Higher dimensional black holes and supersymmetry

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(Received 4 March 2003; published 25 July 2003)

It has recently been shown that the uniqueness theorem for stationary black holes cannot be extended to five dimensions. However, uniqueness is an important assumption of the string theory black hole entropy calculations. This paper justifies this assumption by proving a uniqueness theorem for supersymmetric black holes in five dimensions. Some remarks concerning general properties of nonsupersymmetric higher dimensional black holes are made. It is conjectured that there exist new families of stationary higher dimensional black hole solutions with fewer symmetries than any known solution.

DOI: 10.1103/PhysRevD.68.024024 PACS number(s): 04.70.Bw, 04.50.+h, 04.65.+e

I. INTRODUCTION

One of the most impressive successes of string theory is a microscopic derivation of the entropy of certain supersymmetric black holes [1]. The idea is that a weakly coupled system of strings and branes wrapped around some compact dimensions turns into a black hole in the noncompact dimensions as the string coupling is increased. For fixed asymptotic charges (mass, angular momenta and gauge charges), the degeneracy of microstates can be calculated in the weakly coupled description. Provided sufficient supersymmetry is preserved, this is found to correctly reproduce the Bekenstein-Hawking entropy of the black hole (at least for black holes much larger than the string length).

These calculations were first performed for static supersymmetric black holes in five dimensions [1]. They were subsequently extended to static supersymmetric holes in four dimensions [2,3], to rotating supersymmetric holes in five dimensions [4] and to nearly supersymmetric generalizations of all of these [5–7].

A key assumption made in this work is that the relevant black hole solutions are uniquely specified by their asymptotic charges. If this turned out to be untrue, i.e., if there existed distinct supersymmetric black hole solutions with the same asymptotic charges, then there would be a problem with the conventional interpretation of the entropy calculations. The problem would be in identifying which sets of microstates should correspond to each black hole, as would be necessary in order to compute their respective entropies. The black holes would be distinguished macroscopically by their differing gravitational fields. However, there is no gravitational field present in the weakly coupled description used for the entropy calculations. Hence, at weak coupling, there would be no way of telling which microstates corresponded to which black hole. The distinction between different sets of microstates would only become apparent as the microscopic description became strongly coupled.

Given the importance of this assumption, one might ask how it was originally motivated. It seems that the only evidence in its favor is the existence of the black hole uniqueness theorems in four dimensions. These establish that stationary four dimensional black holes are indeed uniquely specified by their asymptotic charges, at least in Einstein-Maxwell theory. The uniqueness theorems assume a nondegenerate event horizon and therefore do not apply to supersymmetric black holes. Nevertheless, it would be very surprising if supersymmetric black holes turned out to be nonunique in four dimensions.[1]

In five dimensions, the only evidence for the uniqueness assumption seems to have been that higher dimensional black holes appeared to have very similar properties to four dimensional ones. However, this is not really evidence at all because all known higher dimensional black hole solutions were derived using *Ansätze* based on simple generalizations of four dimensional black hole solutions, or were related by dualities to solutions based on such *Ansätze*. Therefore it is not very surprising that the known higher dimensional black holes had similar properties to four dimensional ones.

The situation has changed with the recent discovery of a class of five dimensional vacuum black holes that are completely unlike anything encountered in four dimensions—"black rings" [9]. These are stationary black holes with event horizons of topology $S^1\times S^2$. They can be regarded as rotating loops of black string, with the centrifugal force balancing the tendency of the ring to collapse under gravity. The existence of black rings implies that the uniqueness theorem for stationary black holes does not extend to five dimensions. This is because black rings can carry the same asymptotic charges as the vacuum black holes of spherical topology discovered by Myers and Perry [10].[2]

In the first part of this paper, it will be suggested that there should be many more exotic black hole solutions in higher dimensions. Examining the steps that go into proving the uniqueness theorems in four dimensions suggests that a *general* stationary asymptotically flat black hole in higher dimensions should admit only two commuting Killing vector fields. However, all *known* higher dimensional black hole solutions have more symmetry. So there may exist large families of higher dimensional black hole solutions in addition to the known ones. This would imply that black hole uniqueness would always be badly violated in higher

[1]This paper will only discuss spacetimes containing a single black hole. Otherwise multi-black hole solutions [8] would be an example of nonuniqueness.

[2]*Static* higher dimensional black hole solutions were first obtained in [11].

dimensions,[3] and emphasizes the importance of justifying the uniqueness assumption for supersymmetric black holes.

A uniqueness theorem *has* been proved for nondegenerate higher dimensional *static* black holes in Einstein-Maxwell [13] and Einstein-Maxwell-dilaton theory [14], and for Einstein gravity coupled to a σ-model [15]. The uniqueness assumption for static supersymmetric black holes in higher dimensions therefore seems plausible.

For rotating holes, it is not at all clear whether this assumption is correct. Rotating supersymmetric black holes seem to exist only in five dimensions—the first example was found by Breckenridge, Myers, Peet and Vafa (BMPV) [4]. It seems rather likely that charged black ring solutions should also exist, and if these had a regular supersymmetric limit then uniqueness of supersymmetric rotating black holes might be violated. Also, if there do exist higher dimensional black holes with fewer symmetries than any known solution then why not supersymmetric black holes with fewer symmetries than BMPV? It is clearly desirable to know whether this happens and, if not, whether a uniqueness theorem for supersymmetric black holes can be proved.

The main goal of this paper is to provide the first example of such a uniqueness theorem, and thereby justify the uniqueness assumption made in the black hole entropy calculations. This is therefore a check on the consistency of the entropy calculations that can be performed at the level of classical supergravity.

The supergravity theory that will be considered is minimal $N=1$, $D=5$ supergravity [16] because it is the simplest theory in which black hole uniqueness is known to be violated (the theory admits black ring solutions). Furthermore, the BMPV supersymmetric rotating black hole solution can be embedded in this theory [17]. In fact, this theory is sufficiently simple that it is possible to find *all* supersymmetric solutions [18]. Previously, the only theories for which this had been done were minimal $N=2$, $D=4$ supergravity [19] and some simple $D=4$ generalizations [20].

The general supersymmetric solution obtained in [18] is sufficiently complicated that it is far from obvious which solutions correspond to black holes. In fact, the solution given in [18] is only valid away from any horizons that may be present in the spacetime. In this paper, it will be shown how a local analysis of the constraints imposed by supersymmetry in the neighborhood of the horizon can be combined with global information about the black hole exterior provided by the general solution of [18] to prove that the BMPV solution is the only supersymmetric black hole solution of minimal $N=1$, $D=5$ supergravity.

It is reassuring that a uniqueness theorem can be proved for supersymmetric black holes. However, this theorem also serves to emphasize how special such black holes are, in the sense that they fail to exhibit features that are expected of *general* black holes, e.g., nonuniqueness in five dimensions. So the results of this paper highlight how far string theory is from providing a complete understanding of black holes.

This paper is organized as follows. Section II discusses general properties of higher dimensional black holes. Section III contains the uniqueness theorem. There is one Appendix dealing with a special case that arises in the analysis.

II. HIGHER DIMENSIONAL BLACK HOLES

A. Black holes with fewer symmetries

All known stationary D-dimensional black hole solutions have at least $[(D+1)/2]$ commuting isometries. The purpose of this section is to point out that this seems to be "too many," i.e., in general one would expect fewer symmetries. Before explaining this, it is helpful to recall what happens for the analagous case of black string solutions.

Consider the uniform black string solution of the five dimensional vacuum Einstein equations. The metric is the product of the four dimensional Schwarzschild solution with a flat direction, so there are three commuting Killing vector fields, corresponding to time translations, rotations and spatial translations. If the string is compactified on a circle of asymptotic radius L then one can define a dimensionless parameter

$$\eta = \frac{GM}{L^2}, \tag{2.1}$$

where M is the mass of the string. There is a particular value $\eta=\eta_c$ for which the uniform string solution admits a static zero-mode that breaks the translational symmetry [21]. This led to the conjecture [21,22] that exact static black string solutions without translational symmetry should also exist. There is good perturbative [23] and numerical [24,25] evidence that this is indeed the case, but the solutions are not known analytically.[4] These solutions have only *two* commuting Killing vector fields, which is one fewer than for the solutions that are known analytically.

To understand why there might also exist stationary black *holes* with fewer symmetries than any known solution, it is worth reviewing the steps that go into proving the uniqueness theorem for four dimensional black holes, and asking which steps can be generalized to higher dimensions. For simplicity, only vacuum black holes will be considered, although similar remarks should apply to nondegenerate charged black holes. It is probably also worth emphasizing that only asymptotically flat black holes will be considered in this paper.

The first step is the proof that the event horizon of a stationary black hole must have S^2 topology [28,29]. This relies on the Gauss-Bonnet theorem applied to the (two dimensional) horizon and therefore does not generalize to higher dimensions. An alternative proof in four dimensions is based on the notion of "topological censorship" [30]. Consider a spacelike slice Σ that intersects the future event ho-

[3]It is tempting to conjecture that adding the requirement of *stability* would guarantee uniqueness [12], but there is no evidence for this since stability of higher dimensional black holes has never been studied.

[4]See [26,27] for attempts to construct such solutions analytically.

rizon and let H denote the intersection. Topological censorship requires that Σ be simply connected. Note that Σ has two boundaries, namely H and the sphere at spatial infinity. Hence topological censorship requires that H be cobordant to a sphere via a simply connected cobordism. For a stationary black hole, this can be shown to imply that H is a sphere [31].

Topological censorship is also valid for $D>4$ but it is much less restrictive. First, if $D>4$ and there exists a cobordism from H to the sphere then there also exists a simply connected cobordism.[5] Secondly, a cobordism from H to the sphere exists if, and only if, H has vanishing Pontrjagin and Stiefel-Whitney numbers. For $D=5$, H is an oriented 3-manifold and hence automatically has vanishing Pontrjagin and Stiefel-Whitney numbers so topological censorship does not restrict the topology of the event horizon for $D=5$ black holes [33]. For $D=6$, H is a 4-manifold and topological censorship excludes, for example, $H=CP^2$ because it has nonvanishing Pontrjagin and Stiefel-Whitney numbers.

In summary, there are very few useful restrictions on the topology of the event horizon of a general stationary black hole in higher dimensions. However, black rings are the only known example of stationary black holes with nonspherical horizons.

The next step in the four dimensional uniqueness proof is that a stationary black hole must either be static or have an ergoregion [29]. This theorem is straightforward to extend to higher dimensions. In the static case, it can then be shown that the only solution is the Schwarzschild solution [34,35], and this theorem has recently been extended to higher dimensions [13]. A simple corollary is that a static higher dimensional black hole must have a spherical horizon.

The possibility of an ergoregion disjoint from the event horizon was excluded in [36] for four dimensional black holes. This proof relies on a technical theorem concerning maximal hypersurfaces [37]; it will be assumed here that it can be generalized to higher dimensions. This implies that the stationary Killing vector field of a stationary, nonstatic, higher dimensional black hole is spacelike on the event horizon.

In four dimensions, it can be argued [28,29] that the tangent vector to the null geodesic generators of the event horizon can be extended to give a Killing vector field ξ of the full spacetime, which commutes with the stationary Killing vector field. The latter cannot be equal to ξ since it is spacelike on the horizon. One can therefore write (after appropriately scaling ξ)

$$\xi=\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}, \tag{2.2}$$

with $\partial/\partial\phi$ spacelike and Killing. It seems likely that this theorem could be extended to higher dimensions although, since the topology of the horizon is not known, the geometrical interpretation of ϕ is not clear. Roughly speaking, this Killing vector should correspond to a symmetry in the direction of rotation.

In four dimensions, the existence of two commuting Killing vector fields implies that the metric has to take a fairly simple form, and it can then be argued that any such solution to the Einstein equations is uniquely determined by its mass and angular momentum [38,39] and must belong to the Kerr family of solutions. In higher dimensions, two Killing vector fields is not enough symmetry to write the metric in a useful form, and the existence of black rings shows that uniqueness should not be expected even when more symmetry is present.

These general arguments suggest that all stationary higher dimensional black holes must have two commuting symmetries. However, no known higher dimensional black hole solution has *only* two commuting symmetries. This suggests that higher dimensional black holes may be similar to black strings in the sense that there may exist undiscovered stationary solutions with fewer symmetries than the presently known solutions. More precisely:

Conjecture. There exist stationary, asymptotically flat black hole solutions of the $D>4$ dimensional vacuum Einstein equations that admit exactly two commuting Killing vector fields.

These solutions would have to be nonstatic (because of the uniqueness theorem for static black holes [13]). If such solutions do exist then it seems unlikely that the Schwarzschild solution would be recovered as a limit. This would imply that such solutions must have an angular momentum that is bounded below (in terms of their mass), just as occurs for black rings.

If the above conjecture is correct then higher dimensional black holes would exhibit similar behavior to black strings. There would be known solutions with lots of symmetry and new solutions with less symmetry. It is tempting to push this analogy further. Consider the case of five dimensions with a single nonvanishing angular momentum. Define a dimensionless parameter η by

$$\eta\equiv\frac{27\pi J^2}{32GM^3}, \tag{2.3}$$

where J and M are the angular momentum and mass of a black hole. The known solutions are the Myers-Perry solutions [10] (which exist for $\eta<1$) and black rings [9] ($\eta>\eta_*\approx0.84$). These solutions have three commuting Killing vector fields $\partial/\partial t$, $\partial/\partial\phi$ and $\partial/\partial\psi$ where ϕ is the direction of rotation. The above conjecture suggests looking for new solutions without symmetry in the ψ direction. The analogy with black strings suggests that there might be some critical value $\eta=\eta_c$ for which the Myers-Perry solution (or black ring) admits a stationary zero-mode that breaks the symmetry in the ψ direction. Finding such a mode would therefore

[5] See [32] for a recent review of this, and other results from cobordism theory, with references to the original literature.

be evidence in favor of the above conjecture.[6] However, the absence of such a mode would not rule out the existence of new solutions. For example, the topology of the new solutions might differ from that of the Myers-Perry solutions and black rings, in which case they would not be seen in perturbation theory about the known solutions.

B. Magnetic rings

The existence of black rings implies that stationary black holes in five dimensions are not uniquely specified by their asymptotic charges. If the above conjecture is correct then there exist further black hole solutions, and therefore black hole uniqueness is more severely violated. It is clearly desirable to know *how many* stationary higher dimensional black hole solutions have a given set of asymptotic charges. Are there finitely many or infinitely many? The purpose of the present subsection is to suggest that there may be a *continuous* infinity of solutions with a given set of asymptotic charges.

The black ring solutions obtained in [9] are solutions of the vacuum Einstein equations in five dimensions. It is interesting to ask whether electromagnetic generalizations exist. Consider Einstein-Maxwell theory in five dimensions, possibly with a Chern-Simons term. This theory admits two types of static black string solution: electric and magnetic. The electric solution becomes nakedly singular in the extremal limit. The extremal solution is best viewed as a smeared distribution of black holes. The magnetic solution has a regular extremal limit: this is the supersymmetric black string of minimal $N=1$, $D=5$ supergravity.

Black rings can be regarded as rotating loops of black string. Consider a rotating loop of magnetic black string. If such a solution exists then it would have vanishing electric charge.[7] The magnetic charge of a localized configuration must vanish in four spatial dimensions [41]. Therefore the only asymptotic charges that would be carried by such a solution are its mass and angular momentum. However, the solution would presumably be characterized by a third parameter α measuring the strength of the magnetic field. Therefore, if magnetic black rings exist, then they would be an example of a continuous family (labeled by α) of solutions with the same asymptotic charges.

III. A UNIQUENESS THEOREM

A. Introduction

The above considerations highlight how little is known about general properties of higher dimensional stationary black holes, and suggest that such black holes are highly nonunique, if nonstatic. This casts doubt on the uniqueness assumption that underlies the entropy calculations for supersymmetric rotating black holes [4]. A uniqueness theorem is required in order to justify this assumption. In this section, the following theorem will be proved.

Theorem. The only supersymmetric, asymptotically flat, black hole solutions of the minimal $N=1$, $D=5$ supergravity theory are the BMPV solutions, which are uniquely specified by their mass and angular momentum.

Proving a uniqueness theorem for supersymmetric black holes is much easier than, say, attempting to generalize the known black hole uniqueness theorems to include degenerate horizons. This is because the existence of a globally defined supercovariantly constant spinor highly constrains the form of the spacetime. In fact, for minimal $N=2$, $D=4$ supergravity, it fully determines the local form of the metric [19]. For minimal $N=1$, $D=5$ supergravity, a simple algorithm can be given for the construction of all supersymmetric solutions [18]. This will be reviewed in Sec. III B.

The method of [18] yields the general supersymmetric solution in a coordinate system that does not cover any event horizons in the spacetime. Therefore, the first step in the uniqueness proof is to introduce a coordinate system valid in the neighborhood of a Killing horizon (Sec. III C), and to repeat some of the analysis of [18] in these coordinates (Sec. III D). It turns out that this fully determines the local form of the near-horizon geometry (Secs. III E and III F). The final step (Sec. III G) is to show that knowing the local form of the near-horizon geometry, together with asymptotic flatness, is sufficient to select a unique solution from the general solution of [18], which must therefore be the known BMPV solution.

B. Minimal five dimensional supergravity

Minimal $N=1$, $D=5$ supergravity was constructed in [16]. The bosonic sector has action[8]

$$S=\frac{1}{4\pi G}\int\left(\frac{1}{4}R*1-\frac{1}{2}F\wedge *F-\frac{2}{3\sqrt{3}}F\wedge F\wedge A\right). \tag{3.1}$$

All purely bosonic supersymmetric solutions of this theory were obtained in [18] as follows. Starting from a commuting super-covariantly constant (Dirac) spinor ϵ, one can construct a real scalar field f, a real vector field V and three real two-form fields $X^{(i)}$:[9]

$$f\sim i\bar{\epsilon}\epsilon,\quad V^{\alpha}\sim\bar{\epsilon}\gamma^{\alpha}\epsilon,$$

$$(X^{(1)}+iX^{(2)})_{\alpha\beta}\sim\epsilon^{T}C\gamma_{\alpha\beta}\epsilon,\quad X^{(3)}_{\alpha\beta}\sim\bar{\epsilon}\gamma_{\alpha\beta}\epsilon. \tag{3.2}$$

[6]Examining perturbations of Myers-Perry solutions would also be of interest in view of the conjecture [9] that a five dimensional Myers-Perry black hole with a single nonvanishing angular momentum is classically unstable for η close to 1.

[7]Hence it could not saturate the Bogomol'nyi bound appropriate to an asymptotically flat spacetime [40] and therefore would not be supersymmetric.

[8]Conventions: the metric has positive signature, curvature is defined so that de Sitter space has positive Ricci scalar. Curved indices are denoted by $\mu,\nu\ldots$ and tangent space indices by $\alpha,\beta,\ldots$.

[9]The precise definition of these objects is given in [18] in terms of symplectic-Majorana spinors. Converting to Dirac spinors may introduce numerical factors, which have not been calculated here.

Fierz identities imply various algebraic identities between these quantities, for example

$$f^2 = -V^2, \tag{3.3}$$

$$i_V X^{(i)} = 0, \tag{3.4}$$

$$i_V * X^{(i)} = -f X^{(i)}, \tag{3.5}$$

$$X^{(i)}_{\gamma\alpha} X^{(j)\gamma}_{\beta} = \delta_{ij}(f^2 \eta_{\alpha\beta} + V_\alpha V_\beta) - f \epsilon_{ijk} X^{(k)}_{\alpha\beta}, \tag{3.6}$$

where $\epsilon_{123} = +1$ and, for a p-form A and a vector Y, $i_Y A$ denotes the $(p-1)$-form obtained by contracting Y with the first index of A. Equation (3.3) implies that the vector field V is timelike, null or zero. Since $V_0 \sim \epsilon^\dagger \epsilon$, the latter possibility occurs if, and only if, ϵ vanishes. Since ϵ is super-covariantly constant, the above quantities must also satisfy certain differential constraints [18]:

$$df = -\frac{2}{\sqrt{3}} i_V F, \tag{3.7}$$

$$D_{(\alpha} V_{\beta)} = 0, \tag{3.8}$$

$$dV = -\frac{4}{\sqrt{3}} f F - \frac{2}{\sqrt{3}} *(F \wedge V), \tag{3.9}$$

and

$$D_\alpha X^{(i)}_{\beta\gamma} = \frac{1}{\sqrt{3}} [2 F_\alpha{}^\delta (*X^{(i)})_{\delta\beta\gamma} - 2 F_{[\beta}{}^\delta (*X^{(i)})_{\gamma]\alpha\delta} + \eta_{\alpha[\beta} F^{\delta\epsilon} (*X^{(i)})_{\gamma]\delta\epsilon}], \tag{3.10}$$

which implies

$$dX^{(i)} = 0. \tag{3.11}$$

These equations imply that V is a Killing vector field that preserves the field strength (i.e. $\mathcal{L}_V F = 0$ where $\mathcal{L}$ denotes the Lie derivative), i.e., V generates a symmetry of the full solution.

If p is a point at which V vanishes then consider a timelike geodesic through p. Let U denote the tangent vector to this geodesic. V is a Killing vector field so $V \cdot U$ is conserved along the geodesic, and must therefore vanish because it vanishes at p. Therefore U and V are orthogonal along the geodesic. However, U is timelike and V is nonspacelike so this implies that V must vanish everywhere along the geodesic, and therefore so must ϵ. This applies to all timelike geodesics through p. Hence ϵ vanishes in open regions to the future and past of p. By analyticity, ϵ must then vanish everywhere, which contradicts the assumption that the spacetime admits a super-covariantly constant spinor. Hence there cannot exist any point in the spacetime at which V or ϵ vanishes.

Either f vanishes throughout the spacetime or there is some point p at which $f \neq 0$. These will be referred to as the "null case" and "timelike case" respectively. In the null case, V is a globally defined null Killing vector field V. In fact the general solution in this case is a plane-fronted wave [18]. Special cases of this general solution include the magnetic black string solution [42] and its near horizon geometry, $AdS_3 \times S^2$. The existence of a globally defined null Killing vector field implies that these solutions cannot describe black holes.

In the timelike case, by continuity, there is some topologically trivial neighborhood $\mathcal{U}$ of p in which $f \neq 0$. Therefore V is a timelike Killing vector field in $\mathcal{U}$. It will be assumed that $f > 0$ without loss of generality [18]. Coordinates can be introduced so that the metric in $\mathcal{U}$ can be written [18]

$$ds^2 = -f^2(dt + \omega) + f^{-1} ds_4^2, \tag{3.12}$$

where $V = \partial/\partial t$ and ds_4^2 is the metric on a four dimensional Riemannian "base space" orthogonal to the orbits of V. Note that all metric components must be independent of t. ω is a 1-form that is defined by the equations

$$i_V \omega = 0, \quad d\omega = -d(f^{-2} V). \tag{3.13}$$

This determines ω up to a gradient, which reflects the freedom to choose the $t = 0$ hypersurface. Supersymmetry requires that the base space be hyper-Kähler, with $X^{(i)}$ the three complex structures and a volume form η_4 chosen so that these are anti-self-dual. This volume form is related to the volume form η on the five dimensional spacetime by

$$\eta_4 = f i_V \eta. \tag{3.14}$$

$d\omega$ can be regarded as a 2-form on the base space and can therefore be decomposed into self-dual and anti-self-dual parts with respect to the base space:

$$f d\omega = G^+ + G^-. \tag{3.15}$$

It is then possible to solve for the field strength [18]:

$$F = -\frac{\sqrt{3}}{2} d[f^{-1} V] - \frac{1}{\sqrt{3}} G^+. \tag{3.16}$$

The Bianchi identity for F yields

$$dG^+ = 0, \tag{3.17}$$

and the equation of motion for F gives

$$\Delta f^{-1} = \frac{4}{9} (G^+)^2, \tag{3.18}$$

where Δ is the Laplacian associated with the base space metric and

$$(G^+)^2 \equiv \frac{1}{2} (G^+)_{mn} (G^+)^{mn}, \tag{3.19}$$

where m,n are indices on the base space, raised with the base space metric. The above equations guarantee that Eqs. (3.12) and (3.16) yield a supersymmetric solution of the supergravity theory [18].

Any supersymmetric black hole solution must belong to the timelike class. Therefore the full black hole spacetime is determined by analytic continuation of a solution of the above form. The only known supersymmetric black hole solution of this theory is the BMPV black hole [4,17], which has base space R^4, with metric

$$ds_4^2 = d\rho^2 + \frac{\rho^2}{4}[(\sigma_R^1)^2 + (\sigma_R^2)^2 + (\sigma_R^3)^2], \qquad (3.20)$$

where σ_R^i are left invariant 1-forms on $SU(2)$—see [18] for details. The solution has

$$\omega = \frac{j}{2\rho^2}\sigma_R^3, \qquad (3.21)$$

which implies $G^+ = 0$. The solution for f is

$$f^{-1} = 1 + \frac{\mu}{\rho^2}. \qquad (3.22)$$

The global properties of this solution were investigated in detail in [17,43]. The solution describes a black hole provided $j^2 < \mu^3$. If this bound is violated then it instead describes a regular spacetime with naked closed causal curves [43] and the microscopic description becomes nonunitary [44]. There exists evidence [43,45] that it is physically impossible to add angular momentum to the black hole and violate the above bound.

C. Introduction of coordinates

The coordinate system introduced above is only valid locally, and does not cover regions in which f vanishes, for example the event horizon of a black hole. In this section, a new set of coordinates will be introduced that covers such a horizon. However, before doing this, it is necessary to argue that, for a supersymmetric black hole solution, the Killing vector field V has the usual properties associated with the stationary Killing vector field of an equilibrium black hole spacetime.

First, consider the possibility that V becomes null at some point p outside the black hole. Equation (3.7) implies $V \cdot \partial f = 0$, so f is constant along the orbits of V. Since f vanishes at p, it must vanish along the orbit through p. Hence V is null on this orbit. However, one would not expect a spacetime describing the rest frame of a *single* black hole to admit a Killing vector field with a null orbit *outside* the black hole. This is because such an orbit would correspond to an observer moving at the speed of light for whom the gravitational field would appear unchanging. This is only possible on the event horizon. Therefore, V cannot become null outside the black hole so it will be assumed that $f > 0$ everywhere outside the black hole.

Now consider the behavior of V at infinity. If V were to vanish at some point p on $\mathcal{I}^+$ then consider an affinely parametrized outgoing null geodesic with an end point at p. Let k denote the tangent vector to this geodesic. $V \cdot k$ is constant along the geodesic and vanishes at infinity. Hence $V \cdot k$ vanishes everywhere on the geodesic. Therefore V must be proportional to k, and therefore null, along this geodesic. But the argument above excludes the possibility of V being null outside the black hole. Hence V cannot vanish on $\mathcal{I}^+$. Similarly, V cannot vanish on $\mathcal{I}^-$.

If V were to become null at some point p on $\mathcal{I}^+$ then it is easy to see that V must be everywhere tangent to the null geodesic generator of $\mathcal{I}^+$ through p. Once again, such a null symmetry would not be expected of a spacetime describing a black hole at rest so V cannot be null anywhere on $\mathcal{I}^+$. Similarly, V cannot be null anywhere on $\mathcal{I}^-$.

These considerations establish that V must be timelike everywhere outside the black hole and also on $\mathcal{I}^\pm$. If f were to diverge anywhere on $\mathcal{I}^\pm$ then V would be behaving as a boost symmetry, which is not expected for a black hole in its rest frame. Hence f must be nonzero and bounded on $\mathcal{I}^\pm$.

It will be assumed that the future event horizon $\mathcal{H}^+$ has a single connected component. Since V is an isometry, it must leave this horizon invariant and must therefore be null on $\mathcal{H}^+$.

Let Σ be a Cauchy surface for the exterior region of the black hole such that Σ has a boundary H on the future event horizon. A null Gaussian coordinate system can be set up in a neighborhood of H as follows (see [46] for more details). Introduce local coordinates x^A $(A = 1,2,3)$ on H. Let p be a point on H with coordinates x^A. Consider the future directed null geodesic generator of $\mathcal{H}^+$ that passes through p, with tangent vector V. The coordinates of a point affine parameter distance u from p along this generator will be defined to be (u, x^A). This defines coordinates on a neighborhood $\mathcal{U}$ of H in $\mathcal{H}^+$ with $V = \partial/\partial u$. Now let n be the unique past directed null vector field defined on $\mathcal{U}$ by $V \cdot n = 1$ and $n \cdot X = 0$ for all X tangent to surfaces of constant u. Finally, consider the null geodesic from a point $p \in \mathcal{U}$ with tangent n. Let the coordinates of a point affine parameter distance r along this geodesic be (u, r, x^A) where (u, x^A) are the coordinates of p.

It is easy to check that $\mathcal{L}_V n = 0$ on $\mathcal{H}^+$. Moreover, V is a Killing vector field and hence geodesics are mapped to geodesics under the flow of V. Putting these facts together, under the flow of V through a parameter distance δ, the point with coordinates (u, r, x^A) is mapped to the point with coordinates $(u + \delta, r, x^A)$. Hence

$$V = \partial/\partial u \qquad (3.23)$$

everywhere, not just on the horizon.

Since f vanishes at $r = 0$, differentiability implies

$$f = r\Delta(r, x^A), \qquad (3.24)$$

for some function Δ independent of u (as $V \cdot \partial f = 0$). The exterior of the black hole is $r > 0$ so Δ must be positive for $r > 0$. $\partial/\partial x^A$ is tangent to surfaces of constant u in $\mathcal{H}^+$ and hence orthogonal to V at $r = 0$. Therefore $g_{uA} = r h_A(r, x^B)$ for some functions h_A independent of u (as V is Killing). The full metric must take the form [46]

$$ds^2 = -r^2\Delta^2 du^2 + 2du dr + 2r h_A du dx^A + \gamma_{AB} dx^A dx^B, \qquad (3.25)$$

where γ_{AB} is a function of r and x^A. It was argued above that black holes must belong to the timelike family of solutions, for which $\Delta>0$ for $r>0$. However, the above line element is clearly also valid in the neighborhood of a Killing horizon of V in the *null* family, for which $\Delta\equiv 0$. Also note that the form of this metric guarantees the existence of a regular near horizon geometry, defined by the limit $r=\epsilon\tilde{r}$, $u=\tilde{u}/\epsilon$ and $\epsilon\to 0$.

D. Supersymmetry near the horizon

The next step in the proof is to examine the constraints imposed by supersymmetry in the above coordinate system.

Using the above form for the metric, Eqs. (3.4) and (3.5) imply that the two forms can be written

$$X^{(i)}=dr\wedge Z^{(i)}+r(h\wedge Z^{(i)}-\Delta *_3 Z^{(i)}), \qquad (3.26)$$

where $Z^{(i)}\equiv Z_A^{(i)}dx^A$, $h\equiv h_A dx^A$ and $*_3$ denotes the Hodge dual with respect to γ_{AB}. $X^{(i)}$ is globally defined so $Z^{(i)}$ are well-defined in a neighborhood of H. The algebraic relations satisfied by $X^{(i)}$ imply

$$\langle Z^{(i)},Z^{(j)}\rangle=\delta^{ij}, \quad Z^{(i)}\wedge Z^{(j)}=\epsilon_{ijk}*_3 Z^{(k)}, \qquad (3.27)$$

where $\langle,\rangle$ denotes the inner product defined by γ_{AB}. Closure of $X^{(i)}$ [Eq. (3.11)] yields

$$\hat{d}Z^{(i)}=-\frac{1}{2}\partial_r(r\Delta)\epsilon_{ijk}Z^{(i)}\wedge Z^{(j)}+\partial_r(rh)\wedge Z^{(i)} -r\Delta\epsilon_{ijk}\partial_r Z^{(j)}\wedge Z^{(k)}+rh\wedge\partial_r Z^{(i)}, \qquad (3.28)$$

and

$$*_3\hat{d}h-\hat{d}\Delta-\Delta h+r\partial_r\Delta h-2r\Delta\partial_r h-r*_3(h\wedge\partial_r h) -r\Delta^2\epsilon_{ijk}Z^{(i)}\langle Z^{(j)},\partial_r Z^{(k)}\rangle =0, \qquad (3.29)$$

where, for a p-form Y with only A,B,C indices,

$$(\hat{d}Y)_{ABC\ldots}\equiv(p+1)\partial_{[A}Y_{BC\ldots]}. \qquad (3.30)$$

For $r>0$, ω can be defined as in Eq. (3.13), giving

$$\omega=-\frac{dr}{r^2\Delta^2}-\frac{1}{r\Delta^2}h, \qquad (3.31)$$

where an arbitrary gradient can be absorbed by shifting the $u=0$ surface. The definition of G^+ can be rewritten as

$$G^+=\frac{1}{2}(fd\omega+i_V*d\omega). \qquad (3.32)$$

Computing G^+ then gives

$$G^+=dr\wedge\mathcal{G}+r(h\wedge\mathcal{G}+\Delta*_3\mathcal{G}), \qquad (3.33)$$

where

$$\mathcal{G}=-\frac{3}{2r\Delta^2}\hat{d}\Delta+\frac{3}{2\Delta^2}\partial_r\Delta h-\frac{3}{2\Delta}\partial_r h -\frac{1}{2}\epsilon_{ijk}Z^{(i)}\langle Z^{(j)},\partial_r Z^{(k)}\rangle. \qquad (3.34)$$

Equation (3.29) was used to eliminate $\hat{d}h$ from this expression. Note the similarity between Eqs. (3.26) and (3.33): this structure is a consequence of (anti)-self-duality on the base space. Equation (3.16) now gives

$$F=\frac{\sqrt{3}}{2}\left[-\partial_r(r\Delta)du\wedge dr-rdu\wedge\hat{d}\Delta+\frac{1}{3}\epsilon_{ijk}dr\wedge Z^{(i)}\langle Z^{(j)},\partial_r Z^{(k)}\rangle-*_3 h-r*_3\partial_r h +\frac{r}{3}\epsilon_{ijk}(-2\Delta*_3 Z^{(i)}+h\wedge Z^{(i)})\langle Z^{(j)},\partial_r Z^{(k)}\rangle\right]. \qquad (3.35)$$

Note that this is well-defined at $r=0$, and has a well-defined near-horizon limit, even though G^+ need not be regular at $r=0$.

The ABC component of the Bianchi identity for F [or equivalently, Eq. (3.17)] now yields

$$\hat{d}*_3 h=\mathcal{O}(r). \qquad (3.36)$$

Now Eqs. (3.29) and (3.36) give

$$\hat{d}\Delta\wedge*_3\hat{d}\Delta=\hat{d}\left[\Delta\hat{d}h-\frac{1}{2}\Delta^2*_3 h\right]+\mathcal{O}(r). \qquad (3.37)$$

Integrating this equation over the compact 3-manifold H (which is at $r=0$) then implies

$$\hat{d}\Delta=0 \quad \text{on} \quad H, \qquad (3.38)$$

hence Δ is constant on the event horizon. Equations (3.29) and (3.36) now yield

$$\hat{d}h=\Delta *_3 h, \quad \hat{d}*_3 h=0 \quad \text{on} \quad H. \tag{3.39}$$

The ArB component of Eq. (3.10) gives

$$\nabla_A Z_B^{(i)} = -\frac{1}{2}\Delta(*_3 Z^{(i)})_{AB} + \gamma_{AB}\langle h, Z^{(i)}\rangle - Z_A^{(i)} h_B + \mathcal{O}(r), \tag{3.40}$$

where ∇ is the connection associated with γ_{AB}. Taking another derivative and antisymmetrizing then yields an expression for the Riemann tensor of H. From this, the Ricci tensor of H is [using Eq. (3.39)]

$$R_{AB} = \left(\frac{\Delta^2}{2} + h^2\right)\gamma_{AB} - \nabla_{(A}h_{B)} - h_A h_B, \tag{3.41}$$

where $h^2 = h_A h^A$, raising indices with γ^{AB} on H. Equations (3.39) imply

$$\nabla^2 h_A = R_{AB}h^B - \Delta^2 h_A \quad \text{on} \quad H. \tag{3.42}$$

Now consider

$$I = \int_H \nabla_{(A}h_{B)}\nabla^{(A}h^{B)}. \tag{3.43}$$

Integrating by parts and using Eq. (3.42) and $\nabla_A h^A = 0$ [from Eq. (3.39)] gives

$$I = \int_H (\Delta^2 h^2 - 2R_{AB}h^A h^B). \tag{3.44}$$

Finally, substituting in Eq. (3.41) and integrating by parts yields $I=0$. Hence

$$\nabla_{(A}h_{B)} = 0 \quad \text{on} \quad H. \tag{3.45}$$

Therefore, if nonzero, then h is a Killing vector field on H. Substituting into Eq. (3.41) gives the Ricci tensor of H

$$R_{AB} = \left(\frac{\Delta^2}{2} + h^2\right)\gamma_{AB} - h_A h_B. \tag{3.46}$$

This completely determines the curvature of H because H is a 3-manifold. Combining Eqs. (3.39) and (3.45) gives

$$\nabla_A h_B = \frac{1}{2}\Delta\,\eta_{ABC}h^C, \tag{3.47}$$

where η_{ABC} is the volume form of H. Note that this implies that h^2 is constant on H. Furthermore, combining Eqs. (3.40) and (3.47) gives

$$[h, Z^{(i)}] = 0 \quad \text{on} \quad H. \tag{3.48}$$

E. A special case

This subsection will consider the case in which Δ vanishes on H. If this happens then, on H,

$$\hat{d}Z^{(i)} = h \wedge Z^{(i)}, \tag{3.49}$$

which implies that the 1-forms $Z^{(i)}$ are hypersurface orthogonal, i.e., there exist functions z^i and $K^{(i)}$ defined on H so that $Z^{(i)} = K^{(i)}dz^i$ (no summation on i). Equation (3.49) then requires that the functions $K^{(i)}$ be proportional. The constants of proportionality can be absorbed into z^i, so $K^{(i)} = K$ for $i=1,2,3$, i.e.,

$$\hat{d}Z^{(i)} = K dz^i. \tag{3.50}$$

Equation (3.49) also implies

$$h = \hat{d}\log K. \tag{3.51}$$

The functions z^i can be used as local coordinates on H, i.e., $\{x^A\} = \{z^i\}$. Orthonormality of $Z^{(i)}$ implies that the metric on H is conformally flat:

$$\gamma_{AB}dx^A dx^B = K^2 dz^i dz^i. \tag{3.52}$$

Equation (3.47) says that h is covariantly constant. In these coordinates, this gives

$$K^{-1}\partial_i\partial_j K - 3K^{-2}\partial_i K\partial_j K + K^{-2}\partial_k K\partial_k K\,\delta_{ij} = 0. \tag{3.53}$$

This equation was encountered in [18]. By shifting the origin and rescaling the coordinates z^i, the solutions can be written

$$K=1, \quad \text{or} \quad K = \frac{L}{R}, \tag{3.54}$$

where $R = \sqrt{z^i z^i}$. In the first case, this implies that the metric near H can be written

$$ds^2 = 2drdu + [\delta_{ij} + \mathcal{O}(r)]dz^i dz^j + \mathcal{O}(r^4)du^2 + \mathcal{O}(r^2)du dz^i. \tag{3.55}$$

The near horizon limit of this solution is locally isometric to flat space with vanishing gauge field. Globally it must differ by some discrete identifications because H is assumed compact. The metric on H is flat so H must be some quotient of R^3 with its flat metric [47], i.e., R^3 identified with respect to some subgroup of its isometry group. However, these identifications have to preserve the 1-forms $Z^{(i)}$, which implies that they must be translations. So H is a compact manifold obtained by identifying R^3 with respect to certain translations, i.e., H must be a 3-torus T^3.

In the second case, it is convenient to use spherical polar coordinates $\{x^A\} = (R, \theta, \phi)$. The solution can be written

$$ds^2 = 2drdu - 2\frac{r}{R}du dR + L^2\left(\frac{dR^2}{R^2} + d\theta^2 + \sin^2\theta d\phi^2\right) + \mathcal{O}(r^4)du^2 + \mathcal{O}(r^2)du dx^A + \mathcal{O}(r)dx^A dx^B. \tag{3.56}$$

The near horizon limit of such a solution is locally isometric to the (maximally supersymmetric) $AdS_3 \times S^2$ solution (to see this, let $r = vR/L$ for some new coordinate v). Globally, it must differ because H is compact. The metric on H can be written

$$ds_3^2 = L^2(dZ^2 + d\theta^2 + \sin^2\theta d\phi^2), \tag{3.57}$$

where $Z = \log R$ (note that R must be bounded away from zero because the 1-forms $Z^{(i)}$ are well-defined on H). So H is locally isometric to the standard metric on $R \times S^2$, which presumably implies that H can be obtained as a quotient of $R \times S^2$. However, the only elements of the isometry group of $R \times S^2$ which preserve the 1-forms $Z^{(i)}$ are translations $Z \to Z + \text{const}$. Hence H must be globally isometric to the standard metric on $S^1 \times S^2$ with $Z \sim Z + l$ for some l.

It seems highly unlikely that either of these solutions could arise as the near horizon geometry of a black hole. In fact, the $AdS_3 \times S^2$ solution arises as the near-horizon geometry of a momentum-carrying black *string* wrapped around a compact Kaluza-Klein direction.[10] A proof that $AdS_3 \times S^2$ cannot arise as the near horizon limit of a black hole solution is given in the Appendix. The flat case can probably be excluded using a modification of this argument. It will therefore be assumed henceforth that $\Delta > 0$ on H.

F. Near horizon geometry

Having eliminated the case in which $\Delta = 0$ on H, it will now be shown that the near horizon geometry of a supersymmetric black hole must be locally isometric to that of the BMPV black hole. First define a set of 1-forms on H by[11]

$$\sigma_L^i = \frac{\Delta^2 + h^2}{\Delta} Z^{(i)} + \frac{1}{\Delta} d(h^A Z_A^{(i)}). \tag{3.58}$$

Using Eqs. (3.40) and (3.47), it can be shown that these 1-forms obey

$$d\sigma_L^i = -\frac{1}{2}\epsilon_{ijk}\sigma_L^j \wedge \sigma_L^k, \tag{3.59}$$

and

$$[h, \sigma_L^i] = 0. \tag{3.60}$$

The vector fields dual to these 1-forms are

$$\xi_L^i = \frac{\Delta}{\Delta^2 + h^2} Z^{(i)} - \frac{1}{\Delta^2 + h^2} *_3 (h \wedge Z^{(i)}). \tag{3.61}$$

Equations (3.40) and (3.47) imply that these are Killing vector fields:

$$\nabla_{(A}(\xi_L^i)_{B)} = 0, \tag{3.62}$$

and satisfy the commutation relations of $SU(2)$:

$$[\xi_L^i, \xi_L^j] = \epsilon_{ijk}\xi_L^k. \tag{3.63}$$

Furthermore, these Killing vector fields commute with h:

$$[h, \xi_L^i] = 0. \tag{3.64}$$

These considerations show that, if $h \neq 0$ then H admits four globally defined Killing vector fields satisfying the commutation relations of $SU(2) \times U(1)$.

If $h \neq 0$ then local coordinates can now be introduced as follows. Let $\hat{h} = h/\sqrt{h^2}$, and

$$\hat{x}^i = \hat{h}^A Z_A^{(i)} \quad \text{so} \quad \hat{x}^i \hat{x}^i = 1. \tag{3.65}$$

Equations (3.47) and (3.40) imply

$$(d\hat{x}^i d\hat{x}^i)_{AB} = (\Delta^2 + h^2)(\gamma_{AB} - \hat{h}_A \hat{h}_B). \tag{3.66}$$

Note that $\mathcal{L}_h \hat{x}^i = 0$ so $\hat{x}^i$ is constant along integral curves of h. In some open set it is therefore possible to use $\hat{x}^i$ and the parameter along these curves as coordinates. It is convenient to define

$$\mu = \frac{4}{\Delta^2 + h^2}, \quad j = \pm \frac{8\sqrt{h^2}}{(\Delta^2 + h^2)^2}, \tag{3.67}$$

where the sign of j will be left arbitrary. Note that $j^2 < \mu^3$. To bring the metric to standard form, introduce polar coordinates

$$\hat{x}^1 = -\cos\phi \sin\theta,$$

$$\hat{x}^2 = \sin\phi \sin\theta,$$

$$\hat{x}^3 = \cos\theta, \tag{3.68}$$

and let ψ be the parameter along the integral curves of h, normalized so that

$$h = -4j\mu^{-5/2}\left(1 - \frac{j^2}{\mu^3}\right)^{-1/2} \frac{\partial}{\partial\psi}. \tag{3.69}$$

The metric must take the form

$$ds_3^2 = \gamma_{AB} dx^A dx^B = \frac{\mu}{4}\left[\left(1 - \frac{j^2}{\mu^3}\right)(d\psi + \mathcal{A})^2 + d\theta^2 + \sin^2\theta d\phi^2\right], \tag{3.70}$$

for some locally defined 1-form $\mathcal{A}$ on H. Equation (3.47) determines $\mathcal{A}$ up to a gradient, which is just the freedom to choose the $\psi = 0$ surface. A convenient choice is

$$\mathcal{A} = \cos\theta d\phi, \tag{3.71}$$

which also fixes the orientation of H so that $d\theta \wedge d\psi \wedge d\phi$ is positively oriented. The metric on H now takes the form

$$ds_3^2 = \frac{\mu}{4}\left[\left(1 - \frac{j^2}{\mu^3}\right)(d\psi + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right], \tag{3.72}$$

[10] If the string does not carry momentum then it cannot be identified to yield a regular compact event horizon [42].

[11] All equations in this subsection are evaluated on H, so hats will not be included on exterior derivatives.

which is the standard form of the metric on a squashed S^3. However, this is a local result—globally, H could differ from a squashed S^3 by discrete identifications.

Writing h as a 1-form gives

$$h = -j\mu^{-3/2}\left(1-\frac{j^2}{\mu^3}\right)^{1/2}(d\psi+\cos\theta d\phi). \quad (3.73)$$

The case $h=0$ (i.e. $j=0$) is much simpler. Equation (3.46) establishes that H is a three dimensional compact Einstein space of positive curvature and hence locally isometric to S^3 with its round metric. Therefore the above local coordinates can also be introduced in this case, and the metric is as above with $j=0$.

It is worth summarizing what has been shown. Local coordinates have been introduced in a neighborhood of the horizon ($r=0$) and explicit expressions for the local behavior of Δ, h_A and γ_{AB} at $r=0$ have been obtained. Using the expressions for the metric and field strength [Eqs. (3.25) and (3.35)], it is now clear that the above analysis has fully determined the local form of the near-horizon solution. Since this local form is unique, it must agree with that of the BMPV solution. So the above analysis proves that the near horizon geometry of any supersymmetric black hole solution must be locally isometric to that of the BMPV solution.

It has been proved that H is locally isometric to a squashed S^3 when $j\neq 0$ and a round S^3 when $j=0$. In the latter case, H must be globally isometric to a discrete quotient of a round S^3 (since H is a positive Einstein 3-manifold) and in the former case, H is presumably globally isometric to a discrete quotient of a squashed S^3. The question of which quotients are consistent with supersymmetry can be deduced from the existence of the vector fields ξ_L^i and h. Whatever quotient is taken must preserve these vector fields.

If $j=0$ then ξ_L^i generate the $SU(2)_L$ subgroup of the $SU(2)_L\times SU(2)_R$ isometry group of S^3. The allowed quotients must therefore be subgroups of $SU(2)_R$. So H is of the form S^3/Γ, where Γ is a discrete subgroup of $SU(2)_R$.

If $j\neq 0$ then ξ_L^i generate the $SU(2)$ factor, and h the $U(1)$ factor of the $SU(2)\times U(1)$ isometry group of a squashed S^3. The allowed quotients must be subgroups of $U(1)$, and therefore cyclic groups. Hence H must be a quotient of a squashed S^3 by a cyclic group, i.e., H is a squashed lens space.

G. Global constraints

The final step is to prove that the black hole must actually be globally isometric to the BMPV solution. This is where detailed knowledge of the general supersymmetric solution [18] of minimal $N=1$, $D=5$ supergravity is required.

The first step is to write the near-horizon solution in the form of Eq. (3.12). Doing so, the metric on the base space is

$$ds_4^2 = r\Delta\left(\gamma_{AB}+\frac{1}{\Delta^2}h_A h_B\right)dx^A dx^B + \frac{dr^2}{r\Delta} + \frac{2}{\Delta}dr h_A dx^A. \quad (3.74)$$

A priori, there is no reason why this metric should be regular at $r=0$ because the form (3.12) is only valid for $r>0$. It has just been demonstrated that coordinates $x^A=(\psi,\theta,\phi)$ can be introduced so that the metric on H is locally isometric to that of a squashed S^3. In these coordinates,

$$\begin{aligned} ds_4^2 = & \frac{r\mu^{1/2}}{2}\left(1-\frac{j^2}{\mu^3}\right)^{1/2}\Bigg\{\Bigg[d\psi+\cos\theta d\phi - jr^{-1}\mu^{-3/2} \\ & \times\left(1-\frac{j^2}{\mu^3}\right)^{-1/2}dr\Bigg]^2 + d\theta^2+\sin^2\theta d\phi^2\Bigg\} + \frac{dr^2}{2r}\mu^{1/2} \\ & \times\left(1-\frac{j^2}{\mu^3}\right)^{1/2} + \mathcal{O}(r^0)dr^2 + \mathcal{O}(r)dr dx^A \\ & + \mathcal{O}(r^2)dx^A dx^B. \end{aligned} \quad (3.75)$$

Now let

$$R = 2^{1/2}r^{1/2}\mu^{1/4}\left(1-\frac{j^2}{\mu^3}\right)^{1/4}, \quad (3.76)$$

and

$$\psi' = \psi - 2j\mu^{-3/2}\left(1-\frac{j^2}{\mu^3}\right)^{-1/2}\log R, \quad (3.77)$$

so

$$\begin{aligned} ds_4^2 = & dR^2 + \frac{R^2}{4}[(d\psi'+\cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2] \\ & + \mathcal{O}(R^2)dR^2 + \mathcal{O}(R^3)dR dx'^A + \mathcal{O}(R^4)dx'^A dx'^B, \end{aligned} \quad (3.78)$$

where $x'^A=(\psi',\theta,\phi)$. It should be emphasized that this is only valid locally—no assumptions have been made about the ranges of the coordinates (ψ,θ,ϕ). However, it can be seen that the base space metric is locally flat near $R=0$. Near $R=0$, the surfaces of constant R are positive Einstein spaces and must therefore be globally isometric to S^3 with its round metric, identified under some subgroup Γ of its $SU(2)\times SU(2)$ isometry group. Generally, this implies that there will be a conical singularity at $R=0$. The metric has to be hyper-Kähler for $R>0$, so the holonomy has to be a subgroup of $SU(2)$. Therefore the singularity at $R=0$ has to be an $A-D-E$ orbifold singularity (see [48] for a review).[12]

Now, f approaches a constant at infinity (since V is the stationary Killing vector field), and in Eq. (3.12), ω must vanish fast enough at infinity for the ADM angular momentum to be well-defined. Asymptotic flatness then implies that the base space has to be asymptotically Euclidean. Near $R=0$, the metric is well-approximated by flat space with an $A-D-E$ singularity at the origin, and it is known [48] that

[12]This result also follows from the properties of H deduced in the previous subsection.

such singularities can be resolved by "blowing up" the singularity. This produces a complete asymptotically Euclidean hyper-Kähler space. However, the only such space is R^4 with its standard metric [49]. In other words, there could not have been a singularity at $R=0$ after all.[13]

It follows that the metric (3.78) describes a portion of global flat space, i.e., the coordinates (θ,ϕ,ψ) must have their standard ranges $0\leqslant\theta\leqslant\pi$, $\phi\sim\phi+2\pi$ and $\psi'\sim\psi'+4\pi$ (which implies $\psi\sim\psi+4\pi$). Hence H has S^3 topology. The base space is globally flat, with metric

$$ds^2=d\rho^2+\frac{\rho^2}{4}[(\sigma_R^1)^2+(\sigma_R^2)^2+(\sigma_R^3)^2], \tag{3.79}$$

where

$$R=\rho+\mathcal{O}(\rho^3), \tag{3.80}$$

and

$$\sigma_R^1=-\sin\psi' d\theta+\cos\psi'\sin\theta d\phi$$
$$\sigma_R^2=\cos\psi' d\theta+\sin\psi'\sin\theta d\phi \tag{3.81}$$
$$\sigma_R^3=d\psi'+\cos\theta d\phi.$$

Having established that the base space is flat, the next step is to show that the solution must have $G^+=0$. Consider Eq. (3.33). Δ is constant and nonzero at $r=0$, which implies that G^+ is well-defined in a neighborhood of H. (In fact, G^+ vanishes when restricted to H.) It is easy to see from the behavior of G^+ near $r=0$ that G^+ is regular at the origin of the base space. Hence G^+ is globally defined on R^4. G^+ must vanish at infinity in R^4 (because ω has to decay fast enough for the solution to be asymptotically flat). Furthermore G^+ is closed and must therefore belong to $H^2_{cpt}(R^4,R)$, the second compactly supported cohomology group on R^4. However, this group is trivial, so there exists a 1-form Γ globally defined on R^4 such that $G^+=d\Gamma$ with Γ vanishing at infinity. Now consider

$$0=\int_{S^3_\infty}\Gamma\wedge G^+=\int_{R^4}G^+\wedge G^+=\int_{R^4}d^4x(G^+)^2, \tag{3.82}$$

where the first integral is taken over the three sphere at infinity, and the final equality follows from the self-duality of G^+. Hence G^+ must vanish everywhere on the base space and therefore everywhere in the spacetime.

The vanishing of G^+ implies [Eq. (3.18)] that f^{-1} is harmonic on the base space. Near the origin,

$$f^{-1}=\frac{1}{r\Delta}=\frac{\mu}{\rho^2}+g, \tag{3.83}$$

where g is $\mathcal{O}(\rho^0)$ near $\rho=0$. As mentioned above, for a single black hole, it can be assumed that f is positive everywhere outside the black hole, and hence f^{-1} must be finite for $\rho>0$. Furthermore, f must approach a constant at infinity. This implies that g is regular and bounded on R^4. But g must be harmonic (as f^{-1} is) and hence constant. By rescaling the coordinates, g can be set to 1, so

$$f^{-1}=1+\frac{\mu}{\rho^2}. \tag{3.84}$$

The final step is to prove that ω is uniquely determined. Topologically, it is clear that ω can be globally defined on $R^4-\{0\}$ by Eq. (3.13). Now

$$-f^{-2}V=du-\frac{dr}{r^2\Delta^2}-\frac{h}{r\Delta^2}=dt+\frac{j}{4r}\mu^{-1/2}\left(1-\frac{j^2}{\mu^3}\right)^{-1/2}\sigma_R^3$$
$$+\mathcal{O}(r^{-1})dr+\mathcal{O}(r^0)dx^A, \tag{3.85}$$

where

$$t=u+\frac{\mu}{4r} \tag{3.86}$$

is a time coordinate defined for $r>0$. Hence, up to a gradient (which can be absorbed into t), ω is given by

$$\omega=\frac{j}{4r}\mu^{-1/2}\left(1-\frac{j^2}{\mu^3}\right)^{-1/2}\sigma_R^3+\mathcal{O}(r^{-1})dr+\mathcal{O}(r^0)dx^A,$$
$$=\frac{j}{2\rho^2}\sigma_R^3+\mathcal{O}(\rho^{-1})d\rho+\mathcal{O}(\rho^0)dx^A. \tag{3.87}$$

For $\rho>0$, ω has to satisfy the following criteria. First, the vanishing of G^+ implies that $d\omega$ has to be anti-self-dual with respect to the metric on the base space. Secondly, asymptotic flatness requires $\omega=\mathcal{O}(\rho^{-3})$ as $\rho\to\infty$. Finally, as $\rho\to0$, ω must be given by Eq. (3.87). To prove uniqueness of ω, assume that there were two solutions ω_1 and ω_2 satisfying these criteria. Let $\tilde{\omega}=\omega_1-\omega_2$. Hence $\tilde{\omega}$ is $\mathcal{O}(\rho^{-3})$ as $\rho\to\infty$, and, near $\rho=0$,

$$\tilde{\omega}=\mathcal{O}(\rho^{-1})d\rho+\mathcal{O}(\rho^0)dx^A. \tag{3.88}$$

Therefore $\tilde{\omega}$ can be written, on $R^4-\{0\}$, as

$$\tilde{\omega}=\frac{\alpha}{\rho}d\rho+\nu, \tag{3.89}$$

where $\nu\equiv\nu_A dx^A$ is a (ρ-dependent) 1-form defined on S^3. The quantities α and ν are well-behaved as $\rho\to0$. Anti-self-duality of $d\tilde{\omega}$ reduces to

$$\hat{d}\alpha=*_3\hat{d}\nu+\rho\partial_\rho\nu, \tag{3.90}$$

where $\hat{d}$ and $*_3$ are now defined on the round unit S^3. This equation implies that α becomes a harmonic function on S^3

[13]If one relaxes the condition of asymptotic flatness then an $A-D-E$ singularity *can* occur at $R=0$. Such behavior can by obtained by taking appropriate quotients of the BMPV solution.

as $\rho \to 0$. However, the only harmonic functions on S^3 are constant, so α must become constant as $\rho \to 0$, i.e., $\hat{d}\alpha \to 0$ as $\rho \to 0$. Therefore $\hat{d}\nu \to 0$ as $\rho \to 0$. However this implies

$$0 = \lim_{\epsilon \to 0} \int_{\rho=\epsilon} \tilde{\omega} \wedge d\tilde{\omega} = \int_{R^4 - \{0\}} d\tilde{\omega} \wedge d\tilde{\omega}$$

$$= -\int_{R^4 - \{0\}} d^4x (d\tilde{\omega})^2, \tag{3.91}$$

where the surface term at infinity vanishes because of the boundary conditions on $\tilde{\omega}$, and the final equality is a consequence of anti-self-duality. It follows then, that $\tilde{\omega}$ is closed and hence $\tilde{\omega} = d\lambda$ for some function λ defined for $\rho > 0$. Therefore, ω_1 and ω_2 differ at most by a gradient. Hence the general solution for ω must agree with the BMPV solution up to a gradient:

$$\omega = \frac{j}{2\rho^2} \sigma_R^3 + d\lambda. \tag{3.92}$$

Finally, this gradient can be absorbed into the time coordinate t, and then

$$V = -f^2 \left(dt + \frac{j}{2\rho^2} \sigma_R^3 \right) \tag{3.93}$$

for $r > 0$. Since the base space is flat, and f is given by Eq. (3.84), the solution is identical to the BMPV solution for $r > 0$. Therefore the BMPV solution is the unique supersymmetric black hole solution of minimal $N=1$, $D=5$ supergravity.

H. Discussion

The above results constitute the first example of a uniqueness theorem for supersymmetric black holes. The proof involved two steps. The first (local) step was to examine the constraints imposed by supersymmetry in a neighborhood of the event horizon. It was shown that the near-horizon geometry is completely determined. The second (global) step was to use the fact that, away from the Killing horizon, a general form for all supersymmetric solutions exists [18]. Asymptotic flatness and the boundary conditions obtained from the near-horizon geometry select a unique solution from this class.

The first step in the proof actually determines the near-horizon geometry of *any* (i.e. not necessarily asymptotically flat) supersymmetric solution that admits a compact Killing horizon preserved by V. The near-horizon geometry of such a solution has to be locally isometric to flat space, to $AdS_3 \times S^2$, or the near-horizon geometry of the BMPV solution (of which $AdS_2 \times S^3$ is a special case). Furthermore, in each case, the allowed possibilities for the spatial geometry of the event horizon (i.e. of H) have been determined. In the flat case, H must be T^3 with its flat metric and in the $AdS_3 \times S^2$ case, H must be $S^1 \times S^2$ with the usual metric. In the case of a BMPV near-horizon geometry there are more possibilities. If $j \neq 0$ then H must be a squashed lens space. If $j=0$ then the near-horizon BMPV geometry reduces to $AdS_2 \times S^3$, and in this case H must be a quotient of a round S^3 by a discrete subgroup of $SU(2)$.

Even if a spacetime has a noncompact Killing horizon, it is often possible to make identifications to render the horizon compact. For example, $AdS_3 \times S^2$ arises as the near-horizon geometry of a magnetic black string wrapped around a compact Kaluza-Klein direction, with momentum around this direction. A supersymmetric spacetime admitting a Killing horizon for which the analysis of this paper does *not* determine the near-horizon geometry would have to satisfy one of two criteria. Either the horizon would not be preserved by V, or the horizon would be noncompact and could not be rendered compact by identifications without breaking supersymmetry. The former case is not of much physical interest since one is usually interested in event horizons, which *must* be preserved by all Killing vector fields.

It would be interesting to see whether the above method could be extended to prove a uniqueness theorem for other supergravity theories. For example, proving uniqueness of supersymmetric black holes in minimal $N=2$, $D=4$ supergravity amounts to proving the long-standing conjecture [8] that the only black holes in the Israel-Wilson-Perjés (IWP) class of solutions are the Mujumdar-Papapetrou multi-black hole solutions. This is because all supersymmetric solutions of this theory are known [19], and fall into a timelike and a null class, as for the minimal $N=1$, $D=5$ theory. The IWP solutions constitute the timelike class. It seems very likely that this conjecture could be proved easily using the methods of this paper, i.e., first constructing bosonic objects from the super-covariantly constant spinor, using these to determine the form of the near horizon geometry (presumably either flat space or $AdS_2 \times S^2$), and then showing that this information together with asymptotic flatness determines a unique member of the IWP family.

Of more physical interest would be the extension of the above results to more complicated supergravity theories, for example the maximally supersymmetric theories in $D=4,5$. It seems rather unlikely that the general supersymmetric solution of these theories could be obtained using the methods of [18,19], so a complete uniqueness proof is probably not possible using the methods of this paper. However, it might be possible to determine all possible near-horizon geometries of solutions with compact Killing horizons [one would start with the metric (3.25) and assume that Δ, h_A and γ_{AB} are independent of r, since this is what happens in the near-horizon limit]. This information might lead to an understanding (in classical supergravity) of why supersymmetric rotating black hole solutions only seem to exist in $D=5$. Finally, it might be possible to use the methods of the present paper to classify possible near-horizon geometries of supersymmetric solutions of $D=10,11$ supergravity theories.

ACKNOWLEDGMENTS

I would like to thank Fay Dowker, Steven Gubser, Gary Horowitz, Chris Hull, Juan Maldacena, Robert Wald, Daniel Waldram, Toby Wiseman, and Edward Witten for discussions, and Roberto Emparan, Jerome Gauntlett, Gary Gibbons, and especially James Sparks for discussions and comments on the manuscript. I am grateful to Amanda Peet for pointing out a typo in the first version of this paper. This work was supported by PPARC.

APPENDIX

The purpose of this appendix is to exclude the possibility of supersymmetric black holes with near-horizon geometry $AdS_3 \times S^2$. It is best read after Sec. III G because it relies on results developed there.

Since Δ vanishes at $r=0$, but is not identically zero (this would correspond to the null class) then by analyticity it must be possible to write

$$\Delta = r^p \tilde{\Delta}, \tag{A1}$$

where p is a positive integer, $\tilde{\Delta}$ is not identically zero on H, and $\tilde{\Delta} > 0$ for $r > 0$. The Maxwell equation (3.18) then gives

$$\nabla^2 \tilde{\Delta} - (2p-1) h^A \nabla_A \tilde{\Delta} = -p(p-1) \tilde{\Delta} h^2 + \mathcal{O}(r). \tag{A2}$$

Integrating this equation over H (using $\nabla_A h^A = 0$) implies

$$p(p-1)\tilde{\Delta} h^2 = 0 \quad \text{on} \quad H. \tag{A3}$$

Since $h^2 \neq 0$ on H for the solution (3.56), it follows that this solution must have $p=1$. Substituting this back into Eq. (A2), multiplying by $\tilde{\Delta}$ and integrating over H gives

$$\tilde{\Delta} = \tilde{\Delta}_0 \quad \text{on} \quad H, \tag{A4}$$

where $\tilde{\Delta}_0$ is a positive constant. Next, from Sec. III E, h can be written as

$$h = -\frac{dR}{R} + r h_1 + r^2 h_2, \tag{A5}$$

where h_1 is independent of r and h_2 is smooth at $r=0$. Equation (3.29) implies

$$\hat{d}(R h_1) = 0. \tag{A6}$$

Hence there is some function λ defined locally on H such that

$$h_1 = \frac{L}{R} d\lambda. \tag{A7}$$

The base space metric can be calculated from Eq. (3.74):

$$ds_4^2 = \tilde{\Delta}^{-1} \left(\frac{r}{R} \right)^2 \left[d\left(L\lambda - \frac{R}{r} \right) + R r h_2 \right]^2 + r^2 \tilde{\Delta} \gamma_{AB} dx^A dx^B. \tag{A8}$$

Define a new coordinate ρ by

$$r = \frac{\rho}{X}, \tag{A9}$$

where

$$X = 1 + \frac{L\lambda\rho}{R}. \tag{A10}$$

The base space metric is then

$$ds_4^2 = X^{-2} \left\{ \tilde{\Delta}^{-1} \left[\frac{d\rho}{\rho} - \frac{dR}{R} + \frac{\rho^2 h_2}{X} \right]^2 + L^2 \rho^2 \tilde{\Delta} \left[\frac{dR^2}{R^2} + d\Omega^2 + \mathcal{O}(\rho) dx^A dx^B \right] \right\}, \tag{A11}$$

where the metric on H deduced in Sec. III E has been used and $x^A = (R, \theta, \phi)$. Completing the square on dR/R gives

$$ds_4^2 = X^{-2} \left\{ \tilde{\Delta}^{-1} Y \left[\frac{dR}{R} - Y^{-1} \left(\frac{d\rho}{\rho} + \frac{\rho^2 h_2}{X} \right) \right]^2 + \tilde{\Delta} Y^{-1} L^2 \rho^2 \left[\frac{d\rho}{\rho} + \frac{\rho^2 h_2}{X} \right]^2 + L^2 \rho^2 \tilde{\Delta} d\Omega^2 + \mathcal{O}(\rho^3) dx^A dx^B \right\}, \tag{A12}$$

where

$$Y \equiv 1 + L^2 \rho^2 \tilde{\Delta}^2. \tag{A13}$$

Hence

$$ds_4^2 = [1 + \mathcal{O}(\rho)] \left\{ \tilde{\Delta}_0^{-1} \left[\frac{dR}{R} - \frac{d\rho}{\rho} + \mathcal{O}(\rho) d\rho + \mathcal{O}(\rho^2) dx^A \right]^2 + L^2 \tilde{\Delta}_0 (d\rho^2 + \rho^2 d\Omega^2) + \mathcal{O}(\rho) d\rho^2 + \mathcal{O}(\rho^3) d\rho dx^A + \mathcal{O}(\rho^3) dx^A dx^B \right\}. \tag{A14}$$

Now define

$$x^4 = \tilde{\Delta}_0^{-1/2} \log \frac{R}{\rho}, \qquad \tilde{\rho} = L \tilde{\Delta}_0^{1/2} \rho. \tag{A15}$$

Then

$$ds_4^2 = d\tilde{\rho}^2 + \tilde{\rho}^2 d\Omega^2 + (dx^4)^2 + \mathcal{O}(\tilde{\rho}) d\tilde{\rho}^2 + \mathcal{O}(\tilde{\rho}) d\tilde{\rho} dx^4 + \mathcal{O}(\tilde{\rho}^3) d\tilde{\rho} dx'^A + \mathcal{O}(\tilde{\rho})(dx^4)^2 + \mathcal{O}(\tilde{\rho}^2) dx^4 dx'^A + \mathcal{O}(\tilde{\rho}^3) dx'^A dx'^B, \tag{A16}$$

where $x'^A = (x^4, \theta, \phi)$. The identifications inherited from H imply that θ and ϕ parametrize a two-sphere and x^4 is periodically identified. The base space is therefore regular at $\tilde{\rho} = 0$. As in Sec. III G, asymptotic flatness requires the base space to be asymptotically Euclidean and hence it must be global R^4 with its usual metric. However, the above metric (with periodic x^4) cannot arise from global R^4. Hence $AdS_3 \times S^2$ cannot arise as the near-horizon geometry of a black hole.

Of course $AdS_3 \times S^2$ does arise as the near horizon geometry of a black string. Such strings exist in both the null class and the timelike class. To get such a string from the timelike class, one can take the base space to be R^4 parametrized as in Eq. (A16) (neglecting the corrections) and then follow the method of Sec. 3.7 of [18] (with $H = 1$, $\chi_i = \omega_i = 0$, and point sources for the harmonic functions).

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