On the apparent convergence of perturbative QCD at high temperature

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The successive perturbative estimates of the pressure of QCD at high temperature T show no sign of convergence, unless the coupling constant g is unrealistically small. Exploiting known results of an effective field theory which separates hard (order $2\pi T$) and soft (order gT) contributions, we explore the accuracy of simple resummations which at a given loop order systematically treat hard contributions strictly perturbatively, but soft contributions without truncations. This turns out to improve significantly the two-loop and the three-loop results in that both remain below the ideal-gas value, and the degree of renormalization scale dependence decreases as one goes from two to three loop order, whereas it increases in the conventional perturbative results. Including the four-loop logarithms recently obtained by Kajantie *et al.*, we find that this trend continues and that with a particular sublogarithmic constant the untruncated four-loop result is close to the three-loop result, which itself agrees well with available lattice results down to temperatures of about $2.5T_c$. We also investigate the possibility of optimization by using a variational ("screened") perturbation theory in the effective theory. At two loops, this gives a result below the ideal gas value and also closer to lattice results than the recent two-loop hard-thermal-loop-screened result of Andersen *et al.* While at three-loop order the gap equation of dimensionally reduced screened perturbation theory does not have a solution in QCD, this is remedied upon inclusion of the four-loop logarithms.

I. INTRODUCTION

One could expect weak coupling calculations to lead to reasonable estimates of the QCD free energy at high temperature T, a regime where indeed the gauge coupling becomes small because of asymptotic freedom. But explicit perturbative calculations, which have been pushed in recent years up to order g^5 [1–7], do not exhibit any sign of convergence, as depicted in Fig. 1; they rather show increasing ambiguities due to the dependence on the renormalization point, signaling a complete loss of predictive power.

Various mathematical extrapolation techniques have been tried, such as Padé approximants [8,9] and Borel resummation [10]. The resulting expressions are indeed smooth functions of the coupling, better behaved than polynomial approximations truncated at order g^5 or lower, with a weaker dependence on the renormalization scale. However, while these methods do improve the situation somewhat, it is fair to say that they offer little physical insight into the source of the difficulty.

Recognizing that an important effect of thermal fluctuations is to give mass to the excitations, thereby screening long range interactions, it has been suggested to incorporate such screening effects in the tree-level Lagrangian and correct for double counting by adding suitable counterterms to the interactions. Such a scheme has been implemented with some success in scalar field theory under the name of screened perturbation theory (SPT) [11–14]. It has been extended to QCD by including at the tree level the entire nonlocal Lagrangian of the hard thermal loops (HTL) [15–18], which is referred to as HTL perturbation theory (HTLPT).

A different approach, motivated physically by the success of the quasiparticle picture, is based on the so-called Φ-derivable approximations [19–21] (see also Peshier [22]). This approach takes advantage of remarkable simplifications which occur in the calculation of the entropy at two loop in the skeleton expansion. Together with further approximations[1] for the self-energies based on hard thermal loops, this led to results for the thermodynamical functions which are consistent with lattice calculations for temperatures above $2.5T_c$.

This paper reconsiders the known results up to order g^5 in the light of the simple observation that the accuracy of perturbation theory is not the same at all momentum scales. Thus, while perturbation theory at the scale T is an expansion in powers of g^2, the perturbation theory at the scale gT is an expansion in powers of g and is therefore less accurate. It is when they are treated strictly perturbatively that the soft contributions turn out to completely spoil apparent convergence, as exemplified by the (soft) contribution of order g^3 which leads to a pressure exceeding the ideal gas value. The main idea that we want to pursue is to decouple approximations in the hard and the soft sectors: in the hard sector, we shall use perturbation theory since it is accurate; in the soft sector we shall go (minimally) beyond perturbation theory, with the remarkable result that the apparent convergence of perturbative QCD at high temperatures is dramatically improved. This, in our opinion, lends support to the more ambitious

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[1] A full Φ-derivable approximation has been worked out successfully in scalar field theory [23], for which recently also the question of renormalizability could be answered affirmatively in Refs. [24,25].
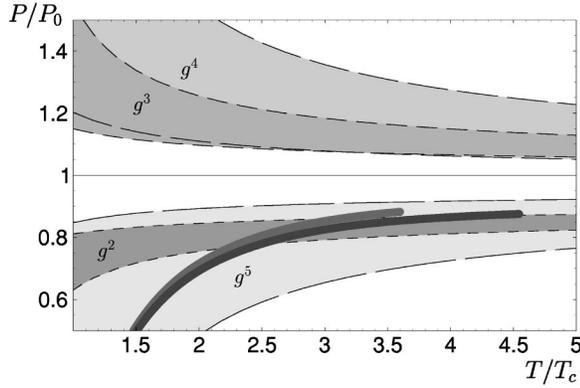
FIG. 1. Strictly perturbative results for the thermal pressure of pure glue QCD normalized to the ideal-gas value, as a function of T/T_c (assuming $T_c/\Lambda_{\overline{\mathrm{MS}}} = 1.14$). The various gray bands bounded by differently dashed lines show the perturbative results to order g^2, g^3, g^4, and g^5, with modified minimal subtraction ($\overline{\mathrm{MS}}$) renormalization point $\bar{\mu}$ varied between πT and $4\pi T$. The thick dark-gray line shows the continuum-extrapolated lattice results from Ref. [26]; the lighter one behind that of a lattice calculation using an renormalization-group- (RG-) improved action [27].

attempts to reorganize perturbation theory by novel resummation techniques.

The tools to deal with various momentum scales in field theory are well developed and involve the construction of effective theories. In fact, the perturbative results through order g^5, first calculated in Refs. [4,5], have been rederived and confirmed by Braaten and Nieto [7] by an elegant and efficient effective-field-theory method which separates contributions from hard, soft, and supersoft momentum scales, $2\pi T$, gT, and $g^2 T$, respectively. In Euclidean space, the only soft modes are static ones, and the effective field theories for these are therefore dimensionally reduced [28–30] to three-dimensional ones. The dimensionally reduced theory consists of massive adjoint scalar fields A_0^a and massless three-dimensional Yang-Mills fields (with coupling g_E) and, following Braaten and Nieto, we shall refer to it as electrostatic QCD (EQCD) in the following. The corresponding effective three-dimensional Lagrangian is

$$\mathcal{L}_{EQCD} = \frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2} (D_i A_0)^a (D_i A_0)^a + \frac{1}{2} m_E^2 A_0^a A_0^a$$
$$+ \frac{1}{8} \lambda_E (A_0^a A_0^a)^2 + \delta\mathcal{L}_{EQCD} \tag{1}$$

where the parameters are determined perturbatively by matching [7]. In lowest order we have

$$m_E^2 = (1 + N_f/6) g^2 T^2, \qquad g_E^2 = g^2 T, \tag{2}$$

and [31]

$$\lambda_E = \frac{9 - N_f}{12\pi^2} g^4 T. \tag{3}$$

In fact, λ_E starts to contribute to the pressure only at order g^6. For this reason we shall ignore this particular coupling in most of the following, as well as all the other vertices contained in $\delta\mathcal{L}_{EQCD}$.

The thermal pressure of the 4-dimensional theory can be decomposed into contributions from the hard modes $\sim T$, calculable by standard perturbation theory, and soft contributions governed by Eq. (1) which involves both perturbatively calculable contributions up to order $g^5 T^4$ and nonperturbative ones coming from the fact that the effective theory for the modes $A_i(\vec{x})$ is a confining theory. However, the latter magnetic contributions start at order g^6 and cannot *a priori* be made responsible for the poor apparent convergence that is seen up to order g^5.

We shall then focus in this paper on the soft contributions, i.e. those coming from the momentum scale gT. In principle these can be calculated using perturbation theory, but as clear from Eqs. (1) and (2) the corresponding expansion parameter is $g_E^2/m_E \sim g$, so that the perturbative expansion converges only slowly, more slowly than the perturbative expansion in the hard sector. As a minimal step towards a nonperturbative treatment of the soft sector, we shall perform a simple loop expansion of Eq. (1), keeping the parameters m_E and g_E as given in terms of g by the matching conditions and not expanding them out in powers of g in the final result. As we shall see, this simple method leads to a significant improvement over the strict perturbative results. We also consider the effect of including the four-loop logarithms recently obtained by Kajantie *et al.*, and again find that strict perturbation theory has large scale dependences which are drastically reduced when keeping soft contributions untruncated. For a particular choice of the sublogarithmic constant the untruncated four-loop result is moreover close to the three-loop result, which itself agrees well with available lattice results down to temperatures of about $2.5T_c$.

In Sec. III we consider a simple variational improvement of perturbation theory in the form of dimensionally reduced screened perturbation theory (DRSPT). This turns out to be much simpler than HTLPT, while also allowing us to resum screening masses in a gauge invariant manner. At two-loop order it leads to a result significantly closer to lattice data than two-loop HTLPT, which suggests that the partial failure of the latter as observed in [17,18] is due to spurious hard contributions. At three-loop order, the gap equation of DRSPT has no real solution, but this is remedied upon inclusion of the four-loop logarithms.

In the final section we summarize and discuss our results and try to put them into perspective with other techniques.

II. DIMENSIONAL REDUCTION BEYOND STRICT PERTURBATION THEORY

In the following we adhere to Ref. [7] in the treatment of the dimensionally reduced effective theory responsible for the contributions from the scale gT, but we deviate in that we shall not treat the soft sector strictly perturbatively. We shall organize our presentation by considering the successive approximations obtained by expanding the contributions of the hard modes to the pressure in powers of $\alpha_s = g^2/(4\pi)$.

Each order in this expansion defines also the accuracy with which the parameters of the effective Lagrangian are determined through perturbative matching conditions. However, at each level of approximation we shall consider the effect of treating the contributions of the soft sector more completely by refraining from the specific truncations usually employed in a strictly perturbative expansion.

A. One-loop order

In massless QCD at one-loop order, the only contribution to the thermal pressure (identical to minus the free energy) is coming predominantly from hard momenta. Introducing a momentum cutoff Λ_E to separate the hard scale $2\pi T$ from the scale gT, one gets a contribution from the soft sector proportional to $T\Lambda_E^3$, compensating a similar term in the interaction-free one-loop contribution from hard modes. One ends up with the standard ideal-gas result $P_0 = \pi^2 T^4(8/45 + 7N_f/60)$ for N_f quarks and 3 colors.

Reference [7] avoids the introduction of momentum cutoffs by using dimensional regularization for the purpose of both ultraviolet and infrared regularizations. Doing so, the one-loop result exclusively comes from the hard modes: At this level of approximation where all interactions are neglected, the soft (zero) modes are to be taken as massless, and their contribution vanishes in dimensional regularization where scaleless integrals are set to zero.

B. Two-loop order

The two-loop (order α_s) contribution of hard modes to the pressure is still independent of a cutoff Λ_E, if this is handled by dimensional regularization. It reads [1]

$$P_{\text{hard}}^{(2)} = -\frac{2\pi}{3}\left(1 + \frac{5}{12}N_f\right)\alpha_s T^4. \tag{4}$$

At this level of approximation, the soft modes described by EQCD are massive, the mass being given by the leading order Debye mass (2). The one-loop contribution to the pressure from EQCD is now, on dimensional grounds, proportional to m_E^3 times an overall factor of T. If only this contribution is added to Eq. (4), the result is the ill-behaved strictly perturbative result to order g^3 displayed in Fig. 1. However, in the soft sector, we need not restrict ourselves to this one-loop approximation, but rather treat more completely the interactions which are present in $\mathcal{L}_{EQCD}$. Thus, with the parameters of EQCD determined by matching with the perturbative calculation at the present level of accuracy, we shall consider also the two-loop contributions from the dimensionally reduced effective theory. This yields the following contribution to the pressure:

$$P_{\text{EQCD}}^{(1)+(2)} = -Tf_M^{(1)+(2)} \tag{5}$$

with $f_M^{(1)+(2)}$ given by [7]

$$f_M^{(1)+(2)} = -\frac{2}{3\pi}m_E^3 + \frac{3}{8\pi^2}\left(\frac{1}{\epsilon} + 4\ln\frac{\Lambda_E}{2m_E} + 3\right)g_E^2 m_E^2 + \delta f_E, \tag{6}$$

where Λ_E is the scale of dimensional regularization in the soft sector, which may be loosely associated with the separation scale between the soft and the hard momenta. Choosing the counterterm δf_E by minimal subtraction,

$$\delta f_E = -\frac{3}{8\pi^2}g_E^2 m_E^2 \frac{1}{\epsilon}, \tag{7}$$

leaves behind a dependence on Λ_E. Eventually, $\ln(\Lambda_E)$ has to combine with a matching logarithm arising from the hard scales for which Λ_E provides the infrared cutoff. This in fact happens after a careful perturbative matching at three-loop order [see Eq. (9)].

Together Eqs. (4) and (5) are accurate through order $g^4\log(1/g)$, with an error of order g^4 as to be expected from a two-loop calculation. The coefficient of the $g^4\log(1/g)$ term is correct provided[2] Λ_E is set to a constant times T. From its role in dimensional reduction, it should be smaller than $2\pi T$ but larger than gT. However, as remarked in Ref. [7], the introduction of cutoffs through dimensional regularization leaves their relationship to momentum cutoffs undetermined, and there could be a different relationship between the scale of dimensional regularization and effective momentum cutoffs depending on whether the latter act as IR or as UV cutoffs. Reference [7] even found that the scale Λ_E should be chosen *larger* than the UV scale $\bar{\mu} \sim 2\pi T$ in order to have optimal convergence of the hard contributions. For simplicity and to facilitate the comparison with other calculations where a similar identification is made, we shall put $\Lambda_E = \bar{\mu}$ in the following, i.e. introduce only one scale for dimensional regularization, and refrain from modifying it by hand depending on whether the various logarithms can be identified as arising from regularization in the IR or in the UV.[3]

In Fig. 2 we give the numerical evaluation of the full 2-loop result obtained as indicated above, for pure-glue QCD and T between T_c and $5T_c$ in analogy to the strictly perturbative results of Fig. 1. We always use the standard[4] perturbative solution to the two-loop renormalization group equation for α_s assuming $T_c/\Lambda_{\overline{\text{MS}}} = 1.14$. The UV renormalization scale $\bar{\mu}$ is varied about a central value $2\pi T$ by a factor of 2, and it should be kept in mind that this variation also traces some of the ambiguity in choosing Λ_E. The resulting total scale dependence is comparable to the

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[2]If Λ_E is chosen to be parametrically smaller than $2\pi T$ by multiplying it by a fractional power g^c with $0 < c < 1$, then the coefficient of $g^4\ln(1/g)$ is wrong by a factor of c.

[3]The ambiguity of the choice of Λ_E can alternatively be understood as arising from the freedom to renormalize the effective 3-dimensional theory differently than by minimal subtraction. Because in the next subsection we shall compare with HTLPT [15–18] where only minimal subtraction of additional divergences has been considered, we stick to minimal subtraction in the following. It should be kept in mind, however, that there is a source of additional renormalization scheme dependence.

[4]This is practically indistinguishable from the full 2-loop solution as soon as $T \gtrsim 2T_c$ (see the Appendix of Ref. [32]).
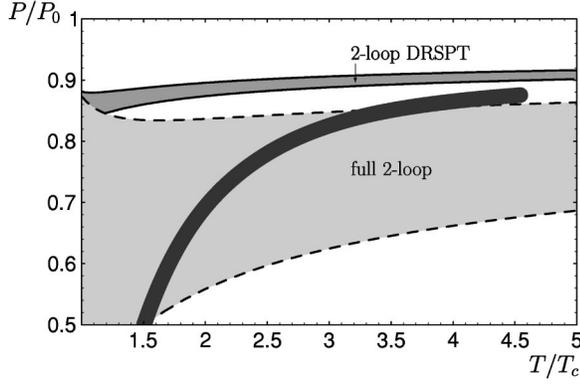
FIG. 2. Two-loop pressure in pure-glue QCD with untruncated EQCD contributions when $\bar{\mu}$ is varied between πT and $4\pi T$ (broad gray band) in comparison with the lattice result from Ref. [26] (thick dark-gray curve). The narrow darker-gray band above the former is the result of 2-loop DRSPT considered in Sec. III A; its lower boundary corresponds to the extremal value when varying $\bar{\mu}$.

scale dependence of the perturbative order-g^4 result. But in contrast to both the result to order g^3, to which it is perturbatively equivalent, and the order-g^4 result, the untruncated 2-loop result of dimensional reduction remains below the ideal-gas and thus has overlap with the lattice results, which the former do not have.

Remarkably, the partial inclusion of order-g^4 effects arising from a two-loop evaluation of EQCD is superior to a complete order-g^4 evaluation in strict perturbation theory. Although the former has an uncanceled Λ_E dependence in addition to the normal $\bar{\mu}$ dependence, numerically the total scale dependence is not worse but even slightly better than that of the perturbative order-g^4 result.

If we had not put λ_E to zero on grounds that it starts contributing only at order g^6, we would have obtained the additional 2-loop term

$$f_M^{(2)}\big|_{\lambda_E} = \frac{5}{8\pi^2} m_E^2 \lambda_E. \tag{8}$$

Inserting the leading-order value of λ_E, Eq. (3), this contribution is not only of order g^6, but it is also numerically quite small in comparison with the other two-loop contributions (6) even when $g \sim 1$.

C. Three-loop order

The three-loop contribution of the hard modes to the pressure is no longer Λ_E independent, because it has to be matched with the minimally subtracted, and thus Λ_E-dependent, EQCD theory. This has been calculated in [7] as[5]

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[5]In Eq. (54) of Ref. [7] the terms corresponding to the second and fourth terms on the right-hand side (RHS) of Eq. (9) have a different sign due to a typographical error.
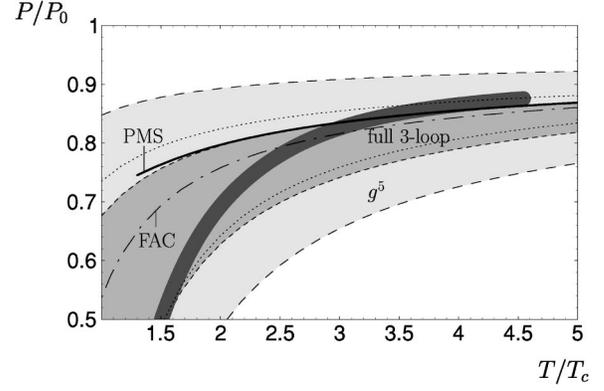

FIG. 3. Three-loop pressure in pure-glue QCD with untruncated EQCD contributions when $\bar{\mu}$ is varied between πT and $4\pi T$ (medium-gray band); the dotted lines indicate the position of this band when only the leading-order result for m_E is used. The broad light-gray band underneath is the strictly perturbative result to order g^5 with the same scale variations. The solid line gives the result upon extremalization (PMS) with respect to $\bar{\mu}$ (which does not have solutions below $\sim 1.3 T_c$); the dash-dotted line corresponds to fastest apparent convergence (FAC) in m_E^2, which sets $\bar{\mu} \approx 1.79 \pi T$.

$$P_{\text{hard}}^{(3)} = \frac{8\pi^2}{45} T^4 \left\{ 244.9 + 17.24 N_f - 0.415 N_f^2 \right.$$

$$+ 135 \left(1 + \frac{N_f}{6} \right) \ln \frac{\Lambda_E}{2\pi T} - \frac{165}{8} \left(1 + \frac{5}{12} N_f \right)$$

$$\left. \times \left(1 - \frac{2}{33} N_f \right) \ln \frac{\bar{\mu}}{2\pi T} \right\} \left(\frac{\alpha_s}{\pi} \right)^2. \tag{9}$$

Similarly, the mass parameter of EQCD can be obtained by a matching calculation of two-loop self-energies as [7]

$$m_E^2 = (2\pi T)^2 \frac{\alpha_s}{\pi} \left\{ \left(1 + \frac{N_f}{6} \right) + \frac{\alpha_s}{4\pi} \left[5 + 22\gamma \right. \right.$$

$$+ 22 \ln \frac{\bar{\mu}}{4\pi T} + \frac{N_f}{3} \left(\frac{1}{2} - 8 \ln 2 + 7\gamma + 7 \ln \frac{\bar{\mu}}{4\pi T} \right)$$

$$\left. \left. + \frac{N_f^2}{9} \left(1 - 2\gamma - 2 \ln \frac{\bar{\mu}}{\pi T} \right) \right] \right\}. \tag{10}$$

As for the three-loop contribution from the soft sector, this is given by the finite and thus Λ_E-independent expression calculated in Ref. [7] (neglecting λ_E contributions now):

$$f_M^{(3)} = \frac{9}{8\pi^3} \left(\frac{89}{24} - \frac{11}{6} \ln 2 + \frac{1}{6} \pi^2 \right) g_E^4 m_E. \tag{11}$$

In a strictly perturbative evaluation which drops all terms of order g^6, the sum of the hard and soft contributions are Λ_E independent as they should be. That is, at order g^4, the term $\propto \ln \Lambda_E$ in Eq. (9) cancels against the corresponding one in Eq. (5). These perturbative contributions, evaluated numerically, give the result marked "g^5" in Figs. 1 or 3.

However, our aim is to go beyond such perturbative results, and as a simple approximation in this direction, we consider keeping the soft contributions (6) and (11) untruncated when the perturbatively determined m_E^2 is inserted. This then corresponds to a selective summation of higher-order effects that may help improve the convergence of perturbation theory, although these higher order terms are Λ_E dependent.[6]

Setting $\Lambda_E = \bar{\mu}$ the ambiguity in choosing Λ_E contributes to the scale dependence and is included in our error bands as $\bar{\mu}$ is varied around its central value.[7] The result is shown in Fig. 3. Despite the incomplete cancellation of Λ_E in the untruncated evaluation of the three-loop result, we observe a considerable reduction of the total scale dependence compared to strict perturbation theory.

Furthermore, compared to the full two-loop result, the three-loop result stays within the (rather large) uncertainties of the former. While the uncertainties remain sizable even at three-loop order, the overall picture is a remarkable improvement over strict perturbation theory, where the results jump about the ideal-gas value and the renormalization scale dependence *increases* steadily with the highest power of g reached.

It is interesting to observe that very similar results are obtained if only the lowest order Debye mass (2) is used in this calculation, in place of the full order g^4 expression (10). This is displayed in Fig. 3 by the dotted lines. It indicates that what matters here is the accuracy with which one treats the soft sector, more than the accuracy with which the parameters of $\mathcal{L}_{EQCD}$ are determined.

Whereas in the untruncated two-loop result the scale dependence is monotonic and does not allow for its elimination by a principle of minimal sensitivity (PMS), such an elimination turns out to be possible in the three-loop result.[8] Choosing $\alpha_s(\bar{\mu})$ according to the 2-loop renormalization group equation, we find an extremum of the untruncated three-loop result as a function of $\bar{\mu}$. As shown in Fig. 3, the corresponding pressure values are in fact remarkably close to the lattice data for $T \gtrsim 3T_c$.

The extremum is only a local one with respect to $\bar{\mu}$. For large temperatures $T \gtrsim 10T_c$ it occurs at $\bar{\mu} \sim 2\pi T$, which following Ref. [7] we have taken as a central value because it is the spacing of the Matsubara frequencies. For smaller temperatures, the required value of $\bar{\mu}$ increases and exceeds $4\pi T$ below $T \sim 3T_c$, where the lattice data start to deviate from the three-loop pressure. For still smaller temperatures

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[6]Specifically, they lead to a $g^6\ln(g)$ contribution with the constant under the logarithm carrying the ambiguity in Λ_E if the latter is proportional to T. Numerically, however, this $g^6\ln(g)$ contribution is completely negligible compared to the recently determined [33] perturbative $g^6\ln(g)$ contribution appearing at 4-loop order.

[7]These error bands would of course be widened by an independent variation, so the scheme dependences displayed in our figures are certainly underestimated to some extent.

[8]This is possible in fact only if the complete order-g^4 expression for the Debye mass is used.
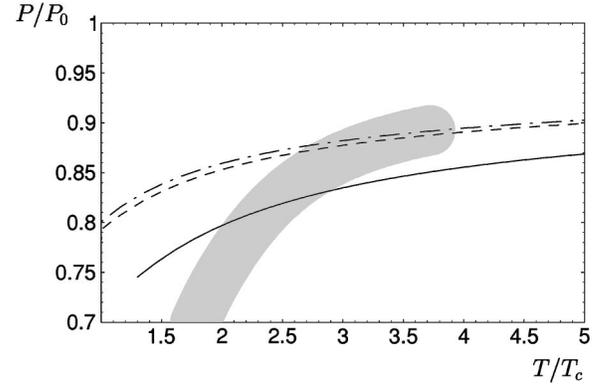

FIG. 4. PMS-extremalized full-three-loop pressure in QCD with $N_f=0$ (solid line), $N_f=2$ (dashed line), and $N_f=3$ (dash-dotted line) in comparison with the estimated continuum extrapolation of QCD with 2 light quark flavors of Ref. [34].

$T \lesssim 2T_c$ the required $\bar{\mu}$ becomes unreasonably large, and finally for $T \lesssim 1.3T_c$ the local extremum disappears completely. (The strictly perturbative result, on the other hand, has a monotonic, i.e. runaway, scale dependence for all T.)

For completeness, we also give the numerical results obtained by including $N_f=2$ and 3 massless quark flavors. The scale selected by the minimal sensitivity turns out to be slightly higher than at $N_f=0$; it also becomes large for $T \lesssim 2T_c$, but the extremum exists down to T_c now. There exist no reliable continuum extrapolated lattice data for this case yet, but in Ref. [34] an *estimated* continuum extrapolation has been given for $N_f=2$ light quark flavors. This is compared with the extremalized full 3-loop results in Fig. 4. The fact that the cases $N_f=2$ and 3 are nearly degenerate [when normalized to the respective ideal-gas values and plotted as a function of T/T_c with the respective critical temperature (assumed to be $1.14\Lambda_{\overline{MS}}$)] is consistent with lattice results [34] and is very similar to the pattern observed in the "next-to-leading approximation" (NLA) results of Ref. [20] for the entropy.

D. Four-loop order

In an impressive four-loop calculation, the authors of Ref. [33] have recently determined the last coefficient in the perturbative expansion of thermal pressure of QCD that can be computed analytically. At four-loop order in the effective theory (1) there appear two logarithmic terms whose coefficents have been obtained as [33]

$$P_{\text{soft}}^{(4)}/T = N_g \frac{(N g_E^2)^3}{(4\pi)^4} \left(\left[\frac{43}{12} - \frac{157\pi^2}{768} \right] \ln \frac{\Lambda_E}{g_E^2} \right.$$
$$\left. + \left[\frac{43}{4} - \frac{491\pi^2}{768} \right] \ln \frac{\Lambda_E}{m_E} + c \right), \quad (12)$$

where the first logarithm is in fact from the magnetostatic sector. To obtain the complete $g^6\ln g$ contribution in the QCD pressure one now also needs g_E^2 to order g^4, given in Ref. [35] as (for $N_f=0$)

$$g_E^2 = 4\pi\alpha_s\left\{1 + \frac{\alpha_s}{4\pi}\left[22\left(\ln\frac{\bar{\mu}}{4\pi T} + \gamma\right) + 1\right]\right\}. \quad (13)$$

The constant c in Eq. (12), however, is strictly nonperturbative and can in principle be determined by lattice simulations of the dimensionally reduced theory, but requires even more (also analytical) work [33].

Also, the four-loop contribution to P_{hard} is not yet known, but all the terms proportional to g^6 and involving explicit logarithms of $\bar{\mu}$ or Λ_E are determined by the $\bar{\mu}$ and Λ_E independence of the total pressure to order g^6 and have been given explicitly in Ref. [33]—only a constant times g^6 thus remains undetermined. Equating again Λ_E with $\bar{\mu}$, the four-loop contribution to be added to the above three-loop one when evaluated with Eq. (13) can be written as (again for $N_f = 0$ only)

$$P^{(4)} = \frac{8\pi^2}{45}T^4\left\{\frac{21945}{16}\ln^2\frac{\bar{\mu}}{2\pi T} + 2676.4\ln\frac{\bar{\mu}}{2\pi T}\right\}\left(\frac{\alpha_s}{\pi}\right)^3$$
$$+ \frac{27}{32\pi^4}\left(\left[\frac{43}{12} - \frac{157\pi^2}{768}\right]\ln\frac{T}{g_E^2} + \left[\frac{43}{4} - \frac{491\pi^2}{768}\right]\ln\frac{T}{m_E}\right)$$
$$+ 7.57\,\delta\right)g_E^6 T, \quad (14)$$

where for easier comparison with Ref. [33] we have collected all undetermined constants to this order in a new constant δ, chosen such that the $g_E^6 T$ contribution in Eq. (14), when expanded in g, is proportional to $\ln(1/g) + \delta$.

The unknown coefficient δ of course leaves the numerical outcome completely open until the required analytical and numerical calculations will have been performed. However, it has been observed in Ref. [33] that this coefficient could well be such that the perturbative result follows closely the 4D lattice results. In obtaining numerical results, Ref. [33] in fact used a particular optimized renormalization scheme, introduced in [35], which also involves keeping the parameters of the effective theory unexpanded. (References [36,33] also mentioned that this reduces the scale dependence.)

In Fig. 5 we present our numerical results obtained by adding the 4-loop terms (14) to the full 3-loop result of the previous section, but now evaluated with Eq. (13) in all terms involving g_E, for the possibilities $\delta = 0$, 1/3, 2/3. This differs from [33] in that we treat hard contributions strictly perturbatively and only soft ones without truncations. Furthermore, we use the perturbative 2-loop running coupling[9] and vary $\bar{\mu}$ about $2\pi T$ by a factor of 2 as above. The results, which are displayed by the medium-gray areas, show a remarkably small scale variation. By contrast, the strictly per-

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[9]Reference [33] found $\delta \approx 0.7$ to give results which agree well with the 4D lattice results. The difference to our central value of 1/3 is mainly due to the fact that we used 2-loop rather than 1-loop running coupling. Like Ref. [33] we included also the contribution (8), which is however fairly small.
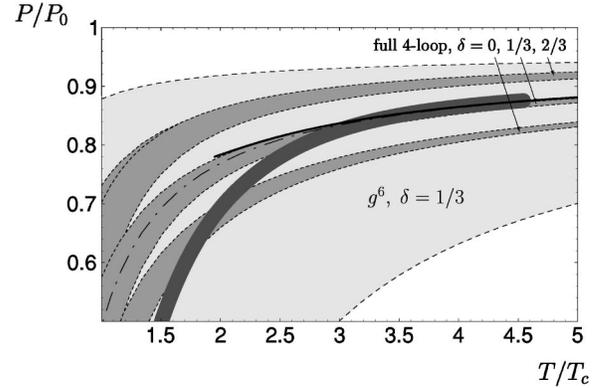

FIG. 5. Four-loop pressure in pure-glue QCD including the recently determined $g^6\ln(1/g)$ contribution of [33] together with three values for the undetermined constant δ in Eq. (14) when evaluated fully with $\bar{\mu}$ varied between πT and $4\pi T$ (medium-gray bands). The broad light-gray band underneath is the strictly perturbative result to order g^6 corresponding to the central value $\delta = 1/3$. The solid line gives the untruncated result with $\delta = 1/3$ extremalized with respect to $\bar{\mu}$ (which does not have solutions below $\sim 1.9T_c$); the dash-dotted line corresponds to fastest apparent convergence (FAC) in m_E^2, which sets $\bar{\mu} \approx 1.79\pi T$.

turbative evaluation (given for $\delta = 1/3$ only) shows an *increased* scale dependence when compared to the 3-loop results.

Also like in the previous 3-loop case, we find that the untruncated result has again a nonmonotonic scale dependence which makes it possible to fix the scale by PMS.[10] The result is again close to the FAC choice considered previously as well. The strictly perturbative result on the other hand, has a runaway scale dependence for almost all T.

III. SCREENED PERTURBATION THEORY IN THE SOFT SECTOR

In massless scalar field theories, which have a poorly convergent perturbative series for the thermal pressure similar to what is found in QCD, Karsch, Patkós and Petreczky and others [11–14] have proposed to ameliorate the situation by a variationally improved perturbation theory which uses a simple mass term as variational parameter. In this so-called screened perturbation theory the mass term is part of the tree-level Lagrangian as well as the interactions, where it is counted like a one-loop counterterm.

This approach has been extended to QCD by Andersen *et al.* [15–18] by using the hard-thermal-loop action in place of a simple mass term, turning its prefactor, which is proportional to the Debye mass squared, into a variational parameter. This HTL-screened perturbation theory has been recently carried through to two-loop order in QCD in Refs. [17,18]. The result is perturbatively correct to order $g^4\ln(g)$ and has been found to give rather stable results which are

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[10]This has been observed also before in Ref. [37], but using one-loop running and the particular parametrization of Ref. [33].

smaller than the ideal-gas pressure, but significantly above the lattice results.

In this section we investigate the possibility of improving the soft contributions considered above to two and three loop order by a dimensionally reduced screened perturbation theory for EQCD defined by trivially rewriting the EQCD Lagrangian according to

$$\mathcal{L}_{\text{EQCD}} = \frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2} (D_i A_0)^a (D_i A_0)^a + \frac{1}{2} (m_E^2 + \delta m^2) A_0^a A_0^a$$

$$- \frac{1}{2} \delta m^2 A_0^a A_0^a. \tag{15}$$

As above we take m_E^2 to be determined by perturbative matching to order g^2 and g^4 when calculating the pressure to two and three loop order, respectively, and we neglect λ_E because, as we have seen above, it starts to contribute only at order g^6 with small numerical effects.

A. Two-loop order

In DRSPT to two-loop order, the result for the pressure is given by the sum of Eqs. (4) and (5) except that the latter now involves

$$f_M^{(1)+(2)} = -\frac{2}{3\pi} m^3 + \frac{3}{8\pi^2} \left(\frac{1}{\epsilon} + 4 \ln \frac{\Lambda_E}{2m} + 3 \right) g_E^2 m^2 + \frac{m}{\pi} \delta m^2$$

$$+ \delta f_E^{\text{DRSPT}}, \tag{16}$$

where $m^2 = m_E^2 + \delta m^2$ according to Eq. (15). The term $m/\pi \delta m^2$ arises from a one-loop diagram where the counterterm δm^2, which itself has to be counted as a one-loop quantity, has been inserted.

SPT generally produces additional UV divergences and associated scheme dependences, which can be seen here in that the pole term in the second term on the right-hand side of Eq. (16) no longer matches Eq. (7). Following Refs. [11–14,17,18] we treat those by minimal subtraction. This means that we introduce a counterterm $\delta f_E^{\text{DRSPT}}$ with $\delta f_E^{\text{DRSPT}} - \delta f_E \propto g_E^2 \delta m^2 (1/\epsilon)$, which can be discarded in the perturbative matching as being of higher order and will disappear in fact at the next loop order (see below). However, the replacement of m_E^2 by $m_E^2 + \delta m^2$ has the effect of modifying the dependence on Λ_E at the present loop order.

The untruncated two-loop result, including now Eq. (16) in place of Eq. (6), can be optimized by a variational principle (principle of minimal sensitivity) for δm^2. This leads to the variational mass

$$m^2 = m_E^2 - 3\alpha_s(\bar{\mu}) Tm \left(4 \ln \frac{\Lambda_E}{2m} + 1 \right). \tag{17}$$

As before and similar to what is done in Refs. [17,18] we equate Λ_E and $\bar{\mu}$. Since the latter is always set proportional to T, this gives rise to a term $\delta m^2 \propto g^3 T^2 \ln(g)$, like in the actual next-to-leading order Debye mass in QCD [38]; however, both coefficient and sign are different (and the loga-
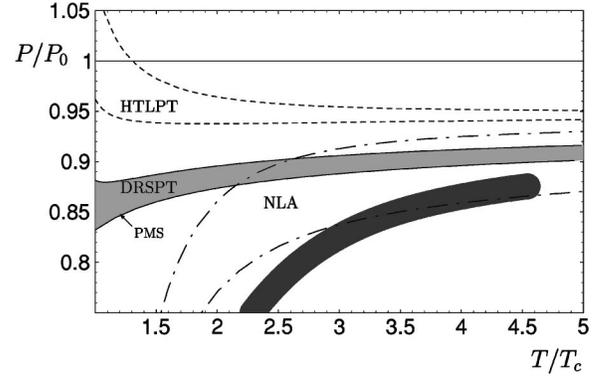

FIG. 6. The two-loop DRSPT result for pure-glue QCD (gray band) in comparison with the two-loop HTLPT result of Ref. [17] (dashed lines) and the "NLA" result of Ref. [20] (dash-dotted lines), all with $\bar{\mu}$ varied around $2\pi T$ by a factor of 2.

rithm involves an UV rather than an IR cutoff). This is no contradiction, however, since the variational mass (like the effective mass parameter m_E) is not identical with the Debye mass responsible for the exponential screening behavior of the full electrostatic propagator. In fact, we shall see below that at higher loop orders the gap equation does not contain a logarithmic term.

The simple "gap equation" (17) is different from the one obtained in 2-loop HTLPT [17,18].[11] It is much simpler in form, and it turns out to have a numerical solution that behaves differently from that of 2-loop HTLPT. While the latter increases sharply at large coupling, the solution of Eq. (17) saturates at the value $m \approx 0.503T$ as α_s increases when $N_f = 0$ and $\Lambda_E = \bar{\mu} = 2\pi T$ [for larger (smaller) $\bar{\mu}$ the saturation occurs at smaller (larger) values; for $\bar{\mu} < 1.4\pi T$ there is a maximum value of α_s beyond which solutions no longer connect continuously to the perturbative leading-order result—for instance, for our lowest value $\bar{\mu} = \pi T$ this restricts α_s to less than 0.34, which however presents no problem to the following application].

The thermal pressure at two-loop DRSPT is obtained by evaluating Eq. (16) at the solution of Eq. (17) and combining it with the hard contribution (4). The result for pure-glue QCD is given in Fig. 2 where it is compared with the untruncated two-loop result which uses the perturbative leading-order mass instead of the solution of the gap equation (17). In Fig. 6 the two-loop DRSPT result is furthermore compared to the result of the two-loop HTLPT calculation of Ref. [17] and the "NLA" result of Ref. [20] which is based on the Φ-derivable two-loop expression for the entropy evaluated with HTL propagators that include next-to-leading order corrections to asymptotic thermal masses. We find that the two-loop DRSPT result has a rather small scale dependence like the two-loop HTLPT result, but is significantly below the latter and thus closer to the lattice data. The non-

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[11] When fermions are included in 2-loop HTLPT [18] the latter gives rise to a gap equation for fermions as well, whereas in DRSPT fermions contribute only through the parameter m_E.
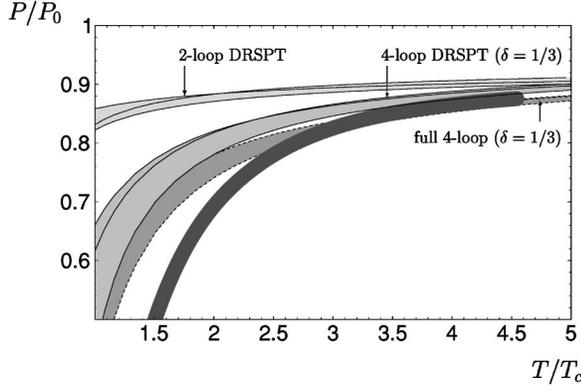
P/P_0

FIG. 7. The 4-loop DRSPT result for pure-glue QCD (3 solid lines corresponding to $\bar{\mu} = \pi T$, $2\,\pi T$, $4\,\pi T$ and $\delta = 1/3$) in comparison with the two-loop DRSPT result [now evaluated with the contribution (8) included] and the untruncated 4-loop with $\delta = 1/3$.

linear scale dependence of the DRSPT result in fact makes it possible to eliminate the scale dependence by a principle of minimal sensitivity. The result is given by the lower boundary of the DRSPT band in Fig. 6. This optimized DRSPT result happens to lie right at the center of the estimated error of the NLA result of Ref. [20] for $T > 3T_c$ [the fact that the latter sharply drops close to T_c is in fact not a prediction of the NLA result but comes from the necessity to fix an integration constant which has been chosen such that $P(T_c) = 0$].

Here we have neglected the soft 2-loop contribution involving λ_E of Eq. (8) because HTLPT does not include a comparable term. When Eq. (8) is included, the 2-loop gap equation is modified by an extra contribution $(-5/3\pi)(9 - N_f)\alpha_s^2 T m$ on the right-hand side of Eq. (17). Its (rather small) effect is displayed on the occasion of the comparison with 4-loop DRSPT in Fig. 7.

B. Three-loop order

The three-loop free energy of EQCD in DRSPT can be easily derived from the results of Ref. [7]:

$$f_M^{(1)+(2)+(3)} = -\frac{2}{3\pi} m^3 + \frac{3}{8\pi^2}\left(\frac{1}{\epsilon} + 4\ln\frac{\Lambda_E}{2m} + 3\right) g_E^2 m_E^2$$

$$+ \frac{9}{8\pi^3}\left(\frac{89}{24} - \frac{11}{6}\ln 2 + \frac{1}{6}\pi^2\right) g_E^4 m + \frac{m}{\pi}\delta m^2$$

$$+ \frac{3}{4\pi^2} g_E^2 \delta m^2 - \frac{1}{4\pi m}(\delta m^2)^2, \tag{18}$$

where $m^2 = m_E^2 + \delta m^2$ is now defined with the order-g^4 result for m_E^2, Eq. (10), and where up to two insertions of the SPT counterterm δm^2 have to be included.

At this order the δm^2 counterterm of DRSPT restores the UV divergent part to be proportional to m_E^2 [second term on

the RHS of Eq. (18)], and because no further pole terms appear at three-loop order, the additional UV divergences of SPT drop out altogether.

The attempt to determine m by a variational principle leads to the simple quartic gap equation

$$G^{(3)}(m) \equiv \frac{1}{8}(m^2 - m_E^2)^2 + \frac{g_E^2 N_c}{4\pi}(m^2 - m_E^2)m$$

$$+ \left(\frac{g_E^2 N_c}{4\pi}\right)^2\left(\frac{89}{24} - \frac{11}{6}\ln 2 + \frac{1}{6}\pi^2\right)m^2$$

$$= 0, \tag{19}$$

which we have written out for general color number N_c. However, one can readily prove that this equation has no solution that connects continuously to the perturbative result m_E as $\alpha_s \to 0$ [for any value of N_c; also inclusion of the term (8) does not change this situation].

The same problem in fact occurs in SPT in scalar ϕ^4 theory to three-loop order [13]. Like Ref. [13] one may look for alternative prescriptions to set up a gap equation. The simplest conceivable option, however, clearly is to set $\delta m^2 = 0$. This trivially connects to perturbation theory and amounts to our previous suggestion of keeping the 3-loop contributions of EQCD untruncated, which is in fact most natural since m_E in EQCD is just a mass parameter. Fortunately, the now nonlinear scale dependences can be eliminated by a principle of minimal sensitivity, as we have discussed above, so in a sense our above improvement may be taken as a trivial implementation of DRSPT, with the variational parameter being the renormalization and separation scale rather than an additional mass term.

C. Four-loop order

In DRSPT to four-loop order (neglecting λ_E contributions) we have

$$f_M^{(4)} = -\frac{27 g_E^6}{32\pi^4}\left(\left[\frac{43}{12} - \frac{157\pi^2}{768}\right]\ln\frac{\Lambda_E}{g_E^2} + \left[\frac{43}{4} - \frac{491\pi^2}{768}\right]\right.$$

$$\times \ln\frac{\Lambda_E}{m} + c\right) - \frac{9 g_E^4}{16\pi^3}\left(\frac{89}{24} - \frac{11}{6}\ln 2 + \frac{1}{6}\pi^2\right)\frac{\delta m^2}{m}$$

$$- \frac{3 g_E^2}{8\pi^2}\frac{(\delta m^2)^2}{m^2} - \frac{(\delta m^2)^3}{24\pi m^3}, \tag{20}$$

which is to be added to Eq. (18), evaluated now also with g_E^2 to order g^4 as given by Eq. (13).

The variational gap equation associated with the sum of Eqs. (18) and (20) does not involve the unknown constant c and is therefore known completely to order g_E^6 [in fact, the contribution (8) proportional to λ_E does not enter either, while λ_E^2 contributions to Eq. (20) are already of order g^9 or higher]. This equation is now of sixth order in m and reads

$$G^{(4)}(m) = (m^2 - m_E^2) G^{(3)}(m) + 2 \left(\frac{g_E^2 N_c}{4\pi} \right)^3 \left[\frac{43}{4} - \frac{491\pi^2}{768} \right] m^3$$

$$= 0. \tag{21}$$

As it turns out, the inclusion of the 4-loop logarithm lifts the impasse encountered with DRSPT at 3-loop order: there is now a (unique) solution to the 4-loop gap equation which does connect continuously to perturbation theory and which is given by the quadratic gap equation[12]

$$m^2 = m_E^2 - 0.33808 \, g_E^2 N_c m, \tag{22}$$

which appears as a factor of $G^{(4)}(m)$ and whose only solution with real and positive m is given by

$$m = \sqrt{m_E^2 + (0.16904 \, g_E^2 N_c)^2} - 0.16904 \, g_E^2 N_c. \tag{23}$$

It is intriguing that the quadratic gap equation (22) is of the same form as the one adopted in the NLA approximation of Ref. [20] for the asymptotic thermal masses, and also the coefficients in the two gap equations happen to be very close ($0.338 N_c$ versus N_c/π). However, it should be emphasized that the gap equation of DRSPT has no physical meaning outside of DRSPT. Indeed there is no reason to expect the leading correction to m_E as prescribed by the 4-loop DRSPT gap equation to remain the same at 5-loop level, if at that order solutions exist at all. At any given loop order, the deviation of m from m_E only influences the orders beyond the perturbative accuracy.

Numerically, the deviation of m as given by Eq. (22) from the perturbative value m_E has some effect as displayed in Fig. 7. It turns out that the 4-loop DRSPT differs from the untruncated 4-loop result of the previous section in that it has a significantly larger g^7 coefficient (by a factor of almost 6 when $\bar{\mu} = 2\pi T$). As a consequence, this gives a slightly different "prediction" for the unknown g^6 constant δ, but otherwise the results are quite similar to those obtained in the simple untruncated evaluation.

In fact, at 4-loop order the order-g^7 coefficient could in principle be calculated completely, if m_E^2 is determined to 3-loop accuracy and relevant higher-dimension operators in the effective theory are included. (The order g^6 coefficient of course remains beyond the reach of perturbation theory.) In this case, however, 4-loop DRSPT would spoil the achieved perturbative accuracy, because it changes the g^7 coefficient without correcting these changes through the SPT counterterms, which only take care of orders up to and including $m^3 T[(\delta m^2)/m^2]^3 \sim g^6 T^4$. Thus, beginning at 4-loop order, (DR)SPT ceases to be a possible improvement over (truncated or untruncated) perturbation theory.

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[12]The numerical coefficient therein is given by the (real) root of a cubic equation involving the somewhat unwieldy constants appearing in Eqs. (18) and (20) and could in principle be given in closed (but lengthy) form.
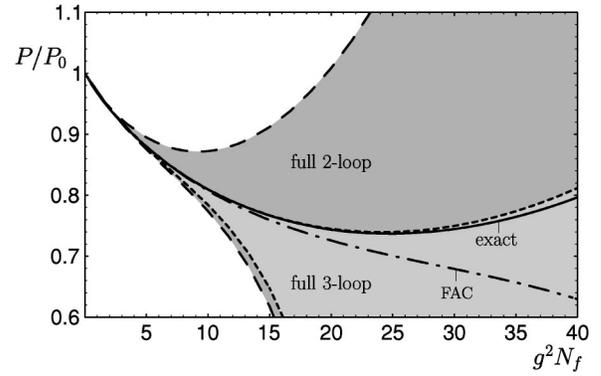

FIG. 8. The exact result of the thermal pressure in the limit of large N_f from Ref. [40], normalized to that of free gluons and as a function of $g^2 N_f(\bar{\mu} = \pi e^{-\gamma} T)$ (solid line), in comparison with the untruncated two-loop and three-loop results (darker and lighter gray areas, respectively), each with $\bar{\mu}$ varied around a central value of $2\pi T$ by a factor of e. The dash-dotted line corresponds to fastest apparent convergence.

IV. LARGE-N_f LIMIT

We finally consider also the recently solved large-N_f limit of QCD [39–41] which has been proposed as a testing ground for improvements of perturbation theory. In this limit only terms involving products $\alpha_s N_f$ are kept in the above results and α_s itself is taken to zero. The dimensionally reduced theory is therefore non-interacting in the large-N_f limit and the soft contributions are given exactly by $f_M = -(2/3\pi)m_E^3$. Still, at a given loop order for the hard contributions, we can investigate the difference between a strictly perturbative evaluation of f_M versus an untruncated one which resums an infinite number of terms with odd powers in g. Also the gap equations of DRSPT become trivial (but solvable): they all amount to setting $m^2 = m_E^2$.

The results of a numerical evaluation of the untruncated two-loop and three-loop results are compared with the exact result of Ref. [40] in Fig. 8. Again one can observe a great reduction of the scale dependence by going from two-loop to three-loop order. Compared to the strictly perturbative result to order g^5 (not displayed in Fig. 8) the reduction of the size of the scheme dependence is less important than in the pure-glue case; e.g., at $g^2 N_f = 10$ the reduction is about 16%.

Also in contrast to the pure-glue case, the three-loop result has a monotonic scale dependence, so the scale cannot be fixed by minimal sensitivity. As remarked in Ref. [40], $\bar{\mu}$ could instead be fixed by fastest apparent convergence. Requiring e.g. that the α_s^2 term in m_E^2 vanish leads to the choice $\bar{\mu}/T = \pi e^{1/2-\gamma} \approx 0.93\pi T$. For this value the untruncated 3-loop result coincides with the result to order g^5 in strict perturbation theory, which in turn agrees quite well with the exact result up to rather large coupling [40]. So while this comparison does not favor one over the other, it shows that with an optimal choice of the renormalization scale both the perturbative result to order g^5 and the untruncated 3-loop result fare rather well in the large-N_f limit.

If one applies the same prescription to the 3-loop result of EQCD in pure-glue QCD, one is lead to setting $\bar{\mu}/T$

$=4\pi e^{-\gamma-5/22}\approx 1.79\pi T$. This is lower but close to the scale selected by minimal sensitivity of the untruncated 3-loop result. Correspondingly, the numerical result following from the FAC choice is fairly close to the one obtained from minimal sensitivity (dash-dotted and solid line in Fig. 3, respectively), at least for $T\gtrsim 3T_c$, which appears to validate the PMS results.[13]

V. DISCUSSION AND CONCLUSION

We have found that the scale dependence and convergence of the results for the thermal pressure from perturbative QCD at high temperature are significantly improved when the contributions from soft scales as given by the effective dimensionally reduced theory EQCD are not treated in strict perturbation theory.

In particular, we have explored the predictions of a simple loop expansion of EQCD, in which, after strictly perturbative matching of the parameters of the effective Lagrangian, the results are not subsequently expanded out in powers of g and truncated. The result obtained at the two-loop level in this scheme includes contributions to order $g^4\ln g$ completely while being incomplete to order g^4 and is such that the pressure no longer exceeds the ideal-gas limit. The scale dependence is large; however, the three-loop result is within the estimated boundaries of the two-loop result. This three-loop result has a smaller scale dependence than that of strict perturbation theory to order g^5, and moreover the scale dependence is nonmonotonic so that it can be eliminated by a principle of minimal sensitivity. The correspondingly optimized result is rather close to the lattice data on the continuum limit of pure-glue QCD for $T\gtrsim 3T_c$. When including 4-loop effects, in particular the recently determined $g^6\ln g$ contribution of Ref. [33], we find that this trend continues and, in line with [33,37], that it is quite possible that all higher-order contributions add up to a very small correction above $\sim 3T_c$.

We have also considered variationally improved "screened" perturbation theory in the dimensionally reduced theory (DRSPT), where it is a much simpler, gauge invariant alternative to HTLPT (though not extensible to dynamic quantities, as HTLPT in principle is). The result for the pure-glue pressure when improved through 2-loop DRSPT turns out to be significantly lower than that of HTLPT and for $T\gtrsim 3T_c$ fairly close to the lattice results as well as to the results of Ref. [20].

An obvious advantage of DRSPT over HTLPT is that it does not modify the theory at hard momentum scales, where

the HTL approximation in general breaks down. The latter continues to be a good approximation at hard momentum scales only at soft virtuality. In the entropy-based HTL resummations of Ref. [20] it turns out that the contributions are predominantly coming from hard momenta close to the light cone. In the HTLPT approach, on the other hand, spurious contributions at hard momenta occurring at a given loop order are corrected for by the specific counterterms of HTLPT at next loop order, so this may present a problem at low loop orders. Our two-loop DRSPT results seem to indicate that this is indeed the reason for the difficulties of two-loop HTLPT [17,18].

An unsatisfactory feature of DRSPT, as observed before in SPT applied to scalar field theories and thus not unlikely to affect HTLPT as well, is that at three-loop order the mass gap equation does not have solutions which connect to perturbation theory. This impasse happens to be lifted by the inclusion of soft four-loop logarithms, and the result is then close to that obtained by a simple untruncated evaluation of all soft contribtions. Nevertheless, the gap equations of (DR)SPT have no particular physical interpretation [as discussed after Eqs. (17) and (23)], which casts some doubt on the systematics of SPT and its usefulness in improving perturbation theory.

Evidently, our main result is that the convergence behavior of successive approximations to the pressure is dramatically improved by abandoning strict perturbation theory in the soft sector. Treating this sector beyond strict perturbation theory is in fact closer in spirit to the so-called Φ-derivable approximations [42] which are the basis for the resummation techniques developed in Refs. [19,20]. Such approximations, when implemented in the soft sector, may represent an interesting alternative to DRSPT. DRSPT has a single variational parameter, the mass of electrostatic gluons. While this has the advantage of great simplicity as well as gauge invariance, the full self-energy of electrostatic gluons is a nonlocal quantity. One might consider a Φ-derivable approach which does not have the need for the specific counterterms of SPT and try to construct improved approximations, which are in principle gauge dependent, though such gauge dependences are strongly suppressed at the variational point [43].

It would be interesting to compare such a dimensionally reduced Φ-derivable approach with the one based on the entropy formalism of Refs. [19,20]. In the latter, the emphasis is on dynamical quasiparticles, which at two-loop order are interaction free. It should be noted that this approach is dependent on a real-time formalism which does not lend itself to dimensional reduction. Indeed, it involves differentiating thermal distribution functions at the stationary point, where the temperature dependence of spectral functions drops out. However, the relevant theory for the soft modes (including hard ones with soft virtuality) is known: at leading order this involves the non-local hard-thermal-loop effective action [44–46]). In fact, the real-time approach might have advantages when it comes to including the effects of high chemical potentials μ. In this case dimensional reduction does not occur. The quasiparticle approach on the other hand appears promising for covering the thermodynamics in the entire T-μ plane [47–49].

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[13]The upper boundary of the range of the three-loop results shown in Fig. 8 follows the exact result to even much larger values. There is in fact a choice of $\bar\mu\approx 0.75\pi T$ for both the untruncated three-loop result and the strictly perturbative result, where they become almost indistinguishable from the exact result. A renormalization scale close to this turns out to be favored by applying FAC to the perturbative result to order g^6, which has been extracted numerically in the large-N_f limit [41], but which is strictly nonperturbative in real QCD.

We intend to investigate these matters in future work. From the present study we conclude that perturbative QCD at high temperature is not limited to $T \gg 10^5 T_c$ as previously thought [4,6], but when supplemented by appropriate resummation techniques for soft physics, seems to be capable of remarkably good quantitative predictions down to $T \sim 2.5 T_c$.

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