving $\partial/\partial k_\mu$ and $\partial/\partial p_\mu$ is very similar to the one given above for current conservation; therefore, we will consider only the case of the last term in Eq. (45).

Any function F of the four-vector P can be considered a function of $|P| = \sqrt{P^2}$ and any three independent components of $n = P/|P|$. In this case it is clear that

$$n_\mu \frac{\partial F(P)}{\partial P_\mu} = \frac{\partial F(|P|n)}{\partial |P|}, \qquad (46)$$

so that

$$n_\mu \frac{\partial}{\partial P_\mu} = \frac{\partial}{\partial |P|} = 2|P|\frac{\partial}{\partial P^2}. \qquad (47)$$

To determine the contribution of the last term of Eq. (45) to $n_\mu \delta J^\mu$, we need to consider the contractions

$$n_\mu \Lambda_1^\mu(P,\underline{k};P,\underline{p}) = -n_\mu[\tilde{G}_1^{-1}]^\mu(P,\underline{k};P,\underline{p})$$
$$\to e\frac{\partial\tilde{G}_1^{-1}(P,\underline{k},\underline{p})}{\partial |P|},$$

$$n_\mu \Delta^\mu(P,\underline{k};P,\underline{p}) \to -e\frac{\partial\Delta(P,\underline{k},\underline{p})}{\partial |P|}, \qquad (48)$$

where $e = e_1 + e_2$ and the dependence on $|P|$ is found by writing $P^\mu = |P|n^\mu$. Then

$$n_\mu \bar{\Phi} \Lambda_1^\mu \delta\Phi \to 2e|P| \bar{\Phi} \frac{\partial \tilde{G}_1^{-1}}{\partial P^2} \left[\tilde{G}^b \Delta + \frac{i}{2} \left(\bar{\Phi} \frac{\partial \Delta}{\partial P^2} \Phi \right) \right] \Phi, \tag{49}$$

which for our purpose needs to be evaluated at $P^2 = M^2$. Using the fact that

$$\bar{\Phi} \frac{\partial \tilde{G}_1^{-1}}{\partial M^2} \tilde{G}_1^b = -\frac{i}{2} \bar{\Phi} \left(\bar{\Phi} \frac{\partial^2 \tilde{G}_1^{-1}}{(\partial M^2)^2} \Phi \right), \tag{50}$$

which follows from the derivation given in the Appendix, and the normalization condition of Eq. (41), we obtain that

$$n_\mu \bar{\Phi} \Lambda_1^\mu \delta\Phi \to -ieM \bar{\Phi} \Delta \Phi \left(\bar{\Phi} \frac{\partial^2 \tilde{G}_1^{-1}}{(\partial M^2)^2} \Phi \right) + eM \left(\bar{\Phi} \frac{\partial \Delta}{\partial M^2} \Phi \right), \tag{51}$$

$$n_\mu \delta M^2 \bar{\Phi} \frac{\partial \Lambda_1^\mu}{\partial M^2} \Phi \to 2ieM \bar{\Phi} \Delta \Phi \left(\bar{\Phi} \frac{\partial^2 \tilde{G}_1^{-1}}{(\partial M^2)^2} \Phi \right), \tag{52}$$

$$n_\mu \bar{\Phi} \Delta^\mu \Phi \to -2eM \left(\bar{\Phi} \frac{\partial \Delta}{\partial M^2} \Phi \right). \tag{53}$$

Using Eq. (51), the corresponding expression for $n_\mu \delta\bar{\Phi} \Lambda_1^\mu \Phi$, Eqs. (52) and (53) in Eq. (33), one obtains that $n_\mu \delta J^\mu = 0$.

IV. CONCLUSIONS

The equality of the three-dimensional LF expression for the current, Eq. (9), and the corresponding four-dimensional BS expression, Eq. (5), has been known for a long time [9]. However, this result is not very practical for calculational purposes as both the LF vertex function, defined by Eq. (8), and the potential generating the LF wave function, Eq. (4), are represented by infinite series even if the underlying BS kernel K is simple. In addition, it has so far not been noticed that between these two operators there is a direct connection (even though they are given by series); namely, Λ^μ can be obtained from V by the procedure of gauging if the latter is properly defined in terms of two-time Green functions. A simple and natural definition of gauging in this paper is summarized by Eqs. (14) and (18), and makes the last statement clear. Our definition of gauging enables us to construct the current operator corresponding to any term of the series of Eq. (4). In particular, we have given the explicit expression for the gauge invariant current [Eqs. (26) and (24)] corresponding to the first term of Eq. (4). This expression can be used, for example, in studies of one-particle exchange models. Finally, we have shown how to account perturbatively for the remainder of the terms in Eq. (4) by explicit construction of the current at NLO [Eq. (33)]. Close examination shows that all terms in Eq. (33) are important for current conservation, which is a result of the cancellation between their longitudinal parts.

The perturbation theory presented in this paper, in particular the expression of Eq. (33), could be applied, for example, to the calculation of meson cloud effects on the electromagnetic form factors, which are known to be important [1,20]. Although a similar program for the Nambu–Jona-Lasinio (NJL) model has been demonstrated in Ref. [19], this should also be done in the LF approach. In this case, first the NLO potential Δ should be constructed to incorporate one-meson exchange in all possible ways within the LO model. Next, such a Δ should be gauged in order to derive the meson exchange electromagnetic current operator Δ^μ, etc.

The gauge invariant perturbation theory proposed in this note is not specific to the LF approach and can, for example, be applied to the spectator approach [7,8]. The spectator potential corresponding to the example of Eq. (23) reads

$$V_1 = K_1,$$
$$V_2 = K_2 + K_1 G_0 K_1 - K_1 \delta d K_1, \tag{54}$$

where δd is the product of the single-particle propagator d and the spectator on-mass-shell δ function. Currents would again be given in LO by Eq. (26) and in NLO by Eq. (33); however, rather than gauging the equal-time LF propagators, one would gauge the on-mass-shell propagators instead [7,8].

APPENDIX

Here we derive the following three useful expressions for the case of $P^2 = M^2$:

$$\tilde{G}_1^{-1} \tilde{G}_1^b = 1 - i \frac{\partial \tilde{G}_1^{-1}}{\partial M^2} \Phi \bar{\Phi}, \quad \tilde{G}_1^b \tilde{G}_1^{-1} = 1 - i \Phi \bar{\Phi} \frac{\partial \tilde{G}_1^{-1}}{\partial M^2}, \tag{A1}$$

$$\tilde{G}_1^b \frac{\partial \tilde{G}_1^{-1}}{\partial M^2} \Phi = -\frac{i}{2} \left(\bar{\Phi} \frac{\partial^2 \tilde{G}_1^{-1}}{(\partial M^2)^2} \Phi \right) \Phi. \tag{A2}$$

To carry out the necessary algebra it is useful to introduce the following notation:

$$G_P \equiv \tilde{G}_1(P, \underline{k}, \underline{p}), \quad G_M \equiv \tilde{G}_1(P, \underline{k}, \underline{p})|_{P^2 = M^2}. \tag{A3}$$

Note that our bound-state wave function Φ is covariant and does not depend on P^2, as discussed in Ref. [18]. Using this notation and the definition of G_1^b given in Eq. (31), it follows that

$$G_P^{-1} G_P^b = G_P^{-1} \left(G_P - \frac{i \Phi \bar{\Phi}}{P^2 - M^2} \right) = 1 - \frac{i G_P^{-1}}{P^2 - M^2} \Phi \bar{\Phi}. \tag{A4}$$

As $G_M^{-1} \Phi = 0$, we obtain the first of the equations in Eqs. (A1) in the limit $P^2 = M^2$; the second equation follows similarly. The last of the equations, Eq. (A2), results from the following algebra:

$$
\begin{aligned}
G_M^b \frac{\partial G_M^{-1}}{\partial M^2}\Phi &= \frac{\partial G_M^b G_M^{-1}}{\partial M^2}\Phi - \frac{\partial G_M^b}{\partial M^2} G_M^{-1}\Phi = \frac{\partial G_M^b G_M^{-1}}{\partial M^2}\Phi = \left.\frac{\partial G_P^b G_P^{-1}}{\partial P^2}\right|_{P^2=M^2}\Phi = \left[\frac{\partial}{\partial P^2}\left(G_P - \frac{i\Phi\bar{\Phi}}{P^2-M^2}\right)G_P^{-1}\right]_{P^2=M^2}\Phi \\
&= \left[\frac{\partial}{\partial P^2}\left(1 - \frac{i\Phi\bar{\Phi}G_P^{-1}}{P^2-M^2}\right)\right]_{P^2=M^2}\Phi = i\Phi\bar{\Phi}\left[\frac{G_P^{-1}}{(P^2-M^2)^2} - \frac{1}{(P^2-M^2)}\frac{\partial G_P^{-1}}{\partial P^2}\right]_{P^2=M^2}\Phi \\
&= i\Phi\bar{\Phi}\left\{\frac{1}{(P^2-M^2)^2}\left[G_M^{-1} + (P^2-M^2)\frac{\partial G_M^{-1}}{\partial M^2} + \frac{(P^2-M^2)^2}{2}\frac{\partial^2 G_M^{-1}}{(\partial M^2)^2}\right]\right. \\
&\quad \left. - \frac{1}{(P^2-M^2)}\left[\frac{\partial G_M^{-1}}{\partial M^2} + (P^2-M^2)\frac{\partial^2 G_M^{-1}}{(\partial M^2)^2}\right] + O(P^2-M^2)\right\}_{P^2=M^2}\Phi \\
&= -\frac{i}{2}\left(\bar{\Phi}\frac{\partial^2 G_M^{-1}}{(\partial M^2)^2}\Phi\right)\Phi. \qquad \text{(A5)}
\end{aligned}
$$

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