

Hydrodynamic fluctuations, long-time tails, and supersymmetry

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Hydrodynamic fluctuations at a nonzero temperature can cause slow relaxation toward equilibrium even in observables which are not locally conserved. A classic example is the stress-stress correlator in a normal fluid, which, at zero wave number, behaves at large times as $t^{-3/2}$. A novel feature of the effective theory of hydrodynamic fluctuations in supersymmetric theories is the presence of Grassmann-valued classical fields describing macroscopic supercharge density fluctuations. We show that hydrodynamic fluctuations in supersymmetric theories generate essentially the same long-time power-law tails in real-time correlation functions that are known in simple fluids. In particular, a $t^{-3/2}$ long-time tail must exist in the stress-stress correlator of $\mathcal{N}=4$ supersymmetric Yang-Mills theory at non-zero temperature, regardless of the value of the coupling. Consequently, this feature of finite-temperature dynamics can provide an interesting test of the AdS/CFT correspondence. However, the coefficient of this long-time tail is suppressed by a factor of $1/N_c^2$. On the gravitational side, this implies that these long-time tails are not present in the classical supergravity limit; they must instead be produced by one-loop gravitational fluctuations.

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I. INTRODUCTION

Holographic AdS/CFT duality implies that properties of strongly coupled large- N_c quantum field theories can be deduced by doing calculations in classical (super)gravity [1–4]. The correspondence is believed to hold at non-zero temperatures, and can be helpful in extracting information about both supersymmetric and non-supersymmetric theories [5]. The most well-known example of AdS/CFT duality is $\mathcal{N}=4$ supersymmetric Yang-Mills theory with the gauge group $SU(N_c)$, in 3 + 1 dimensions, which is believed to be dual to type IIB string theory on the $AdS_5 \times S^5$ background [4]. However, to date, there are very few physical properties which can be independently calculated on both sides of the duality and thus used as non-trivial tests of the finite temperature version of the correspondence.

In this paper we focus on the low-frequency real-time dynamics (rather than static thermodynamics) of a finite temperature field theory. The relevant degrees of freedom are hydrodynamic fluctuations, by which one means those degrees of freedom whose relaxation time diverges with the wavelength.¹ A suitable version of hydrodynamics is the appropriate form for an effective theory characterizing these degrees of freedom. As we will discuss in detail, one consequence of hydrodynamic fluctuations at a non-zero temperature is the presence of long-time power-law tails in real-time correlation functions of conserved currents [6–8]. For correlations in the spatial parts of these currents at zero wave

number, one finds $t^{-d/2}$ behavior in d spatial dimensions, or equivalently, nonanalytic $\omega^{d/2}$ terms in the small-frequency behavior of the associated finite-temperature spectral densities.² These long-time tails are a generic feature of systems which behave as fluids on arbitrarily long time and large distance scales; their existence, and the value of the power-law exponent, are insensitive to the microscopic details of the theory. Hence, long-time tails in real-time thermal correlators may be added to the small set of observables about which one can make firm predictions even in strongly coupled theories.

We will apply our general results to the long-time behavior of thermal correlation functions in $SU(N_c)$ $\mathcal{N}=4$ supersymmetric Yang-Mills theory. Because this is a scale invariant quantum field theory, it cannot have phase transitions at any non-zero temperature (in infinite volume).³ In particular, it must act like a fluid at all non-zero temperatures, and hence should have a valid hydrodynamic description of its long-time dynamics. Consequently, one should (for reasons we will discuss) expect the zero-wave-number stress-stress correlator to show power-law relaxation,

$$\int d^3 \mathbf{x} \langle \{ T^{ij}(t, \mathbf{x}), T^{kl}(0) \} \rangle \sim t^{-3/2}, \quad (1)$$

as $t \rightarrow \infty$.

²More precisely, the spectral density divided by ω , which is related to the power spectrum of thermal fluctuations, has a non-analytic $|\omega|^{(d-2)/2}$ term at low frequency, unless $d=4n+2$ in which case an additional $\ln|\omega|$ is present. In many respects, long-time hydrodynamic tails are analogous to the non-analytic chiral logarithms which appear in one-loop chiral perturbation theory (at zero temperature) due to quantum fluctuations of massless Goldstone bosons.

³The $N_c \rightarrow \infty$ theory has a phase transition in finite volume. This is irrelevant for our considerations. See Ref. [5] for more detailed discussion of the phase diagram.

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¹In normal fluids, the relevant hydrodynamic degrees of freedom are fluctuations in energy and momentum density, or equivalently temperature and local fluid velocity. Hydrodynamic variables can also include fluctuations in charge densities associated with any globally conserved currents, as well as order parameter phase fluctuations (Goldstone modes) in theories with spontaneously broken continuous symmetries.

On the gravitational side of the AdS/CFT correspondence, the stress-stress correlator can be extracted from the absorption cross section $\sigma(\omega)$ for scattering of gravitons by non-extremal three-branes [9,10]. This calculation was performed by Policastro and Starinets [11]. In the case when the graviton frequency ω was small, Policastro, Son and Starinets [12] interpreted the zero-frequency limit as the shear viscosity of $\mathcal{N}=4$ supersymmetric Yang-Mills plasma. In Ref. [11], the small frequency behavior of the cross section was found to have the form $\sigma(\omega) = \sigma(0) + O(\omega^2)$. However, if the stress-stress correlator does exhibit the power-law tail (1), then AdS/CFT duality implies that the corresponding graviton cross section should be non-analytic at small ω and behave as $\sigma(\omega) - \sigma(0) \sim |\omega|^{1/2}$.

This apparent contradiction is the motivation for this paper. Our goal is to determine the coefficient of the long-time tail in the stress-stress correlator (1), and in analogous correlators involving spatial parts of other conserved currents. To do so, we will have to construct the correct effective description of low-frequency, long-distance dynamics in supersymmetric theories. We will find that supersymmetry (or superconformal) invariance of a quantum field theory does not preclude the existence of long-time tails, in accord with hydrodynamic expectations. However, in large N_c gauge theories, we will find that a $1/N_c^2$ suppression appears in the amplitude of long-time tails. Thus, on the gravity side of the correspondence, the small-frequency non-analyticity cannot be seen in classical supergravity (considered in Ref. [11]), but should emerge from a one-loop calculation on a plane-symmetric AdS black hole background.

The paper is organized as follows. Section II reviews what is meant by an effective hydrodynamic theory and summarizes the properties of hydrodynamic fluctuations in typical high temperature relativistic theories. Included is a discussion of why hydrodynamics predicts that fluctuations in the stress tensor T^{ij} , at zero wave number, decay as $t^{-3/2}$ at long times due to their coupling to both sound waves and transverse momentum density fluctuations. In Sec. III we construct effective hydrodynamics for supersymmetric theories, and argue that long-time tails are necessarily still present. The treatment of hydrodynamic fluctuations in Secs. II and III is applicable to general field theories. In Sec. IV we specialize the general results to $\mathcal{N}=4$ supersymmetric Yang-Mills theory in the context of the AdS/CFT correspondence.

II. EFFECTIVE HYDRODYNAMICS

Hydrodynamics may be viewed as an effective theory describing the dynamics of a thermal system on length and time scales which are large compared to any relevant microscopic scale [13,14]. It is a classical field theory, whose fields can be regarded as expectation values in some non-equilibrium thermal ensemble of microscopic quantum operators averaged over spatial volumes large compared to microscopic length scales. The beauty of a hydrodynamic description is that it applies to any system which acts like a fluid on sufficiently long distance scales, regardless of the strength of microscopic interactions. We will be concerned specifically with near-equilibrium behavior and the resulting

dynamics of small perturbations about some equilibrium state.

A. Hydrodynamic variables

To construct the appropriate effective hydrodynamics for a particular physical system, one has to identify the relevant low-frequency degrees of freedom (hydrodynamic variables) and find the equations of motion which govern their dynamics.⁴ As in any effective field theory, the symmetries, and symmetry realizations, of the underlying microscopic theory will constrain the form of the effective theory. All remaining dependence on the microscopic theory will be isolated in the particular values of some finite set of adjustable parameters appearing in the effective theory. In the case of hydrodynamics, these input parameters include transport coefficients (viscosity, diffusivity, conductivity, etc.) and equilibrium thermodynamic functions.

The degrees of freedom in a hydrodynamic description will include a minimal set of fields, whose thermal expectation values distinguish the possible equilibrium states of the theory. Among these are the energy and momentum densities, which we will denote as

$$\varepsilon \equiv T^{00}, \quad \pi^i \equiv T^{i0}, \quad (2)$$

as well as the charge densities $n_a \equiv J_a^0$ of any other conserved currents J_a^μ .⁵ The argument for this is as follows.

The usual grand canonical ensemble, with statistical density operator⁶

$$\hat{\rho} = Z^{-1} e^{\beta(u_\nu P^\nu + \mu_a N_a)} \quad (3)$$

built from the conserved charges

$$P^\nu \equiv \int d^3\mathbf{x} T^{\nu 0}(\mathbf{x}), \quad N_a \equiv \int d^3\mathbf{x} J_a^0(\mathbf{x}), \quad (4)$$

describes a manifold of time-independent equilibrium states in which variations in the thermodynamic parameters β , u^ν , and μ_a produce space-independent variations in the energy density, momentum density, and other charge densities. In other words, infinite wavelength variations in these densities have infinite relaxation time (precisely because they are densities of conserved charges). But in any local, causal theory, this implies that arbitrarily long wavelength variations in these densities must have relaxation times which diverge

⁴Because the effective hydrodynamic theory describes dissipative dynamics, it is easier to work directly with the equations of motion than with a classical Lagrangian formulation.

⁵Until otherwise stated, we assume that such symmetry currents are ordinary bosonic vector fields (i.e., not supercurrents).

⁶We use a $(-+++)$ metric convention. $\beta \equiv 1/T$ is the inverse temperature, u^ν is the rest frame 4-velocity (satisfying $u^2 = -1$), and μ_a are chemical potentials. We assume that translation invariance is a symmetry of the theory, and that this symmetry is not spontaneously broken. We work in three (flat, infinite) spatial dimensions throughout our analysis, but all results generalize trivially to $d > 3$ spatial dimensions.

with wavelength. Consequently, fluctuations in conserved charge densities remain relevant variables on arbitrarily long time scales, and must be retained in an effective hydrodynamic theory.⁷

If the theory under consideration is in a phase with spontaneously broken continuous symmetry, then the phase (or orientation) of the relevant order parameter is also needed to uniquely characterize equilibrium states, and fluctuations in the order parameter orientation will become another hydrodynamic degree of freedom.⁸ Well known examples include superfluid helium, where the phase of the condensate wave function becomes a hydrodynamic variable and is responsible for the appearance of second sound [14]. Another example is the chiral limit of QCD, whose hydrodynamic variables include the $SU(N_f)$ orientation of the chiral condensate, fluctuations in which describe pions [16]. At a second order phase transition, long wavelength fluctuations in the magnitude of an order parameter acquire divergent relaxation times (due to critical slowing down), and also become relevant hydrodynamic degrees of freedom. Finally, for gauge theories in a Coulomb phase, long wavelength magnetic fields have divergent relaxation times and must be retained in a hydrodynamic description; the effective theory in this case is termed magneto-hydrodynamics.

In the following discussion we will assume, for simplicity, that all these complications are absent. That is, we assume that the theory under consideration is in a phase of unbroken global symmetry, is not in a Coulomb phase, and is not sitting precisely at a second order phase transition. Also for simplicity, we will assume that the theory possesses charge conjugation symmetries under which any global conserved charges N_a transform non-trivially. We further assume that the equal-time (or more generally, imaginary-time) thermal correlation functions of the theory exhibit a finite correlation length. This is basically just a restatement of our assumed unbroken global symmetry, the absence of Coulomb phase gauge fields, and non-critical behavior. Finally, we assume that the theory under consideration is an interacting theory which describes a sensible equilibrating thermodynamic system—no discussion of hydrodynamic behavior is applicable to a non-interacting theory. All these assumptions hold in typical field theories without U(1) gauge fields, at sufficiently high temperatures.⁹

⁷For theories with approximate symmetries, an approximately conserved charge density will only act like a hydrodynamic degree of freedom on time scales which are short compared to the mean time between charge non-conserving reactions.

⁸Despite the pervasive misuse of the phrase “spontaneously broken gauge symmetry,” Higgs phases of gauge theories are not examples of spontaneous breaking of any physical symmetry [15], and do not imply the existence of an enlarged manifold of physical equilibrium states, or the presence of additional hydrodynamic degrees of freedom.

⁹This includes non-Abelian gauge theories with most any matter field content, because interacting scalar fields, fermions, and non-Abelian gauge fields all develop finite spatial correlation lengths at high temperature.

B. Constitutive relations

The resulting hydrodynamic description for this general class of theories takes the form of exact local conservation laws for the conserved currents,

$$\partial_\mu T^{\mu\nu}(x) = 0, \quad (5)$$

$$\partial_\mu J_a^\mu(x) = 0, \quad (6)$$

together with constitutive relations, valid on sufficiently long time and distance scales, expressing the fluxes of conserved quantities (i.e., spatial parts of conserved currents) as local functionals of the hydrodynamic variables themselves (the densities of conserved charges).¹⁰ As in any effective theory, possible terms appearing in these constitutive relations must be consistent with the symmetries of the underlying theory, and may be classified according to the number of spatial derivatives, and powers of fields representing departures from equilibrium. This is the analogue of the usual power counting in an effective field theory.

We will focus on the dynamics of fluctuations away from some charge conjugation invariant equilibrium state, for which all chemical potentials vanish. Therefore, the charge density $n_a(x)$ will only be non-vanishing due to some perturbation away from the equilibrium state of interest. Similarly, if we work in the rest frame of the system, then the momentum density $\boldsymbol{\pi}(x)$ will also be non-vanishing only due to a departure from our given equilibrium state.

The only terms, linear in deviations from our equilibrium state, which can appear in the resulting constitutive equation for the charge fluxes have the form

$$\mathbf{j}_a = -D_{ab} \nabla n_b, \quad (7)$$

where $D = \|D_{ab}\|$ is, in general, a matrix of diffusion coefficients characterizing, in a linear response approximation, the flux induced by a spatially non-uniform charge density. These diffusion coefficients are input parameters to the effective hydrodynamics.¹¹ Terms with three or more spatial derivatives are also allowed by symmetry but, for sufficiently

¹⁰More generally, one may formulate constitutive relations which, in addition to terms involving hydrodynamic variables, also contain noise terms representing the influence of short-distance degrees of freedom on the quantity of interest. Such noise terms convert the hydrodynamic equations of motion into Langevin equations, and allow the effective hydrodynamic theory to generate the correct equal time correlations of long wavelength equilibrium fluctuations. However, these noise terms will not be relevant for our purposes, and will be ignored throughout our discussion.

¹¹In terms of equilibrium correlation functions in the underlying microscopic theory, diffusion coefficients are given by the Kubo formula $(D\chi)_{ab} = \frac{1}{6} \lim_{\omega \rightarrow 0} \int d^4x e^{i\omega x^0} \langle \frac{1}{2} \{J_a^i(x), J_b^i(0)\} \rangle$, where $\langle \dots \rangle = Z^{-1} \text{tr}[e^{-\beta H} \dots]$ is an equilibrium thermal average, $\chi = \|\chi_{ab}\|$ is the charge susceptibility matrix, $\chi_{ab} = \partial n_a / \partial(\beta \mu_b) = [\langle N_a N_b \rangle - \langle N_a \rangle \langle N_b \rangle] / \mathcal{V}$, and \mathcal{V} is the spatial volume. The Kubo formula makes manifest the Onsager relation stating that $D\chi$ is a symmetric matrix.

long-wavelength fluctuations, will be negligible compared to the above single derivative terms. It should be emphasized that the constitutive relation (7) is not an operator identity, unlike Eq. (6). Rather it is a relation between expectation values of operators in non-equilibrium states, valid only when those operators are averaged over a volume large compared to relevant microscopic scales.

Of course, terms can also appear in the constitutive equation for \mathbf{j}_a which are non-linear in the departure from the given equilibrium state. At quadratic order¹² it is possible to add a term of the right symmetry which involves no spatial gradients, and is proportional to the product of charge and momentum density fluctuations,

$$\mathbf{j}_a = -D_{ab} \nabla n_b + \kappa n_a \boldsymbol{\pi}. \quad (8)$$

Since this new term involves no spatial derivatives, the coefficient κ is completely determined by thermodynamic derivatives which stay within the manifold of equilibrium states. Namely, an equilibrium state with non-vanishing chemical potentials, when viewed in an arbitrary reference frame, will have an energy-momentum tensor of the perfect fluid form,

$$T^{\mu\nu} = (\varepsilon + \mathcal{P}) u^\mu u^\nu + \mathcal{P} g^{\mu\nu}, \quad (9)$$

and conserved currents

$$J_a^\mu = n_a u^\mu, \quad (10)$$

where n_a , ε , and \mathcal{P} are the equilibrium charge densities, energy density, and pressure, respectively, in the rest frame of the fluid, and u^μ is the rest-frame 4-velocity.¹³ Hence, for infinitesimal boosts away from the rest frame, the momentum density and charge flux are related to the flow velocity via

$$\boldsymbol{\pi} = (\varepsilon + \mathcal{P}) \mathbf{v}, \quad \mathbf{j}_a = n_a \mathbf{v}, \quad (11)$$

implying that the coefficient κ appearing in the constitutive relation (8) equals $(\varepsilon + \mathcal{P})^{-1}$. For later convenience, we will denote the enthalpy density, which is the sum of energy density and pressure, as

$$w \equiv \varepsilon + \mathcal{P}, \quad (12)$$

so $\kappa = w^{-1}$ and $\boldsymbol{\pi} = w \mathbf{v}$.

In the constitutive relation for the spatial stress T^{ij} , terms without spatial derivatives are also completely determined by equilibrium thermodynamics. Let $\bar{\beta}$, $\bar{\varepsilon}$, and $\bar{\mathcal{P}}$ denote the inverse temperature, energy density, and pressure, respec-

tively, of the chosen equilibrium state with vanishing charge and momentum densities. Then the stress tensor of a nearby equilibrium state with energy density $\bar{\varepsilon} + \delta\varepsilon$, momentum density $\boldsymbol{\pi}$, and charge densities n_a , expanded to second order in deviations away from the reference state, has the form

$$T^{ij}|_{\text{equil.}} = \delta^{ij} \left[\bar{\mathcal{P}} + v_s^2 \delta\varepsilon + \frac{1}{2} \xi (\delta\varepsilon)^2 + \frac{1}{2} \Xi_{ab} n_a n_b - v_s^2 \frac{\boldsymbol{\pi}^2}{\bar{w}} \right] + \frac{\boldsymbol{\pi}^i \boldsymbol{\pi}^j}{\bar{w}}, \quad (13)$$

where $\delta\varepsilon \equiv T^{00} - \bar{\varepsilon}$ is the fluctuation in energy density, $v_s \equiv (\partial \bar{\mathcal{P}} / \partial \bar{\varepsilon})^{1/2}$ is the speed of sound, $\xi \equiv \partial^2 \bar{\mathcal{P}} / \partial \bar{\varepsilon}^2$, and $\Xi_{ab} \equiv \partial^2 \bar{\mathcal{P}} / \partial n_a \partial n_b$. (Energy derivatives are to be evaluated at constant charge density, and vice versa.) No terms linear in n_a can be present due to the assumed charge conjugation invariance of the reference equilibrium state. An exercise in thermodynamic derivatives shows that

$$v_s^2 = \frac{\partial \bar{\mathcal{P}}}{\partial \bar{\varepsilon}} = \frac{\partial \bar{\mathcal{P}}}{\partial \bar{\beta}} \left(\frac{\partial \bar{\varepsilon}}{\partial \bar{\beta}} \right)^{-1} = \frac{\bar{w}}{\bar{\beta}} C^{-1}, \quad (14)$$

$$\xi = [(1 + 2v_s^2) C^{-1} - \bar{w} \Gamma^{HHH}] / \bar{\beta}, \quad (15)$$

$$\Xi_{ab} = [(\chi^{-1})_{ab} - \bar{w} \Gamma_{ab}^{NNH}] / \bar{\beta}. \quad (16)$$

We have defined C and χ as the mean square fluctuation in energy or charge per unit volume,

$$C \equiv \overline{\langle H^2 \rangle}_{\text{conn}} / \mathcal{V}, \quad (17)$$

$$\chi_{ab} \equiv \overline{\langle N_a N_b \rangle}_{\text{conn}} / \mathcal{V}, \quad (18)$$

and Γ^{HHH} and Γ^{NNH} as ‘‘amputated’’ third order connected correlators of energy and charge,

$$\Gamma^{HHH} \equiv C^{-3} \overline{\langle H^3 \rangle}_{\text{conn}} / \mathcal{V}, \quad (19)$$

$$\Gamma_{ab}^{NNH} \equiv C^{-1} (\chi^{-1})_{aa'} (\chi^{-1})_{bb'} \overline{\langle N_{a'} N_{b'} H \rangle}_{\text{conn}} / \mathcal{V}, \quad (20)$$

with \mathcal{V} the spatial volume, $\langle \dots \rangle$ denoting an expectation in the reference equilibrium state, and χ^{-1} the matrix inverse of $\chi \equiv \|\chi_{ab}\|$. Note that the heat capacity per unit volume $c_v \equiv \partial \varepsilon / \partial T = \beta^2 C$, while the charge susceptibility $\partial n_a / \partial \mu_b = \beta \chi_{ab}$.

Terms linear in momentum density can also appear in the constitutive relation for stress, but only when combined with a spatial gradient so as to yield a rank-2 tensor. Two linearly independent structures are possible, proportional to the shear or divergence of the vector field, so that

¹²Terms higher than quadratic are irrelevant for our purposes. This will be discussed further at the end of Sec. II E.

¹³The local rest frame of any thermal system, at a particular event x , may always be defined as the frame in which the momentum density $\boldsymbol{\pi}(x)$ vanishes. The local flow velocity in any other frame is then defined as the velocity needed to boost to the local rest frame. This implies that the local 4-velocity $u^\nu(x)$ is, in general, the time-like eigenvector of $T^\mu{}_\nu(x)$.

$$T^{ij} = \delta^{ij} \left[\bar{\mathcal{P}} + v_s^2 \delta\varepsilon + \frac{1}{2} \xi (\delta\varepsilon)^2 + \frac{1}{2} \Xi_{ab} n_a n_b - v_s^2 \frac{\pi^2}{w} \right] + \frac{\pi^i \pi^j}{w} - \gamma_\zeta \delta^{ij} \nabla \cdot \boldsymbol{\pi} - \gamma_\eta (\nabla^i \pi^j + \nabla^j \pi^i - \frac{2}{3} \delta^{ij} \nabla \cdot \boldsymbol{\pi}). \quad (21)$$

The coefficients γ_ζ and γ_η are conventionally written as

$$\gamma_\zeta \equiv \frac{\zeta}{w}, \quad \gamma_\eta \equiv \frac{\eta}{w}, \quad (22)$$

with ζ and η the bulk and shear viscosities, respectively.¹⁴

All other terms, consistent with symmetries, which could be added to the constitutive relations (8) or (21) involve either more gradients, or more powers of fluctuations away from equilibrium relative to the terms included in Eq. (8) or (21), and so are negligible for sufficiently small, long-wavelength fluctuations.¹⁵

The above constitutive relation for the stress tensor simplifies considerably in scale-invariant theories. The tracelessness of the energy-momentum tensor, $T^\mu{}_\mu = 0$, is an operator identity in such theories, valid in both equilibrium and non-equilibrium states (at vanishing chemical potentials). Consequently, in any scale invariant theory the equation of state is exactly $\bar{\varepsilon} = 3\bar{\mathcal{P}}$, the speed of sound $v_s = 1/\sqrt{3}$, and the thermodynamic curvatures ξ and Ξ vanish identically,¹⁶ as does the bulk viscosity ζ . Therefore, the constitutive relation for the stress tensor (21) in a scale invariant theory takes the simpler form

$$T^{ij} = \delta^{ij} \left(\bar{\mathcal{P}} + \frac{1}{3} \delta\varepsilon \right) + \frac{1}{w} \left(\pi^i \pi^j - \frac{1}{3} \delta^{ij} \pi^2 \right) - \gamma_\eta \left(\nabla^i \pi^j + \nabla^j \pi^i - \frac{2}{3} \delta^{ij} \nabla \cdot \boldsymbol{\pi} \right). \quad (23)$$

C. Linearized hydrodynamics

If one retains only terms linear in fluctuations in the constitutive relations (8) and (21), then it is trivial to solve the resulting linearized hydrodynamic equations. For charge

¹⁴Kubo formulas for the bulk and shear viscosities are $\zeta = (\beta/2) \lim_{\omega \rightarrow 0} \int d^4x e^{i\omega t} \langle \frac{1}{2} \{ \mathcal{P}(x), \mathcal{P}(0) \} \rangle$, and $\eta = (\beta/20) \lim_{\omega \rightarrow 0} \int d^4x e^{i\omega x^0} \langle \frac{1}{2} \{ s^{ij}(x), s^{ij}(0) \} \rangle$, where $\mathcal{P} \equiv \frac{1}{3} T^{ii}$ and $s^{ij} \equiv T^{ij} - \mathcal{P} \delta^{ij}$ are the trace and traceless parts of the stress tensor, respectively.

¹⁵It turns out that the non-linear interactions of hydrodynamic fluctuations cause the coefficients of higher order terms to become scale dependent quantities, just like couplings in a typical effective field theory [17].

¹⁶Hence, for scale invariant theories $\bar{w} = \frac{4}{3} \bar{\varepsilon}$, $C = 4\bar{\varepsilon}/\bar{\beta}$, $\Gamma^{HHH} = \frac{5}{16} \bar{\beta}/\bar{\varepsilon}^2$, and $\Gamma^{NNH} = \chi^{-1}/\bar{w}$. These relations also follow by simple thermodynamics, given a free energy $F \sim \mathcal{V} T^4$ and susceptibility $\chi_{ab} \sim T^3$.

density fluctuations, inserting $\mathbf{j} = -D \nabla n$ into the conservation law $\partial_t n = -\nabla \cdot \mathbf{j}$ yields the usual diffusion equation $(\partial_t - D \nabla^2) n = 0$. (We have suppressed indices distinguishing multiple charge densities, but in general D is a matrix of diffusion constants.) Inserting a spatial Fourier transform,

$$n(t, \mathbf{x}) \equiv \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} n(t, \mathbf{k}), \quad (24)$$

immediately gives

$$n(t, \mathbf{k}) = e^{-k^2 D t} n(0, \mathbf{k}), \quad (25)$$

showing that fluctuations with wave vector \mathbf{k} relax diffusively at a rate $k^2 D$ which vanishes quadratically as $\mathbf{k} \rightarrow 0$. Momentum density fluctuations may be decomposed into longitudinal and transverse parts,

$$\boldsymbol{\pi}(t, \mathbf{x}) = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} [\hat{\mathbf{k}} \pi_{\parallel}(t, \mathbf{k}) + \boldsymbol{\pi}_{\perp}(t, \mathbf{k})], \quad (26)$$

where $\hat{\mathbf{k}} \cdot \boldsymbol{\pi}_{\perp}(t, \mathbf{k}) \equiv 0$. For transverse momentum fluctuations, the conservation relation $\partial_t \pi^i = -\nabla_j T^{ij}$ plus the constitutive relation (21) again combine to give a diffusion equation, $(\partial_t - \gamma_\eta \nabla^2) \boldsymbol{\pi}_{\perp} = 0$, so that

$$\boldsymbol{\pi}_{\perp}(t, \mathbf{k}) = e^{-k^2 \gamma_\eta t} \boldsymbol{\pi}_{\perp}(0, \mathbf{k}). \quad (27)$$

Because energy and momentum densities are both conserved quantities, fluctuations in energy and longitudinal momentum density are coupled. One finds that both quantities satisfy the damped oscillator equation $(-\partial_t^2 - \gamma_s k^2 \partial_t - v_s^2 k^2) \delta\varepsilon(t, \mathbf{k}) = 0$, where $\gamma_s \equiv \gamma_\zeta + \frac{4}{3} \gamma_\eta$, and that $|\mathbf{k}| \pi_{\parallel}(t, \mathbf{k}) = i \partial_t \delta\varepsilon(t, \mathbf{k})$. In the long wavelength limit ($\gamma_s |\mathbf{k}| \ll v_s$), one has weakly damped sound waves,

$$\begin{pmatrix} i \pi_{\parallel}(t, \mathbf{k}) \\ v_s \delta\varepsilon(t, \mathbf{k}) \end{pmatrix} = e^{-(1/2)k^2 \gamma_s t} \begin{pmatrix} \cos(|\mathbf{k}| v_s t) & \sin(|\mathbf{k}| v_s t) \\ -\sin(|\mathbf{k}| v_s t) & \cos(|\mathbf{k}| v_s t) \end{pmatrix} \times \begin{pmatrix} i \pi_{\parallel}(0, \mathbf{k}) \\ v_s \delta\varepsilon(0, \mathbf{k}) \end{pmatrix}, \quad (28)$$

propagating at the sound speed v_s , and whose energy attenuates at a rate of $k^2 \gamma_s$.

D. Real-time correlators

The non-equilibrium linear response results (25), (27) and (28), describing the relaxation of specific perturbations away from equilibrium, may be converted into equivalent results for real-time correlation functions characterizing the spectrum of fluctuations in the equilibrium thermal ensemble.¹⁷

¹⁷This is sometimes referred to as Onsager's postulate.

Consider, for example,¹⁸ the two-point correlation of charge density fluctuations,

$$A_{n_a n_b}(t, \mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \langle \frac{1}{2} \{n_a(t, \mathbf{x}), n_b(0, \mathbf{0})\} \rangle. \quad (29)$$

For $t > 0$, one may insert the linear response result (25) and find¹⁹

$$A_{nn}(t, \mathbf{k}) \equiv \|A_{n_a n_b}(t, \mathbf{k})\| = e^{-k^2 D t} \chi(\mathbf{k}), \quad (30)$$

where $\chi(\mathbf{k}) \equiv \|\chi_{ab}(\mathbf{k})\|$ is the variance matrix of equal-time charge density fluctuations,

$$\chi_{ab}(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \langle \frac{1}{2} \{n_a(\mathbf{x}), n_b(\mathbf{0})\} \rangle. \quad (31)$$

In the small- \mathbf{k} regime where a hydrodynamic description is valid, one may replace this variance by its small \mathbf{k} limit, which is the charge susceptibility matrix χ introduced previously,

$$\lim_{\mathbf{k} \rightarrow 0} \chi_{ab}(\mathbf{k}) = [\langle N_a N_b \rangle - \langle N_a \rangle \langle N_b \rangle] / \mathcal{V} = \chi_{ab}. \quad (32)$$

Analogous results for the real-time correlations of energy and momentum densities are

$$A_{\varepsilon\varepsilon}(t, \mathbf{k}) = e^{-(1/2)k^2 \gamma_s |t|} \cos(|\mathbf{k}|v_s t) C, \quad (33a)$$

$$iA_{\pi^i \varepsilon}(t, \mathbf{k}) = \hat{k}^i v_s e^{-(1/2)k^2 \gamma_s |t|} \sin(|\mathbf{k}|v_s t) C, \quad (33b)$$

$$iA_{\varepsilon \pi^j}(t, \mathbf{k}) = (\hat{k}^j / v_s) e^{-(1/2)k^2 \gamma_s |t|} \sin(|\mathbf{k}|v_s t) \langle \frac{1}{3} \mathbf{P}^2 \rangle / \mathcal{V}, \quad (33c)$$

¹⁸The effective hydrodynamic theory, in which fields are classical, cannot distinguish between the symmetrized correlator $\langle \frac{1}{2} \{n(t), n(0)\} \rangle$ and the Wightman correlator $\langle n(t)n(0) \rangle$. In frequency space, this difference is smaller than the correlators themselves by a factor of $\hbar\omega/T$, which is negligible in the low frequency domain where the hydrodynamic description is valid. We have chosen to write expressions involving the symmetrized correlator just because this correlation function is always real (for Hermitian operators).

¹⁹For $t < 0$, one may use time reversal (or *CPT* plus rotation invariance) to show that $A_{nn}(-t, \mathbf{k}) = A_{nn}(t, \mathbf{k})$. Consequently, a Fourier transform in time gives $A_{nn}(\omega, \mathbf{k}) = 2k^2 D [\omega^2 + (k^2 D)^2]^{-1} \chi$. Combined with the Ward identity $\partial_\mu \langle \frac{1}{2} \{J^\mu(x), J^\nu(0)\} \rangle = 0$, one finds associated hydrodynamic forms for the charge-current correlator, $A_{nj^i}(\omega, \mathbf{k}) = 2\omega k^i D [\omega^2 + (k^2 D)^2]^{-1} \chi$, and the current-current correlator, $A_{j^i j^j}(\omega, \mathbf{k}) = 2\omega^2 \delta^{ij} D [\omega^2 + (k^2 D)^2]^{-1} \chi$. As required, this current-current correlator is consistent with the Kubo formula of footnote 11.

$$A_{\pi^i \pi^j}(t, \mathbf{k}) = [(\delta^{ij} - \hat{k}^i \hat{k}^j) e^{-k^2 \gamma_s |t|} + \hat{k}^i \hat{k}^j e^{-(1/2)k^2 \gamma_s |t|} \cos(|\mathbf{k}|v_s t)] \langle \frac{1}{3} \mathbf{P}^2 \rangle / \mathcal{V}, \quad (33d)$$

where $A_{\pi^i \pi^j}(t, \mathbf{k}) \equiv \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \langle \frac{1}{2} \{\pi^i(t, \mathbf{x}), \pi^j(0, \mathbf{0})\} \rangle$, etc., and C is the mean square fluctuation in energy per unit volume defined in Eq. (17). The mean-square fluctuation in total spatial momentum may be deduced from the equilibrium relation (11) between momentum density and flow velocity, which implies

$$\frac{\langle P^i P^j \rangle}{\mathcal{V}} = \frac{1}{\beta} \frac{\partial \pi^i}{\partial v^j} = \bar{w} T \delta^{ij}, \quad (34)$$

or $\langle \frac{1}{3} \mathbf{P}^2 \rangle / \mathcal{V} = \bar{w} T$. Hence the mean square fluctuations in momentum and energy are related by the velocity of sound,

$$\frac{\langle \frac{1}{3} \mathbf{P}^2 \rangle}{\mathcal{V}} = T \bar{w} = T^2 \frac{\partial \bar{P}}{\partial T} = v_s^2 T^2 \frac{\partial \bar{\varepsilon}}{\partial T} = v_s^2 C. \quad (35)$$

E. Long-time tails

Consider a spatial charge flux integrated over all space,

$$\mathbf{J}_a(t) \equiv \int d^3x \mathbf{j}_a(t, \mathbf{x}). \quad (36)$$

If non-linear terms in the constitutive relation for \mathbf{j}_a are ignored, this zero wave number charge flux has no coupling to hydrodynamic variables due to the gradient in the linear $-D\nabla n$ part of its constitutive relation. Hence, a linear response analysis predicts that the zero wave number flux should relax on a short, microscopic time scale.²⁰ This conclusion is wrong, because the charge flux does couple to hydrodynamic degrees of freedom, even at zero wave number, through the non-linear $n\boldsymbol{\pi}/\bar{w}$ term in its constitutive relation (8). Although this term is quadratic in deviations away from the chosen equilibrium state, because it contains no spatial gradients it will always dominate over the single gradient $-D\nabla n$ term of linear response for sufficiently long wavelengths.

To evaluate the zero-wave-number $\langle J_a^i(t) J_b^j(0) \rangle$ correlator correctly for large times, one needs only to express these currents in terms of the product of charge and momentum density fluctuations, and then insert the results (30) and (33d) for the spectrum of these fluctuations. One finds the well-known result for correlations in simple fluids [6]:

²⁰This may be seen explicitly from the linear response form of the current-current correlator $A^{il}(\omega, \mathbf{k})$ shown in footnote 19, whose $\mathbf{k} \rightarrow 0$ limit is frequency independent, $\lim_{\mathbf{k} \rightarrow 0} A^{il}(\omega, \mathbf{k}) = 2\delta^{il} D \chi$.

$$\begin{aligned}
 \mathcal{V}^{-1} \langle \frac{1}{2} \{ J_a^i(t), J_b^j(0) \} \rangle &= \frac{1}{\bar{w}^2} \int d^3 \mathbf{x} \langle \frac{1}{2} \{ n_a(t, \mathbf{x}) \pi^i(t, \mathbf{x}), n_b(0, \mathbf{0}) \pi^j(0, \mathbf{0}) \} \rangle \\
 &= \frac{1}{\bar{w}^2} \int d^3 \mathbf{x} \langle \frac{1}{2} \{ n_a(t, \mathbf{x}), n_b(0, \mathbf{0}) \} \rangle \langle \frac{1}{2} \{ \pi^i(t, \mathbf{x}), \pi^j(0, \mathbf{0}) \} \rangle \\
 &= \frac{1}{\bar{w}^2} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} A_{n_a n_b}(t, \mathbf{k}) A_{\pi^i \pi^j}(t, -\mathbf{k}) \\
 &= \frac{T}{\bar{w}} \delta^{ij} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left[\frac{2}{3} e^{-k^2(D+\gamma_\eta)|t|} \chi + \frac{1}{3} e^{-k^2[D+(1/2)\gamma_s]|t|} \cos(|\mathbf{k}|v_s t) \chi \right]_{ab} \\
 &= \frac{T}{\bar{w}} \frac{\delta^{ij}}{12} \left\{ \frac{1}{[(D+\gamma_\eta)\pi|t|]^{3/2}} \chi \right\}_{ab} + (\text{exponential decay}). \tag{37}
 \end{aligned}$$

Factorizing the first correlation function into a product of two-point correlators is justified because, in the small- \mathbf{k} hydrodynamic regime, the distribution of fluctuations is arbitrarily close to Gaussian.²¹ After inserting the explicit forms of $C_{n_a n_b}(t, \mathbf{k})$ and $C_{\pi^i \pi^j}(t, \mathbf{k})$ in the third step, an angular average over the direction of \mathbf{k} was performed. In the final step, the term involving longitudinal momentum fluctuations does not contribute to the $t^{-3/2}$ long-time tail because of the oscillating cosine; this term only gives rise to an exponentially falling $O[\exp\{-\frac{1}{4}|t|v_s^2/(D+\frac{1}{2}\gamma_s)\}]$ contribution. The above treatment is essentially perturbation theory for small fluctuations in the long-lived degrees of freedom; upon factorization, the two-point correlators of conserved densities are taken to evolve according to linear response.

This calculation is illustrated graphically in Fig. 1. It is important to understand that the one-loop diagram in this figure is a depiction of the origin of the long-time tail in the effective hydrodynamic theory; this is not a Feynman diagram of the underlying microscopic theory.²²

²¹The essential point is that hydrodynamic variables represent microscopic fields averaged over spatial volumes which are large compared to the ‘‘correlation volume’’ ξ^3 (where ξ is the static correlation length). Hence, a hydrodynamic fluctuation with wavelength $\lambda \gg \xi$ may be regarded as averaging over roughly $(\lambda/\xi)^3$ essentially independent correlation volumes. By the central limit theorem, the distribution of such an average will approach a Gaussian distribution as the number of independent samples becomes large. Residual non-Gaussian correlations in fluctuations with wave numbers of order \mathbf{k} will generate corrections to the result (37) which are suppressed by an additional factor of $|\mathbf{k}|^3 \xi^3$, leading to sub-dominant $O(t^{-3})$ corrections to the final result.

²²In terms of Feynman diagrams of the fundamental theory, each propagator of Fig. 1 involves a resummation of an infinite series of ladder-like graphs in non-gauge theories [18,19], or a vastly more complicated set of diagrams in gauge theories [20]. Attempting to calculate the coefficient of the long-time tail by evaluating and summing the required set of perturbative graphs in the underlying theory is extremely ill advised; using the appropriate effective theory is infinitely easier.

A completely analogous long-time tail must also be present in the zero wave number stress-stress correlator due to the coupling of the stress tensor to products of energy, momentum, and charge density fluctuations. Inserting the constitutive relation (21), and evaluating all the terms which can contribute to a long-time tail is straightforward, but slightly tedious. This calculation is summarized in the Appendix. Here we will specialize to the simpler case of a scale invariant theory, for which

$$\begin{aligned}
 T^{ij}(t) &\equiv \int d^3 \mathbf{x} T^{ij}(t, \mathbf{x}) \\
 &= \int d^3 \mathbf{x} \left\{ \delta^{ij} \left[\bar{P} + \frac{1}{3} \delta \varepsilon(t, \mathbf{x}) \right] \right. \\
 &\quad \left. + \frac{1}{\bar{w}} H_{mn}^{ij} \pi^m(t, \mathbf{x}) \pi^n(t, \mathbf{x}) \right\}, \tag{38}
 \end{aligned}$$

where $H_{mn}^{ij} \equiv \frac{1}{2} \delta_m^i \delta_n^j + \frac{1}{2} \delta_n^i \delta_m^j - \frac{1}{3} \delta^{ij} \delta_{mn}$ is a projector onto traceless symmetric tensors. Therefore the zero-wave-number stress-stress correlator in the hydrodynamic regime is

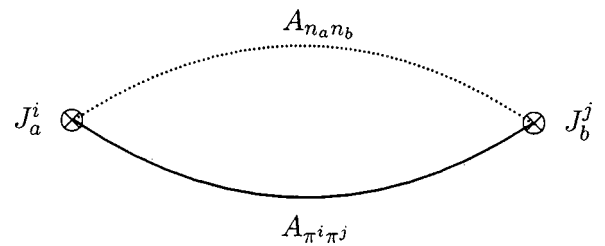


FIG. 1. Graphical depiction of the effective hydrodynamic calculation of the long-time tail in the $\langle J_a^i(t) J_b^j(0) \rangle$ correlator. Propagators in the effective theory characterize the relaxation of conserved densities according to linearized hydrodynamics. External vertex factors represent the coefficients of non-linear terms in the constitutive relations for the currents. Only the charge diffusion and transverse shear modes contribute to the leading long-time tail in current-current correlators.

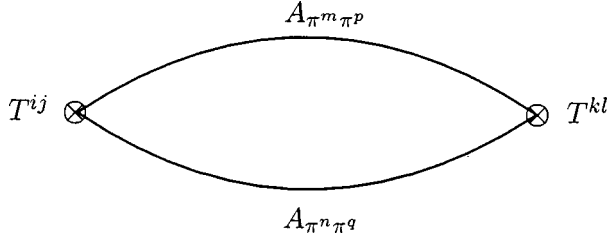


FIG. 2. Illustration of the effective hydrodynamic calculation of the long-time tail in the $\langle T^{ij}(t)T^{kl}(0) \rangle$ correlator. External vertex factors represent the coefficients of non-linear terms in the constitutive relation for the stress tensor, and are proportional to the projector H_{mn}^{ij} defined in the text. Both transverse shear and propagating sound modes contribute to the long-time tail in the stress-stress correlator.

$$\begin{aligned} & \mathcal{V}^{-1} \langle \frac{1}{2} \{ T^{ij}(t), T^{kl}(0) \} \rangle \\ &= \int d^3 \mathbf{x} \left\{ \frac{1}{w^2} H_{mn}^{ij} H_{pq}^{kl} \right. \\ & \quad \times \langle \frac{1}{2} \{ \pi^m(t, \mathbf{x}) \pi^n(t, \mathbf{x}), \pi^p(0, \mathbf{0}) \pi^q(0, \mathbf{0}) \} \rangle \\ & \quad \left. + \frac{1}{9} \delta^{ij} \delta^{kl} \langle \frac{1}{2} \{ \varepsilon(t, \mathbf{x}), \varepsilon(0, \mathbf{0}) \} \rangle \right\}. \end{aligned} \quad (39)$$

The first term may be evaluated using the same logic as above. Retaining only pieces which contribute to the connected correlator [i.e. $\langle \frac{1}{2} \{ T^{ij}(t), T^{kl}(0) \} \rangle - \langle T^{ij}(t) \rangle \langle T^{kl}(0) \rangle$], one finds

$$\begin{aligned} & \frac{1}{w^2} \int d^3 \mathbf{x} H_{mn}^{ij} H_{pq}^{kl} \langle \frac{1}{2} \{ \pi^m(t, \mathbf{x}) \pi^n(t, \mathbf{x}), \pi^p(0, \mathbf{0}) \pi^q(0, \mathbf{0}) \} \rangle_{\text{conn}} \\ &= \frac{2}{w^2} H_{mn}^{ij} H_{pq}^{kl} \int d^3 \mathbf{x} \langle \frac{1}{2} \{ \pi^m(t, \mathbf{x}), \pi^p(0, \mathbf{0}) \} \rangle \\ & \quad \times \langle \frac{1}{2} \{ \pi^n(t, \mathbf{x}), \pi^q(0, \mathbf{0}) \} \rangle \\ &= \frac{2}{w^2} H_{mn}^{ij} H_{pq}^{kl} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} A_{\pi^m \pi^p}(t, \mathbf{k}) A_{\pi^n \pi^q}(t, -\mathbf{k}) \\ &= \frac{2T^2}{15} H_{kl}^{ij} \left[\left(\frac{3}{2} \right)^{3/2} + 7 \right] \frac{1}{(8\pi\gamma_\eta |t|)^{3/2}}. \end{aligned} \quad (40)$$

This calculation is depicted graphically in Fig. 2. Once again, contributions which fall exponentially with time have been dropped in the last step. The final term in Eq. (39) gives a time-independent contribution:

$$\begin{aligned} & \frac{1}{9} \delta^{ij} \delta^{kl} \int d^3 \mathbf{x} \langle \frac{1}{2} \{ \varepsilon(t, \mathbf{x}), \varepsilon(0, \mathbf{0}) \} \rangle = \frac{1}{9} \delta^{ij} \delta^{kl} A_{\varepsilon\varepsilon}(t, \mathbf{k}=\mathbf{0}) \\ &= \frac{4}{9} \varepsilon T \delta^{ij} \delta^{kl}. \end{aligned} \quad (41)$$

This reflects the direct coupling between stress and fluctuations in energy density. At zero wave number, these become fluctuations in the total energy (in the equilibrium grand canonical ensemble), which are strictly conserved in time. Hence, one finds that the leading long-time behavior of the connected zero-wave-number stress-stress correlator in a scale invariant theory is²³

$$\begin{aligned} & \mathcal{V}^{-1} \langle \frac{1}{2} \{ T^{ij}(t), T^{kl}(0) \} \rangle_{\text{conn}} \\ & \sim \frac{4}{9} \delta^{ij} \delta^{kl} \varepsilon T + \frac{2}{15} \left[\left(\frac{3}{2} \right)^{3/2} + 7 \right] \\ & \quad \times \left(\frac{1}{2} \delta^{ik} \delta^{jl} + \frac{1}{2} \delta^{il} \delta^{jk} - \frac{1}{3} \delta^{ij} \delta^{kl} \right) \frac{T^2}{(8\pi\gamma_\eta |t|)^{3/2}}. \end{aligned} \quad (42)$$

Adding any higher-order terms to the stress tensor constitutive relation (23) will generate sub-dominant power-law tails in Eq. (42) proportional to $|t|^{-5/2}$, $|t|^{-7/2}$, etc. This is because such higher-order terms will contain either additional spatial gradients or additional powers of fluctuations away from equilibrium. In either case, one finds contributions which scale with higher powers of \mathbf{k} as $\mathbf{k} \rightarrow 0$, implying additional powers of $1/|t|$.

F. Low frequency behavior

The long-time tails present in the current-current correlator (37) or the stress-stress correlator (42) automatically imply that the Fourier transforms of these correlation functions cannot be analytic at zero frequency.²⁴ A little analysis shows that if $G(t) \sim \alpha/|t|^{3/2}$ at large times then²⁵

²³Once again, this result is for three spatial dimensions. In $d > 3$ spatial dimensions, the same analysis leads to a $|t|^{-d/2}$ long-time tail. In two or fewer spatial dimensions, hydrodynamic fluctuations are sufficiently infrared-singular that the $\omega \rightarrow 0$ limits in the Kubo formulas in footnotes 11 and 14 fail to exist. Instead, one finds that transport coefficients exhibit non-trivial scale dependence on arbitrarily long length scales [21].

²⁴This follows from the standard argument showing that if an integrable function $\tilde{G}(\omega)$ is analytic in a strip of width 2δ surrounding the real axis, then its Fourier transform $G(t)$ falls off exponentially as $O(e^{-\delta|t|})$, as may be seen by shifting the contour of integration.

²⁵More generally, if $G(t) \sim \alpha|t|^{-d/2}$, then the leading non-analyticity in $\tilde{G}(\omega)$ for small frequency is $\tilde{G}(\omega) \sim \alpha [i^{(d-1)(d+1)/4} \sqrt{2} \pi / \Gamma(d/2)] |\omega|^{(d-2)/2}$ for d odd, $\tilde{G}(\omega) \sim \alpha [\pi(-1)^{d/4} / \Gamma(d/2)] |\omega|^{(d-2)/2}$ for $d=4n$, and $\tilde{G}(\omega) \sim \alpha [(-1)^{(d+2)/4} / \Gamma(d/2)] |\omega|^{(d-2)/2} \ln|\omega|$ for $d=4n+2$.

$$\tilde{G}(\omega) = \int dt e^{i\omega t} G(t) \sim -\sqrt{8\pi} \alpha |\omega|^{1/2} + (\text{analytic terms}), \quad (43)$$

as $\omega \rightarrow 0$. Consequently, in three spatial dimensions, the Fourier transforms of the real-time zero-wave-number correlators $\langle \frac{1}{2} \{J_a^i(t), J_b^j(0)\} \rangle$ and $\langle \frac{1}{2} \{T^{ij}(t), T^{kl}(0)\} \rangle$ must have square root branch points at zero frequency whose coefficients are related via Eq. (43) to the amplitudes of their long-time tails.

In thermal equilibrium, the spectral density of any two operators A and B ,

$$\rho_{AB}(\omega) \equiv \int dt e^{i\omega t} \langle [A(t), B(0)] \rangle, \quad (44)$$

is directly related to the Fourier transform of the corresponding symmetrized correlator

$$\rho_{AB}(\omega) = 2 \tanh(\beta\omega/2) \int dt e^{i\omega t} \langle \frac{1}{2} \{A(t), B(0)\} \rangle. \quad (45)$$

Consequently, the current-current or stress-stress spectral densities, divided by $\beta\omega$, must have the same non-analyticity at $\omega=0$ as do the Fourier transformed real-time correlators. For example, in a scale-invariant theory one has

$$\begin{aligned} & \frac{1}{\omega} \int dx^0 d^3\mathbf{x} e^{i\omega x^0} \langle [T_{ij}(x), T_{kl}(0)] \rangle \\ &= \left(\frac{1}{2} \delta_{ik} \delta_{jl} + \frac{1}{2} \delta_{il} \delta_{jk} - \frac{1}{3} \delta_{ij} \delta_{kl} \right) \\ & \times [s^{(0)} + s^{(1)} |\omega|^{1/2} + O(|\omega|^{3/2})], \end{aligned} \quad (46)$$

with

$$s^{(0)} = 4\eta, \quad s^{(1)} = -\frac{T}{60\pi} \left[\left(\frac{3}{2} \right)^{3/2} + 7 \right] \left(\frac{\bar{w}}{\eta} \right)^{3/2}. \quad (47)$$

G. Large N_c scaling

The stress-stress correlator will, of course, have short time transient contributions in addition to its long time power-law tail. In large N_c gauge theories (or large N matrix models) if the 't Hooft coupling $g^2 N_c$ is held fixed as N_c becomes large, then the short time contribution to the (connected) stress-stress correlator will scale as N_c^2 , just because there are $O(N_c^2)$ degrees of freedom. In contrast, from Eq. (42) one sees that the coefficient of the $t^{-3/2}$ long time tail will be $O(1)$ as $N_c \rightarrow \infty$ if the shear viscosity divided by the enthalpy, $\gamma_\eta = \eta/w$, has a finite, non-zero large N_c limit. This ratio is effectively a microscopic relaxation time, the mean free time for large angle scattering of elementary excitations in the system. In weakly coupled high temperature gauge theories, this ratio does have a finite large N_c limit [22]. Presumably this is generally true, at least in theories like $\mathcal{N}=4$ supersymmetric Yang-Mills theory which are strongly believed not to have any phase transition separating their

weak and strong coupling regimes. Assuming this is the case, the cross-over time (beyond which the long-time tail dominates over the short time transients) will become arbitrarily large as $N_c \rightarrow \infty$.²⁶ For any fixed time t , the $t^{-3/2}$ tail will not be visible in the leading large N_c behavior of the stress-stress correlator, but will instead appear as a sub-leading $1/N_c^2$ relative correction:

$$\langle T_{ij}(t) T_{kl}(0) \rangle \sim O(N_c^2) e^{-t/\tau} + O(1) t^{-3/2}.$$

Exactly the same result is true of the long-time tail in current-current correlators. Diffusion constants are $O(1)$ as $N_c \rightarrow \infty$ (at least in weakly coupled hot gauge theories [22]) and therefore for fixed time t , the long time tail in the current correlator (37) also scales as $O(1/N_c^2)$ relative to the short time transients (which scale the same as the susceptibility χ).

In both cases, the result that long-time tails become sub-leading effects at large N_c may be regarded as a consequence of the fact that equilibrium velocity fluctuations are ‘‘anomalously’’ small when $N_c \rightarrow \infty$. As noted earlier in Sec. II B, the flow velocity equals the momentum density divided by the enthalpy. Therefore, the mean square fluctuations in average flow velocity are directly related to the mean square fluctuations in total momentum. Using the previous result (34), one has

$$\langle \bar{v}^i \bar{v}^j \rangle = \frac{\langle P^i P^j \rangle}{\bar{w}^2 \mathcal{V}^2} = \frac{\delta^{ij} T}{\bar{w} \mathcal{V}} = O(N_c^{-2}) \quad (48)$$

as $N_c \rightarrow \infty$ for fixed volume. More generally, this means that the fluctuations in flow velocity averaged over a patch of fluid of linear size L have a $1/N_c$ suppression (on top of the expected $L^{-3/2}$ behavior). Consequently, in the spatial current density $\mathbf{j} = -D\nabla n + n\mathbf{v}$, the non-linear $n\mathbf{v}$ contribution is $1/N_c$ suppressed relative to the linear $-D\nabla n$ term. Exactly the same conclusion holds for the terms quadratic in flow velocity in the stress tensor (23) as compared to the linear terms involving the gradient of the velocity.

III. SUPERSYMMETRIC HYDRODYNAMICS

We now wish to generalize the preceding discussion of hydrodynamic fluctuations to the case of supersymmetric theories. For ease of presentation, in this section we consider theories with $\mathcal{N}=1$ supersymmetry in four dimensions. Such theories possess a set of conserved supercurrents, which we will denote by $S_\alpha^\mu(x)$ and $\bar{S}_\alpha^\mu(x)$, which are Lorentz vector-

²⁶Completely analogous nonuniformities between the large distance and large N limits are well known in other contexts. For example, adjoint representation Wilson loops in $SU(N_c)$ Yang-Mills theory have a cross-over from area-law to perimeter-law behavior on a distance scale which diverges as $N_c \rightarrow \infty$. And various $SU(N)$ symmetric two-dimensional models have correlators which fall with distance like $|x|^{-1/N}$ [23].

spinors. The corresponding conserved supercharges satisfy the supersymmetry algebra,²⁷

$$\{Q_\beta, \bar{Q}_{\dot{\gamma}}\} = 2\sigma_{\beta\dot{\gamma}}^\mu P_\mu. \quad (49)$$

The theory may also have a chiral $U(1)$ symmetry (R symmetry), whose charge R does not commute with the supercharges,

$$[Q_\alpha, R] = Q_\alpha, \quad [\bar{Q}_{\dot{\alpha}}, R] = -\bar{Q}_{\dot{\alpha}}, \quad (50)$$

as well as various additional internal symmetries whose charges N_a do commute with the supercharges. For definiteness, we will assume that the theory does have R symmetry, and that this symmetry is not spontaneously broken.²⁸

Since the supercurrents are conserved, one might expect that fluctuations in supercharge densities will be unable to relax locally, just as ordinary current conservation directly leads to diffusive relaxation of bosonic conserved charge densities. And slow relaxation of supercharge densities may in turn generate long-time tails in *bosonic* observables which can couple quadratically to supercharge densities. In other words, even though supercharge densities are fermionic operators with vanishing expectation value in any conventional statistical ensemble, the dynamics of supercharge fluctuations may produce distinctive effects in ordinary bosonic observables. We will see that these expectations are largely correct.

A. Supercharge fluctuations

The hydrodynamic nature of supercharge densities is reflected in the long-time behavior of the corresponding correlation function,

$$C_{\dot{\alpha}\dot{\beta}}^{\mu\nu}(x) \equiv \langle \bar{S}_{\dot{\alpha}}^\mu(x) S_{\dot{\beta}}^\nu(0) \rangle. \quad (51)$$

Finite-temperature thermal ensembles are not invariant under

²⁷We follow the notations of Wess and Bagger [24] for spinors. In particular, undotted early greek indices α, β, \dots label $(\frac{1}{2}, 0)$ two-component Weyl spinors, while corresponding dotted indices label $(0, \frac{1}{2})$ conjugate Weyl spinors. These indices are raised or lowered using the antisymmetric tensor $\epsilon_{21} = \epsilon^{12} = \epsilon_{\dot{2}\dot{1}} = \epsilon^{\dot{1}\dot{2}} = 1$, so that $\xi^\alpha Q_\alpha = -\xi_\alpha Q^\alpha$ and $\bar{\xi}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}} = -\bar{\xi}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}$. Extended Pauli matrices are $\sigma_{\alpha\dot{\beta}}^\mu \equiv (-1, \boldsymbol{\sigma})$ and $\bar{\sigma}^{\mu\dot{\alpha}\dot{\beta}} \equiv (-1, -\boldsymbol{\sigma})$, and the spinor representation generators of the Lorentz group are $(\sigma^{\mu\nu})_{\alpha\dot{\beta}} \equiv \frac{1}{4}(\sigma_{\alpha\dot{\alpha}}^\mu \bar{\sigma}^{\nu\dot{\alpha}\dot{\beta}} - \sigma_{\alpha\dot{\alpha}}^\nu \bar{\sigma}^{\mu\dot{\alpha}\dot{\beta}})$ and $(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} \equiv \frac{1}{4}(\bar{\sigma}^{\mu\dot{\alpha}\dot{\alpha}} \sigma_{\dot{\alpha}\dot{\beta}}^\nu - \bar{\sigma}^{\nu\dot{\alpha}\dot{\alpha}} \sigma_{\dot{\alpha}\dot{\beta}}^\mu)$.

²⁸In $\mathcal{N}=1$ supersymmetry, conservation of the $U(1)$ R symmetry current requires scale invariance. Non-scale invariant theories will have anomalies in the R symmetry current, although it is often then possible to combine the anomalous R symmetry with an anomalous chiral symmetry to yield a non-anomalous R symmetry. If there is no non-anomalous R symmetry, then the R -charge density will not be a hydrodynamic degree of freedom. For such theories, ignore all mention of the R -symmetry current in the following discussion.

supersymmetry transformations.²⁹ This is reflected in the fact that at finite temperature bosons and fermions obey different statistics (Bose-Einstein or Fermi-Dirac), as well as in the fact that bosonic and fermionic fields obey different boundary conditions (periodic versus anti-periodic) in Euclidean functional integrals representing equilibrium thermal systems. But it is important to understand that the non-invariance of thermal ensembles under supersymmetry does not imply that the supercurrent fails to be conserved. The conservation of the supercurrent,

$$\partial_\mu S_\alpha^\mu(x) = 0 \quad (52)$$

is an operator identity, valid in any physical state. Since thermal averages are just linear combinations of expectation values in physical states, the conservation of the supercurrent implies that the thermal supercurrent correlation function satisfies the Ward identity³⁰

$$\partial_\mu C_{\dot{\alpha}\dot{\beta}}^{\mu\nu}(x) = 0. \quad (53)$$

Similarly, taking thermal expectations of both sides of the supersymmetry algebra (49) shows that mean square fluctuations in supercharge are directly related to the total energy of the thermal ensemble. A spatial Fourier transform of the Ward identity (53) gives

$$\partial_0 \bar{C}_{\dot{\alpha}\dot{\beta}}^{0\nu}(\mathbf{k}, t) = -ik_j \bar{C}_{\dot{\alpha}\dot{\beta}}^{j\nu}(\mathbf{k}, t) = O(k), \quad (54)$$

which directly shows that long wavelength fluctuations in supercharge density must relax arbitrarily slowly.

²⁹Formally, the transformation of a statistical density matrix ρ under any infinitesimal symmetry transformation is defined by the condition that $\text{Tr}(\delta\rho \mathcal{O}) = \text{Tr}(\rho \delta\mathcal{O})$, where \mathcal{O} is any observable. This is a generalization of the relation between the Schrödinger and Heisenberg pictures, and expresses the equivalence between the active and passive views of symmetry transformations. If \mathcal{G} is the generator of the transformation, then the infinitesimal transformation of the observable is $\delta\mathcal{O} = [\mathcal{G}, \mathcal{O}]$. For normal (bosonic) symmetries, this of course implies that $\delta\rho = -[\mathcal{G}, \rho]$, so invariance under the symmetry means the density matrix ρ commutes with the symmetry generator \mathcal{G} . But for an infinitesimal supersymmetry transformation, $\mathcal{G} = \xi^\alpha Q_\alpha$ with ξ an external Grassmann spinor which anticommutes with all fermionic operators. In this case, one has $\text{Tr}(\rho [\xi Q, \mathcal{O}]) = \text{Tr}(\xi \rho [Q, \mathcal{O}]) = \text{Tr}([\xi \rho, Q] \mathcal{O}) = \text{Tr}(\{\rho, \xi Q\} \mathcal{O})$, so the variation of the density matrix is the anticommutator with the supersymmetry generator, $\delta\rho = \{\rho, \xi Q\} = \xi \{\rho, Q\}$, not the commutator. (We assume throughout that ρ and \mathcal{O} are bosonic operators—possibly constructed by multiplying odd numbers of fermionic operators by independent external Grassmann spinors.) Therefore, even though the equilibrium canonical ensemble $\rho = Z^{-1} e^{-\beta H}$ commutes with the supercharge Q , this does not mean that the canonical ensemble is invariant under supersymmetry.

³⁰This assumes one suitably defines the short-distance subtraction in the renormalization of the product of supercurrents to remove what would otherwise be contact terms on the right hand side. One can always do so, provided the supersymmetric theory under discussion exists.

B. Super-thermodynamics

Having realized that supercharge densities are hydrodynamic degrees of freedom in supersymmetric theories, we would like to construct the appropriate hydrodynamic equations in complete analogy with the earlier non-supersymmetric treatment. To do so, we must construct the appropriate constitutive relations for the spatial part of the supercurrent (and the R -symmetry current), and also understand possible new terms involving products of supercharge densities which might appear in the constitutive relations for the stress tensor and other ordinary symmetry currents.

As discussed previously, non-derivative terms in constitutive relations are completely determined by equilibrium thermodynamic derivatives of the free energy. This is true provided one has introduced chemical potentials conjugate to all conserved densities of interest. To do this for a supersymmetric theory, we must first understand the generalization of ordinary thermodynamics to “equilibrium” ensembles in which there is a non-zero expectation value of the supercharge, produced by turning on a super-chemical potential.

To construct an equilibrium state with non-zero supercharge density, consider generalizing the usual statistical operator (3) by adding supercharges multiplied by Grassmann-valued super-chemical potentials,³¹

$$\hat{\rho}_s \equiv Z^{-1} \exp[\beta(u_\nu P^\nu + \mu^\alpha Q_\alpha + \bar{\mu}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}})]. \quad (55)$$

To simplify expressions, we are temporarily suppressing any chemical potentials for bosonic conserved charges. Note that the addition of spinorial chemical potentials coupled to the supercharges Q_α and $\bar{Q}_{\dot{\alpha}}$ maintains translation invariance of the equilibrium state but explicitly breaks rotation invariance.

In the ensemble described by $\hat{\rho}_s$, the supercharges Q_α and $\bar{Q}_{\dot{\alpha}}$ have non-zero expectation values proportional to $\bar{\mu}$ and μ , respectively. More precisely, to first order in μ and $\bar{\mu}$,

$$\begin{aligned} \mathcal{V}^{-1} \langle \xi^\alpha Q_\alpha \rangle_{\mu, \bar{\mu}} &\equiv \text{Tr}(\hat{\rho}_s \xi^\alpha Q_\alpha) / \mathcal{V} \\ &= -\beta \langle \bar{\mu}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}} \xi^\alpha Q_\alpha \rangle / \mathcal{V} \\ &= -\beta \xi^\alpha \bar{\mu}^{\dot{\beta}} \langle \frac{1}{2} \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} \rangle / \mathcal{V} \end{aligned}$$

³¹Though fermionic chemical potentials may seem peculiar, one can regard them as a non-dynamical gravitino field, to which the theory is coupled. It should be stressed that these generalized density operators do not have the usual statistical interpretation. Their diagonal matrix elements, although commuting, are not real numbers, and therefore cannot be viewed as statistical probabilities. Nevertheless, these generalized density operators may formally be regarded as equilibrium ensembles, in which generalized thermodynamic relations hold. The logical alternative of introducing real-valued chemical potentials for supercharges is formally inconvenient and also fails to yield any satisfactory physical interpretation. In particular, a density matrix containing a real-valued chemical potential for supercharges, when applied to the union of two disjoint physical systems, does not factorize, even if the two systems are causally disconnected [25].

$$\begin{aligned} &= -\beta \xi^\alpha \sigma_{\alpha\dot{\beta}}^\nu \bar{\mu}^{\dot{\beta}} \langle T_\nu^0 \rangle \\ &= \beta \xi^\alpha (\bar{\epsilon} \sigma^0 - \bar{w} \boldsymbol{\sigma} \cdot \mathbf{u})_{\alpha\dot{\beta}} \bar{\mu}^{\dot{\beta}}, \end{aligned} \quad (56)$$

where $O(\mathbf{u}^2)$ corrections have been dropped in the last equality. Here ξ is an independent Grassmann spinor. The intermediate expectations in Eq. (56) denote expectation values in the usual canonical ensemble (with 4-velocity u), and trace cyclicity and the supersymmetry algebra (49) have been used. Expanding to higher order in μ and $\bar{\mu}$ would lead to corrections of order $\mu \bar{\mu}^2$ which will not be relevant for our purposes. Similarly,

$$\begin{aligned} \mathcal{V}^{-1} \langle \bar{\xi}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}} \rangle_{\mu, \bar{\mu}} &\equiv \mathcal{V}^{-1} \text{Tr}(\hat{\rho}_s \bar{\xi}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}) \\ &= \beta \mu^\alpha (\bar{\epsilon} \sigma^0 - \bar{w} \boldsymbol{\sigma} \cdot \mathbf{u})_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}}, \end{aligned} \quad (57)$$

up to corrections of order $\mu^2 \bar{\mu}$ [and $O(\mathbf{u}^2)$].

Naively, one would expect the generalized partition function

$$Z(\beta, u, \mu, \bar{\mu}) \equiv \text{Tr} \exp[\beta(u_\nu P^\nu + \mu^\alpha Q_\alpha + \bar{\mu}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}})] \quad (58)$$

to be a generating function for expectation values of supercharges and products of supercharges. However, this is not true. Although the two terms we have added to the exponential defining $\hat{\rho}_s$ both commute with the total momentum P^ν , they do not commute with each other. As a result, the derivative $(\partial/\partial\mu) \ln Z(\beta, u, \mu, \bar{\mu})$ does not equal $\beta \langle Q \rangle$. In fact, the generalized partition function is completely independent of the super-chemical potentials, and simply equals the usual (physical) partition function,

$$Z(\beta, u, \mu, \bar{\mu}) = Z(\beta, u, 0, 0) = \text{Tr} e^{\beta u_\nu P^\nu}. \quad (59)$$

To see this, first note that

$$\text{Tr}[\mu^\alpha e^{\beta u_\nu P^\nu} \dots] = \mu^\alpha \text{Tr}[(-1)^F e^{\beta u_\nu P^\nu} \dots], \quad (60)$$

and therefore

$$\frac{\partial}{\partial \mu^\alpha} \text{Tr}[e^{\beta u_\nu P^\nu} \dots] = \text{Tr}\left[(-1)^F \frac{\partial}{\partial \mu^\alpha} e^{\beta u_\nu P^\nu} \dots\right]. \quad (61)$$

Here $(-1)^F \equiv e^{2\pi i J_z}$ is the operator which multiplies all bosonic states by $+1$ and all fermionic states by -1 . This operator commutes with P^ν and anticommutes with the supercharges. Next, factorize the exponentials in $\hat{\rho}_s$ using the Baker-Campbell-Hausdorff formula,

$$\begin{aligned} e^{\beta(u_\nu P^\nu + \mu^\alpha Q_\alpha + \bar{\mu}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}})} &= e^{\beta u_\nu P^\nu} e^{\beta \mu^\alpha Q_\alpha} e^{\beta \bar{\mu}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}} \\ &\quad \times e^{-(1/2)\beta^2 [\mu^\alpha Q_\alpha, \bar{\mu}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}]} \\ &= e^{\beta \mu^\alpha Q_\alpha} e^{\beta \bar{\mu}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}} e^{\beta P_\nu (u^\nu - \beta \mu^\alpha \sigma_{\alpha\dot{\alpha}}^\nu \bar{\mu}^{\dot{\alpha}})}. \end{aligned} \quad (62)$$

Consequently,

$$\begin{aligned}
& \frac{1}{\beta} \frac{\partial}{\partial \mu^\gamma} Z(\beta, u, \mu, \bar{\mu}) \\
&= \frac{1}{\beta} \text{Tr} \left[(-1)^F \frac{\partial}{\partial \mu^\gamma} e^{\beta \mu^\alpha Q_\alpha} e^{\beta \bar{\mu}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}} e^{\beta P_\nu (u^\nu - \beta \mu^\alpha \sigma_{\alpha\dot{\alpha}}^\nu \bar{\mu}^{\dot{\alpha}})} \right] \\
&= \text{Tr} [(-1)^F (Q_\gamma - \beta \sigma_{\gamma\dot{\alpha}}^\nu \bar{\mu}^{\dot{\alpha}} P_\nu) e^{\beta \mu^\alpha Q_\alpha} e^{\beta \bar{\mu}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}} \\
&\quad \times e^{\beta P_\nu (u^\nu - \beta \mu^\alpha \sigma_{\alpha\dot{\alpha}}^\nu \bar{\mu}^{\dot{\alpha}})}] \\
&= \text{Tr} [(-1)^F e^{\beta \mu^\alpha Q_\alpha} e^{\beta \bar{\mu}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}} e^{\beta P_\nu (u^\nu - \beta \mu^\alpha \sigma_{\alpha\dot{\alpha}}^\nu \bar{\mu}^{\dot{\alpha}})} \\
&\quad \times (-Q_\gamma - \beta \sigma_{\gamma\dot{\alpha}}^\nu \bar{\mu}^{\dot{\alpha}} P_\nu)] \\
&= \text{Tr} [(-1)^F (-Q_\gamma + \beta \sigma_{\gamma\dot{\alpha}}^\nu \bar{\mu}^{\dot{\alpha}} P_\nu) e^{\beta \mu^\alpha Q_\alpha} e^{\beta \bar{\mu}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}} \\
&\quad \times e^{\beta P_\nu (u^\nu - \beta \mu^\alpha \sigma_{\alpha\dot{\alpha}}^\nu \bar{\mu}^{\dot{\alpha}})}]. \tag{63}
\end{aligned}$$

The penultimate step uses the anticommutation of Q with $(-1)^F$ and trace cyclicity to move Q from the left to the right end of the trace. The final step uses the supersymmetry algebra to commute Q through the exponential of \bar{Q} :

$$\begin{aligned}
[Q_\gamma, e^{\beta \bar{\mu}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}}] &= \beta \bar{\mu}^{\dot{\gamma}} \{Q_\gamma, \bar{Q}_{\dot{\gamma}}\} e^{\beta \bar{\mu}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}} \\
&= 2\beta \sigma_{\gamma\dot{\gamma}}^\nu \bar{\mu}^{\dot{\gamma}} P_\nu e^{\beta \bar{\mu}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}}. \tag{64}
\end{aligned}$$

Since the last line in Eq. (63) is minus the second line, this shows that $(\partial/\partial \mu^\gamma)Z(\beta, u, \mu, \bar{\mu})=0$. One may show that $(\partial/\partial \bar{\mu}_{\dot{\gamma}})Z(\beta, u, \mu, \bar{\mu})=0$ in a completely similar fashion.

In other words, although adding fermionic chemical potentials to the statistical operator induces non-zero expectation values for the supercharge densities, it does not change the partition function at all. This means that the ‘‘thermodynamic pressure,’’ defined as $(\beta \mathcal{V})^{-1} \ln Z$, is not affected by the presence of non-zero equilibrium supercharge densities. In fact, this is required if the thermodynamic pressure is to agree with the microscopic pressure, defined in terms of the expectation value of the stress tensor, $\mathcal{P} \equiv \frac{1}{3} \text{Tr}(\hat{\rho}_s T_{ii})$. To see that the microscopic pressure is also μ independent, one may repeat the steps shown in Eq. (63) when there is an additional insertion of the stress-energy tensor. This yields

$$\frac{1}{\beta} \frac{\partial}{\partial \mu^\alpha} \langle T^{\lambda\rho} \rangle_{\mu, \bar{\mu}} = \frac{1}{2} \langle (-1)^F [Q_\alpha, T^{\lambda\rho}] \rangle_{\mu, \bar{\mu}}. \tag{65}$$

But the supersymmetry transformation of the stress-energy tensor involves spacetime derivatives of the supercurrent (see, e.g., [26]),

$$[Q_\alpha, T^{\mu\nu}] = -\frac{i}{2} \{ (\sigma^{\mu\rho})_\alpha{}^\beta \partial_\rho S_\beta^\nu + (\sigma^{\nu\rho})_\alpha{}^\beta \partial_\rho S_\beta^\mu \}, \tag{66}$$

whose expectation (65) vanishes in the translationally invariant equilibrium ensemble.

None of the above results are affected by the presence of non-zero chemical potentials for ordinary conserved charges

which commute the the supercharges. But if a non-zero chemical potential μ_R for the R charge is present, then the generalized partition function does acquire dependence on the fermionic chemical potentials. The lowest such term is proportional to $\mu \bar{\mu} \mu_R$; this term is directly related to the equilibrium form of the R -charge current shown below.

In a similar fashion, one may evaluate the generalized equilibrium expectation values of the bosonic symmetry currents and the supercurrents. Ordinary bosonic conserved currents commute with the supercharges, and hence are unaffected by super-chemical potentials. In the presence of non-zero ordinary chemical potentials,

$$\langle J_a^\mu \rangle_{\mu, \bar{\mu}} = \bar{n}_a u^\mu \tag{67}$$

as usual, where \bar{n}_a is the equilibrium density of the conserved charge N_a . For the supercurrents one finds

$$\langle \xi^\alpha S_\alpha^\lambda \rangle_{\mu, \bar{\mu}} = -\beta \bar{T}^\lambda{}_\nu (\xi^\alpha \sigma_{\alpha\dot{\beta}}^\nu \bar{\mu}^{\dot{\beta}}) + O(\mu \bar{\mu}^2), \tag{68a}$$

$$\langle \bar{\xi}^{\dot{\alpha}} \bar{S}_{\dot{\alpha}\lambda} \rangle_{\mu, \bar{\mu}} = -\beta \bar{T}^\lambda{}_\nu (\mu^\alpha \sigma_{\alpha\dot{\beta}}^\nu \bar{\xi}^{\dot{\beta}}) + O(\mu^2 \bar{\mu}), \tag{68b}$$

where $\bar{T}^{\mu\nu} = \bar{w} u^\mu u^\nu + \bar{P} g^{\mu\nu}$ is the usual equilibrium form of the stress-energy tensor. These results reflect the supersymmetry transformations

$$\{Q_\beta, \bar{S}_\alpha^\lambda\} = 2 T^\lambda{}_\nu \sigma_{\beta\dot{\alpha}}^\nu + (\text{space-time gradients}), \tag{69a}$$

$$\{\bar{Q}_{\dot{\beta}}, S_\alpha^\lambda\} = 2 T^\lambda{}_\nu \sigma_{\alpha\dot{\beta}}^\nu + (\text{space-time gradients}), \tag{69b}$$

and $\{Q_\beta, S_\alpha^\lambda\}$, $\{\bar{Q}_{\dot{\beta}}, \bar{S}_\alpha^\lambda\}$ being pure space-time gradients. (The expectation value of the space-time gradient terms vanish in all translationally invariant equilibrium states.) For the $\lambda=0$ components of these commutators, the unspecified space-time gradients are strictly spatial gradients which vanish on spatial integration, so that the anti-commutation relations (69) are consistent with the supersymmetry algebra (49). For the R -symmetry current, one finds

$$\langle J_R^\lambda \rangle_{\mu, \bar{\mu}} = \bar{n}_R u^\lambda - \frac{1}{2} \beta^2 \bar{T}^\lambda{}_\nu (\mu^\alpha \sigma_{\alpha\dot{\beta}}^\nu \bar{\mu}^{\dot{\beta}}) + O(\mu^2 \bar{\mu}^2), \tag{70}$$

where \bar{n}_R is the equilibrium R -charge density in the absence of super-chemical potentials. The term quadratic in super-chemical potentials arises from the non-trivial commutators of the supercharges with the R current,

$$[Q_\alpha, J_R^\mu] = S_\alpha^\mu + (\text{space-time gradients}), \tag{71a}$$

$$[\bar{Q}_{\dot{\alpha}}, J_R^\mu] = -\bar{S}_{\dot{\alpha}}^\mu + (\text{space-time gradients}). \tag{71b}$$

Note that the relation (70) implies that non-zero super-chemical potentials induce a non-zero R -charge density even when the R -charge chemical potential vanishes.

C. Supersymmetric constitutive relations

We can now construct the required constitutive relations for the stress tensor and supercurrent. As before, we will need to include all possible terms which are linear in fluctuations away from a reference equilibrium state (chosen to have vanishing charge, supercharge, and spatial momentum densities) involving one spatial derivative, as well as terms without derivatives which are linear or quadratic in fluctuations. Neglected terms involving either more derivatives or more fluctuations will not affect the leading long-time tails in stress-stress or current-current correlators.

We will use ρ_α and $\bar{\rho}_{\dot{\alpha}}$ to denote the (Grassmann valued) supercharge densities in the non-equilibrium ensemble of interest. In equilibrium, the relations (68) imply that the supercharge densities are related to the super-chemical potentials via

$$\begin{aligned}\rho_\alpha &= \beta(\bar{\varepsilon}\sigma^0 - \bar{w}\boldsymbol{\sigma}\cdot\mathbf{u})_{\alpha\dot{\beta}}\bar{\mu}^{\dot{\beta}}, \\ \bar{\rho}_{\dot{\alpha}} &= -\beta\mu^\beta(\bar{\varepsilon}\sigma^0 - \bar{w}\boldsymbol{\sigma}\cdot\mathbf{u})_{\beta\dot{\alpha}},\end{aligned}\quad (72)$$

neglecting $O(u^2)$ corrections. For the reasons discussed in the preceding subsection, to quadratic order in deviations away from the reference equilibrium state, the pressure contains no coupling to supercharge densities. Consequently, the constitutive relation for the stress is identical to the previous result (21), except for the inclusion of R -charge density fluctuations along with other bosonic charge fluctuations,³²

$$\begin{aligned}T^{ij} &= \delta^{ij}\left[\bar{P} + v_s^2\delta\varepsilon + \frac{1}{2}\xi(\delta\varepsilon)^2 + \frac{1}{2}\Xi_{ab}n_a n_b + \frac{1}{2}\Xi_R n_R^2\right. \\ &\quad \left. - v_s^2\frac{\boldsymbol{\pi}^2}{\bar{w}}\right] + \frac{\pi^i\pi^j}{\bar{w}} - \gamma_\xi\delta^{ij}\boldsymbol{\nabla}\cdot\boldsymbol{\pi} \\ &\quad - \gamma_\eta\left(\nabla^i\pi^j + \nabla^j\pi^i - \frac{2}{3}\delta^{ij}\boldsymbol{\nabla}\cdot\boldsymbol{\pi}\right).\end{aligned}\quad (73)$$

The constitutive relations for bosonic internal symmetry currents are unchanged from the non-supersymmetric case,

$$\mathbf{j}_a = -D_{ab}\boldsymbol{\nabla}n_b + \frac{n_a\boldsymbol{\pi}}{\bar{w}},\quad (74)$$

while the new constitutive relation for the R -symmetry current is

$$\mathbf{j}_R = -D_R\boldsymbol{\nabla}n_R + \frac{n_R\boldsymbol{\pi}}{\bar{w}} - \frac{\bar{P}}{2\varepsilon^2}\bar{\rho}_{\dot{\alpha}}\bar{\sigma}^{\dot{\alpha}\beta}\rho_\beta.\quad (75)$$

³²In writing the constitutive relation (73), we have assumed, for simplicity, that the global bosonic symmetry currents \mathbf{j}_a are all vectors, rather than pseudo-vectors, and that parity is a symmetry of the theory. This prevents a $\Xi_{aR}n_a n_R$ cross term from appearing in the constitutive relation (73) for the stress, and likewise ensures that the R -charge density does not mix with other global charge densities in the constitutive relations (74) and (75) for the currents.

The quadratic coupling to supercharge densities follows directly from the (generalized) equilibrium form (70) for the expectation of the R current, combined with relation (72).

The remaining constitutive relations are those of the supercurrents. The appropriate form is more complicated than for bosonic currents because the supercharge density is a spinor. There are two independent rotationally covariant forms in which a spatial gradient can be applied to a spinor ρ_α to form a vector spinor, namely $\nabla^i\rho_\alpha$ and $(\sigma^{ij})_\alpha{}^\beta\nabla_j\rho_\beta$. Consequently, the supercurrent constitutive relations have the form

$$\begin{aligned}S_\alpha^i &= -D_s\nabla^i\rho_\alpha - D_\sigma(\sigma^{ij})_\alpha{}^\beta\nabla_j\rho_\beta - \frac{\bar{P}}{\varepsilon}(\sigma^i\bar{\sigma}^0)_\alpha{}^\beta\rho_\beta \\ &\quad + \frac{1}{\varepsilon}\rho_\alpha\pi^i + \frac{\bar{P}}{\varepsilon^2}\pi^k(\sigma^i\bar{\sigma}^k)_\alpha{}^\beta\rho_\beta,\end{aligned}\quad (76a)$$

$$\begin{aligned}\bar{S}_{\dot{\alpha}}^i &= -D_s\nabla^i\bar{\rho}_{\dot{\alpha}} - D_\sigma\nabla_j\bar{\rho}_{\dot{\beta}}(\bar{\sigma}^{ji})_{\dot{\alpha}}{}^{\dot{\beta}} - \frac{\bar{P}}{\varepsilon}\bar{\rho}_{\dot{\beta}}(\bar{\sigma}^0\sigma^i)_{\dot{\alpha}}{}^{\dot{\beta}} \\ &\quad + \frac{1}{\varepsilon}\bar{\rho}_{\dot{\alpha}}\pi^i + \frac{\bar{P}}{\varepsilon^2}\pi^k\bar{\rho}_{\dot{\beta}}(\bar{\sigma}^k\sigma^i)_{\dot{\alpha}}{}^{\dot{\beta}}.\end{aligned}\quad (76b)$$

Here D_s, D_σ are two new diffusion constants, which multiply the two structures allowed by rotation invariance.³³ The remaining non-derivative terms are dictated by the equilibrium results (68), when expanded to first order in flow velocity. The third term, which is linear in supercharge density, reflects the fact that a constant supercharge density is not rotationally invariant, and hence induces a non-zero constant spatial supercurrent. The convective terms are more complicated than just $\rho_\alpha\pi^i/\bar{w}$, because of the presence of this linear non-derivative term. If the supersymmetric theory in question is also scale invariant, then $\bar{\sigma}^\mu S_\mu$ must vanish, as it is in the same supersymmetry multiplet as the trace of the energy-momentum tensor. Applying this constraint to the constitutive relation (76), one finds that the two diffusion constants must coincide, $D_\sigma = D_s$, in scale-invariant supersymmetric theories.

D. Relaxation of supercharge fluctuations

The constitutive relation (76), combined with conservation of the supercurrent, allows one to determine how long-wavelength fluctuations in supercharge density relax. Inserting the constitutive relation into the continuity equation $\partial_t\rho_\alpha = -\boldsymbol{\nabla}\cdot\mathbf{S}_\alpha$, neglecting the non-linear convective terms, and Fourier transforming in space yields

$$[(\partial_t + D_s\mathbf{k}^2)\delta_\alpha^\beta - ic_s(\mathbf{k}\cdot\boldsymbol{\sigma}\bar{\sigma}^0)_\alpha{}^\beta]\rho_\beta(t, \mathbf{k}) = 0,\quad (77)$$

³³Assuming parity invariance of the theory ensures that the diffusion constants for the left and right handed supercurrents are the same. The Kubo formula for the supercharge diffusion constant D_s is $\bar{\varepsilon}D_s = -\lim_{\omega\rightarrow 0}\frac{1}{i2}\int d^4x e^{i\omega x^0}\bar{\sigma}^{0\dot{\alpha}\beta}C_{\dot{\alpha}\beta}^{ii}(x)$.

where $c_s \equiv \bar{\mathcal{P}}/\bar{\varepsilon}$. The solution in the small- \mathbf{k} limit shows weakly damped, propagating behavior,

$$\rho_\alpha(t, \mathbf{k}) = e^{-D_s k^2 t} [\delta_\alpha^\beta \cos(|\mathbf{k}|c_s t) + i(\hat{\mathbf{k}} \cdot \boldsymbol{\sigma} \bar{\sigma}^0)_\alpha^\beta \sin(|\mathbf{k}|c_s t)] \rho_\beta(0, \mathbf{k}). \quad (78)$$

This is a propagating collective excitation which is distinct from normal sound. In some respects, it is analogous to second sound in a superfluid. Note that the velocity $c_s = \bar{\mathcal{P}}/\bar{\varepsilon}$ of these ‘‘supersound waves’’ is always less than the ordinary speed of sound $v_s = \sqrt{\partial \bar{\mathcal{P}}/\partial \bar{\varepsilon}}$. In scale invariant theories, $c_s = 1/3$ while $v_s = 1/\sqrt{3}$. These collective excitations in supercharge density were previously identified (and found to propagate at speed c_s) in Ref. [27].

Just as in the previous non-supersymmetric analysis, the linearized non-equilibrium relaxation (78) may be converted into a statement about the time dependence of the equilibrium supercharge density correlator. Let

$$C(t, \mathbf{k})_{\alpha\beta} \equiv \int d^3 \mathbf{x} e^{-i\mathbf{k} \cdot \mathbf{x}} \langle S_\alpha^0(t, \mathbf{x}) \bar{S}_\beta^0(0, \mathbf{0}) \rangle, \quad (79)$$

where the expectation value is in the reference equilibrium state with vanishing chemical potentials. The small \mathbf{k} limit of the equal time correlator is fixed by the supersymmetry algebra,³⁴

$$C(0, \mathbf{k}=0)_{\alpha\beta} = -\bar{\varepsilon} \sigma_{\alpha\beta}^0. \quad (80)$$

Consequently, the linear response result (78) is equivalent to the statement that, in the small \mathbf{k} limit, the equilibrium supercharge correlator for large times has the form

$$C(t, \mathbf{k})_{\alpha\beta} = -\bar{\varepsilon} e^{-D_s k^2 |t|} [\sigma_{\alpha\beta}^0 \cos(|\mathbf{k}|c_s t) + i(\hat{\mathbf{k}} \cdot \boldsymbol{\sigma})_{\alpha\beta} \sin(|\mathbf{k}|c_s t)]. \quad (81)$$

One finds the same result for the small \mathbf{k} behavior of the supercharge correlator with the opposite ordering,

$$\begin{aligned} \bar{C}(t, \mathbf{k})_{\alpha\beta} &\equiv \int d^3 \mathbf{x} e^{-i\mathbf{k} \cdot \mathbf{x}} \langle \bar{S}_\beta^0(t, \mathbf{x}) S_\alpha^0(0, \mathbf{0}) \rangle \\ &= -\bar{\varepsilon} e^{-D_s k^2 |t|} [\sigma_{\alpha\beta}^0 \cos(|\mathbf{k}|c_s t) \\ &\quad + i(\hat{\mathbf{k}} \cdot \boldsymbol{\sigma})_{\alpha\beta} \sin(|\mathbf{k}|c_s t)]. \end{aligned} \quad (82)$$

E. Supersymmetric long-time tails

As in the non-supersymmetric case, second order terms in the constitutive relations are sufficient to understand the leading long-time tails in $\mathbf{k}=0$ correlators of all conserved currents. Proceeding in exactly the same way as in Sec. II E, one sees from the absence of any $\bar{\rho}$ term in the stress tensor

constitutive relation (73) that supercharge density fluctuations have no effect on the leading long-time tail in the stress-stress correlator. Consequently, the stress-stress correlator has the same form as in the non-supersymmetric case [i.e., Eq. (42) for scale-invariant theories, or Eq. (A11) for the general case] provided the R charge is included as one of the global bosonic charges. Similarly, the leading long-time tails (37) in correlators of ordinary bosonic symmetry currents are unaffected by the presence of supercharge density fluctuations.

In contrast, the presence of a quadratic supercharge density term in the constitutive relation (75) for the R -symmetry current means that the long-time tail in the R -symmetry current correlator is affected by supercharge fluctuations. One finds that the leading large time behavior is

$$\begin{aligned} &\int d^3 \mathbf{x} \langle \frac{1}{2} \{ j_R^k(t, \mathbf{x}), j_R^l(0, \mathbf{0}) \} \rangle \\ &\sim \frac{\delta^{kl}}{12} \left\{ \frac{(T/\bar{w}) \chi_R}{[(D_R + \gamma_\eta) \pi |t|]^{3/2}} + \frac{\frac{1}{4} c_s^2}{[2D_s \pi |t|]^{3/2}} \right\}. \end{aligned} \quad (83)$$

If the same analysis is applied to the spatial supercurrent correlation function, one finds that the second order $\pi\rho$ terms in the constitutive relation (76) do not generate long-time tails. Because the speed of ordinary sound (v_s) and supersound (c_s) differ, fluctuations (with the same wave vector) in momentum density and supercharge density cannot remain in phase. This is analogous to the fact that only transverse momentum density fluctuations, not longitudinal sound waves, contribute to the long-time tail (37) of ordinary current-current correlators. So the supercurrent correlator $\langle \bar{S}_\alpha^i(t) S_\alpha^j(0) \rangle$, at $\mathbf{k}=0$, should decay exponentially in time.³⁵

The differing long-time behaviors of the stress tensor and supercurrent correlators does not conflict with the fact that the stress-energy tensor and supercurrents belong to the same supersymmetry multiplet. The supersymmetry algebra implies simple relations between $(-1)^F$ inserted thermal correlation functions of conserved currents, but not between conventional thermal correlation functions, for the reasons noted in footnote 29.

IV. LONG-TIME TAILS IN $\mathcal{N}=4$ SUPERSYMMETRIC YANG-MILLS THEORY

As a specific application, the above discussion (trivially generalized) may be applied to $\mathcal{N}=4$ supersymmetric $SU(N_c)$ Yang-Mills theory, at the origin of the moduli space and at non-zero temperature.³⁶ Once again, hydrodynamic

³⁴In our conventions, $\sigma^0 = -1$, so $C(t=0, \mathbf{k}=0)$ is strictly positive, as the equal-time correlation function of any operator with its Hermitian conjugate must be.

³⁵Because $c_s < v_s$, one finds that higher-order $\rho\pi^n$ terms in the supercurrent constitutive relation also do not generate (higher-order) power-law tails.

³⁶The vacuum degeneracy associated with a non-trivial moduli space is lifted at non-zero temperature. For sufficiently high temperatures, the only stable equilibrium states lie at the origin of moduli space.

degrees of freedom are densities of conserved charges. In $\mathcal{N}=4$ supersymmetric Yang-Mills theory, the supersymmetry algebra³⁷ is generated by the charges of the dilation current $j_D^\mu = x_\nu T^{\mu\nu}$, the special conformal currents $K_\lambda^\mu = 2x_\lambda x^\nu T_\nu^\mu - x^2 T_\lambda^\mu$, and the special supersymmetry currents $S_{A\alpha}^\mu = x_\nu \sigma_{\alpha\dot{\alpha}}^\nu \bar{S}_{A\dot{\alpha}}^\mu$ and $\bar{S}_{A\dot{\alpha}}^\mu = x_\nu \bar{\sigma}_{\dot{\alpha}\alpha}^\nu S_{A\alpha}^\mu$, in addition to the stress-energy tensor $T^{\mu\nu}$ and supercurrents $S_{A\alpha}^\mu, \bar{S}_{A\dot{\alpha}}^\mu$. The index $A=1, \dots, 4$ is the extended supersymmetry label. There are also 15 bosonic R -symmetry currents, whose charges generate a non-anomalous global $SU(4)_R$ symmetry.

The currents $j_D^\mu, K_\lambda^\mu, S_{A\alpha}^\mu$, and $\bar{S}_{A\dot{\alpha}}^\mu$ are all constructed from the energy-momentum tensor and ordinary supercurrents. Therefore, the time evolution of the charge densities of these currents is completely determined by the time evolution of the energy-momentum tensor and the supercurrent. In other words, $j_D^0, K_\lambda^0, S_{A\alpha}^0$, and $\bar{S}_{A\dot{\alpha}}^0$ are not new hydrodynamic degrees of freedom, independent from $T^{0\nu}, S_{A\alpha}^0$, and $\bar{S}_{A\dot{\alpha}}^0$. No Abelian magnetic fields appear as hydrodynamic degrees of freedom, since the gauge group of the theory is $SU(N_c)$, not $U(N_c)$. Moreover, holographic arguments imply that the theory at non-zero temperature has a mass gap [5], in accord with the assumption of finite correlation length made in Sec. II.

The discussion of supercharge fluctuations in the previous section generalizes easily to the case of $\mathcal{N}=4$ supersymmetry. In particular, the constitutive relation for the stress tensor has the same form (73), except for the addition of a fundamental $SU(4)_R$ index on the supercharges $\rho_{A\alpha}$ and an adjoint index on the R -charge densities n_{Ra} (i.e., $a=1, \dots, 15$). The theory is scale invariant, and therefore $\bar{\varepsilon}=3\bar{\mathcal{P}}$, and the bulk viscosity vanishes, as does the coefficient $\bar{\Xi}_R$. Hence, fluctuations in the R -charge densities do not contribute to constitutive relation for stress to second order, which therefore must have the same form (23) as in a non-supersymmetric scale-invariant theory. Consequently, the long-time tail for the stress tensor correlator (at $\mathbf{k}=0$) is given by the previous result (42), and the equivalent small frequency form of the spectral density is given by Eq. (46).

The constitutive relation for supercurrent (76) stays unchanged to second order (except for the addition of the index A , and the specialization to $D_s=D_\sigma$ and $\bar{\mathcal{P}}/\bar{\varepsilon}=1/3$). No long-time tail is present in the supercurrent correlator, for the reasons discussed in the previous section. Hence, the corresponding spectral density is analytic at zero frequency.

The constitutive relations for the R -symmetry currents become

$$\dot{j}_{Ra} = -D_R \nabla n_{Ra} + \frac{n_{Ra} \pi}{\bar{w}} - \frac{1}{3\bar{\varepsilon}} \bar{\rho}_{A\dot{\alpha}}(t_a)^{AB} \bar{\sigma}^{\dot{\alpha}\beta} \rho_{B\beta}, \quad (84)$$

where t_a are the generators of $SU(4)$, normalized so that

³⁷See, for example, Ref. [26], for an introduction to the $\mathcal{N}=4$ algebra.

$\text{tr}(t_a t_b) = \frac{1}{2} \delta_{ab}$. The resulting long-time tails in the R -symmetry current correlators (at $\mathbf{k}=0$) are

$$\int d^3 \mathbf{x} \langle \frac{1}{2} \{ J_{Ra}^k(t, \mathbf{x}), j_{Rb}^l(0, \mathbf{0}) \} \rangle \sim \frac{\chi_R T}{16 \bar{\varepsilon}} \frac{\delta^{kl} \delta_{ab}}{[(D_R + \gamma_\eta) \pi |t|]^{3/2}} + \frac{1}{216} \frac{\delta^{kl} \delta_{ab}}{[2D_s \pi |t|]^{3/2}}. \quad (85)$$

The equivalent small-frequency expansion of the spectral density is given by

$$\frac{1}{\omega} \int dt d^3 \mathbf{x} e^{i\omega t} \langle [J_{Ra}^k(t, \mathbf{x}), j_{Rb}^l(0, \mathbf{0})] \rangle = \delta_{ab} \delta^{kl} (c^{(0)} + c^{(1)} \sqrt{|\omega|}) + O(|\omega|^{3/2}), \quad (86)$$

where

$$c^{(0)} = \frac{2D_R \chi_R}{T}, \quad (87a)$$

$$c^{(1)} = -\frac{\chi_R}{2\pi \bar{\varepsilon}} [2(D_R + \gamma_\eta)]^{-3/2} - \frac{1}{216\pi T} D_s^{-3/2}. \quad (87b)$$

In writing expressions (84) and (85) we have used the fact that in a thermal state without chemical potentials for the $SU(4)_R$ charges, the matrix of diffusion constants, as well as the R charge susceptibility, must be invariants of the group: $(D_R)_{ab} = D_R \delta_{ab}, (\chi_R)_{ab} = \chi_R \delta_{ab}$.

The amplitudes of the long-time tails for the stress tensor correlator (42), and the R -current correlator (85), depend on the values of the equilibrium energy density $\bar{\varepsilon}$, the R -charge susceptibility χ_R , the shear viscosity η , and R -charge diffusion constant D_R . These parameters must be treated as input from short-distance physics. In a weakly coupled theory, these quantities can be computed in perturbation theory, but no field-theoretic calculation is available when the coupling constant is large. However, the AdS/CFT correspondence predicts specific values for these parameters in $\mathcal{N}=4$ supersymmetric Yang-Mills theory in the limit of large N_c and large 't Hooft coupling. The equilibrium energy density is evaluated from the thermodynamics of the dual anti-de Sitter black hole, and is predicted to be

$$\bar{\varepsilon} = \frac{3\pi^2}{8} N_c^2 T^4. \quad (88)$$

The shear viscosity is evaluated as the zero-frequency limit of the absorption cross-section of gravitons by the black three-brane (with gravitons polarized parallel to the brane) [12], and is predicted to be

$$\eta = \frac{\pi}{8} N_c^2 T^3. \quad (89)$$

Hence, in this limit the ratio $\gamma_\eta \equiv \eta/\bar{w}$ is predicted to equal $1/(4\pi T)$. The diffusion constant and susceptibility for

$SU(4)_R$ charges are extracted from the pole structure of the thermal correlation function of $SU(4)_R$ charge densities [28], and are predicted to be

$$D_R = \frac{1}{2\pi T}, \quad (90)$$

and

$$\chi_R = \frac{1}{8} N_c^2 T^3. \quad (91)$$

The supercharge diffusion constant D_s has, to our knowledge, not yet been evaluated, but should be computable by the methods of Ref. [28].

Although these strong coupling (and large N_c) limits of transport coefficients, susceptibility, and energy density cannot be compared with direct field-theoretic calculations, they may be used as input into the effective hydrodynamic theory (whose validity is independent of coupling and N_c). Inserting the AdS/CFT predictions (88),(89) into the result (46) yields the completely explicit form

$$\begin{aligned} & \frac{1}{\omega} \int d^3x d^3x e^{i\omega x^0} \langle [T_{ij}(x), T_{kl}(0)] \rangle \\ &= \left(\frac{1}{2} \delta_{ik} \delta_{jl} + \frac{1}{2} \delta_{il} \delta_{jk} - \frac{1}{3} \delta_{ij} \delta_{kl} \right) T^3 \\ & \times \left[\frac{\pi}{2} N_c^2 - \frac{56 + 6^{3/2}}{60} \sqrt{\frac{\pi|\omega|}{T}} + O(|\omega|^{3/2}) \right] \end{aligned} \quad (92)$$

for the spectral density of the stress-stress correlator in the strong coupling, large N_c limit. This correlator is a non-trivial probe of the real-time dynamics of strongly coupled supersymmetric Yang-Mills plasma. Because the AdS/CFT prediction of the supercharge diffusion constant D_s is currently lacking, we cannot give an equally explicit result for the R -symmetry current spectral density (86).

The result (92) displays the expected $1/N_c^2$ relative suppression of the non-analytic part, and also shows that the ratio of the leading analytic to the leading non-analytic piece does not depend on the coupling constant g_{YM} of the gauge theory in the strong coupling limit. In the language of the AdS/CFT correspondence, $\alpha'/R_{AdS}^2 = 1/\sqrt{g_{YM}^2 N_c}$, and $4\pi g_s = g_{YM}^2$. The ten-dimensional gravitational constant κ scales as $g_s \alpha'^2 \sim R_{AdS}^4/N_c$. Thus, it is string/supergravity loops, proportional to κ^2 , which are responsible for $1/N_c^2$ corrections in field theory correlators. The corresponding one-loop amplitudes are not straightforward to evaluate, given that even the scalar propagator on the plane-symmetric AdS

black hole background is not known analytically.³⁸ However, it would be quite interesting to reproduce hydrodynamic results like Eq. (92) directly from supergravity loop corrections. Just verifying the form of the small-frequency non-analyticity from the gravity amplitudes, without computing the precise coefficient, would be a worthwhile goal.

Finally, given that in various AdS/CFT-like scenarios, finite-temperature field theories are believed to be exactly equivalent to string theories on backgrounds with thermal horizons, it is natural to expect that there must exist a correspondence between appropriate effective theories as well. Namely, string theories on thermal backgrounds should have low-energy descriptions which are dual to effective long distance descriptions of finite-temperature field theories, specifically hydrodynamics and kinetic theory (in weakly coupled regimes). As we have seen, constructing a string theory dual of hydrodynamics will necessarily require incorporating string loop effects in order to reproduce aspects of long distance dynamics, such as long-time tails, which are sensitive to the non-linearities of hydrodynamics.

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APPENDIX: LONG-TIME TAIL IN STRESS-STRESS CORRELATOR

In a theory, which is not necessarily scale invariant, inserting the stress tensor constitutive relation (21) into the zero-wave-number connected stress-stress correlator gives

$$\begin{aligned} & \mathcal{V}^{-1} \langle \frac{1}{2} \{ T^{ij}(t), T^{kl}(0) \} \rangle_{\text{conn}} \\ &= \delta^{ij} \delta^{kl} v_s^2 \bar{w} T + \frac{1}{w^2} H_{mn}^{ij} H_{pq}^{kl} B_{\pi^m \pi^n, \pi^p \pi^q}(t) \\ &+ \frac{\xi}{2w} \delta^{ij} H_{pq}^{kl} B_{\varepsilon\varepsilon, \pi^p \pi^q}(t) + \frac{\xi}{2w} H_{mn}^{ij} \delta^{kl} B_{\pi^m \pi^n, \varepsilon\varepsilon}(t) \\ &+ \frac{\xi^2}{4} \delta^{ij} \delta^{kl} B_{\varepsilon\varepsilon, \varepsilon\varepsilon}(t) + \frac{\Xi_{ab} \Xi_{cd}}{4} \delta^{ij} \delta^{kl} B_{\rho_a \rho_b, \rho_c \rho_d}(t), \end{aligned} \quad (A1)$$

where $H_{mn}^{ij} \equiv \frac{1}{2} \delta_m^i \delta_n^j + \frac{1}{2} \delta_n^i \delta_m^j - v_s^2 \delta^{ij} \delta_{mn}$, and

³⁸For progress in this direction, see Ref. [29].

$$B_{\pi^i \pi^j, \pi^k \pi^l}(t) \equiv \int d^3 \mathbf{x} \langle \frac{1}{2} \{ \pi^i(t, \mathbf{x}) \pi^j(t, \mathbf{x}), \pi^k(0, \mathbf{0}) \pi^l(0, \mathbf{0}) \} \rangle_{\text{conn}}, \quad (\text{A2})$$

$$B_{\pi^i \pi^j, \varepsilon \varepsilon}(t) \equiv \int d^3 \mathbf{x} \langle \frac{1}{2} \{ \pi^i(t, \mathbf{x}) \pi^j(t, \mathbf{x}), \varepsilon(0, \mathbf{0})^2 \} \rangle_{\text{conn}}, \quad (\text{A3})$$

$$B_{\varepsilon \varepsilon, \pi^i \pi^j}(t) \equiv \int d^3 \mathbf{x} \langle \frac{1}{2} \{ \varepsilon(t, \mathbf{x})^2, \pi^i(0, \mathbf{0}) \pi^j(0, \mathbf{0}) \} \rangle_{\text{conn}}, \quad (\text{A4})$$

$$B_{\varepsilon \varepsilon, \varepsilon \varepsilon}(t) \equiv \int d^3 \mathbf{x} \langle \frac{1}{2} \{ \varepsilon(t, \mathbf{x})^2, \varepsilon(0, \mathbf{0})^2 \} \rangle_{\text{conn}}, \quad (\text{A5})$$

$$B_{\rho_a \rho_b, \rho_c \rho_d}(t) \equiv \int d^3 \mathbf{x} \langle \frac{1}{2} \{ \rho_a(t, \mathbf{x}) \rho_b(t, \mathbf{x}), \rho_c(0, \mathbf{0}) \rho_d(0, \mathbf{0}) \} \rangle_{\text{conn}}. \quad (\text{A6})$$

Evaluating these correlators in the hydrodynamic regime, as described in Sec. II E, yields

$$\begin{aligned} B_{\pi^i \pi^j, \pi^k \pi^l}(t) &= \int \frac{d^3 \mathbf{k}}{(2\pi)^3} [A_{\pi^i \pi^k}(t, \mathbf{k}) A_{\pi^j \pi^l}(t, -\mathbf{k}) + A_{\pi^i \pi^l}(t, \mathbf{k}) A_{\pi^j \pi^k}(t, -\mathbf{k})] \\ &\sim \frac{\bar{w}^2 T^2}{15} \left[\frac{\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}}{(4\pi \gamma_s |t|)^{3/2}} + \frac{2 \delta^{ij} \delta^{kl} + 7 (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk})}{(8\pi \gamma_\eta |t|)^{3/2}} \right], \end{aligned} \quad (\text{A7})$$

$$B_{\pi^i \pi^j, \varepsilon \varepsilon}(t) = B_{\varepsilon \varepsilon, \pi^i \pi^j}(t) = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} 2 A_{\pi^i \varepsilon}(t, \mathbf{k}) A_{\pi^j \varepsilon}(t, -\mathbf{k}) \sim \frac{\bar{w}^2 T^2}{3 v_s^2} \frac{\delta^{ij}}{(4\pi \gamma_s |t|)^{3/2}}, \quad (\text{A8})$$

$$B_{\varepsilon \varepsilon, \varepsilon \varepsilon}(t) = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} 2 A_{\varepsilon \varepsilon}(t, \mathbf{k}) A_{\varepsilon \varepsilon}(t, -\mathbf{k}) \sim \frac{\bar{w}^2 T^2}{v_s^4} \frac{1}{(4\pi \gamma_s |t|)^{3/2}}, \quad (\text{A9})$$

$$\begin{aligned} B_{\rho_a \rho_b, \rho_c \rho_d}(t) &= \int \frac{d^3 \mathbf{k}}{(2\pi)^3} [A_{\rho_a \rho_c}(t, \mathbf{k}) A_{\rho_b \rho_d}(t, -\mathbf{k}) + A_{\rho_a \rho_d}(t, \mathbf{k}) A_{\rho_b \rho_c}(t, -\mathbf{k})] \\ &\sim \chi_{aa'}^{1/2} \chi_{bb'}^{1/2} \chi_{cc'}^{1/2} \chi_{dd'}^{1/2} \frac{\delta_{a'c'} \delta_{b'd'} + \delta_{a'd'} \delta_{b'c'}}{[4\pi(D_{a'} + D_{b'})|t|]^{3/2}}. \end{aligned} \quad (\text{A10})$$

In writing the result (A10), we have chosen to use a basis for conserved charges in which the symmetric matrix $\chi^{-1/2} D \chi^{1/2}$ is diagonal and has eigenvalues $\{D_a\}$.

Setting everything together yields

$$\begin{aligned} \mathcal{V}^{-1} \langle \frac{1}{2} \{ T^{ij}(t), T^{kl}(0) \} \rangle &= \delta_{ij} \delta_{kl} T \bar{w} v_s^2 + \frac{1}{2} \delta_{ij} \delta_{kl} \sum_{a,b} \frac{[(\chi^{1/2} \Xi \chi^{1/2})_{ab}]^2}{[4\pi(D_a + D_b)|t|]^{3/2}} + \frac{2T^2}{15} \left(\frac{1}{2} \delta_{ik} \delta_{jl} + \frac{1}{2} \delta_{il} \delta_{jk} - \frac{1}{3} \delta_{ij} \delta_{kl} \right) \\ &\times \left[\frac{1}{(4\pi \gamma_s |t|)^{3/2}} + \frac{7}{(8\pi \gamma_\eta |t|)^{3/2}} \right] + \frac{T^2}{9} \delta_{ij} \delta_{kl} \left[\frac{\left(1 - 3v_s^2 + \frac{3}{2} v_s^{-2} \xi \bar{w} \right)^2}{(4\pi \gamma_s |t|)^{3/2}} + \frac{4(1 - 3v_s^2)^2}{(8\pi \gamma_\eta |t|)^{3/2}} \right], \end{aligned} \quad (\text{A11})$$

up to terms vanishing faster than $|t|^{-3/2}$ as $|t| \rightarrow \infty$.

[1] J.M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998).

[2] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, *Phys. Lett. B* **428**, 105 (1998).

[3] E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998).

[4] For a review of the subject, see O. Aharony, S.S. Gubser, J.M.

Maldacena, H. Ooguri, and Y. Oz, *Phys. Rep.* **323**, 183 (2000).

[5] E. Witten, *Adv. Theor. Math. Phys.* **2**, 505 (1998).

[6] Y. Pomeau and P. Résibois, *Phys. Rep.* **19**, 63 (1975).

[7] R. Balescu, *Equilibrium and nonequilibrium statistical mechanics* (Wiley, New York, 1975).

- [8] P. Arnold and L.G. Yaffe, Phys. Rev. D **57**, 1178 (1998).
[9] I.R. Klebanov, Nucl. Phys. **B496**, 231 (1997).
[10] S.S. Gubser, I.R. Klebanov, and A.A. Tseytlin, Nucl. Phys. **B499**, 217 (1997).
[11] G. Policastro and A. Starinets, Nucl. Phys. **B610**, 117 (2001).
[12] G. Policastro, D.T. Son, and A.O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001).
[13] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin/Cummings, New York, 1975).
[14] L.D. Landau and E.M. Lifshitz, *Statistical Physics, Part 2* (Pergamon, New York, 1980).
[15] See, for example, G. 't Hooft, in *Field Theory and Strong Interactions*, edited by P. Urban (Springer, New York, 1980); S. Elitzur, Phys. Rev. D **12**, 3978 (1975).
[16] D.T. Son, Phys. Rev. Lett. **84**, 3771 (2000).
[17] M.H. Ernst and J.R. Dorfman, J. Stat. Phys. **12**, 311 (1975).
[18] S. Jeon and L.G. Yaffe, Phys. Rev. D **53**, 5799 (1996).
[19] S. Jeon, Phys. Rev. D **52**, 3591 (1995).
[20] G.D. Moore, hep-ph/0211281.
[21] S.W. Lovesey, *Condensed Matter Physics. Dynamic Correlations* (Benjamin/Cummings, New York, 1980).
[22] P. Arnold, G.D. Moore, and L.G. Yaffe, J. High Energy Phys. **11**, 001 (2000).
[23] E. Witten, Nucl. Phys. **B145**, 110 (1978).
[24] J. Wess and J. Bagger, *Supersymmetry and Supergravity* (Princeton University Press, Princeton, NJ, 1992).
[25] M.B. Paranjape, A. Taormina, and L.C.R. Wijewardhana, Phys. Rev. Lett. **50**, 1350 (1983).
[26] M.F. Sohnius, Phys. Rep. **128**, 39 (1985).
[27] V.V. Lebedev and A.V. Smilga, Nucl. Phys. **B318**, 669 (1989).
[28] G. Policastro, D.T. Son, and A. Starinets, J. High Energy Phys. **09**, 043 (2002).
[29] A. Starinets, Phys. Rev. D **66**, 124013 (2002).