General class of brane-world black holes

K. A. Bronnikov* and V. N. Melnikov

VNIIMS, 3-1 M. Ulyanovoy St., Moscow 117313, Russia
and Institute of Gravitation and Cosmology, PFUR, 6 Miklukho-Maklaya St., Moscow 117198, Russia

Heinz Dehnen[†]

Fachbereich Physik, Universität Konstanz, Postfach M 677 78457 Konstanz, Germany

We use the general solution to the trace of the 4-dimensional Einstein equations for static, spherically symmetric configurations as a basis for finding a general class of black hole (BH) metrics, containing one arbitrary function $g_{tt} = A(r)$ which vanishes at some $r = r_h > 0$, the horizon radius. Under certain reasonable restrictions, BH metrics are found with or without matter and, depending on the boundary conditions, can be asymptotically flat or have any other prescribed asymptotic. It is shown that our procedure generically leads to families of globally regular BHs with a Kerr-like global structure as well as symmetric wormholes. Horizons in space-times with zero scalar curvature are shown to be either simple or double. The same is generically true for horizons inside a matter distribution, but in special cases there can be horizons of any order. A few simple examples are discussed. A natural application of the above results is the brane world concept, in which the trace of the 4D gravity equations is the only unambiguous equation for the 4D metric, and its solutions can be continued into the 5D bulk according to the embedding theorems.

I. INTRODUCTION

The brane world concept, which describes our four-dimensional world as a surface (brane), supporting all or almost all matter fields and embedded in a higher-dimensional space-time (bulk), leads to a great variety of models both in the cosmological context and in the description of local self-gravitating objects (see, e.g., [1] for reviews and further references). In particular, black hole (BH) physics on the brane turns out to be considerably richer than in general relativity, though only a few special examples of brane-world BHs have been considered in detail by now [2–10] (see also references therein). Thus, in the spherically symmetric vacuum case, in addition to Schwarzschild BHs (which lead to a black-string singularity in the bulk [2,6]), there are BHs nonsingular on the brane [7] and having a pancake-shaped event horizon in the bulk [9]; some of them have been shown to possess unusual quantum properties potentially observable on the brane [10].

Most of the results have been obtained in the simplest framework: a single brane in a $\mathbb{Z}_2$-symmetric 5-dimensional, asymptotically anti–de Sitter bulk, with all fields except gravity confined on the brane. It is the so-called RS2 framework, generalizing the second model suggested by Randall and Sundrum, with a single Minkowski brane in an anti–de Sitter bulk [11]. Let us also adhere to this class of models.

The gravitational field on the brane is then described by the modified Einstein equations derived by Shiromizu, Maeda and Sasaki [12] from 5-dimensional gravity with the aid of the Gauss and Codazzi equations [29]:

$$G_\mu^\nu = -\Lambda_4 \delta_\mu^\nu - \kappa_4^2 T_\mu^\nu - \kappa_5^4 \Pi_\mu^\nu - E_\mu^\nu, \tag{1}$$

where $G_\mu^\nu = R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R$ is the 4D Einstein tensor, Λ_4 is the 4D cosmological constant expressed in terms of the 5D cosmological constant Λ_5 and the brane tension λ:

$$\Lambda_4 = \frac{1}{2}\kappa_5^2 \left(\Lambda_5 + \frac{1}{6}\kappa_5^2 \lambda^2 \right); \tag{2}$$

$\kappa_4^2 = 8\pi G_N = \kappa_5^4 \lambda/(6\pi)$ is the 4D gravitational constant (G_N is Newton's constant of gravity); T_μ^ν is the stress energy tensor (SET) of matter confined on the brane; Π_μ^ν is a tensor quadratic in T_μ^ν, obtained from matching the 5D metric across the brane:

$$\Pi_\mu^\nu = \frac{1}{2}T_\mu^\alpha T_\alpha^\nu - \frac{1}{2}TT_\mu^\nu - \frac{1}{2}\delta_\mu^\nu (T_{\alpha\beta}T^{\alpha\beta} - \frac{1}{2}T^2) \tag{3}$$

where $T = T_\alpha^\alpha$; lastly, E_μ^ν is the "electric" part of the 5D Weyl tensor projected onto the brane: in proper 5D coordinates, $E_{\mu\nu} = \delta_\mu^A \delta_\nu^C {}^{(5)}C_{ABCD} n^B n^D$ where $A, B, \ldots$ are 5D indices and n^A is the unit normal to the brane. By construction, E_μ^ν is traceless, $E_\mu^\mu = 0$ [12].

Other characteristics of E_μ^ν are unknown without specifying the properties of the 5D metric, hence the set of equations (1) is not closed. In isotropic cosmology this leads to an additional arbitrary constant in the field equations, connected with the density of "dark radiation" [1]. For static, spherically symmetric systems to be discussed in the present paper, this freedom is expressed in the existence of one arbitrary function of the radial coordinate. Despite this arbitrariness, the trace of Eqs. (1) may be integrated in a general form [13,14].

Our interest here is in selecting a general class of static, spherically symmetric BH solutions to Eqs. (1) without specifying E_μ^ν. In particular examples we mostly deal with

---

*Electronic address: kb@rgs.mccme.ru

[†]Electronic address: Heinz.Dehnen@uni-konstanz.de

asymptotically flat vacuum solutions, such that $\Lambda_4 = T^\nu_\mu = 0$, but the BH construction procedure is formulated in the general case when both matter and the cosmological constant are present and the space-time asymptotic properties are not specified.

We will not discuss the possible bulk properties of models in question and only note that the existence of the corresponding solutions to the higher-dimensional equations of gravity (in our case, the 5D vacuum Einstein equations with a cosmological term) is guaranteed by the Campbell-Magaard type embedding theorems [15]. A recent discussion of these theorems, applied, in particular, to brane world scenarios, and more references can be found in Ref. [16].

The paper is organized as follows. In Sec. II we present some common relations and the general solution to the trace of Eqs. (1), containing an arbitrary generating function $A(r)$.

In Sec. III we analyze the properties of the metric near a Killing horizon in a static, spherically symmetric space-time described by the general solution. A conclusion of general significance is that a space-time with $R \equiv 0$ can only have horizons of orders one (simple, like Schwarzschild's) and two (double, as in the extremal Reissner-Nordström metric) and no higher. This analysis is used for formulation of two BH construction algorithms. It is shown that a generic choice of $A(r)$ leads to a one-parameter family of solutions which, in a certain range of the parameter (integration constant) C, unifies globally regular non-extremal BHs with a Kerr-like causal structure, extremal BHs and symmetric wormholes. Singular nonextremal BHs can be found outside this range of C.

Section IV contains some simple examples, illustrating different features of the present formalism. Examples 1 and 3 reproduce already known BH solutions from the viewpoint of our algorithms. Example 2 is a BH solution with zero Schwarzschild mass, illustrating violation of Thorne's hoop conjecture possible in a brane world. Example 4 shows that well-behaved special solutions can be found even for such choices of $A(r)$ that the trace equation (9) has a singular point. Example 5 illustrates the smoothness properties of some BH metrics at the horizon in different coordinate frames. Section V is a discussion.

We will assume that all relevant functions are analytic unless otherwise explicitly indicated. The symbol $\sim$, as usual, connects quantities of the same order of magnitude in a certain limit.

II. THE GENERAL SOLUTION

The general static, spherically symmetric metric in 4 dimensions in the curvature coordinates has the form

$$ds^2 = A(r)dt^2 - \frac{dr^2}{B(r)} - r^2 d\Omega^2 \tag{4}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the linear element on a unit sphere.

Let us write down the scalar curvature and the Kretschmann scalar for the metric (4):

$$R = \frac{2}{r^2}(1-B) - B\left[\frac{A_{rr}}{A} - \frac{A_r^2}{2A^2} + \frac{A_r B_r}{2AB} + \frac{2}{r}\left(\frac{A_r}{A} + \frac{B_r}{B}\right)\right];$$
$$\tag{5}$$

$$\mathcal{K} = R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\mu\nu} = 4K_1^2 + 8K_2^2 + 8K_3^2 + 4K_4^2,$$

$$K_1 = \frac{B}{4}\left(\frac{2AA_{rr} - A_r^2}{A^2} + \frac{A_r B_r}{AB}\right),$$

$$K_2 = \frac{B}{2r}\frac{A_r}{A}, \quad K_3 = \frac{B_r}{2r}, \quad K_4 = \frac{1-B}{r^2}, \tag{6}$$

where the subscript r means d/dr. The finiteness of $\mathcal{K}$ is a natural regularity criterion for the geometries to be discussed since $\mathcal{K}$ is a sum of squares of all components $R_{\mu\nu}{}^{\varepsilon\sigma}$ of the Riemann tensor for the metric (4), therefore $\mathcal{K} < \infty$ is a necessary and sufficient condition for the finiteness of all algebraic curvature invariants. Meanwhile, $\mathcal{K}$ is finite if and only if all K_i defined in Eq. (6) are finite.

If we treat Eqs. (1) as the conventional Einstein equations with an effective SET $T^{\nu\,\text{eff}}_\mu$, i.e., $G^\nu_\mu = -\kappa_4^2 T^{\nu\,\text{eff}}_\mu$, then the effective density ρ^{eff}, radial pressure $p^{\text{eff}}_{\text{rad}}$ and tangential pressure $p^{\text{eff}}_\perp$ are expressed in terms of A and B as follows:

$$G^t_t = -\kappa_4^2 \rho^{\text{eff}} = \frac{B-1}{r^2} + \frac{B_r}{r},$$

$$G^r_r = \kappa_4^2 p^{\text{eff}}_{\text{rad}} = \frac{B-1}{r^2} + \frac{BA_r}{Ar},$$

$$G^\theta_\theta = G^\phi_\phi = \kappa_4^2 p^{\text{eff}}_\perp$$

$$= \frac{B}{4}\left[\frac{2A_{rr}}{A} - \frac{A_r^2}{A^2} - \frac{A_r B_r}{AB} + \frac{2}{r}\left(\frac{A_r}{A} + \frac{B_r}{B}\right)\right]. \tag{7}$$

The only combination of the Einstein equations (1) in a brane world written unambiguously without specifying E^ν_μ, is their trace:

$$R = 4\Lambda_4 + \kappa_4^2 T^\alpha_\alpha + \kappa_5^4 \Pi^\alpha_\alpha. \tag{8}$$

Assuming that the right-hand side is a known function of the radial coordinate, i.e., that $R = R(r)$ is known, Eq. (8) may be written as a linear first-order equation with respect to $f(r) := rB(r)$ [13,14]:

$$A(rA_r + 4A)f_r + [r(2AA_{rr} - A_r^2) + 3AA_r]f$$
$$= 2A^2[2 - r^2 R(r)]. \tag{9}$$

Its general solution is

$$f(r) = \frac{2A e^{3\Gamma}}{(4A + rA_r)^2}\int (4A + rA_r)[2 - r^2 R(r)]e^{-3\Gamma}\, dr \tag{10}$$

where

$$\Gamma(r) = \int \frac{A_r dr}{4A + rA_r}. \tag{11}$$

Thus, choosing any smooth function $A(r)$, we obtain $f(r)$ from Eq. (10), and, after fixing the integration constant, the metric is known completely.

If the function $R(r)$ is not specified, Eq. (10) is simply another form of the trace of the Einstein equations. It is valid for *any* static, spherically symmetric metric, at least in ranges of r where the quantity $A(4A + rA_r)$ is finite and nonzero and where $r = \sqrt{-g_{\theta\theta}}$ is an admissible coordinate. The latter means, in particular, that Eq. (10) is applicable to a wormhole metric only on one (but either) side of a wormhole throat (see [13] for details) and is evidently invalid for flux-tube metrics, characterized by $g_{\theta\theta} = $ const.

For a given SET T_μ^ν, the dependence $R(r)$ is not always known, but in the vacuum case $T_\mu^\nu = 0$, so that $R(r) = 4\Lambda_4$, the solution to the Einstein equations can always be written in the form (10) under the above evident restrictions.

III. BLACK HOLE CONSTRUCTION

A. Conditions at horizons

Before singling out BH solutions on the basis of Eq. (10), let us first formulate the conditions under which the generating function $A(r)$ leads to a metric with a Killing horizon. The latter is a surface where a timelike or spacelike Killing vector becomes null. To describe Killing horizons (to be called horizons for short) in spherically symmetric space-times, it is helpful to use the so-called quasiglobal coordinate u specified by the condition $g_{tt}g_{uu} = -1$. The metric (4) is then rewritten in the form

$$ds^2 = \mathcal{A}(u)dt^2 - \frac{du^2}{\mathcal{A}(u)} - r^2(u)d\Omega^2, \tag{12}$$

where the variables are connected with those in Eq. (4) as follows:

$$\mathcal{A}(u) = A(r), \quad r(u) = r, \quad \mathcal{A}(u)\left(\frac{dr}{du}\right)^2 = B(r). \tag{13}$$

The reason for using this coordinate is that, in a close neighborhood of a horizon (a sphere where $g_{tt} = 0$), it varies like manifestly well-behaved Kruskal-like coordinates used for an analytic continuation of the metric [17,18]. Using this coordinate, one can "cross the horizons" preserving the formally static expression for the metric. Both A and r must be smooth functions of u near the horizon $u = h$, so that

$$\mathcal{A}(u) \sim (u - h)^k, \quad r(u) \approx r_h + \text{const}(u - h)^s, \tag{14}$$

where $k = 1, 2, \ldots$ is the order of the horizon (the horizon is simple if $k = 1$, double if $k = 2$, etc.) while the number $s = 1, 2, \ldots$ characterizes the possible behavior of $r(u)$; $r_h > 0$ is the horizon radius. (We leave aside possible horizons of infinite radius which can in principle appear as well [17].) Generically but not necessarily one has $s = 1$. When $s = 1$ (i.e., dr/du is finite at the horizon), the coordinate r can be

used for continuing the metric through the horizon on equal grounds with u. The continuation will be analytic if both $\mathcal{A}(u)$ and $r(u)$ are analytic at $u = h$.

Assuming Eq. (14) and directly employing the relations (13), we find that

$$f(r) \sim B(r) \sim (r - r_h)^{(k + 2s - 2)/s} \text{ as } u \to h. \tag{15}$$

On the other hand, substituting $A = \mathcal{A}(u) \sim (r - r_h)^{k/s}$ into the solution (10) and assuming that the quantity $Q(r) = 2 - r^2 R(r)$ is finite at $r = r_h$ (which is generically the case), it is easy to obtain that near r_h

$$f(r) \sim B(r) \sim (r - r_h)^{2 - k/s}[(r - r_h)^{k/s} + C], \tag{16}$$

where C is an integration constant and $C = 0$ corresponds to the case when integration in Eq. (10) is performed from r_h to r. Comparing the exponents in Eqs. (15) and (16), we see that (i) in case $C = 0$: $k = 2$, s is not restricted; (ii) in case $C \neq 0$: $k = 1$, s is not restricted.

Thus, to obtain a solution with a horizon at $r = r_h$, we should take $A(r)$ behaving as $(r - r_h)^{k/s}$ with $k = 1$ or 2 and $s \in \mathbb{N}$.

An important point is that, under the condition $R(r_h) \neq 2/r_h^2$, a horizon can be either simple ($k = 1$) or double ($k = 2$); horizons of higher orders k do not appear. This is true, in particular, for all static, spherically symmetric metrics with $R = 0$.

Let us now look at what changes when the function $2 - r^2 R(r)$ vanishes at $r = r_h$. One can write

$$Q(r) := 2 - r^2 R(r) \sim (r - r_h)^p,$$

$$p = 0, 1, 2, \ldots, \quad \text{near} \quad r = r_h, \tag{17}$$

preserving the assumptions (14). So $p = 0$ corresponds to the above generic case $Q(r_h) \neq 0$ and $p > 0$ means that $Q(r)$ has a zero of order p. Then the expression (15) for $f(r)$ remains the same but Eq. (16) must be replaced with

$$f(r) \sim B(r) \sim (r - r_h)^{2 - k/s}[(r - r_h)^{k/s + p} + C]. \tag{18}$$

Consequently, in case $C \neq 0$ the metric behaves as before, whereas for $C = 0$ we obtain near $r = r_h$

$$A(r) \sim (r - r_h)^{p + 2/s}, \quad B(r) \sim (r - r_h)^{p + 2},$$

$$\mathcal{A}(u) \sim (u - h)^{ps + 2}, \tag{19}$$

that is, a horizon of order $ps + 2$.

Now, assuming $A = 0$ at $r = r_h > 0$, one can rewrite Eq. (10) in the form

$$f(r) \equiv rB(r) = \frac{2Ae^{3\Gamma}}{(4A + rA_r)^2}\left\{\int_{r_h}^r (4A + rA_r)\right.$$

$$\left. \times [2 - r^2 R(r)]e^{-3\Gamma} dr + C \right\}. \tag{20}$$

The above analysis shows that this relation leads to a metric with a horizon at $r = r_h$ in two cases:

(i) $A \sim (r-r_h)^{1/s}$, as $r \to r_h$, $s \in \mathbb{N}$. Then Eq. (20) leads to a metric with a simple horizon in case $C \neq 0$ and a metric with a horizon of order $2 + ps$ in case $C = 0$.

(ii) $A \sim (r-r_h)^{2/s}$ as $r \to r_h$, s odd. Then Eq. (20) leads to a metric with a horizon of order $2 + ps$ in case $C = 0$.

Here, as before, the parameter p characterizes the behavior of $Q(r)$ according to Eq. (17).

Item (ii) does not include solutions with $C \neq 0$. The point is that in case $A \sim (r-r_h)^2$, $C \neq 0$, the metric is singular at $r = r_h$, as is confirmed by calculating the Kretschmann scalar (6): its constituent K_2 blows up at $r \to r_h$. For odd numbers $s > 1$, the corresponding metric with $A \sim (r-r_h)^{2/s}$, $C \neq 0$ has a finite Kretschmann scalar but loses analyticity at $r = r_h$ and therefore cannot be analytically continued through this sphere. Indeed, in this case $r - r_h \sim (u-h)^{s/(ps+2)}$, where the exponent is a fraction for any odd s and $p = 0,1,2,\dots$. The metric is thus only continuous (C^0-smooth) at $u = h$ in case $p > 0$ and $C^{(s-1)/2}$-smooth in case $p = 0$.

B. Definition and algorithms

We have been so far discussing local conditions at possible Killing horizons. Let us now turn to space-time properties at large and try to select BH metrics. We shall not need a general rigorous definition of a BH [19] which in turn needs such notions as strong asymptotic predictability, trapped regions, etc. The following working definition will be appropriate for our purposes.

Definition. The metric (12) is said to describe a black hole if (a) the functions $\mathcal{A}(u)$ and $r(u)$ are analytic in the range $\mathcal{R}[u]: h \leq u < u_{\max}$ where $u_{\max}$ may be finite or infinite; (b) $r(u) > 0$ in $\mathcal{R}[u]$, and $r(u_{\max}) > r(h) = r_h$; (c) $\mathcal{A}(u) > 0$ at $u > h$, and $\mathcal{A}(h) \sim (u-h)^k, k \in \mathbb{N}$ as $r \to r_h$.

Item (c) means that $u > h$ is a static region (R region) of a static, spherically symmetric space-time while the sphere $u = h$, a boundary of this region, is a Killing horizon of a certain order k. So, in usual terms, our working definition describes the domain of outer communication of a BH, and $u = h$ is its event horizon.

The definition uses the u coordinate rather than r, due to its advantage in horizon description, discussed in the previous subsection. The difference is really essential: there are metrics which behave nonanalytically in terms of r at $r = r(h) = r_h$ but analytically in terms of u at $u = h$ (see variants $s > 1$ in Sec. III and example 5 in the next section).

The analyticity requirement rejects possible cases of restricted smoothness (see the end of Sec. III A). It is not only a matter of simplicity: in our view, if we are dealing with a field configuration, its nonanalyticity at a certain surface must have a physical reason, e.g., a phase transition, and it seems too artificial, a kind of perfect fine-tuning, to assume that the phase transition occurs precisely at a horizon.

We do not require $r(u_{\max}) = \infty$ since we do not want to rule out metrics with cosmological horizons like the Schwarzschild–de Sitter space-time where an R region is situated between a BH horizon and a cosmological horizon. We, however, adopt the requirement $r(u_{\max}) > r_h$ to exclude configurations with only cosmological horizons.

Let us return to the solution (10), or Eq. (20). Due to its generality, it certainly describes all BH metrics, at least piecewise. We can, however, formulate explicit requirements to the generating function $A(r)$ under which Eq. (10) leads *algorithmically* to a BH metric. Namely, let there be a range

$$\mathcal{R}[r]: r_{\max} > r > r_h, \quad r_h > 0, \tag{21}$$

in which the r.h.s. of Eq. (9) is positive:

$$Q(r) = 2 - r^2 R(r) > 0. \tag{22}$$

Then the above items (i) and (ii) lead to the following BH construction algorithms.

(BH1). Specify a function $A(r)$, positive and analytical in $\mathcal{R}[r]$, in such a way that $g(r) = 4A + rA_r > 0$ in $\mathcal{R}[r]$ and $A \sim (r-r_h)^{1/s}$, $s \in \mathbb{N}$, as $r \to r_h$. Then the functions $A(r)$ and $B(r)$ given by Eq. (20) with $C \geq 0$ determine a black hole metric (4) with a horizon at $r = r_h$. The horizon is simple if $C > 0$; in case $C = 0$ it is of the order $2 + ps$ if $Q(r)$ behaves according to Eq. (17).

(BH2). Specify a function $A(r)$, positive and analytical in $\mathcal{R}[r]$, in such a way that $g(r) = 4A + rA_r > 0$ in $\mathcal{R}[r]$ and $A \sim (r-r_h)^{2/s}$ as $r \to r_h$, s being an odd positive integer. Then the functions $A(r)$ and $B(r)$ given by Eq. (20) with $C = 0$ determine a black hole metric (4) with a horizon at $r = r_h$ of the order $2 + ps$ if $Q(r)$ behaves according to Eq. (17).

Both algorithms (BH1) and (BH2) lead to double horizons in case $C = 0$ if $Q(r_h) > 0$.

To obtain *asymptotically flat* BHs, one should assume $r_{\max} = \infty$ and restrict the choice of $A(r)$ to functions compatible with asymptotic flatness. Properly choosing the time scale, we can require a Schwarzschild behavior of A: $A = 1 - 2m/r + o(1/r)$ as $r \to \infty$, where m is the Schwarzschild mass. Equation (20) then leads to the proper behavior of $B(r)$, i.e., $B \to 1$ as $r \to \infty$, provided the curvature R decays quickly enough: $R(r) = o(r^{-3})$ as $r \to \infty$.

One can notice that both algorithms use the r coordinate whereas the BH definition uses u. A transition to u is accomplished with Eq. (13). The conditions of (BH1) and (BH2) guarantee that $B > 0$ at $r > r_h$, therefore, choosing $dr/du > 0$ in Eq. (13), we evidently satisfy the BH definition.

The condition (22) can be weakened: what actually must be required is that the integral in Eq. (20) should remain positive in $\mathcal{R}[r]$.

Another condition of both algorithms, $g(r) = 4A + rA_r > 0$ in $\mathcal{R}[r]$, allows one to avoid a singular point of Eq. (9), where the coefficient of the derivative f_r vanishes. This coefficient also vanishes at horizons, which have been already discussed. The points where $A \neq 0$ but $g = 0$, if any, also deserve special attention. However, the equality $g = 0$ itself has no evident geometric (and hence physical) meaning; it only represents some technical difficulty in our description with the aid of Eq. (9).

Points where $g = 0$ are avoided by most asymptotically flat BH metrics. Indeed, in this case $g \geq 0$ at the horizon and $g = 4$ at infinity; the condition $g(r) > 0$ means that $r^4 A$ is a

strictly increasing function of r, which is the case, e.g., for all functions $A(r)$ monotonically growing from 0 at the horizon to 1 at infinity.

However, one can easily verify that such points inevitably appear in other important situations, e.g., between a regular center and a cosmological horizon or between two simple horizons.

It can be shown that if the choice of the generating function $A(r)$ leads to $g(r)$ with a simple zero at some $r=r_s \in \mathcal{R}[r]$, then there is a unique nonzero value of the constant C [namely, when integration in Eq. (10) starts from r_s] making it possible to avoid a singularity of the metric at $r=r_s$. So algorithm (BH1) survives, but it now gives a single solution with a horizon instead of a family parametrized by C (see example 4 in Sec. IV). Algorithm (BH2) requires $C=0$ and therefore does not work.

C. Generic behavior of the solutions: Wormholes and regular BHs

Let us discuss the properties of the metric in the generic situation that leads to a BH according to algorithm (BH1): let $A(r)$ be a well-behaved positive function at $r>r_h$ and have a simple zero at $r=r_h$, and let also $Q(r_h)>0$. Then, in a small neighborhood of $r=r_h$, one can write

$$A(r)=A_1(r-r_h)+o(r-r_h),$$

$$B(r)=B_1 C(r-r_h)+B_2(r-r_h)^2+o((r-r_h)^2),$$
$$\tag{23}$$

with fixed positive constants A_1, B_1 and B_2. The integration constant C is a family parameter, and $C=0$ is its critical value at which the solution drastically changes its properties.

If $C<0$, B turns to zero at $r=r_{\min}=r_h+|C|B_1/B_2>r_h$. [Here, $|C|$ should be small enough for the solution to remain in a range where Eq. (23) is still approximately valid.] One obtains $B(r)\approx B_2(r-r_h)(r-r_{\min})$, so that $B(r)$ has a simple zero at $r=r_{\min}$, whereas $A(r_{\min})>0$. Such a behavior of the metric functions corresponds to a symmetric wormhole throat at $r=r_{\min}$ [13]. The substitution $r=r_{\min}+x^2$, $x\in\mathbb{R}$, makes the metric (4) regular at $r=r_{\min}$ ($x=0$), and all metric coefficients are even functions of x. Thus our solution does not reach the anticipated horizon $r=r_h$ and describes a symmetric wormhole.

In case $C=0$, as already described, we obtain a double horizon at $r=r_h$, and the geometry is smoothly continued to smaller r, where the further properties of the metric depend on the specific choice of $A(r)$.

If $C>0$, then $B>0$ at $r>r_h$, turns to zero at the horizon $r=r_h$ and again turns to zero in the T region at $r=r_{\min}=r_h-CB_1/B_2<r_h$. This, as before, bounds the range of r from below in a way similar to a wormhole throat. The coordinate singularity at $r=r_0$ is again removed by the transformation $r=r_{\min}+x^2$, but now x is a temporal coordinate in a T region, where the metric (4) describes a Kantowski-Sachs cosmology with two scale factors $r(x)$ and $A[r(x)]$ and the $\mathbb{R}\times\mathbb{S}^2$ topology of spatial sections. Hence, $x=0$ is the time instant at which $r(x)$ experiences a bounce. It can be roughly said
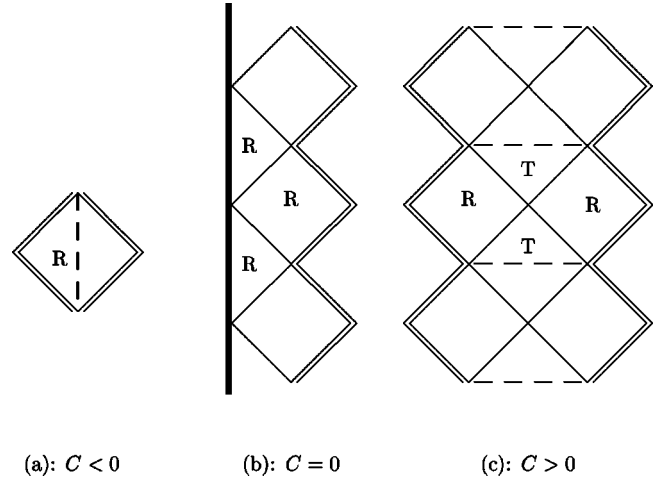

(a): $C<0$ (b): $C=0$ (c): $C>0$

FIG. 1. Carter-Penrose diagrams for a generic family of asymptotically flat solutions: (a) a symmetric wormhole, (b) an extremal BH in case there is a singularity inside the inner R region, and (c) a regular BH. Diagrams (b) and (c) can be infinitely continued upward and downward. The letters R and T mark static (R) and cosmological (T) regions, respectively. Spatial infinity ($r=\infty$) is shown by double lines, horizons ($r=r_h$) by single thin lines and the singularity in (b) by a thick line. Dashed lines show the wormhole throat in diagram (a) and the bouncing time instants in diagram (c).

that the wormhole throat, having moved into a T region, becomes a bouncing time instant of the scale factor r in a Kantowski-Sachs cosmology.

Assuming asymptotic flatness at large r, one can use the standard methodology to obtain the corresponding Carter-Penrose diagram describing the global causal structure: it coincides with that of the Kerr or Kerr-Newman nonextremal BH [19] (but without a ring singularity) and contains an infinite sequence of R and T regions.

By continuity, this behavior is preserved in a finite range of C values (see examples 1 and 2 in the next section). We conclude that each generic choice of $A(r)$ with a simple zero at $r=r_h$ leads to a family of solutions unifying symmetric wormholes ($C<0$), extremal BHs ($C=0$) and regular nonextremal BHs ($C>0$), see Fig. 1.

IV. EXAMPLES

Example 1: $A(r)=1-2m/r$, $m=\text{const}>0$

For this Schwarzschild form of $A(r)$, the solution (10) in the vacuum case $R\equiv0$ can be written as

$$f(r)=\frac{(r-2m)(r-r_0)}{r-3m/2},\tag{24}$$

where r_0 is an integration constant. The metric takes the form

$$ds^2=\left(1-\frac{2m}{r}\right)dt^2-\frac{1-3m/(2r)}{(1-2m/r)(1-r_0/r)}dr^2-r^2d\Omega^2.\tag{25}$$

The Schwarzschild metric is restored in the special case $r_0 = 3m/2$. The metric (25) was obtained by Casadio, Fabbri and Mazzacurati [7] in search for new brane-world black holes and by Germani and Maartens [8] as a possible external metric of a homogeneous star on the brane. Without repeating their more detailed descriptions, we will outline the main points in our notations.

BH metrics appear according to algorithm (BH1), where $C=0$ corresponds to $r_0=2m$ and $C>0$ to $r_0<2m$. In case $r_0>2m$, the metric (25) describes a symmetric traversable wormhole [13].

In case $r_0=2m$ we have a double horizon at $r=2m$: near $r=2m$, the coordinate r is connected with the quasiglobal coordinate u by $r-2m\sim(u-u_h)^2$, u_h being the value of u at the horizon, and $\mathcal{A}(u)\sim(u-u_h)^2$. The Carter-Penrose diagram coincides with that of the extremal Reissner-Nordström metric [Fig. 1(b)] with the only difference that the timelike curvature singularity occurs at $r=3m/2$ instead of $r=0$.

In case $r_0<2m$, as in the Schwarzschild metric, $r=2m$ is a simple horizon, and, as described in Ref. [7], the space-time structure depends on the sign of $\eta=r_0-3m/2$. If $\eta<0$, the structure is that of a Schwarzschild black hole, but the spacelike curvature singularity is located at $r=3m/2$ instead of $r=0$. If $\eta>0$, the solution describes a nonsingular black hole with a wormhole throat at $r=r_0$ inside the horizon, or, more precisely, it is the minimum value of r at which the model bounces. The corresponding global structure [7] is the same as that of a nonextremal Kerr BH [Fig. 1(c)].

Thus the metric properties in the whole range $r_0>3m/2$ of the integration constant r_0 entirely conform to the description in Sec. III C for both positive and negative C.

The components of the effective SET (7) have the form

$$\kappa_4^2\rho^{\text{eff}} = \frac{m(2r_0-3m)}{r^2(2r-3m)^2},$$

$$\kappa_4^2 p_{\text{rad}}^{\text{eff}} = -\frac{2r_0-3m}{r^2(2r-3m)},$$

$$\kappa_4^2 p_\perp^{\text{eff}} = \frac{(r-m)(2r_0-3m)}{r^2(2r-3m)^2}. \tag{26}$$

Example 2: $A(r)=1-h^2/r^2$, $h=\text{const}>0$

This form of $A(r)$ represents a metric with zero Schwarzschild mass.

BH solutions are easily obtained: Eq. (20) with $R\equiv0$ now gives

$$f(r)=rB(r)=r\left(1-\frac{h^2}{r^2}\right)\left(1+\frac{C-h}{\sqrt{2r^2-h^2}}\right). \tag{27}$$

In accord with (BH1), the sphere $r=h$ is a simple horizon if $C>0$ and a double horizon if $C=0$.

In case $C<0$, $B(r)$ has a simple zero at $r=r_{\text{th}}>h$ given by

$$2r_{\text{th}}^2=h^2+(h-C)^2, \tag{28}$$

which is a symmetric wormhole throat [13].

In case $C=0$, $r=h$ is a double horizon, and the Carter-Penrose diagram coincides with that of the extremal Reissner-Nordström metric [Fig. 1(b)], but a timelike singularity due to $B\to\infty$ takes place at $r=h/\sqrt{2}$.

In case $0<C<h$, inside the simple horizon $r=h$, the function $B(r)$ turns to zero at $r=r_{\text{th}}$ given by Eq. (28), which is now between h and $h/\sqrt{2}$, and we obtain a Kerr-like regular BH structure with an infinite sequence of R and T regions [Fig. 1(c)]. We see that the description of Sec. III C is valid in the whole range $C<h$ of the integration constant C.

The value $C=h$ leads to the simplest metric with $A=B=1-h^2/r^2$, which may be identified as the Reissner-Nordström metric with zero mass and pure imaginary charge. The space-time causal structure is Schwarzschild, with a horizon at $r=h$ and a singularity at $r=0$. Lastly, in case $C>h$ the causal structure is again Schwarzschild but the singularity due to $B\to\infty$ occurs at $r=h/\sqrt{2}$.

This example is of certain interest in connection with Thorne's "hoop conjecture," claiming that a BH horizon forms when and only when a mass M gets concentrated in a region whose circumference in every direction is smaller than $4\pi GM$, G being Newton's constant of gravity [20]. Nakamura *et al.* [21] recently found an example of a cylindrical (i.e., infinitely long) matter distribution on the brane able to form a horizon and thus violating the hoop conjecture. The present example of a zero mass BH shows that, in the brane-world context, a BH may exist (at least as a solution to the gravitational equations on the brane) without matter and without mass, solely as a tidal effect from the bulk gravity. The effective SET is in this case certainly quite exotic from the viewpoint of the conventional energy conditions:

$$\kappa_4^2\rho^{\text{eff}} = -\frac{h^2}{r^4} - \frac{h^2(C-h)(3r^2-h^2)}{r^4(2r^2-h^2)^{3/2}},$$

$$\kappa_4^2 p_{\text{rad}}^{\text{eff}} = \frac{h^2}{r^4} + \frac{(C-h)(r^2+h^2)}{r^4(2r^2-h^2)^{1/2}},$$

$$\kappa_4^2 p_\perp^{\text{eff}} = -\frac{h^2}{r^4} - \frac{(C-h)(r^4+2h^2r^2-h^4)}{r^4(2r^2-h^2)^{3/2}}. \tag{29}$$

In the simplest case $C=h$ it has the "anti–Reissner-Nordström" form, $\propto r^{-4}\text{diag}(-1,-1,1,1)$.

Example 3: $A(r)=(1-2m/r)^2$, $m=\text{const}>0$

For this extremal Reissner-Nordström form of $A(r)$, the solution (10) with $R\equiv0$ and the metric can be written as

$$f(r) = \frac{(r-r_0)(r-r_1)}{r}, \quad r_1 := \frac{mr_0}{r_0-m}, \quad (30)$$

$$ds^2 = \left(1 - \frac{2m}{r}\right)^2 dt^2 - \left(1 - \frac{r_0}{r}\right)^{-1}\left(1 - \frac{r_1}{r}\right)^{-1} dr^2$$
$$- r^2 d\Omega^2. \quad (31)$$

The form of $A(r)$ fits algorithm (BH2), and accordingly we obtain a BH solution in the only case $r_0 = r_1 = 2m$, i.e., when the integration in Eq. (20) is conducted from $r_h = 2m$, so that $C = 0$. It is the extremal Reissner-Nordström metric, and accordingly the effective SET is $T_\mu^{\nu\,\mathrm{eff}} \propto r^{-4}\mathrm{diag}(1,1,-1,-1)$.

Other values of r_0 lead either to wormholes (the throat is located at $r = r_0$ if $r_0 > 2m$ or at $r = r_1 > 2m$ in case $2m > r_0 > m$), or to a naked singularity located at $r = 2m$ (when $r_0 < m$) as is confirmed by calculating the Kretschmann scalar—see more detail in Ref. [13].

Example 4: $A(r) = 1 - r^2/a^2$, $a = \mathrm{const} > 0$

The above examples described vacuum asymptotically flat BHs. Now, choosing the de Sitter form of $A(r)$, we will write the solution (10) for a vacuum configuration with a cosmological term, so that $R = 4\Lambda_4 = 12/a^2$, in the region $r < a$. We obtain

$$f(r) = rB(r) = \left(1 - \frac{r^2}{a^2}\right)\left[r + \frac{K}{(2a^2 - 3r^2)^{3/2}}\right], \quad (32)$$

where K is an integration constant such that $K = 0$ corresponds to integration in Eq. (10) from $r = r_s = a\sqrt{2/3}$ to r. The value $r = r_s$ is the one where $g(r) = 4A + rA_r$ vanishes. In full agreement with the description in Sec. III B, $B(r)$ tends to infinity as $r \to r_s$ unless $K = 0$, and thus the only well-behaved solution is de Sitter, with $A = B = 1 - r^2/a^2$. This example illustrates what happens when Eq. (9) has a singular point $g = 0$ in the range of interest.

Example 5: $A(r) = (1 - 2m/r)^{1/s}$, $m = \mathrm{const} > 0$, $s \in \mathbb{N}$

We here try to give an example of a metric behaving nonanalytically at $r = r_h$ in terms of r but analytically in terms of the quasiglobal coordinate u defined by $g_{tt}g_{uu} = -1$, see Sec. III A. A certain difficulty is that the solution (10) for this choice of $A(u)$, even in the simplest case $R = 0$, is expressed with the aid of the hypergeometric function, which can hardly be a very clear illustration. We therefore simply take the following "artificial" example of an asymptotically flat metric (4):

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{1/s} dt^2$$

$$- \left(1 - \frac{2m}{r}\right)^{-2+1/s} dr^2 - r^2 d\Omega^2, \quad (33)$$

as suggested by Eq. (15) for any positive integer s in the case of a simple horizon, and transform it to a coordinate which behaves like u near $r_h = 2m$, namely, put $1 - 2m/r = x^s$ [we do not directly use the transformation (13) since $u(r)$ then looks too cumbersome]. The metric takes the form

$$ds^2 = x\,dt^2 - \frac{4m^2s^2}{x(1-x^s)^4}dx^2 - \frac{4m^2}{(1-x^s)^2}d\Omega^2. \quad (34)$$

Its asymptotic flatness at $x = 1$ is not so evident, but evident is the behavior at $x = 0$ as expected at a simple horizon. In case $s = 1$ it is the Schwarzschild metric. For $s > 1$ it is not a vacuum solution to Eq. (9); the effective SET is easily found according to Eqs. (7); it decays at large r as r^{-4}, in particular, its trace is κ_4^{-2} times the scalar curvature

$$R = \frac{2}{r^2} + \frac{2}{r^3 s}\left(1 - \frac{2m}{r}\right)^{1-1/s}[2m - s(r+2m)].$$

The same substitution $1 - 2m/r = x^s$ applied to the metric

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{2/s} dt^2 - \left(1 - \frac{2m}{r}\right)^{-2} dr^2 - r^2 d\Omega^2 \quad (35)$$

with $m > 0$ and $s \in \mathbb{N}$ reveals a double horizon at $r = 2m$.

V. CONCLUDING REMARKS

Using the trace of the 4D Einstein equations, written as a linear first-order ordinary differential equation and integrated, we have formulated some general requirements to the (arbitrary) generating function $A(r) \equiv g_{tt}$ which are sufficient for obtaining static, spherically symmetric BH metrics — see algorithms (BH1) and (BH2). The BHs may be asymptotically flat or have any other large r behavior.

We have seen that, under some natural restrictions, BH metrics are easily constructed in vacuum or in the presence of matter for which the dependence $R(r)$ may be specified. However, not every kind of matter distribution admits a horizon inside it. No horizon can appear, e.g., in a perfect fluid with the equation of state $\rho = np$, $n \in \mathbb{N}$: the conservation law then implies $\rho = \rho_0 A^{-(n+1)/2}$, $\rho_0 = \mathrm{const}$, so that $\rho \to \infty$ as $A \to 0$. More generally, at a horizon, the effective SET (7) should satisfy the condition $\rho^{\mathrm{eff}} + p_{\mathrm{rad}}^{\mathrm{eff}} = 0$. Indeed, one can write in terms of the metric (12):

$$G_t^t - G_u^u = 2\frac{\mathcal{A}}{r}\frac{d^2r}{du^2} = -\kappa_4^2(\rho^{\mathrm{eff}} + p_{\mathrm{rad}}^{\mathrm{eff}}) \quad (36)$$

[recall that $\mathcal{A}(u) = A(r)$], which leads to $\rho^{\mathrm{eff}} + p_{\mathrm{rad}}^{\mathrm{eff}} = 0$ at regular points where $\mathcal{A} = 0$.

The same quantity is negative at wormhole throats (it is the well-known violation of the null energy condition [22]) but is positive at bounces in T regions. Equation (36) shows that this property is quite general: at a minimum of r, where $d^2r/du^2 > 0$, one has $\rho^{\mathrm{eff}} + p_{\mathrm{rad}}^{\mathrm{eff}} < 0$ if $\mathcal{A} > 0$ (a throat) and $\rho^{\mathrm{eff}} + p_{\mathrm{rad}}^{\mathrm{eff}} > 0$ if $\mathcal{A} < 0$ (a bounce).

A feature of utmost interest is the generic appearance of families of solutions which unify symmetric wormholes and globally regular BHs with a bounce of r in the T region and a Kerr-like global structure. The two qualitatively different branches of any such family are separated by an extremal BH solution.

Certain care should be taken about possible zeros ($r = r_s$) of the function $g(r) = 4A + rA_r$ which is a coefficient of the derivative f_r in Eq. (9). We have shown that even if the choice of $A(r)$ leads to $g(r) = 0$ at some r, a well-behaved solution can generically be obtained.

The whole consideration is quite general and may find application in general relativity (where our effective SET $T_\mu^{\nu\,\mathrm{eff}}$ is simply the matter SET including a possible cosmological term) and alternative theories of gravity that use modified Einstein equations. As usual, for a particular kind of matter, the theory will give one more independent equation for the two metric functions to be found, $A(r)$ and $B(r)$ in our notation, and the set of equations will be determined.

The most natural application of these results is, however, the brane world concept where the trace of the Einstein equations is the only equation of 4D gravity which can be written unambiguously using only 4D quantities. Being a single equation for the two unknown functions $A(r)$ and $B(r)$, it leads to a variety of BH as well as wormhole solutions. This ambiguity reflects the ambiguity of a brane embedding into the bulk, which manifests itself in Eqs. (1) in the arbitrariness of E_μ^ν. Moreover, as remarked in Ref. [13], the tensor E_μ^ν, due to its geometric origin, need not respect the usual energy conditions, and the appearance of wormhole solutions in its presence looks quite natural.

On the other hand, in the brane world context, neither of the particular 4D solutions can be considered as complete or viable before their profile in the bulk is derived and their regularity is ensured. Some known BH solutions [2–4] face the problem of nonlocalized BH singularities. However, in our view, nonsingular 4D metrics, such as the wormhole and regular BH metrics described here, are more likely to have regular continuations into the bulk than singular BH metrics.

The above embedding ambiguity exists even in the simplest brane world models of RS2 type [11], possessing a single extra dimension, $\mathbb{Z}_2$ symmetry with respect to the brane and no matter in the bulk, to say nothing of more complex models. The latter may include scalar fields in the bulk [5,23], multiple (at least two) branes [24,25], more than one extra dimension [26], timelike extra dimensions, lacking $\mathbb{Z}_2$ symmetry, a 4D curvature term [27] etc.; see further references in the cited papers and the reviews [1]. To improve the predictive power of brane world scenarios, it seems necessary to remove the "redundant freedom," applying reasonable physical requirements such as regularity and stability to complete multidimensional models.

Much work in this direction has already been done. Different methods of solving the bulk gravity equations for given brane configurations have been developed [4,9,14,28], and the bulk properties of some particular brane-world BHs have been studied [2,4–6,9]. In particular, in Ref. [9], using a multipole-type expansion, it has been shown that the horizon of the BH metric (25) in case $2m > r_0 > 3m/2$, being continued into the bulk, closes at some value of the fifth coordinate z, and there is no BH singularity at all; still, a singularity at large z is not excluded. Possible quantum properties of such brane-world BHs have also been discussed [10]. It appears to be a necessary, though difficult, task to extend such studies to more general BH configurations.

[1] V.A. Rubakov, Phys. Usp. **44**, 871 (2001); R. Maartens, "Geometry and dynamics of the brane world," gr-qc/0101059; D. Langlois, "Gravitation and cosmology in a brane universe," gr-qc/0207047; S. Nojiri, S.D. Odintsov, and S. Ogushi, Int. J. Mod. Phys. A **17**, 4809 (2002); Ph. Brax and C. van de Bruck, Class. Quantum Grav. **20**, R201 (2003).

[2] A. Chamblin, S.W. Hawking, and H.S. Reall, Phys. Rev. D **61**, 065007 (2000).

[3] N. Dadhich, R. Maartens, P. Papadopoulos, and V. Rezania, Phys. Lett. B **487**, 1 (2000).

[4] A. Chamblin, H.S. Reall, H. Shinkai, and T. Shiromizu, Phys. Rev. D **63**, 064015 (2001).

[5] S. Nojiri, O. Obregon, S.D. Odintsov, and S. Ogushi, Phys. Rev. D **62**, 064017 (2000).

[6] P. Kanti and K. Tamvakis, Phys. Rev. D **65**, 084010 (2002).

[7] R. Casadio, A. Fabbri, and L. Mazzacurati, Phys. Rev. D **65**, 084040 (2002).

[8] C. Germani and R. Maartens, Phys. Rev. D **64**, 124010 (2001).

[9] R. Casadio and L. Mazzacurati, Mod. Phys. Lett. A **18**, 651 (2003).

[10] R. Casadio and B. Harms, Phys. Lett. B **487**, 209 (2000); R. Casadio, "Holography and trace anomaly: What is the fate of (brane-world) black holes?," hep-th/0302171; R. Casadio, "On brane-worldback holes and short scale physics," hep-ph/0304099.

[11] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).

[12] T. Shiromizu, K. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000).

[13] K.A. Bronnikov and S.-W. Kim, Phys. Rev. D **67**, 064027 (2003).

[14] M. Visser and D.L. Wiltshire, Phys. Rev. D **67**, 104004 (2003).

[15] J. Campbell, *A Course of Differential Geometry* (Clarendon, Oxford, 1926); L. Magaard, Ph.D. thesis, University of Kiel, 1963.

[16] S.S. Seahra and P.S. Wesson, Class. Quantum Grav. **20**, 1321 (2003).

[17] K.A. Bronnikov, G. Clément, C.P. Constantinidis, and J.C. Fabris, Phys. Lett. A **243**, 121 (1998); Gravitation Cosmol. **4**, 128 (1998).

[18] K.A. Bronnikov, Phys. Rev. D **64**, 064013 (2001).

[19] R.M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[20] K.S. Thorne, in *Magic without Magic: John Archibald Wheeler*, edited by J. Clauder (Freeman, San Francisco, 1972), p. 231.

[21] K. Nakamura, K. Nakao, and T. Mishima, Prog. Theor. Phys. Suppl. **148**, 291 (2003).

[22] D. Hochberg and M. Visser, Phys. Rev. D **58**, 044021 (1998).

[23] K. Maeda and D. Wands, Phys. Rev. D **62**, 124009 (2000); T. Shiromizu, T. Torii, and D. Ida, J. High Energy Phys. **03**, 007 (2002).

[24] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).

[25] S. Kanno and J. Soda, Phys. Rev. D **66**, 083506 (2002); T. Shiromizu and K. Koyama, *ibid.* **67**, 084022 (2003).

[26] E.I. Guendelman, ''Conformally invariant braneworld and the cosmological constant,'' gr-qc/0303048.

[27] Yu.V. Shtanov, Phys. Lett. B **541**, 177 (2002); V. Sahni and Yu.V. Shtanov, Int. J. Mod. Phys. D **11**, 1 (2002).

[28] T. Wiseman, Phys. Rev. D **65**, 124007 (2002).

[29] The sign conventions are as follows: the metric signature $(+ - - -)$; the curvature tensor $R^{\sigma}{}_{\mu\rho\nu} = \partial_{\nu}\Gamma^{\sigma}_{\mu\rho} - \ldots$, so that, e.g., the Ricci scalar $R > 0$ for de Sitter space-time, and the stress-energy tensor (SET) such that T^{t}_{t} is the energy density.