Decoherence of histories and hydrodynamic equations for a linear oscillator chain

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We investigate the decoherence of histories of local densities for linear oscillators models. It is shown that histories of local number, momentum and energy density are approximately decoherent, when coarse grained over sufficiently large volumes. Decoherence arises directly from the proximity of these variables to exactly conserved quantities (which are exactly decoherent), and not from environmentally induced decoherence. We discuss the approach to local equilibrium and the subsequent emergence of hydrodynamic equations for the local densities.

I. INTRODUCTION

In a large and possibly complex quantum system, which dynamical variables naturally become classical for a wide variety of initial states? This question belongs to the general issue of emergent classicality from quantum theory, and has recently received a considerable amount of attention (see Ref. [1] for an overview). There are a number of different approaches to it, but common to most of them is the demonstration of decoherence: that certain types of quantum states of the system in question exhibit negligible interference, and therefore superpositions of them are effectively equivalent to statistical mixtures.

Decoherence has principally been demonstrated for the situation in which there is a distinguished system, such as a particle, coupled to its surrounding environment [2,3]. More generally, we may expect that decoherence comes about when the variables describing the entire system of interest naturally separate into "slow" and "fast," whether or not this separation corresponds to, respectively, system and environment. (See Ref. [4] for a discussion of the conditions under which the total Hilbert space may be written as a tensor product of system and environment Hilbert spaces.) If the system consists of a large collection of interacting identical particles, as in a fluid for example, the natural set of slow variables are the local densities: energy, momentum, number, charge etc. These variables, in fact, are also the variables which provide the most complete description of the classical state of a fluid at a macroscopic level. The most general demonstration of emergent classicality therefore consists of showing that, for a large collection of interacting particles described microscopically by quantum theory, the local densities become effectively classical. Although decoherence through the system-environment mechanism is expected to play a role since the collection of particles are coupled to each other, it is of interest to explore the possibility that, at least in some regimes, decoherence could come about for a different reason. Namely, because the local densities are almost conserved if averaged over a sufficiently large volume [5]. Hence, the approximate decoherence of local densities would then be due to the fact that they are close to a set of exactly conserved quantities, and exactly conserved quantities obey superselection rules.

We will approach the question using the decoherent histories approach to quantum theory [5–10]. This approach has proved particularly useful for discussing emergent classicality in a variety of contexts. In particular the issues outlined above are most clearly expressed in the language of decoherent histories. The central object of interest is the decoherence functional,

$$D(\underline{\alpha}, \underline{\alpha}') = \text{Tr}[P_{\alpha_n}(t_n) \cdots P_{\alpha_1}(t_1) \rho P_{\alpha_1'}(t_1) \cdots P_{\alpha_n'}(t_n)]. \tag{1}$$

The histories are characterized by the initial state $|\Psi\rangle$ and by the strings of projection operators $P_\alpha(t)$ (in the Heisenberg picture) at times t_1 to t_n (and $\underline{\alpha}$ denotes the string of alternatives $\alpha_1 \cdots \alpha_n$). Intuitively, the decoherence functional is a measure of the interference between pairs of histories $\underline{\alpha}$, $\underline{\alpha}'$. When it is zero for $\underline{\alpha} \neq \underline{\alpha}'$, we say that the histories are decoherent and probabilities $p(\underline{\alpha}) = D(\underline{\alpha}, \underline{\alpha})$ obeying the usual probability sum rules may be assigned to them. One can then ask whether these probabilities are strongly peaked about trajectories obeying classical equations of motion. For the local densities, we expect that these equations will be hydrodynamic equations.

We are generally concerned with a system of N particles interacting through a potential and are therefore described at the microscopic level by a Hamiltonian of the form

$$H = \sum_j \frac{\mathbf{p}_j^2}{2m} + \sum_{\ell > j} V_{j\ell}(\mathbf{q}_j - \mathbf{q}_\ell). \tag{2}$$

We are particularly interested in the number density $n(\mathbf{x})$, the momentum density $\mathbf{g}(\mathbf{x})$ and the energy density $h(\mathbf{x})$, defined by

$$n(\mathbf{x}) = \sum_j \delta(\mathbf{x} - \mathbf{q}_j) \tag{3}$$

$$\mathbf{g}(\mathbf{x}) = \sum_j \mathbf{p}_j \delta(\mathbf{x} - \mathbf{q}_j) \tag{4}$$

$$h(\mathbf{x}) = \sum_j \frac{\mathbf{p}_j^2}{2m} \delta(\mathbf{x} - \mathbf{q}_j) + \sum_{\ell > j} V_{j\ell}(\mathbf{q}_j - \mathbf{q}_\ell) \delta(\mathbf{x} - \mathbf{q}_j). \tag{5}$$

We are generally interested in the integrals of these quantities over small volumes, which will have the effect of smearing out the δ functions. Integrated over an infinite volume, these become the total particle number N, total momentum P and total energy H, which are exactly conserved. It is also often useful to work with the Fourier transforms of the local densities,

$$n(\mathbf{k}) = \sum_j e^{i\mathbf{k}\cdot\mathbf{q}_j} \qquad (6)$$

$$\mathbf{g}(\mathbf{k}) = \sum_j \mathbf{p}_j e^{i\mathbf{k}\cdot\mathbf{q}_j} \qquad (7)$$

$$h(\mathbf{k}) = \sum_j \frac{\mathbf{p}_j^2}{2m} e^{i\mathbf{k}\cdot\mathbf{q}_j} + \sum_{\ell > j} V_{j\ell}(\mathbf{q}_j - \mathbf{q}_\ell) e^{i\mathbf{k}\cdot\mathbf{q}_j}. \qquad (8)$$

These quantities tend to the exactly conserved quantities in the limit $k = |\mathbf{k}| \to 0$.

There is a standard technique for deriving hydrodynamic equations for the local densities [11,12]. It starts with the continuity equations expressing local conservation, which have the form,

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot \mathbf{j} = 0 \qquad (9)$$

where σ denotes n, $\mathbf{g}$ or h (and the current $\mathbf{j}$ is a second rank tensor in the case of $\mathbf{g}$). It is then assumed that, for a wide variety of initial states, conditions of local equilibrium are established after a short period of time. This means that on scales small compared to the overall size of the fluid, but large compared to the microscopic scale, equilibrium conditions are reached in each local region, characterized by a local temperature, pressure etc. which vary slowly in space and time. Local equilibrium is described by the density operator

$$\rho = Z^{-1} \exp\left(-\int d^3x \, \beta(\mathbf{x}) [h(\mathbf{x}) - \bar{\mu}(\mathbf{x}) n(\mathbf{x}) - \mathbf{u}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x})] \right) \qquad (10)$$

where β, $\bar{\mu}$ and $\mathbf{u}$ are Lagrange multipliers and are slowly varying functions of space and time. β is the inverse temperature, $\mathbf{u}$ is the average velocity field, and $\bar{\mu}$ is related to the chemical potential which in turn is related to the average number density. [Note that the local equilibrium state is defined in relation to a particular coarse graining, here, the anticipated calculation of average values of the local densities. Hence it embraces all possible states that are effectively equivalent to the state (10) for the purposes of calculating those averages.] The hydrodynamic equations follow when the continuity equations are averaged in this state. These equations form a closed set because the local equilibrium form depends (in three dimensions) only on the five Lagrange multiplier fields β, $\bar{\mu}$, $\mathbf{u}$ and there are exactly five continuity equations (9) for them. (More generally, it is pos-

sible to have closure up to a set of small terms which may be treated as a stochastic process. See Refs. [4,13], for example.)

The decoherent histories approach to quantum theory offers the possibility of a much more general derivation of emergent classicality than that entailed in the standard derivation of hydrodynamics. The standard derivation is rather akin to the Ehrenfest theorem of elementary quantum mechanics which shows that the averages of position and momentum operators obey classical equations of motion. Yet a description of emergent classicality must involve much more than that [1]. Firstly, it must demonstrate decoherence of the local densities, thereby allowing us to talk about probabilities for their histories. Secondly, it should not be restricted to a special initial state. While it is certainly plausible that many initial states will tend to the local equilibrium state, the standard derivation does not obviously apply to superpositions of macroscopic states, which are exactly the states a description of emergent classicality is supposed to deal with.

This paper is part of a general program, initiated in Refs. [14–16], to obtain a more general derivation of hydrodynamic equations from the underlying quantum theory, using the decoherent histories approach. The aim in particular is to consider reasonably general classes of initial states and to demonstrate decoherence of the local densities, without appealing to environmentally induced decoherence, and to show that the probabilities for histories are peaked about equations of motion of the hydrodynamic type. In this paper the particular system we will apply the program to is a chain of linearly coupled oscillators.

The general sketch of the program, which we will work out in detail in this paper, is as follows. We start from the simple observation that exactly conserved quantities define an exactly decoherent set of histories, essentially because the projectors in the decoherence functional commute with the Hamiltonian [17]. It is therefore expected that the histories will remain approximately decoherent as we go from $k = 0$ to nonzero values of k in the local densities $n(\mathbf{k}), \mathbf{g}(\mathbf{k}), h(\mathbf{k})$. In Ref. [14] it was shown that a useful way to organize this idea is to decompose the initial state of the system into a superposition of states $|n, \mathbf{g}, h\rangle$, which are approximate eigenstates of the local densities. It is then very plausible (and verifiable in specific models) that such states remain approximate eigenstates of the local densities under time evolution, for sufficiently small k (since it is clearly exactly true in the limit $k \to 0$). Here, "sufficiently small" means that k^{-1} should be much greater than the correlation length of each local density eigenstate. The preservation in time of these states means that histories of them will be approximately decoherent.

Given decoherence we may then look at the probabilities for histories. Decoherence also indicates that each element of the superposition of local density eigenstates may be treated separately. We therefore consider the probabilities for histories of local densities with the local density eigenstate as the initial state. For sufficiently coarse-grained histories the probabilities for the local densities are strongly peaked at each time about the average value of the local densities, averaged in the local density eigenstate $|n, \mathbf{g}, h\rangle$. Since the local densities are sums of one-particle operators (to lowest order

in interactions), this is the same as averaging in the one-particle reduced density operator ρ_1 constructed by tracing $|n,\mathbf{g},h\rangle\langle n,\mathbf{g},h|$ over all but one particle states. The density operator ρ_1 is clearly not the same as the (one-particle version of the) local equilibrium state (10), although the two states are clearly very similar, since they are both very localized in the local densities. Hence, to complete the derivation of the hydrodynamic equations, it is necessary to show that ρ_1 tends to the local equilibrium state after a period of time. This is clearly extremely plausible on physical grounds and may be proved in explicit cases, as in this paper. (And indeed, this is much weaker than asserting that *any* initial state tends to the local equilibrium state.)

In brief, the whole story works in particular models contingent only on constructing local density eigenstates and showing that they have the desired properties: that they are preserved in form under time evolution for sufficiently small k, and that they are effectively equivalent to the local equilibrium distribution after a period of time. The point of this paper is to show this for the linear oscillator chain.

The detailed connection between conservation and decoherence is discussed in Sec. II, as is the construction of approximate eigenstates of the local densities. In Sec. III, we describe the dynamics of the linear oscillator chain. We consider two types of chain: the simple chain, where only neighboring particles are coupled, and the bound chain, where each particle is in addition bound to an origin by a harmonic potential. We consider both finite and infinite chains. The most important results are the correlation functions, which establishes the scale on which coarse graining is required for decoherence.

In Sec. IV, as a preparation for proving decoherence of the local densities of the chain of oscillators, we consider a simplified set of variables, namely, the total momentum contained in a subsection of the chain. We show that the eigenstates of this quantity remain approximate eigenstates under time evolution as long as the size of the chain subsection is much greater than the correlation length.

In Sec. V, we consider the local densities of the chain. We prove that approximate eigenstates remain approximate eigenstates, for k^{-1} much larger than the correlation length of the chain.

In Sec. VI, we consider the probabilities for histories. In the case of a finite simple chain, we show that the averages of number and momentum density obey a closed set of equations (although there is no evolution to local equilibrium in this case). For the infinite bound chain, we show that the density operator ρ_1 does indeed tend to the local equilibrium state and we derive the resultant hydrodynamic equations. We discuss our results in Sec. VII.

The idea that local densities should define a natural set of decoherent histories as a result of their approximate conservation was first put forward by Gell-Mann and Hartle [5]. This idea, and the related possibility of deriving hydrodynamic equations, has been developed by numerous authors [13–16,18–20]. This work is perhaps most closely related to that of Brun and Hartle [18], who analyze the linear oscillator chain using the decoherent histories approach. Their approach was rather different in that they considered coarse

grainings in which the center of mass coordinates of chain subsections were specified, rather than the local densities considered here, and they evaluated the decoherence functional explicitly, rather than examine the evolution of eigenstates of the variables of interest.

II. DECOHERENCE AND CONSERVATION

We begin by describing the connection between decoherence and conservation, It is well known that exactly conserved quantities are exactly decoherent [17]. The simple reason for this is that the projectors commute with the unitary evolution operator. The projectors P_{α_k} on one side of the decoherence functional (1) may therefore be brought up against the projectors $P_{\alpha_k'}$ on the other side, hence the decoherence functional is diagonal. (In the situation considered here, in which there are three conserved quantities involved, these quantities must in addition commute with each other, but this is clearly the case.)

We would like to extend this idea to approximate decoherence in the case of approximate conservation. It turns out that the above argument is better formulated in a different way for the purposes of generalization [14]. Suppose the initial state is pure and consider the decoherence functional,

$$D(\underline{\alpha},\underline{\alpha}')=\mathrm{Tr}(P_{\alpha_n}U_{n-1,n}\cdots P_{\alpha_2}U_{12}P_{\alpha_1}|\Psi\rangle$$
$$\times\langle\Psi|P_{\alpha_1'}U_{12}^\dagger P_{\alpha_2'}\cdots U_{n-1,n}^\dagger) \quad (11)$$

where U_{12} is the usual unitary evolution operator between times t_1 and t_2. Suppose the histories are projections onto some conserved quantity, Q. Now let the initial state be a superposition of eigenstates of Q,

$$|\Psi\rangle=\frac{1}{\sqrt{2}}(|Q_1\rangle+|Q_2\rangle) \quad (12)$$

where

$$Q|Q_a\rangle=Q_a|Q_a\rangle \quad (13)$$

and $a=1,2$. Since the P_α's are projections onto Q, we have $P_\alpha|Q_a\rangle=|Q_a\rangle$, if α is suitably chosen, otherwise we get $P_\alpha|Q_a\rangle=0$. Hence the only non-zero off-diagonal terms of the decoherence functional are of the form,

$$D(\underline{\alpha},\underline{\alpha}')=\frac{1}{2}\mathrm{Tr}(P_{\alpha_n}U_{n-1,n}\cdots P_{\alpha_2}U_{12}|Q_1\rangle$$
$$\times\langle Q_2|U_{12}^\dagger P_{\alpha_2'}\cdots U_{n-1,n}^\dagger). \quad (14)$$

But Q is conserved, hence $U_{12}|Q_a\rangle=|Q_a\rangle$ and

$$D(\underline{\alpha},\underline{\alpha}')=\frac{1}{2}\mathrm{Tr}(P_{\alpha_n}U\cdots P_{\alpha_2}|Q_1\rangle\langle Q_2|P_{\alpha_2'}\cdots U^\dagger). \quad (15)$$

Proceeding in this way to the end of the chain,

$$D(\underline{\alpha},\underline{\alpha}')=\langle Q_2|Q_1\rangle=0 \quad (16)$$

for all pairs of distinct histories $\underline{\alpha}, \underline{\alpha}'$. Hence decoherence comes about because neither the projections nor the unitary evolution disturb the states $|Q_a\rangle$, and hence the two orthogonal states $|Q_1\rangle$, $|Q_2\rangle$ are brought together at the final time and overlapped to give zero.

Let us now suppose that we have some operator Q such that under time evolution, its eigenstates are mapped into approximate eigenstates. That is, we initially have Eq. (13), but under evolution to time t,

$$Q(t)|Q_a\rangle \approx \langle Q(t)\rangle|Q_a\rangle \tag{17}$$

(where the average on the right-hand side is in the state $|Q_a\rangle$). More precisely, this can be expressed as

$$\frac{(\Delta Q(t))^2}{\langle Q(t)\rangle^2} \ll 1 \tag{18}$$

where

$$(\Delta Q(t))^2 = \langle Q^2(t)\rangle - \langle Q(t)\rangle^2. \tag{19}$$

Equation (18) means that the state remains strongly peaked in the variable Q under time evolution. The states are then approximate eigenstates of the projectors at each time as long as the widths of the projectors are chosen to be much greater than $[\Delta Q(t)]^2$. The same argument goes through although this time only approximately. Approximate decoherence is therefore assured for sufficiently coarse-grained histories of operators Q and superpositions of initial states each of which have the property that they remain strongly peaked in Q under time evolution [as characterized by Eq. (18)].

A simple example is the case of the coherent states of the harmonic oscillator. These states are preserved in form under time evolution, hence will always be approximate eigenstates of projections onto position, momentum, or phase space, provided that the widths of the projections are chosen to be much greater than the uncertainties in p and q in the coherent states. In this example, there is no obvious local conservation law. For this reason, it is perhaps more accurate to speak of approximate determinism, rather than approximate conservation. So very broadly speaking, approximate decoherence of histories will arise when there is an approximate determinism in the underlying quantum theory.

Returning to the local densities, we require a set of states $|n, \mathbf{g}, h\rangle$ which are eigenstates of all three local densities. Since the local densities do not commute with each other, except in the limit $k \to 0$, we can only find states which are approximate eigenstates. The number and momentum density are both operators of the form,

$$A = \sum_{n=1}^{N} A_n \tag{20}$$

as is the local energy density, if we ignore the interaction term. For such operators it follows that

$$(\Delta A)^2 = \sum_n (\Delta A_n)^2 + \sum_{n \neq m} \sigma(A_n, A_m) \tag{21}$$

and

$$\langle A \rangle^2 = \sum_{n,m} \langle A_n \rangle \langle A_m \rangle. \tag{22}$$

A state will be an approximate eigenstate of the operator A if

$$\frac{(\Delta A)^2}{\langle A \rangle^2} \ll 1. \tag{23}$$

The expression for $\langle A \rangle^2$ potentially involves N^2 terms, as does the expression for $(\Delta A)^2$, but the latter will involve only N terms if the correlation functions $\sigma(A_n, A_m)$ are very small or zero for $n \neq m$. So simple product states will be approximate eigenstates and will have $(\Delta A)^2/\langle A \rangle^2$ of order $1/N$. (See Refs. [14,15] for more detailed examples this argument.)

Under time evolution, the interactions cause correlations to develop. However, the states will remain approximate eigenstates as long as the correlations are sufficiently small that the second term in Eq. (21) is much smaller than order N^2. The interactions and the subsequent correlations are clearly necessary in order to get interesting dynamics and in particular the approach to local equilibrium. The interesting questions is therefore whether there is a regime where the effects of interactions are small enough to permit decoherence but large enough to produce interesting dynamics. The fact that the variables we are interested in are locally conserved indicates that there is such a regime. The important point is that the local densities become arbitrarily close to exactly conserved quantities as $k \to 0$. This means that, at any time, $(\Delta A)^2/\langle A \rangle^2$ becomes arbitrarily close to its initial value (which is of order $1/N$) for sufficiently small k.

In the examples we look at it in the following sections, we will see that an uncorrelated initial state develops correlations with a typical lengthscale (or extending to a certain number of particles down the chain). These correlations typically then decay with time. What we will find is that the second term in Eq. (21) will remain small as long as k^{-1} is much greater than the correlation length. Hence the key physical aspect is the locality of the interactions, meaning that only limited local correlations develop, together with the coarse-graining scale k^{-1} which may be chosen to be sufficiently large that the correlation scale is not seen. Differently put, as k increases from zero, departing from exact decoherence, it introduces a lengthscale k^{-1}. Since the decoherence functional is a dimensionless quantity, clearly nothing significant can happen until k^{-1} becomes comparable with another lengthscale in the system. The natural scale is the correlation length in the local density eigenstates.

The scheme described here would be executed most transparently if we used states which become exact eigenstates of the conserved quantities in the limit $k \to 0$, thereby always maintaining the closest connection with exact conservation. In the next section we will in fact use Gaussians as the approximate eigenstates, because they are the easiest states to work with. These will not be exact eigenstates of the exactly conserved quantities in the $k \to 0$ limit, although this not in fact matter very much, for reasons outlined above. (Fur-

themore, the decoherence functional is always exactly diagonal for any initial state in the $k \to 0$ limit, for the reasons stated at the beginning of this section, but we do not need to exploit this here.)

III. CHAINS OF OSCILLATORS

In this and the following sections, we show how the general program outlined above may be worked out in detail in linear oscillator models. These have the advantage that they can be solved exactly. In particular, the time development of the correlation functions and eigenstates of the local densities can be computed reasonably explicitly.

A. The models and their classical solutions

We consider a chain of point particles which are coupled to each other by a nearest-neighbor linear interaction. We also allow the possibility that each particle is harmonically bound to one of a series evenly distributed points, separated by distance b, say. The Hamiltonian is

$$H = \sum_{n=1}^{N} \left[\frac{p_n^2}{2m} + \frac{\nu^2}{2}(q_n - q_{n-1})^2 + \frac{K}{2}(q_n - b_n)^2 \right] \quad (24)$$

where $b_n = nb$. We will consider the two cases $K = 0$ (the simple chain) and $K \neq 0$ (the harmonically bound chain). In the bound chain case, it is also useful to consider the case $b_n = 0$, which corresponds to the situation in which the whole chain moves in a harmonic potential. (In fact, for the classical solutions, b_n is readily absorbed into q_n, but this makes a difference to the local densities considered below.) We initially consider a finite number N of particles but we also consider the case of N infinite.

The equations of motion are

$$m\ddot{q}_n + K(q_n - b_n) = \nu^2(q_{n+1} - 2q_n + q_{n-1}) \quad (25)$$

where we take $q_{N+1} = q_1$. This system has been discussed and solved in many places. A particularly useful reference for the case of an infinite chain is the treatment by Huerta and Robertson [21,22]. (See also Refs. [23–26].) The solution may be found by introducing the normal modes, Q_α,

$$q_n = b_n + \sum_{\alpha=1}^{N} \frac{e^{2\pi i \alpha n/N}}{N^{1/2}} Q_\alpha \quad (26)$$

which obey

$$\ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0 \quad (27)$$

where

$$\omega_\alpha = \left[\frac{K}{m} + \frac{4\nu^2}{m}\sin^2\left(\frac{\pi\alpha}{N}\right) \right]^{1/2}. \quad (28)$$

The solution may be written

$$q_n(t) = b_n + \sum_{r=1}^{N} \left[f_{r-n}(t)q_r(0) + \frac{g_{r-n}(t)}{m\Omega} p_r(0) \right] \quad (29)$$

where

$$f_n(t) = \frac{1}{N} \sum_{\alpha=1}^{N} e^{2\pi i \alpha n/N} \cos(\omega_\alpha t) \quad (30)$$

$$\frac{g_n(t)}{\Omega} = \frac{1}{N} \sum_{\alpha=1}^{N} e^{2\pi i \alpha n/N} \frac{\sin(\omega_\alpha t)}{\omega_\alpha}. \quad (31)$$

Here, $\Omega^2 = (K + 2\nu^2)/m$. The solution for $p_n(t)$ is given by

$$p_n(t) = m\dot{q}_n(t). \quad (32)$$

In the limit of an infinite number of particles the solution is

$$q_n(t) = b_n + \sum_{r=-\infty}^{\infty} \left[f_{r-n}(t)q_r(0) + \frac{g_{r-n}(t)}{m\Omega} p_r(0) \right]. \quad (33)$$

For the simple chain, $K = 0$, the solution is then given in terms of Bessel functions [21], and we have

$$f_r(t) = J_{2r}(2\omega t) \quad (34)$$

where $\omega^2 = \nu^2/m$, and

$$g_r(t) = \Omega \int_0^t dt' J_{2r}(2\omega t'). \quad (35)$$

The appearance of the time integral in the expression for $g_r(t)$ is in fact related to the motion of the center of mass of the whole chain, which is not of interest for our considerations. (We imagine the whole system is contained somehow so that there is no wave packet spreading.) It also somewhat obscures the discussion of correlations, which is the main thing we are interested in. The relevant behavior in $g_r(t)$ is best exhibited in terms of the difference variables, $g_{r+1} - g_r$. Using a simple recurrence relation for the Bessel functions, these are given by

$$g_{r+1}(t) - g_r(t) = -2\Omega J_{2r+1}(2\omega t). \quad (36)$$

We will discuss this in more detail below.

For the bound chain, $K \neq 0$, it is most useful to work in the regime in which the interaction between particles is much weaker than the binding to their origins, so $\nu^2 \ll K$. In this case, the solution then is [21],

$$f_r(t) \approx J_r(\gamma\Omega t)\cos(\Omega t - \pi r/2) \quad (37)$$

and

$$g_r(t) \approx J_r(\gamma\Omega t)\sin(\Omega t - \pi r/2) \quad (38)$$

where $\gamma = (\omega/\Omega)^2$, so $\gamma \ll 1$.

The general behavior of the solutions in both cases is easily seen. The functions $f_{r-n}(t)$ and $g_{r-n}(t)$ loosely represent the manner in which an initial disturbance of particle r affects particle n after a time t, and is given in both the bound

and unbound case by the properties of Bessel functions. Recall the following forms of the Bessel functions [27]. For small arguments we have

$$J_n(x) = \left(\frac{x}{2}\right)^n \left(\frac{1}{\Gamma(n+1)} - \frac{(x/2)^2}{2!\Gamma(n+3)} + \cdots\right). \quad (39)$$

[This is for $n > 0$. For $n < 0$ we use $J_{-n}(x) = (-1)^n J_n(x)$.] For large arguments we have the asymptotic form

$$J_n(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \cos(x - \pi n/2 - \pi/4). \quad (40)$$

Hence the Bessel functions start out at zero [except for $n = 0$, where $J_0(0) = 1$], grow exponentially, and then go into a slowly decaying oscillation. For large n and fixed x we have

$$J_n(x) \sim (2\pi n)^{-1/2} \left(\frac{ex}{2n}\right)^n. \quad (41)$$

A point not immediately obvious from these standard asymptotic forms, and which will turn out to be important, is that the different Bessel functions each go into their oscillatory regions at different values of x. In particular, one can estimate from the plots of the Bessel functions that $J_n(x)$ goes into its oscillatory regime when x is of order n, by which stage $J_n(x)$ is therefore of order $n^{-1/2}$. In terms of the behavior of the chain, this means that distant pairs of particles never come to influence each other very much, even after long periods of time: at short times, particles separated by n have exponentially suppressed correlation, similar to $x^n/n!$ and at long times, their correlations are also suppressed, similar to $n^{-1/2}$. This particular aspect turns out to be crucial for our purposes.

Another important observation is that in the oscillatory regime, the Bessel function $J_n(x)$ has only a very limited dependence on n, namely it has the form (40) for some n, plus the three possible phase shifts of $\pi/2$. The significance of this for the chain is that when the functions $f_{r-n}(t)$ and $g_{r-n}(t)$ have entered the oscillatory regime, the conditions at particles r and n and everywhere in between are approximately the same. This feature is clearly relevant to the approach to local equilibrium.

B. Correlation functions

As described in Sec. II, we are interested in the time development of eigenstates of the local densities, and this boils down to the behavior of the various correlation functions of the system. We define the correlation function

$$\sigma(A,B) = \frac{1}{2}\langle AB + BA\rangle - \langle A\rangle\langle B\rangle. \quad (42)$$

Because the system is linear the classical solutions described above may be used to discuss the solutions in the Heisenberg picture in the quantum case. In fact, the only quantum calculations in this paper have essentially been done already in

Sec. II, and the remaining analysis is essentially the same as for a classical stochastic system.

For simplicity, we will concentrate on Gaussian initial states (which will of course remain Gaussian, because they system is linear), although these will be sufficient for our purposes. The variances of these Gaussians are restricted by the requirement that the state is pure. (They are also of course restricted by the uncertainty principle.) We will consider two different types of Gaussian initial states which can be approximate eigenstates of the local densities. The first type we consider are product states, so have no initial correlations between different particles. The second type are the coherent states of the normal modes, and are naturally suggested by the normal mode decomposition (26). We will see in the next section that these are eigenstates of the local densities as long as the correlation functions remain sufficiently small.

1. Normal mode coherent states

Taking the second type first, we therefore consider the set of Gaussian states which have

$$\sigma(Q_\alpha, Q_\beta^*) = \frac{\hbar}{2m\omega_\alpha}\delta_{\alpha\beta} \quad (43)$$

$$\sigma(K_\alpha, K_\beta^*) = \frac{1}{2}\hbar m\omega_\alpha\delta_{\alpha\beta} \quad (44)$$

$$\sigma(Q_\alpha, K_\beta) = 0 \quad (45)$$

where K_α is the momentum conjugate to Q_α and note that $Q_\alpha^* = Q_{-\alpha}$. Because these are the coherent states of the harmonic oscillator, these correlation functions all remain of this form under time evolution, and the only time development of the states is in terms of their centers, $Q_\alpha(t)$, $K_\alpha(t)$, which follow the classical equations of motion. For the case of the simple chain, $K = 0$, we have $\omega_\alpha = 0$ when $\alpha = N$. This corresponds to the center of mass of the whole chain, and may be quite simply omitted.

In terms of the original coordinates q_n and p_n, we have the correlation functions

$$\sigma(q_n, q_m) = \frac{1}{N}\sum_{\alpha=1}^{N}\frac{\hbar}{2m\omega_\alpha}e^{2\pi i\alpha(n-m)/N} \quad (46)$$

$$\sigma(p_n, p_m) = \frac{1}{N}\sum_{\alpha=1}^{N}\frac{1}{2}\hbar m\omega_\alpha e^{2\pi i\alpha(n-m)/N} \quad (47)$$

$$\sigma(q_n, p_m) = 0. \quad (48)$$

These correlation functions are constant in time, and this feature makes this case a useful one to study. Also notice that $(\Delta q_n)^2$ and $(\Delta p_n)^2$ are independent of n. The correlation functions will typically decay very rapidly with increasing $|n-m|$, since they are sums of rapidly oscillating terms. This is especially true in the case $K \neq 0$ with $K \gg \nu^2$, since then

ω_α, Eq. (28), is a constant to leading order. (We will see a similar effect in more detail in the next subsection.) In the case $K=0$, we have

$$\omega_\alpha = \frac{2\nu}{m^{1/2}} \sin\left(\frac{\pi\alpha}{N}\right). \tag{49}$$

The correlation function $\sigma(p_n, p_m)$ clearly still decays for large $|n-m|$, but this is less obvious for $\sigma(q_n, q_m)$, Eq. (46), because the denominator becomes very small close to $\alpha = N$. However, if we eliminate by hand a small cluster of modes close to $\alpha = N$, we get satisfactory decay properties for this correlation function, and we will assume that this has been done.

2. Uncorrelated initial states for the infinite chain

We now consider uncorrelated initial states for the infinite chain, so the correlation functions $\sigma(q_n, q_m)$, $\sigma(q_n, p_m)$ and $\sigma(p_n, p_m)$ all vanish at the initial time for $n \neq m$. We are then interested in the behavior of these three types of correlation functions at later times.

From the solution (33), with the assumption of no initial correlation between the particles, we have

$$\sigma[q_n(t), q_m(t)] = \sum_r \left(f_{r-n}(t) f_{r-m}(t)(\Delta q_r)^2 \right.$$

$$+ \frac{1}{m^2\Omega^2} g_{r-n}(t) g_{r-m}(t)(\Delta p_r)^2$$

$$+ \frac{1}{m\Omega} [f_{r-n}(t) g_{r-m}(t)$$

$$\left. + f_{r-m}(t) g_{r-n}(t)] \sigma(q_r, p_r) \right) \tag{50}$$

and similarly for $\sigma[q_n(t), p_m(t)]$ and $\sigma[p_n(t), p_m(t)]$. These expressions simplify if we make the further assumption that the initial values of $(\Delta q_r)^2$, $(\Delta p_r)^2$ and $\sigma(q_r, p_r)$ are independent of r (we will show below how to go beyond this assumption). In this case we obtain

$$\sigma[q_n(t), q_m(t)] = a_{nm}(t)(\Delta q)^2 + 2e_{nm}(t)\sigma(q,p) + d_{nm}(t)$$

$$\times (\Delta p)^2 \tag{51}$$

$$\sigma[q_n(t), p_m(t)] = b_{nm}(t)(\Delta q)^2 + [a_{nm}(t) + k_{nm}(t)]\sigma(q,p)$$

$$+ e_{nm}(t)(\Delta p)^2 \tag{52}$$

$$\sigma[p_n(t), p_m(t)] = c_{nm}(t)(\Delta q)^2 + 2b_{nm}(t)\sigma(q,p) + a_{nm}(t)$$

$$\times (\Delta p)^2 \tag{53}$$

where

$$a_{nm}(t) = \sum_r f_{r-n}(t) f_{r-m}(t) \tag{54}$$

$$b_{nm}(t) = m \sum_r f_{r-n}(t) \dot{f}_{r-m}(t) \tag{55}$$

$$c_{nm}(t) = m^2 \sum_r \dot{f}_{r-n}(t) \dot{f}_{r-m}(t) \tag{56}$$

$$d_{nm}(t) = \frac{1}{m^2\Omega^2} \sum_r g_{r-n}(t) g_{r-m}(t) \tag{57}$$

$$e_{nm}(t) = \frac{1}{m\Omega^2} \sum_r g_{r-n}(t) \dot{g}_{r-m}(t) \tag{58}$$

$$k_{nm}(t) = \frac{1}{\Omega} \sum_r \dot{f}_{r-n}(t) g_{r-n}(t). \tag{59}$$

Since the coefficients $f_n(t)$ and $g_n(t)$ are all given by Bessel functions, these expressions can be evaluated using the following Bessel function addition theorem:

$$J_n(2x) = \sum_{k=-\infty}^{\infty} J_{n-k}(x) J_k(x). \tag{60}$$

For the bound chain, $K \neq 0$, the coefficients are [21],

$$2a_{nm}(t) = \delta_{nm} + J_{n-m}(2\gamma\Omega t)\cos\left[2\Omega t - \frac{1}{2}(n-m)\pi\right] \tag{61}$$

$$2b_{nm}(t) = -m\Omega J_{n-m}(2\gamma\Omega t)\sin\left[2\Omega t - \frac{1}{2}(n-m)\pi\right] \tag{62}$$

$$2c_{nm}(t) = (m\Omega)^2 \delta_{nm} - (m\Omega)^2 J_{n-m}(2\gamma\Omega t)$$

$$\times \cos\left[2\Omega t - \frac{1}{2}(n-m)\pi\right] \tag{63}$$

$$d_{nm}(t) = (m\Omega)^{-4} c_{nm}(t) \tag{64}$$

$$e_{nm}(t) = -(m\Omega)^{-2} b_{nm}(t) \tag{65}$$

$$k_{nm}(t) = -(m\Omega)^{-2} c_{nm}(t). \tag{66}$$

All of these coefficients, and hence all of the correlation functions, decay exponentially for large $|n-m|$. Furthermore, in the limit $t \to \infty$ we have

$$\sigma[q_n(t), q_m(t)] \to \frac{1}{2}\delta_{nm}\left((\Delta q)^2 + \frac{1}{m^2\Omega^2}(\Delta p)^2\right) \tag{67}$$

$$\sigma[q_n(t), p_m(t)] \to 0, \tag{68}$$

$$\sigma[p_n(t), p_m(t)] \to \frac{1}{2}\delta_{nm}[m^2\Omega^2(\Delta q)^2 + (\Delta p)^2]. \tag{69}$$

We also find that $\langle q_n(t)\rangle \to b_n$ and $\langle p_n(t)\rangle \to 0$ as $t\to\infty$. We thus obtain an equilibrium distribution, since the corresponding phase space distribution function for the whole chain is

$$w=\prod_n \exp\left(-\frac{1}{kT}\left[\frac{p_n^2}{2m}+\frac{1}{2}m\Omega^2(q_n-b_n)^2\right]\right) \quad (70)$$

where we identify the temperature as

$$kT=\frac{1}{2m}[m^2\Omega^2(\Delta q)^2+(\Delta p)^2]. \quad (71)$$

For the simple chain, $K=0$, we have [21],

$$2a_{nm}(t)=\delta_{nm}+J_{2n-2m}(4\omega t) \quad (72)$$

$$2b_{nm}(t)=m\omega[J_{2n-2m-1}(4\omega t)-J_{2n-2m+1}(4\omega t)] \quad (73)$$

$$2c_{nm}(t)=(m\omega)^2[J_{2n-2m+2}(4\omega t)+J_{2n-2m-2}(4\omega t)$$

$$-2J_{2n-2m}(4\omega t) \quad (74)$$

$$-\delta_{n,m-1}-\delta_{n,m+1}+2\delta_{nm}] \quad (75)$$

$$d_{nm}(t)=\frac{t}{2\omega m^2}\left[\int_0^{4\omega t}J_0(y)dy-J_1(4\omega t)\right]$$

$$-\frac{1}{(2\omega m)^2}\sum_{j=1}^{|n-m|}\int_0^{4\omega t}J_{2j-1}(y)dy \quad (76)$$

$$e_{nm}(t)=\frac{1}{4\omega m}\int_0^{4\omega t}J_{2n-2m}(y)dy. \quad (77)$$

[The explicit form of $k_{nm}(t)$ is not required.] The first three of these coefficients, as in the bound case, are exponentially suppressed for large $|n-m|$. This means that the behavior of the correlation $\sigma[p_n(t),p_m(t)]$, which depends only on these three coefficients, has the expected behavior, but the other correlation functions do not have this property. In particular, from the behavior of $\sigma[q_n(t),q_m(t)]$, the whole chain appears to become highly correlated. In the long-time limit, we find

$$\sigma[q_n(t),q_m(t)]\to\frac{1}{2}\delta_{nm}(\Delta q)^2+\frac{t}{2\omega m^2}(\Delta p)^2 \quad (78)$$

$$\sigma[q_n(t),p_m(t)]\to\frac{1}{4\omega m}(\Delta p)^2 \quad (79)$$

$$\sigma[p_n(t),p_m(t)]\to\frac{1}{2}(m\omega)^2[2\delta_{nm}-\delta_{n,m-1}-\delta_{n,m+1}]$$

$$\times(\Delta q)^2+\frac{1}{2}\delta_{nm}(\Delta p)^2 \quad (80)$$

[where for simplicity we have taken $\sigma(q_n,p_n)=0$ initially]. This is not an equilibrium distribution, and the behavior of $[\Delta q_n(t)]^2$ at late times indicates diffusive behavior.

This growth of correlations and variances is essentially unphysical, since any realistic system is contained in some way, so the spreading cannot proceed beyond the size of the container (and indeed there is no such effect in the $K\neq 0$ case). It is, however, difficult to factor out this unphysical effect in a convenient way. The long-time limits of Ref. [21], which we have followed closely, are dominated by this spreading effect in the $K=0$ case. In an attempt to understand this, the authors of Ref. [21] considered a different set of models in Ref. [22], in which the chain was fixed at one end. This avoided the diffusive growth encountered above, but still led to significant correlations between all particles on the chain in the long-time limit. This in turn spoils the desired behavior of the local densities, discussed below. The upshot of this is that it is not possible to prove decoherence of the local densities in the case of the infinite chain with $K=0$.

There are two simple ways in which the above results are easily generalized. First of all, note that although we are focusing on Gaussian initial states, expressions for the correlation functions such as Eq. (50) are in fact valid for *any* initial state, because of the linearity of the dynamics.

Secondly, in deriving Eqs. (51)–(53), we assumed that the initial variances are independent of r. This assumption was necessary in order to be able to apply the Bessel function addition theorem (60), and thereby obtain explicit expressions for the coefficients, a_{nm}, b_{nm} etc. This is too restrictive, since it means that certain hydrodynamic variables [such as temperature, which depends on $(\Delta p_r)^2$], are obliged to be constant along the chain. One can see, however, that these results easily extend to the case in which the initial variances $(\Delta q_r)^2$, $(\Delta p_r)^2$ and $\sigma(q_r,p_r)$ have a slow dependence on r along the chain. The point is that because of the decay of the functions f_{n-r}, g_{n-r} for large $|n-r|$, the sum over r in Eq. (50) is effectively restricted to a finite range, namely, over the (actually quite small) range for which significant correlations exist. As long as the initial variances vary significantly with r only on a range larger than the correlation range, then the calculation of correlation functions is effectively equivalent to the case in which the variances are completely independent of r. This means that in place of Eq. (51), for example, we get the more general result

$$\sigma[q_n(t),q_m(t)]\approx a_{nm}(t)(\Delta q_r)^2+2e_{nm}(t)\sigma(q_r,p_r)+d_{nm}(t)$$

$$\times(\Delta p_r)^2 \quad (81)$$

where r on the right-hand side is taken to be mid-way between n and m, and for $n=m$ we get

$$[\Delta q_n(t)]^2\approx a_{nn}(t)(\Delta q_n)^2+2e_{nn}(t)\sigma(q_n,p_n)+d_{nn}(t)$$

$$\times(\Delta p_n)^2. \quad (82)$$

This simple observation is important for obtaining interesting hydrodynamic equations because it allows for the possibility

of the system tending towards *local* equilibrium, rather than equilibrium of the whole chain.

IV. COARSE GRAINING BY CHAIN SUBSECTIONS

Although we are ultimately interested in the local densities for the chain variables, we will first consider some simpler variables whose analysis is highly instructive. Namely, we take the variables of interest to be the total momentum in a subsection of the chain containing M particles, so we define

$$P_M = \sum_{n=1}^{M} p_n \qquad (83)$$

where $M \ll N$. This is not quite the same as a locally conserved quantity, but it is very similar, since, for the simple chain, the total momentum is conserved. Therefore P_M is an exactly conserved quantity when $M = N$, and otherwise we might expect it to be approximately conserved.

As outlined in Sec. II, to show that these variables are approximately decoherent, we need to show that initial states satisfying the condition

$$\frac{(\Delta P_M)^2}{\langle P_M \rangle^2} \ll 1 \qquad (84)$$

will continue to satisfy it under time evolution, hence the initial state remains an approximate eigenstate. We have

$$(\Delta P_M)^2 = \sum_{n=1}^{M} \sum_{m=1}^{M} \sigma(p_n, p_m) \qquad (85)$$

and

$$\langle P_M \rangle^2 = \sum_{n,m}^{M} \langle p_n \rangle \langle p_m \rangle. \qquad (86)$$

We take the case in which the initial correlations are zero and the initial variances are the same all along the chain, and we also take the initial value of $\sigma(q,p)$ to be zero. We thus obtain

$$[\Delta P_M(t)]^2 = C_M (\Delta q)^2 + A_M (\Delta p)^2 \qquad (87)$$

where

$$A_M(t) = \sum_{n=1}^{M} \sum_{m=1}^{M} a_{nm}(t) \qquad (88)$$

$$C_M(t) = \sum_{n=1}^{M} \sum_{m=1}^{M} c_{nm}(t) \qquad (89)$$

and the coefficients $a_{nm}(t)$ and $c_{nm}(t)$ are given by Eqs. (72),(74). The two terms on the right are very similar, so for simplicity will will concentrate on the second one. (Note also that these terms do not suffer from the spreading problem discussed in the previous section. This is an advantage of working with momenta, rather than positions.) From the expression (72), we have

$$A_M(t) = \frac{1}{2} \sum_{n=1}^{M} \sum_{m=1}^{M} J_{2n-2m}(4\omega t) \qquad (90)$$

(where we have assumed that N is sufficiently large that it is effectively equivalent to the $N \to \infty$ case). Our aim is now to show that $A_M \ll M^2$ for all times, for then the condition (84) will be satisfied.

The expression for A_M cannot be evaluated exactly, but its properties may be seen from the asymptotic forms of the Bessel functions (39),(40). For small times, when the Bessel functions are all in the exponential regime, J_{2n-2m} is exponentially suppressed for large $|n-m|$. For larger times, the Bessel functions start to go into their oscillatory form, where they are already small. Furthermore, because in the oscillatory regime they depend on $n-m$ only through a simple phase, most of the terms in the sum over n and m cancel out. Proceeding along these lines one can see that A_M will not come anywhere close to M^2 except for small values of M. These features are easily confirmed by plotting $A_M(t)$ for different values of M. For example, with $M=5$, $A_M(t)$ quickly decays to a value of about 0.1, clearly much less than $M^2 = 25$, thereafter going into a slowly decaying oscillation. As we shall see, the variances of the local densities differ from the momentum of a chain subsection in that they are more complicated functions of the correlation function, for Gaussian states, but the physical understanding of their behavior is essentially the same, which is why this simple example is instructive.

V. DECOHERENCE OF LOCAL DENSITIES

We now come to the main point of this paper, which is to examiner the behavior of the local densities for the oscillator chain. They are

$$n(x) = \sum_{n=1}^{N} \delta(q_n - x) \qquad (91)$$

$$g(x) = \sum_{n=1}^{N} p_n \delta(q_n - x) \qquad (92)$$

$$h(x) = \sum_{n=1}^{N} \left[\frac{p_n^2}{2m} + \frac{\nu^2}{2}(q_n - q_{n-1})^2 \right.$$
$$\left. + \frac{1}{2} K(q_n - b_n)^2 \right] \delta(q_n - x). \qquad (93)$$

Again it will often be very useful to work with the Fourier transformed local densities,

$$n(k) = \sum_{n=1}^{N} e^{ikq_n} \qquad (94)$$

$$g(k) = \sum_{n=1}^{N} p_n e^{ikq_n} \qquad (95)$$

$$h(k) = \sum_{n=1}^{N} \left[\frac{p_n^2}{2m} + \frac{\nu^2}{2}(q_n - q_{n-1})^2 + \frac{1}{2}K(q_n - b_n)^2 \right] e^{ikq_n}. \quad (96)$$

The local number and local energy density are locally conserved. The local momentum density is locally conserved only for the case of the simple chain, $K=0$.

We consider states which are approximate eigenstates of the local densities. We will use Gaussian initial states, and we expect that these will be approximate eigenstates of the local densities if we choose the correlation functions $\sigma(q_n,q_m)$, $\sigma(q_n,p_m)$ and $\sigma(p_n,p_m)$ to be zero, or at least sufficiently small, for $n \neq m$.

For the Gaussian states we consider here, the computation of variances of the local densities is facilitated by the identity,

$$\left\langle \exp\left(i \sum_n \left[\alpha_n(q_n - \langle q_n \rangle) + \beta_n(p_n - \langle p_n \rangle) \right] \right) \right\rangle \quad (97)$$

$$= \exp\left(- \sum_{nj} \left[\frac{1}{2} \alpha_n \alpha_j \sigma(q_n,q_j) + \alpha_n \beta_j \sigma(q_n,p_j) \right. \right.$$

$$\left. \left. + \frac{1}{2} \beta_n \beta_j \sigma(p_n,p_j) \right] \right). \quad (98)$$

All of the variances of interest will therefore be functions of the three basic types of correlation functions $\sigma(q_n,q_m)$, $\sigma(q_n,p_m)$ and $\sigma(p_n,p_m)$ discussed in Sec. III, and the physical discussion will in fact be very closely related to that of the simple case discussed in Sec. IV. Actually, the formula (98) also holds to quadratic order in α_n and β_n for *any* state, so the results derived below on the basis of Gaussian states will be valid for arbitrary states for small k.

We consider first the Fourier transformed number density $n(k)$. In a general Gaussian state, we have

$$\langle n(k) \rangle = \sum_{j=1}^{N} \langle e^{ikq_j} \rangle = \sum_{j=1}^{N} \exp\left(ik\langle q_j \rangle - \frac{1}{2}k^2(\Delta q_j)^2 \right) \quad (99)$$

and

$$[\Delta n(k)]^2 = \langle n^\dagger(k)n(k) \rangle - |\langle n(k) \rangle|^2$$

$$= \sum_{j=1}^{N} \sum_{n=1}^{N} \langle e^{ikq_j} \rangle \langle e^{-ikq_n} \rangle (e^{k^2\sigma(q_j,q_n)} - 1). \quad (100)$$

The latter is to be compared with

$$|\langle n(k) \rangle|^2 = \sum_{j=1}^{N} \sum_{n=1}^{N} \langle e^{ikq_j} \rangle \langle e^{-ikq_n} \rangle. \quad (101)$$

With an initially uncorrelated state we have $\sigma(q_j,q_n)=0$ for $j \neq n$ and we see that

$$[\Delta n(k)]^2 = \sum_j |\langle e^{ikq_j} \rangle|^2 (e^{k^2(\Delta q_j)^2} - 1). \quad (102)$$

From this we expect that

$$(\Delta n(k))^2 \ll |\langle n(k) \rangle|^2 \quad (103)$$

as long as k^{-1} does not probe on scales that are too short (compared to Δq_j), so the state is an approximate eigenstate.

When there are correlations, as will arise over time, we expect that the state will still be an approximate eigenstate on lengthscales k^{-1} which are much greater than the lengthscale of correlation. As k increases from zero we have

$$\frac{(\Delta n(k))^2}{|\langle n(k) \rangle|^2} \sim \frac{k^2(\Delta X)^2}{N^2} \quad (104)$$

where $X = \Sigma_j q_j$ (the center of mass coordinate of the whole chain). This will be very small as long as k^{-1} is much larger than the typical lengthscale of a single particle. $[\Delta n(k)]^2$ starts to grow very rapidly with k, and Eq. (103) is no longer valid, when k^{-1} becomes less than the correlation length indicated by $\sigma(q_j,q_n)$. Hence the state is strongly peaked about the mean as long as the coarse graining lengthscale k^{-1} remains much greater than the correlation length of the local density eigenstates.

Consider now the local momentum density. We have

$$\langle g(k) \rangle = \sum_j [\langle p_j \rangle + ik\sigma(q_j,p_j)] \exp\left(ik\langle q_j \rangle - \frac{1}{2}k^2(\Delta q_j)^2 \right) \quad (105)$$

and at some length, we find

$$[\Delta g(k)]^2 = \sum_{jn} \langle e^{ikq_j} \rangle \langle e^{-ikq_n} \rangle (A_{jn} + B_{jn} + C_{jn}) \quad (106)$$

where

$$A_{jn} = \sigma(p_j,p_n) e^{k^2\sigma(q_j,q_n)} \quad (107)$$

$$B_{jn} = [\langle p_n \rangle - ik\sigma(q_n - q_j,p_n)][\langle p_j \rangle + ik\sigma(q_j - q_n,p_j)]$$

$$\times (e^{k^2\sigma(q_j,q_n)} - 1) \quad (108)$$

$$C_{jn} = -ik[\langle p_n \rangle - ik\sigma(q_n,p_n)] - ik[\langle p_j \rangle + ik\sigma(q_j,p_j)]$$

$$+ k^2\sigma(q_j,p_n)\sigma(q_n,p_j). \quad (109)$$

We require $[\Delta g(k)]^2$ to be small in comparison to

$$|\langle g(k) \rangle|^2 = \sum_{jn} \langle e^{ikq_j} \rangle \langle e^{-ikq_n} \rangle [\langle p_j \rangle \langle p_n \rangle + ik\langle p_n \rangle \sigma(q_j,p_j)$$

$$- ik\langle p_j \rangle \sigma(q_n,p_n) + k^2\sigma(q_j,p_j)\sigma(q_n,p_n)]. \quad (110)$$

Despite the complexity of these terms the interpretation is reasonably simple. As $k \to 0$,

$$\frac{[\Delta g(k)]^2}{|\langle g(k)\rangle|^2} \rightarrow \frac{(\Delta P)^2}{\langle P\rangle^2} \qquad (111)$$

where $P = \Sigma_j p_j$ is the total momentum, and it is easy to confirm that this is small (typically order $1/N$) for the states we are using. As k increases from zero, $[\Delta g(k)]^2$ will grow, and will potentially contain of order N^2 terms, the same as $|\langle g(k)\rangle|^2$. By inspecting the three terms Eqs. (107)–(110), however, one can see that each of them are prevented from generating order N^2 terms as long as, respectively, the correlation functions $\sigma(p_j, p_n)$, $\sigma(q_j, q_n)$ and $\sigma(q_j, p_n)$ are suppressed for $j \neq n$. As in the case of the number density, $(\Delta g(k))^2/|\langle g(k)\rangle|^2$ will start to grow appreciably as k^{-1} approaches the lengthscale indicated by $\sigma(q_j, q_n)$.

Consider now the energy density. The computation of $[\Delta h(k)]^2$ is rather complicated, but since we are considering Gaussian states, the final conditions on the correlation functions will be very similar to those on the variance of the momentum density considered above, so we will not carry out the computation explicitly. Instead, we consider a simpler special case. Take the case where each oscillator is fixed to an origin at b_j, and suppose that the binding to it is so strong that each particle is well localized around b_j. Then the integral over a volume of the energy density $h(x)$ is then approximately equivalent to taking a coarse graining consisting of the energy contained in a chain subsection, similar to Sec. IV. We therefore consider the variable

$$h_M = \sum_{j=1}^{M} h_j \qquad (112)$$

where

$$h_j = \frac{p_j^2}{2m} + \frac{1}{2}Kq_j^2 \qquad (113)$$

(neglecting the interaction term). Following the general discussion of Sec. II, we will have $(\Delta h_M)^2 \ll \langle h_M \rangle^2$ provided that

$$\sum_{jn} \sigma(h_j, h_n) \ll \langle h_M \rangle^2 \qquad (114)$$

so the left-hand side must be much smaller than order M^2. The correlation function $\sigma(h_j, h_n)$ is constructed from terms like $\sigma(q_j^2, q_n^2)$ and similar functions, and we have

$$\sigma(q_j^2, q_n^2) = 2\sigma(q_j, q_n)^2 \qquad (115)$$

(recalling that $\langle q_j \rangle \approx b_j$ since the particles are tightly bound). The discussion is then very similar to the case of Sec. IV, with essentially the same result, which is that the state is an approximate eigenstate as long as $M \gg 1$.

We have now shown that Gaussian states will be approximate eigenstates of the local densities for k sufficiently small compared to the correlation length determine by $\sigma(q_j, q_n)$,

and provided also that the other correlation functions $\sigma(q_j, p_n)$ and $\sigma(p_j, p_n)$ are small for $j \neq n$. (These results also hold for non-Gaussian states for small k.) The results of Sec. II show that all the correlation functions (excluding the infinite chain in the $K = 0$ case) have the desired properties for all time. This proves the desired result that initial eigenstates of the local densities remain approximate eigenstates under time evolution, for sufficiently small k.

VI. HYDRODYNAMIC EQUATIONS FOR THE LOCAL DENSITIES

We have shown in a variety of circumstances that eigenstates of local densities are approximately preserved in form under time evolution, on sufficiently coarse-grained scales, and therefore superpositions of them define decoherent sets of histories. We may now therefore look at the diagonal elements of the decoherence functional, representing probabilities for histories of these variables. These probabilities are peaked about the average values of the local densities, averaged in the approximate eigenstates we have been considering. (This is reasonably obvious, but a more detailed proof appears in the appendix of Ref. [15].) We will now show that these average values obey hydrodynamic equations.

A. Local conservation equations

The local densities satisfy the local conservation laws,

$$\dot{n}(x) = -\frac{1}{m}\frac{\partial g}{\partial x} \qquad (116)$$

$$\dot{g}(x) = -\frac{\partial \tau}{\partial x} - Kxn(x) + K\sum_j b_j \delta(q_j - x) \qquad (117)$$

$$\dot{h}(x) = -\frac{\partial j}{\partial x}. \qquad (118)$$

They are actually more usefully written in momentum space,

$$\dot{n}(k) = \frac{ik}{m}g(k) \qquad (119)$$

$$\dot{g}(k) = ik\tau(k) - K\sum_j (q_j - b_j)e^{ikq_j} \qquad (120)$$

$$\dot{h}(k) = ikj(k) \qquad (121)$$

where the currents $\tau(k)$ and $j(k)$ are given by

$$\tau(k) = \sum_j \frac{p_j^2}{m}e^{ikq_j} + \nu^2\sum_j q_j \frac{(e^{ikq_{j-1}} - 2e^{ikq_j} + e^{ikq_{j+1}})}{ik} \qquad (122)$$

$$j(k) = \sum_j \frac{p_j}{m} \left(\frac{p_j^2}{2m} + \frac{1}{2} K(q_j - b_j)^2 + \frac{1}{2} \nu^2 (q_j - q_{j-1})^2 \right) e^{ikq_j} \tag{123}$$

$$+ \nu^2 \sum_j \frac{p_j}{m} (q_{j+1} - q_j) \frac{(e^{ikq_j} - e^{ikq_{j+1}})}{ik}. \tag{124}$$

(These are clearly finite as $k \to 0$.) These currents are rather complicated in configuration space, except in the case where we neglect the interaction term, when they are given by

$$\tau(x) = \sum_j \frac{p_j^2}{m} \delta(q_j - x) \tag{125}$$

$$j(x) = \sum_j \frac{p_j}{m} \left(\frac{p_j^2}{2m} + \frac{1}{2} K(q_j - b_j)^2 \right) \delta(q_j - x). \tag{126}$$

Equations (116)–(118) do not in general form a closed system, so do not lead to hydrodynamic equations. To get a closed set, it is necessary to average these equations in a set of states depending on just three fields, thereby obtaining three equations for three unknowns. In the standard approach to deriving hydrodynamics, the local equilibrium state is chosen. We will discuss this below in Sec. VI C, but first we consider the simpler and instructive case of the normal mode coherent states.

B. Hydrodynamic equations in the case of normal mode coherent states

We consider the simple chain, $K = 0$, in the normal mode coherent states. As we have shown, these states are strongly peaked in the local densities so such states define a decoherent set of histories. The correlations functions of these states are constant in time, so we do not expect a settling down to local equilibrium. However, it turns out that the averages of the local densities still obey a simple set of hydrodynamic equations, and this case turns out to be particularly transparent and instructive. Because the only dynamics in this case is contained in the motion of the centers $\langle q_j \rangle, \langle p_j \rangle$, we need consider only the local number and momentum densities, not the energy density (which may in fact be calculated from the number and momentum densities in this case). Closure of the averaged conservation equations (119),(120) is obtained in this case because there are two equations and the states depend on just two sets of quantities, the $\langle q_j \rangle$ and the $\langle p_j \rangle$.

In a general Gaussian state we have

$$\langle n(k) \rangle = \sum_j \exp\left(ik\langle q_j \rangle - \frac{1}{2} k^2 (\Delta q_j)^2 \right) \tag{127}$$

$$\langle g(k) \rangle = \sum_j \left[\langle p_j \rangle + ik\sigma(q_j, p_j) \right]$$

$$\times \exp\left(ik\langle q_j \rangle - \frac{1}{2} k^2 (\Delta q_j)^2 \right) \tag{128}$$

$$\langle \tau(k) \rangle = \sum_j \left[\frac{1}{m} (\langle p_j \rangle + ik\sigma(q_j, p_j))^2 + C \right]$$

$$\times \exp\left(ik\langle q_j \rangle - \frac{1}{2} k^2 (\Delta q_j)^2 \right)$$

$$+ \frac{\nu^2}{ik} \sum_j (\langle q_{j+1} \rangle - 2\langle q_j \rangle$$

$$+ \langle q_{j-1} \rangle) \exp\left(ik\langle q_j \rangle - \frac{1}{2} k^2 (\Delta q_j)^2 \right) \tag{129}$$

where

$$C_j = \frac{1}{m} (\Delta p_j)^2 + \nu^2 [\sigma(q_{j+1}, q_j) - 2(\Delta q_j)^2 + \sigma(q_j, q_{j-1})]. \tag{130}$$

Now recall that the normal mode coherent states have the following special simplifying features: $\sigma(q_j, p_j) = 0$, and the variances $(\Delta q_j)^2, (\Delta p_j)^2$ and correlation functions of the form $\sigma(q_{j+1}, q_j)$ are independent of j. Also, we find that $C_j = 0$ [which follows from taking the time derivative of $\sigma(q_j, p_j) = 0$]. We therefore have

$$\langle n(k) \rangle = \sum_j \exp\left(ik\langle q_j \rangle - \frac{1}{2} k^2 (\Delta q)^2 \right) \tag{131}$$

$$\langle g(k) \rangle = \sum_j \langle p_j \rangle \exp\left(ik\langle q_j \rangle - \frac{1}{2} k^2 (\Delta q)^2 \right) \tag{132}$$

$$\langle \tau(k) \rangle = \tau_p(k) + \tau_q(k) \tag{133}$$

where

$$\tau_p(k) = \sum_j \frac{1}{m} \langle p_j \rangle^2 \exp\left(ik\langle q_j \rangle - \frac{1}{2} k^2 (\Delta q)^2 \right) \tag{134}$$

$$\tau_q(k) = \frac{\nu^2}{ik} \sum_j (\langle q_{j+1} \rangle - 2\langle q_j \rangle + \langle q_{j-1} \rangle)$$

$$\times \exp\left(ik\langle q_j \rangle - \frac{1}{2} k^2 (\Delta q)^2 \right). \tag{135}$$

Generally, we do not expect to derive interesting hydrodynamic equations except in the long wavelength regime. Clearly in this case, this means $k^{-2} \gg (\Delta q)^2$. Going to this regime (whose significance will become apparent below), and reverting back to configuration space, we find

$$\langle n(x) \rangle = \sum_j \delta(\langle q_j \rangle - x) \tag{136}$$

$$\langle g(x) \rangle = \sum_j \langle p_j \rangle \delta(\langle q_j \rangle - x) \tag{137}$$

$$\tau_p(x) = \sum_j \frac{1}{m} \langle p_j \rangle^2 \delta(\langle q_j \rangle - x). \tag{138}$$

The quantity $\tau_q(x)$ is more complicated and will be dealt with below. In these expressions, since we are in the long-wavelength regime, the δ-functions are to be thought of as smeared over a volume greater than Δq.

It is now very useful to introduce the velocity field $v(x)$, defined via

$$\langle p_j \rangle = mv(\langle q_j \rangle) = \int dy \, mv(y) \delta(\langle q_j \rangle - y) \quad (139)$$

which inserted in Eq. (137) yields

$$\langle g(x) \rangle = mv(x)\langle n(x) \rangle. \quad (140)$$

(This is in fact the standard relation between the velocity field and the momentum density [11].) Inserted also in Eq. (138) we obtain

$$\tau_p(x) = mv^2(x)\langle n(x) \rangle. \quad (141)$$

It is for deriving these last two equations that the long wavelength assumption is necessary. Most importantly, $\tau_p(x)$ is expressed in terms of the two fields $v(x), \langle n(x) \rangle$, which is crucial for closure of the equations.

We need now to obtain a similarly simple expression for $\tau_q(x)\rangle$. As it stands, Eq. (135) will not lead to a simple expression in terms of $v(x)$ and $\langle n(x) \rangle$. To proceed further with this term we need to make simplifications. We are ultimately interested in deriving a wave equation for the number density (which one might expect on the basis of the classical equations of motion for the q_j). The key to this is to consider *small* displacements of the q_j about uniformly distributed initial values, and then to consider the fluctuation in number density about the constant background. We therefore write

$$\langle q_j \rangle = jd + \delta q_j \quad (142)$$

where d is a constant representing the spacing between each particle and δq_j is a small displacement. The average number density for $k^{-2} \gg (\Delta q)^2$ and to linear order in δq_j is then

$$\langle n(k) \rangle = \sum_j e^{ik\langle q_j \rangle} \quad (143)$$

$$= n_0(k) + n_1(k) + \cdots \quad (144)$$

where

$$n_0(k) = \sum_j e^{ikjd} \quad (145)$$

$$n_1(k) = ik \sum \delta q_j e^{ikjd} \quad (146)$$

and note that $n_0(k)$ is constant. Inserted in the expression for $\tau_q(k)$, Eq. (135), and assuming also that $k^{-1} \gg d$, we find

$$\tau_q(k) = d^2 \nu^2 n_1(k). \quad (147)$$

This now means that in configuration space we have

$$\langle \tau(x) \rangle = mv^2(x)\langle n(x) \rangle + d^2 \nu^2 n_1(x). \quad (148)$$

Inserting all of these results in the local conservation equations (116),(117), we obtain a closed set of equations for $v(x)$ and $\langle n(x) \rangle = n_0(x) + n_1(x)$ where

$$n_0(x) = \sum_j \delta(x - jd) \quad (149)$$

is a fixed background field (again interpreted as coarse grained over a length of order Δq). Explicitly, we have

$$\frac{\partial}{\partial t}\langle n \rangle = -\frac{\partial}{\partial x}(\langle n \rangle v) \quad (150)$$

$$\frac{\partial}{\partial t}(\langle n \rangle v) = -\frac{\partial}{\partial x}\left(\langle n \rangle v^2 + \frac{d^2 \nu^2}{m} n_1\right). \quad (151)$$

The interesting special case is that in which the velocity field is small, in which case we may neglect the v^2 term in Eq. (151). Then combining the two equations yields the wave equation for $n_1(x)$,

$$\frac{\partial^2 n_1}{\partial t^2} = c^2 \frac{\partial^2 n_1}{\partial x^2} \quad (152)$$

where $c^2 = d^2 \nu^2/m$.

C. Local equilibrium and hydrodynamic equations for the $K \neq 0$ case

We now consider the case of the infinite chain with $K \neq 0$. In this case, we expect that the local density eigenstates will settle down to a local equilibrium state after a period of time. We will justify this important step below, but first we consider the consequences of a local equilibrium state, the standard assumption in derivations of hydrodynamics. This state is characterized by the one-particle distribution function (Wigner function)

$$w_j(p_j, q_j) = f(q_j)\exp\left(-\frac{[p_j - mv(q_j)]^2}{2mkT(q_j)}\right) \quad (153)$$

where f, v and T are slowly varying functions of space and time. (This is the one-particle distribution function for particle j—it is labeled by j since the particles are distinguishable.) If we now average the system Eqs. (116)–(118), together with the currents $\tau(x)$, $j(x)$ in the local equilibrium state, we obtain a closed system, since we get three equations for three unknowns. In the case of negligible interactions and $b_j = 0$, we find

$$\langle n(x) \rangle = Nf(x) \quad (154)$$

$$\langle g(x) \rangle = mv(x)Nf(x) \quad (155)$$

$$\langle h(x) \rangle = \left(\frac{1}{2}mv^2 + \frac{1}{2}kT + \frac{1}{2}Kx^2\right)Nf(x) \quad (156)$$

$$\langle \tau(x) \rangle = (mv^2 + kT)Nf(x) \quad (157)$$

$$\langle j(x)\rangle=\left(\frac{3}{2}vkT+\frac{1}{2}mv^3\right)Nf(x)+\frac{K}{2m}x^2\langle g(x)\rangle. \tag{158}$$

Inserted in Eqs. (116)–(118), the above relations give a closed set of equations for the three variables f, v and T. After some rearrangement, these equations are

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}=-f\frac{\partial v}{\partial x} \tag{159}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{m}\frac{\partial\theta}{\partial x}-\frac{\theta}{mf}\frac{\partial f}{\partial x}-\frac{Kx}{m} \tag{160}$$

$$\frac{\partial\theta}{\partial t}+v\frac{\partial\theta}{\partial x}=-2\theta\frac{\partial v}{\partial x} \tag{161}$$

where $\theta=kT$. These are the equations for a one-dimensional fluid moving in a harmonic potential [11]. Note that nontrivial equations are obtained even though we have neglected the interaction terms in deriving them. The role of interactions is to ensure the approach to local equilibrium, as we discuss below.

In these expressions, the definition of the velocity field is equivalent to Eq. (139) and, similarly, the definition of the temperature fields is essentially equivalent to,

$$\sum_j\frac{1}{2m}(\Delta p_j)^2\delta(q_j-x)=\frac{1}{2}kT(x)n(x) \tag{162}$$

(recalling that we are working at long wavelengths, so the δ function is coarse grained over a scale of order k^{-1}). Hence temperature arises not from an environment, but from the momentum fluctuations averaged over a coarse-graining volume.

D. The approach to local equilibrium

Now the key point is that the states we are actually interested in are the approximate eigenstates of the local densities, evolved in time, or more precisely, the one-particle distribution function w_1 constructed from those states. (Since w_1 is the quantity that will enter in the computation of any averages of sums of one-particle quantities, such as the local densities.) This is not necessarily the same as the local equilibrium distribution, Eq. (153), although they are clearly very similar. The averages of the local densities in the approximate eigenstates will therefore obey the hydrodynamic equations derived above as long as we can show that the one-particle distribution function of these states w_1 approaches the local equilibrium form Eq. (153) after some time.

The local equilibrium form Eq. (153) has $\sigma(q_j,p_j)=0$ and all the other averages $\langle p_j\rangle$, $\langle q_j\rangle$, $(\Delta q_j)^2$ and $(\Delta p_j)^2$ vary slowly in time and space (i.e., slowly with j). [Clearly $\sigma(q_j,p_j)$ has to be zero or small for local equilibrium since it is the time derivative of $(\Delta q_j)^2$ and $(\Delta p_j)^2$.] Compare this with the approximate eigenstates for the case $K\neq0$ and the infinite chain. They are Gaussians, so their one-particle distribution function w_1 is entirely determined by $\langle p_j\rangle$, $\langle q_j\rangle$,

$(\Delta q_j)^2$, $(\Delta p_j)^2$ and $\sigma(q_j,p_j)$. From Eqs. (51)–(53), we see that $\sigma(q_j,p_j)$ grows initially from zero, but then becomes small at late times, while $(\Delta q_j)^2$ and $(\Delta p_j)^2$ approach a constant plus a slowly decaying factor. Moreover, $(\Delta q_j)^2$ and $(\Delta p_j)^2$ depend on j only through their initial values, which as discussed [see Eq. (82) for example], vary slowly. In addition, the centers $\langle q_j(t)\rangle$ and $\langle p_j(t)\rangle$ go into a phase of slow time dependence and limited dependence on j for times sufficiently long for the Bessel functions to go into their oscillatory phases. These asymptotic forms are approached on a time scale $(\gamma\Omega)^{-1}$. Therefore, in this case, the local equilibrium form is indeed achieved at late times, and the average values of the local densities obey hydrodynamic equations.

The final picture we have is as follows. We can imagine an initial state for the system which contains superpositions of macroscopically very distinct states. Decoherence of histories indicates that these states may be treated separately and we thus obtain a set of trajectories which may be regarded as exclusive alternatives each occurring with some probability. Those probabilities are peaked about the average values of the local densities. We have argued that each local density eigenstate may then tend to local equilibrium, and a set of hydrodynamic equations for the average values of the local densities then follow. We thus obtain a statistical ensemble of trajectories, each of which obeys hydrodynamic equations. These equations could be very different from one trajectory to the next, having, for example, significantly different values of temperature. (In the most general case they could even be in different phases, for example one a gas, one a liquid.)

Decoherence requires the coarse-graining scale k^{-1} to be much greater than the correlation length of the local density eigenstates, and the derivation of the hydrodynamic equations requires $k^{-2}\gg(\Delta q)^2$. In brief, the emergence of the classical domain occurs on length scales much greater than any of the scales set by the microscopic dynamics.

VII. SUMMARY AND DISCUSSION

We have shown that for the linear oscillator chain the local densities define a decoherent set of histories of sufficiently coarse-grained scales. The key idea is to split the initial state into local density eigenstates and show that they are preserved in form under time evolution. The subsequent probabilities for histories are peaked about the average values of the local densities, and the equations of motion for them form a closed set of hydrodynamic form on sufficiently large scales, provided, in general, that sufficient time has elapsed for the local density eigenstates to settle down to local equilibrium.

It is perhaps of interest to comment on the time scales involved. Decoherence through interaction with an environment involves a time scale, which is typically exceptionally short. Here, however, there is no time scale associated with decoherence by approximate conservation. The eigenstates of the local densities remain approximate eigenstates for all time. There is, however, a time scale involved in obtaining the hydrodynamic equations, namely, the time required for a

local density eigenstate to approach local equilibrium. In this model, this time scale is of order $(\gamma\Omega)^{-1}$ (for the infinite chain in the $K \neq 0$ case).

Another interesting general issue is the question of the relative roles of the decoherence through approximate conservation considered here versus decoherence through interaction with an environment. The point is that it is a question of length scales. We have demonstrated decoherence of the local densities starting with exact conservation at the largest length scales and then moving inwards. In this way we were able to prove decoherence without using an environment, for certain sets of histories at very coarse-grained scales whose probabilities are peaked about classical paths. However, in general we would like to be able to assign probabilities to nonclassical trajectories. For example, what is the probability that a system will follow an approximately classical trajectory at a series of times, but then at one particular time undergoes a very large fluctuation away from the classical trajectory? The approach adopted here indicates that the probability for this is approximately zero, to the level of approximation used. Yet this is a valid question that we can test experimentally. It is at this stage that an environment becomes necessary to obtain decoherence, and indeed it is frequently seen in particular models that when there is decoherence of histories due to an environment, decoherence is obtained for a very wide variety of histories, not just histories close to classical. It is essentially a question of information. Decoherence of histories means that information about the histories of the system is stored somewhere [5,28]. Classical histories need considerably less information to specify than nonclassical ones, and indeed specification of the three local densities at any time is sufficient to specify their entire classical histories. This is not enough for nonclassical histories, so an environment is required to store the information.

Note also that ''environment'' need not necessarily refer to an external environment. It could also include the internal coordinates not fixed by the coarse graining. These did not play a role in the case considered here, but would become important at finer-grained levels, producing fluctuations about the evolution described by the hydrodynamic equations, hand in hand with decoherence. This has been considered in Refs. [4,13].

Given the need for an environment at finer-grained scales, it is then of interest to ask whether the local densities continue to have an important role for many-body systems when an environment becomes necessary for decoherence. Gratifyingly, the answer is that the local densities, and particularly the number density, remain the naturally preferred variables for a many-body system coupled to an external environment, as was recently shown [29]. It is normally claimed that position is the preferred variable in environmentally induced decoherence, but this is for a single particle coupled to an environment and is in any case an approximation. For a many-body system coupled to a scattering environment, with both described by many-body field theory, it was shown in Ref. [29] that number density is the naturally decohering variable (with momentum density, as its time derivative, also entering in a natural way). Hence there is a smooth match

with environmentally induced decoherence models as we go to finer scales.

It would be of interest to generalize to an oscillator chain with nonlinear interactions. This is because in the linear chain, the energy in each mode is conserved, so there is no possibility of exchanging energy between modes, and the approach to local equilibrium is rather artificial.

It would also be of particular interest to look at a gas. Many-body field theory may be the appropriate medium in which to investigate this, following the lead of Ref. [29]. The decoherent histories analysis might confer some interesting advantages over conventional treatments. For example, one-particle dynamics of a gas is described by a Boltzmann equation. One of the assumptions involved in the derivation of the Boltzmann equation is that the initial state of the system contains no correlations, which is clearly very restrictive [11]. However, in the general approach used here it is natural to break up any arbitrary initial state into a superposition of local density eigenstates, and that these may then be treated separately because of decoherence. The local density eigenstates typically have small or zero correlations. Hence, decoherence gives some justification for one of the rather restrictitve assumptions of the Boltzmann equation.

We have not estimated the degree of decoherence in the models considered here, although it could be estimated by looking more closely at the approximations involved in going from exact to approximate decoherence, described in Sec. II. However, there may be a more rigorous (but more difficult) way of proving the results of this paper, which would allow the degree of decoherence to be estimated. This would be to prove a theorem similar to that proved by Omnès for phase space projectors [8]. For a system of N particles with phase space coordinates $\mathbf{z} = (\mathbf{p}, \mathbf{q})$, Omnès considered (approximate) projection operators onto a region Γ ($\gg \hbar$) of phase space, defined by

$$P_\Gamma = \int_\Gamma d^N z \, |\mathbf{z}\rangle\langle\mathbf{z}| \tag{163}$$

where the states $|\mathbf{z}\rangle$ are some form of phase space localized states, such as coherent states. He showed that under certain reasonable conditions, the form of this projector is approximately preserved under unitary evolution, that is,

$$e^{(i/\hbar)Ht} P_\Gamma e^{-(i/\hbar)Ht} \approx P_{\Gamma_t} \tag{164}$$

where Γ_t is the original phase space region evolved along the classical phase space trajectories. It is easy to see that this ensures approximate decoherence of coarse-grained phase space histories and that the probabilities are peaked about classical phase space paths. The result is therefore very similar in spirit to the present paper. It seems very plausible that a similar result may be proved here for projections onto local densities. That is, we would like to construct a set of projectors onto the local densities, P_{ngh} say, and then show that they are approximately mapped into $P_{n_t g_t h_t}$ under unitary evolution, where n_t, g_t, h_t are related to the initial values n, g, h by a closed set of evolution equations. Such a result is not simply obtained by a coarse graining of the Omnès re-

sult, the issue being that n_t, g_t, h_t have to evolve according to a *closed* set of equations, which is not straightforward to accomplish in general. (The phase space coordinates **p,q** evolve according to the Hamilton equations, which is clearly a closed set of equations, but truncations or coarse grainings of this set will generally not be closed.) Moreover, the Omnès result breaks down when the underlying classical dynamics is chaotic. The corresponding hydrodynamic description, however, being coarser grained, will generally not be chaotic and does not obviously break down, so this is a po-

tential advantage of the hydrodynamic approach. These and related issues will be pursued in a future publication.

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