Quasinormal behavior of the *D*-dimensional Schwarzschild black hole and the higher order WKB approach

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We study characteristic (quasinormal) modes of a *D*-dimensional Schwarzschild black hole. It is shown that the real parts of the complex quasinormal modes, representing the real oscillation frequencies, are proportional to the product of the number of dimensions and inverse horizon radius $\sim Dr_0^{-1}$. The asymptotic formula for large multipole number l and arbitrary D is derived. In addition, the WKB formula for computing QN modes, developed to the third order beyond the eikonal approximation, is extended to the sixth order here. This gives us an accurate and economic way to compute quasinormal frequencies.

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I. INTRODUCTION

Within the framework of the brane world models the size of extra spatial dimensions may be much larger than the Planck length, and the fundamental quantum gravity scale may be very low ($\sim$TeV). When considering models with large extra dimensions the black hole mass may be of the order of 1 TeV, i.e., much smaller than the Planck mass. There is a possibility of production of such mini-black-holes in particle collisions in colliders and in cosmic ray experiments [1]. Estimations show that these higher dimensional black holes can be described by classical solutions of vacuum Einstein equations. Thus the investigation of the general properties of these black holes, including perturbations and decay of different fields around them, has attracted considerable interest (see, for example, [2,3] and references therein).

It is well known that when perturbing a black hole it undergoes damping oscillations which are characterized by some complex eigenvalues of the wave equations called *quasinormal frequencies*. Their real parts represent the oscillation frequencies, while the imaginary ones determine the damping rates of the modes. The quasinormal modes (QNM's) of black holes (BH's) depend only on the black hole parameters and not on the way in which they were excited. QNM's are called, therefore, the "footprints" of a black hole. Being a useful characteristic of black hole dynamics, quasinormal modes are studied also within different contexts now: in anti–de Sitter/conformal field theory (AdS/CFT) correspondence (see, for example, [4–15] and references therein), because of the possibility of observing quasinormal ringing of astrophysical BH's (see [16] for a review), when considering thermodynamic properties of black holes in loop quantum gravity [17–20], in the context of a possible connection with critical collapse [4,9,21,22].

Thus it would be interesting to know, from a different standpoint, what happens to a black hole living in *D*-dimensional space-time with a QN spectrum [23,3]. The subject of the present paper is twofold: First we extend the WKB method of Schutz, Will, and Iyer for computing QN modes from the third to the sixth order beyond the eikonal approximation (see Sec. II and Appendix A). In many physical situations this allows us to compute the QNMs accurately and quickly without resorting to complicated numerical methods. In Appendix B, QN modes of $D=4$ Schwarzschild black holes induced by perturbations of different spin are obtained by the sixth order WKB formula, and compared with the numerical values and third order WKB values. Second, motivated by the above reasons, we apply the obtained WKB formula to find the scalar quasinormal modes of multidimensional Schwarzschild black holes (Sec. III). It is shown that the real parts of the quasinormal frequencies are proportional to the product Dr_0^{-1}, where r_0 is the horizon radius and D is the dimension of space-time.

II. SIXTH ORDER WKB ANALYSIS

The first semianalytical method for calculations of BH QNMs was apparently proposed by Bahram Mashhoon, who used the Poschl-Teller potential to estimate the QN frequencies [24]. In [25], there was proposed a semianalytical method for computing QNM's based on the WKB treatment. Then in [26] the first WKB order formula was extended to the third order beyond the eikonal approximation, and, afterwards, was frequently used in a lot of works (see, for example, [9,28–34] and references therein). The accuracy of the third order WKB formula [see Eq. (1.5) in [26]] is better with a larger multipole number l and a smaller overtone n. For the Schwarzschild BH the results practically coincide with accurate numerical results of Leaver [35] at $l\geqslant 4$ when being restricted by lower overtones for which $l>n$. For fewer multipoles, however, the accuracy is worse, and may reach 10% at $l=0$, $n=0$. The numerical approach [35], on the contrary, is very accurate, but dealing with the numerical integration or systems of recurrence relations is very cumbersome, and, often, requires modification to be applied to different effective potentials. At the same time, the WKB approach lets us obtain QNM's for a full range of parameters, thereby giving us the opportunity to examine the physical behavior of a system. Even though the WKB formula gives the best accuracy at $l>n$, it includes the case of astrophysical black hole radiation where only lower over-

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tones are significantly excited [36]. Both the advantages and the deficiencies of the WKB approach motivated us to extend the existent third order WKB formula up to the sixth order.

The perturbation equations of a black hole can be reduced to the Schrödinger wavelike equation

$$\frac{d^2\psi}{dx^2}+Q(x)\psi(x)=0, \tag{1}$$

where "the potential" $-Q(x)$ is constant at the event horizon ($x=-\infty$) and at infinity ($x=+\infty$) and it rises to maximum at some intermediate $x=x_0$. Consider radiation of a given frequency ω incident on the black hole from infinity and let $R(\omega)$ and $T(\omega)$ be the reflection and transmission amplitudes, respectively. Extend $R(\omega)$ to the complex frequency plane such that $\mathrm{Re}(z)\neq 0$, and $T(z)/R(z)$ is regular. Then, the quasinormal modes correspond to the singularities of $R(z)$. We have a direct analogy with the problem of scattering near the pick of the potential barrier in quantum mechanics, where ω^2 plays a role of energy, and the two turning points divide the space into three regions at which boundaries the corresponding solutions should be matched.

To extend the third order WKB formula of [26], we used the technique of Iyer and Will. We shall omit here the technicalities of this approach, which are described in [26]. The only thing we should stress is that since the coefficients M_{ij} that connect amplitudes near the horizon with those at infinity depend only on ν (related to the overtone number n), they may be found at higher orders, simply by solving the interior (between the turning points) problem to higher orders. Thus there is no need to perform an explicit match of the solutions to WKB solutions in the exterior (outside turning points) regions to the same order. The result has the form

$$\frac{\imath Q_0}{\sqrt{2Q_0''}}-\Lambda_2-\Lambda_3-\Lambda_4-\Lambda_5-\Lambda_6=n+\frac{1}{2}, \tag{2}$$

where the correction terms $\Lambda_4,\Lambda_5,\Lambda_6$ can be found in Appendix A. Note that Λ_4 coincides with preliminary formula (A3) of [26] in proper designations.

An alternative, pure algebraic approach to finding higher order WKB corrections was proposed by Zaslavskii [37] using a quantum anharmonic oscillator problem where WKB correction terms come from perturbation theory corrections to the potential anharmonicity.

Thus we have obtained an economic and accurate formula for straightforward calculation of QNM frequencies. The sixth order formula applied to the $D=4$ Schwarzschild BH is as accurate already at $l=1$ as the third order formula is at $l=4$. We show in Appendix B an example of QNM's corresponding to perturbations of fields of different spins: scalar ($s=0$), neutrino ($s=\frac{1}{2}$), electromagnetic ($s=1$), gravitino ($s=\frac{3}{2}$), and gravitational ($s=2$). In addition, looking at the convergence of all sixth WKB values to some unknown true QN mode, we can judge, approximately, how far from the true QN value we are, staying within the framework of the WKB method.

III. QUASINORMAL MODES OF THE *D*-DIMENSIONAL SCHWARZSCHILD BLACK HOLE

The metric of the Schwarzschild black hole in D dimensions has the form

$$ds^2=f(r)dt^2-f^{-1}(r)dr^2+r^2\,d\Omega_{D-2}^2, \tag{3}$$

where

$$f(r)=1-\left(\frac{r_0}{r}\right)^{D-3}=1-\frac{16\pi GM}{(D-2)\Omega_{D-2}r^{D-3}}. \tag{4}$$

Here we used the quantities

$$\Omega_{D-2}=\frac{(2\pi)^{(D-1)/2}}{\Gamma((D-1)/2)},\quad \Gamma(1/2)=\sqrt{\pi},\quad \Gamma(z+1)=z\Gamma(z).$$

The scalar perturbation equation of this black hole can be reduced to the Schrödinger wavelike equation (1) with respect to the "tortoise" coordinate x: $dx=dr/f(r)$, where "the potential" $-Q(x)$ has the form

$$Q(x)=\omega^2-f(r)\left(\frac{l(l+D-3)}{r^2}+\frac{(D-2)(D-4)}{4r^2}f(r)+\frac{D-2}{2r}f'(r)\right). \tag{5}$$

At some fixed D we can set $r_0=2$ and measure ω in units $2r_0^{-1}$. The quasinormal modes satisfy the boundary conditions

$$\phi(x)\sim c_{\pm}e^{i\pm\omega x}\quad \text{as}\quad x\rightarrow\pm\infty. \tag{6}$$

The sixth WKB order formula used here gives very accurate results for low overtones. The previous orders enable us to see the convergence of the WKB values of ω^2 as a WKB order grows to an accurate numerical result. Namely, we can observe that for $l=1,2,3,4,\ldots$ for the fundamental overtone, the sixth order values differ from its fifth order value by

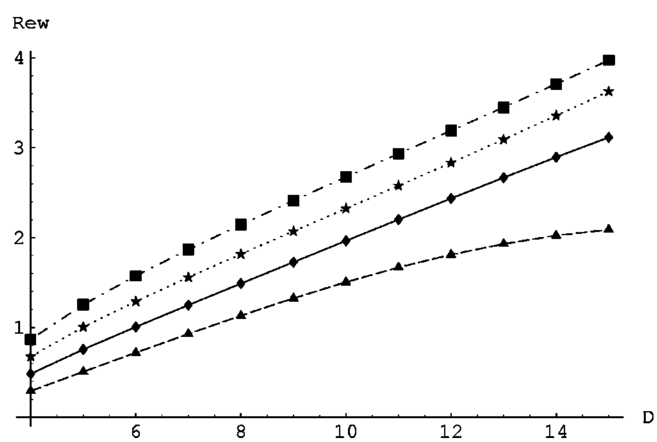

FIG. 1. Re ω for different dimensions D; $l=1$ (bottom), 2,3,4 (top); $n=0$.

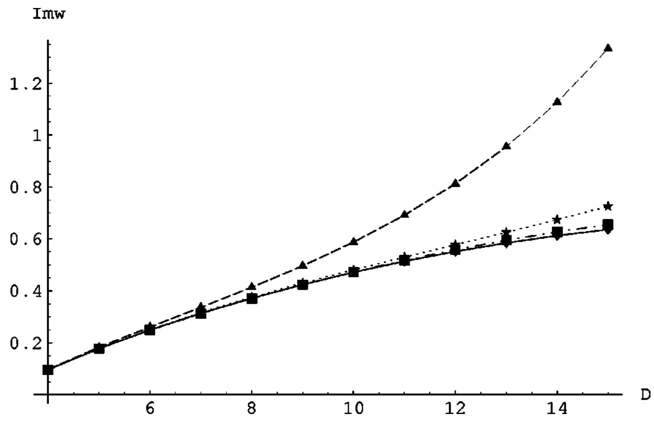

FIG. 2. Im ω for different dimensions D; $l=1$ (bottom), 2,3,4 (top); $n=0$.

fractions of a percent or less at not very large D (we are restricted here by $D=4,5,\ldots,15$).

This proves that if one takes $r_0=2$ for each given D, then the real parts of ω for different D lie on a strict line. That is, ω_{Re} is proportional to the product $r_0 D$ (remember that r_0 depends on D itself). Namely, for the fundamental overtone we obtain the following approximate relations:

$$\omega_{\mathrm{Re}}\sim 0.244D(r_0/2)^{-1}, \quad l=2, \tag{7}$$

$$\omega_{\mathrm{Re}}\sim 0.275D(r_0/2)^{-1}, \quad l=3, \tag{8}$$

$$\omega_{\mathrm{Re}}\sim 0.290D(r_0/2)^{-1}, \quad l=4. \tag{9}$$

Here we take $\omega=\omega_{\mathrm{Re}}-i\omega_{\mathrm{Im}}$. Generally, the higher the multipole number l, the larger is the coefficient before the product Dr_0^{-1}. We observed the same $\sim Dr_0^{-1}$ relation for higher overtone but not higher than l, for which WKB treatment is applicable. In Figs. 1 and 2 we presented the real and imaginary parts of ω measured in $2r_0^{-1}$ for different D. For real parts of $l=1$ modes we see the deviation from the strict line at large D. This, however, is stipulated by the bad accuracy of the WKB approach, and we believe that the true frequencies will lie on a strict line again. Indeed, one can judge by

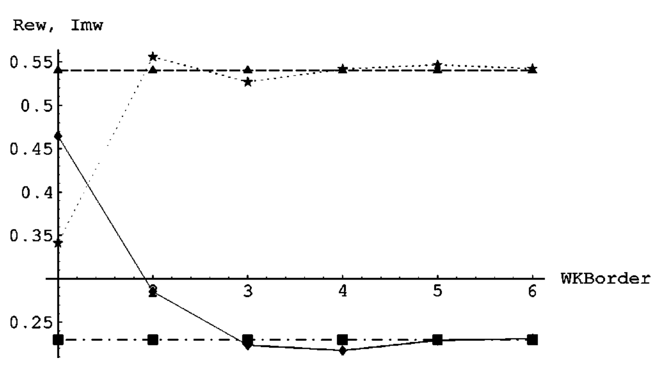

FIG. 3. ω_{Re} (bottom) and ω_{Im} (top) as a function of WKB order of the formula with which it was obtained for $l=1$, $n=2$, $D=4$ modes, and the corresponding numerical value. We see how the WKB values converge to an accurate numerical value as the WKB order increases.

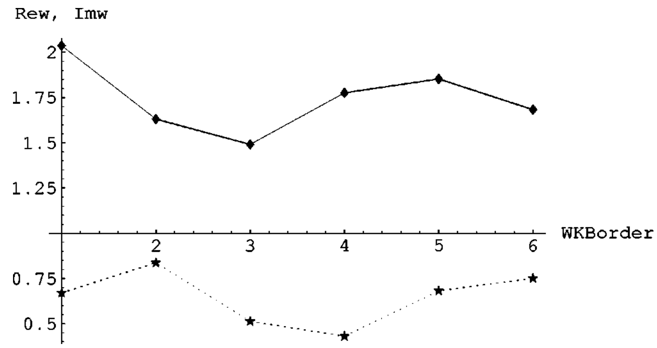

FIG. 4. ω_{Re} (top) and ω_{Im} (bottom) as a function of WKB order of the formula with which it was obtained for $l=0$, $n=0$, $D=12$ modes.

looking at the convergence plot in Figs. 3–6, where the real and imaginary parts of ω are shown as a function of the WKB order. Generally, the accuracy of the WKB formula is better the higher l is and the lower n and D are. Note that the dependence Dr_0^{-1} for lower overtones can be recovered even within the third order formula, provided l is greater than 2, and D is not very large.

Another point is the $l=0$ modes: in this case the lowest overtone implies $l=n$, and the WKB formula has considerable relative error. For a four-dimensional BH, for which the accurate numerical results are known, the error is about 10% for ω_{Im} and 5% for ω_{Re} in the third WKB order, while in the sixth order it reduces to 0% for ω_{Re} and 3% for ω_{Im} (see Appendix B). For greater D the error increases; the difference between the fifth and sixth order WKB values grows and one cannot judge the true quasinormal behavior in this case (see Figs. 3–5). Fortunately, other field perturbations, including gravitational, have the lowest overtone with $l>n$ and the WKB treatment is of good accuracy for all l. In Table I, we compare the third order WKB values of $l=0$, $n=0$ modes for different D [3] with those obtained through the sixth order here.

For large l the well-known approximate formula reads (see [27,38,39] for a proof)

$$\omega_{\mathrm{Re}}=\frac{1}{3\sqrt{3}}\left(l+\frac{1}{2}\right), \quad \omega_{\mathrm{Im}}=\frac{1}{3\sqrt{3}}\left(n+\frac{1}{2}\right). \tag{10}$$

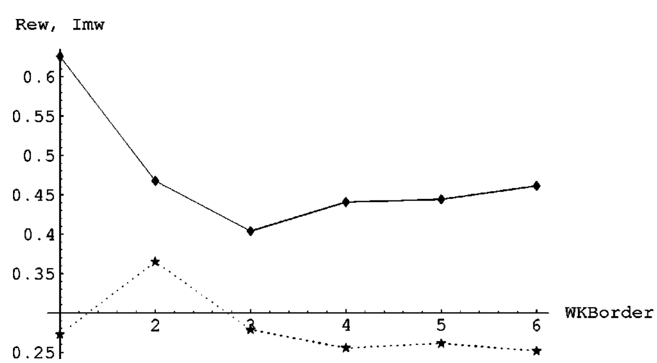

FIG. 5. ω_{Re} (top) and ω_{Im} (bottom) as a function of WKB order of the formula with which it was obtained for $l=0$, $n=0$, $D=6$ modes.

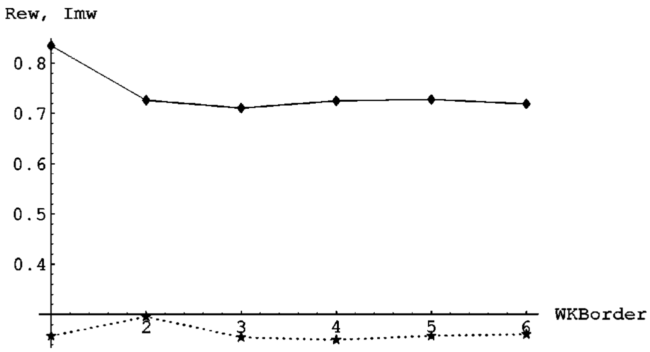

FIG. 6. $\omega_{\rm Re}$ (top) and $\omega_{\rm Im}$ (bottom) as a function of WKB order of the formula with which it was obtained for $l=1$, $n=0$, $D=6$ modes.

To obtain its D-dimensional generalization we find a value $r_{\rm max}$ at which the effective potential V attains its maximum, provided l is large,

$$r_{\rm max}\approx 2^{(D-4)/(D-3)}(D-1)^{1/(D-3)}, \quad D=4,5,6,\ldots . \tag{11}$$

Then let us make use of this value $r_{\rm max}$ when dealing with the first order WKB formula. After expansion in terms of small values of $1/l$, for a fixed D in units of $2r_0^{-1}$ we obtain

$$\omega_{\rm Re}\approx \frac{D+2l-3}{4}\left(\frac{2}{D-1}\right)^{1/(D-3)}\sqrt{\frac{D-3}{D-1}}, \tag{12}$$

$$\omega_{\rm Im}\approx \frac{(D-3)}{4}\left(\frac{2}{D-1}\right)^{1/(D-3)}\frac{2n+1}{\sqrt{D-1}}. \tag{13}$$

When $D=4$, these formulas go over into Eq. (11). We see that when l is much larger than D, the $\sim Dr_0^{-1}$ dependence of $\omega_{\rm Re}$ breaks down.

TABLE I. Schwarzschild QN frequencies for $l=0$, $n=0$ scalar perturbations in various D.

D	Third WKB order	Sixth WKB order	$1/r_0$
4	$0.1046-0.1152i$	$0.1105-0.1008i$	0.5
6	$1.0338-0.7133i$	$1.1808-0.6438i$	1.28
8	$1.9745-1.0258i$	$2.3004-1.0328i$	1.32
10	$2.7828-1.1596i$	$3.2214-1.3766i$	1.25
12	$3.4892-1.2020i$	$3.9384-1.7574i$	1.17

IV. CONCLUSION

We were interested here in the question of how dimensionality effects the quasinormal behavior of black holes. Yet several interesting points are beyond our consideration of low-lying quasinormal modes of multi-dimensional black holes. First of all, one would like to understand the origin of the relation $\sim Dr_0^{-1}$ in $\omega_{\rm Re}$ dependence. In this question it is possible to try to explain it from the interpretations of QN modes as Breit-Wigner type resonances generated by a family of surface waves propagating close to the unstable circular photon orbit [40]. Second, we do not know whether $\sim Dr_0^{-1}$ dependence will be present for perturbations of other fields, and for more general backgrounds, such as multidimensional Reissner-Nordström or Kerr backgrounds. We hope further investigations will clarify these points.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge stimulating discussions with Vitor Cardoso and Oleg Zaslavskii.

APPENDIX A: CORRECTION TERMS FOR WKB FORMULA

Here we shall use the following designations: Q_0 means the value of the potential Q at its peak, while Q_i is the ith derivative of Q with respect to the tortoise coordinate x. Then Q_i^j is the jth power of the ith derivative of Q.

$$\begin{aligned}\Lambda_4=&\frac{1}{597196800\sqrt{2}Q_2^7\sqrt{Q_2}}\{2536975Q_3^6-9886275Q_2Q_3^4Q_4+5319720Q_2^2Q_3^3Q_5-225Q_2^2Q_3^2(-40261Q_4^2+9688Q_2Q_6)\\&+3240Q_2^3Q_3(-1889Q_4Q_5+220Q_2Q_7)-729Q_2^3[1425Q_4^3-1400Q_2Q_4Q_6+8Q_2(-123Q_5^2+25Q_2Q_8)]\}\\&+\frac{(n+1/2)^2}{4976640\sqrt{2}Q_2^7\sqrt{Q_2}}\{348425Q_3^6-1199925Q_2Q_3^4Q_4+57276Q_2^2Q_3^3Q_5-45Q_2^2Q_3^2(-20671Q_4^2+4552Q_2Q_6)\\&+1080Q_2^3Q_3(-489Q_4Q_5+52Q_2Q_7)-27Q_2^3[2845Q_4^3-2360Q_2Q_4Q_6+56Q_2(-31Q_5^2+5Q_2Q_8)]\}\\&+\frac{(n+1/2)^4}{2488320\sqrt{2}Q_2^7\sqrt{Q_2}}\{192925Q_3^6-581625Q_2Q_3^4Q_4+234360Q_2^2Q_3^3Q_5-45Q_2^2Q_3^2(-8315Q_4^2+1448Q_2Q_6)\\&+1080Q_2^3Q_3(-161Q_4Q_5+12Q_2Q_7)-27Q_2^3[625Q_4^3-440Q_2Q_4Q_6+8Q_2(-63Q_5^2+5Q_2Q_8)]\}\end{aligned} \tag{A1}$$

$$\begin{aligned}\Lambda_5=&\frac{(n+1/2)}{57330892800Q_2^{10}}\{2768256Q_{10}Q_2^7-1078694575Q_3^8+5357454900Q_2Q_3^6Q_4-2768587920Q_2^2Q_3^5Q_5\\&+90Q_2^2Q_3^4(-88333625Q_4^2+12760664Q_2Q_6)-4320Q_2^3Q_3^3(-1451425Q_4Q_5+91928Q_2Q_7)-27Q_2^4[7628525Q_4^4\\&-9382480Q_2Q_4^2Q_6+64Q_2^2(19277Q_6^2+37764Q_5Q_7)+576Q_2Q_4(-21577Q_5^2+2505Q_2Q_8)]+540Q_2^3Q_3^2[6515475Q_4^3\\&-3324792Q_2Q_4Q_6+16Q_2(-126468Q_5^2+12679Q_2Q_8)]-432Q_2^4Q_3[5597075Q_4^2Q_5-854160Q_2Q_4Q_7\\&+8Q_2(-145417Q_5Q_6+6685Q_2Q_9)]\}+\frac{(n+1/2)^3}{477757440Q_2^{10}}\{31104Q_{10}Q_2^7-42944825Q_3^8+193106700Q_2Q_3^6Q_4\\&-90039120Q_2^2Q_3^5Q_5+30Q_2^2Q_3^4(-8476205Q_4^2+1102568Q_2Q_6)-4320Q_2^3Q_3^3(-41165Q_4Q_5+2312Q_2Q_7)\\&-9Q_2^4[445825Q_4^4-472880Q_2Q_4^2Q_6+64Q_2^2(829Q_6^2+1836Q_5Q_7)+4032Q_2Q_4(-179Q_5^2+15Q_2Q_8)]\\&+180Q_2^3Q_3^2[532615Q_4^3-241224Q_2Q_4Q_6+16Q_2(-9352Q_5^2+799Q_2Q_8)]-144Q_2^4Q_3[392325Q_4^2Q_5\\&-51600Q_2Q_4Q_7+8Q_2(-8853Q_5Q_6+335Q_2Q_9)]\}+\frac{(n+1/2)^5}{1194393600Q_2^{10}}\{10368Q_{10}Q_2^7-66578225Q_3^8\\&+272124300Q_2Q_3^6Q_4-112336560Q_2^2Q_3^5Q_5+9450Q_2^2Q_3^4(-33775Q_4^2+3656Q_2Q_6)-151200Q_2^3Q_3^3(-1297Q_4Q_5\\&+56Q_2Q_7)-27Q_2^4[89075Q_4^4-83440Q_2Q_4^2Q_6+64Q_2^2(131Q_6^2+396Q_5Q_7)+576Q_2Q_4(-343Q_5^2+15Q_2Q_8)]\\&+540Q_2^3Q_3^2[188125Q_4^3-71400Q_2Q_4Q_6+16Q_2(-3052Q_5^2+177Q_2Q_8)]-432Q_2^4Q_3[118825Q_4^2Q_5-11760Q_2Q_4Q_7\\&+8Q_2(-2303Q_5Q_6+55Q_2Q_9)]\}\end{aligned}\tag{A2}$$

$$\begin{aligned}\Lambda_6=&\frac{-i}{202263389798400Q_2^{12}\sqrt{2Q_2}}(-171460800Q_{12}Q_2^9+1714608000Q_{11}Q_2^8Q_3-10268596800Q_{10}Q_2^7Q_3^2\\&+970010662775Q_3^{10}+3772137600Q_{10}Q_2^8Q_4-62626341755 25Q_2Q_3^8Q_4+137829831961 50Q_2^2Q_3^6Q_4^2\\&-119541481258 50Q_2^3Q_3^4Q_4^3+3449170577475Q_2^4Q_3^2Q_4^4-144528059025Q_2^5Q_4^5+3352602187200Q_2^2Q_3^7Q_5\\&-12300730092000Q_2^3Q_3^5Q_4Q_5+11994129604800Q_2^4Q_3^3Q_4^2Q_5-2624788605600Q_2^5Q_3Q_4^3Q_5\\&+2580769643760Q_2^4Q_3^4Q_5^2-3453909784416Q_2^5Q_3^2Q_4Q_5^2+438440697072Q_2^6Q_4^2Q_5^2+260524397952Q_2^6Q_3Q_5^3\\&-1475306441280Q_2^3Q_3^6Q_6+4329682610400Q_2^4Q_3^4Q_4Q_6-2865128172480Q_2^5Q_3^2Q_4^2Q_6+233443879200Q_2^6Q_4^3Q_6\\&-1660199804928Q_2^5Q_3^3Q_5Q_6+1281705296256Q_2^6Q_3Q_4Q_5Q_6-87403857408Q_2^7Q_5^2Q_6+231105873600Q_2^6Q_3^2Q_6^2\\&-68412859200Q_2^7Q_4Q_6^2+552968700480Q_2^4Q_3^5Q_7-1231789749120Q_2^5Q_3^3Q_4Q_7+470726303040Q_2^6Q_3Q_4^2Q_7\\&+413953400448Q_2^6Q_3^2Q_5Q_7-126242178048Q_2^7Q_4Q_5Q_7-91489305600Q_2^7Q_3Q_6Q_7+5619715200Q_2^8Q_7^2\\&-175752294480Q_2^5Q_3^4Q_8+271759652640Q_2^6Q_3^2Q_4Q_8-39736040400Q_2^7Q_4^2Q_8-73378363968Q_2^7Q_3Q_5Q_8\\&+9773265600Q_2^8Q_6Q_8+47107126080Q_2^6Q_3^3Q_9-43345290240Q_2^7Q_3Q_4Q_9+7400248128Q_2^8Q_5Q_9)\\&-\frac{(n+1/2)^2i}{687970713600Q_2^{12}\sqrt{2Q_2}}(-4551552Q_{12}Q_2^9+60279552Q_{11}Q_2^8Q_3-425036160Q_{10}Q_2^7Q_3^2+73727194625Q_3^{10}\\&+116743680Q_{10}Q_2^8Q_4-443649208275Q_2Q_3^8Q_4+901144103850Q_2^2Q_3^6Q_4^2-711096726150Q_2^3Q_3^4Q_4^3\\&+182164306725Q_2^4Q_3^2Q_4^4-6289615575Q_2^5Q_4^5+222467624400Q_2^2Q_3^7Q_5-746418445200Q_2^3Q_3^5Q_4Q_5\\&+653423900400Q_2^4Q_3^3Q_4^2Q_5-124319674800Q_2^5Q_3Q_4^3Q_5+143980943040Q_2^4Q_3^4Q_5^2-169712521920Q_2^5Q_3^2Q_4Q_5^2\\&+18188188416Q_2^6Q_4^2Q_5^2+11240861184Q_2^6Q_3Q_5^3-91198200240Q_2^3Q_3^6Q_6+241513732080Q_2^4Q_3^4Q_4Q_6\end{aligned}$$

$$-140030897040Q_2^5Q_3^2Q_4^2Q_6+9200103120Q_2^6Q_4^3Q_6-84218693760Q_2^5Q_3^3Q_5Q_6+55248386688Q_2^6Q_3Q_4Q_5Q_6$$
$$-3173043456Q_2^7Q_5^2Q_6+10464952896Q_2^6Q_3^2Q_6^2-2403421632Q_2^7Q_4Q_6^2+31637744640Q_2^4Q_3^5Q_7$$
$$-62649953280Q_2^5Q_3^3Q_4Q_7+20409822720Q_2^6Q_3Q_4^2Q_7+18860532480Q_2^6Q_3^2Q_5Q_7-4693344768Q_2^7Q_4Q_5Q_7$$
$$-3625731072Q_2^7Q_3Q_6Q_7+188054784Q_2^8Q_7^2-9155635200Q_2^5Q_3^4Q_8+12238024320Q_2^6Q_3^2Q_4Q_8$$
$$-1405278720Q_2^7Q_4^2Q_8-2866700160Q_2^7Q_3Q_5Q_8+303295104Q_2^8Q_6Q_8+2210705280Q_2^6Q_3^3Q_9$$
$$-1685525760Q_2^7Q_3Q_4Q_9+235488384Q_2^8Q_5Q_9)-\frac{(n+1/2)^4 i}{20065812480Q_2^{12}\sqrt{2Q_2}}(-66528Q_{12}Q_2^9+1245888Q_{11}Q_2^8Q_3$$
$$-11158560Q_{10}Q_2^7Q_3^2+4668804525Q_3^{10}+2116800Q_{10}Q_2^8Q_4-25898331375Q_2Q_3^8Q_4+47959232650Q_2^2Q_3^6Q_4^2$$
$$-33861927750Q_2^3Q_3^4Q_4^3+7454763225Q_2^4Q_3^2Q_4^4-184988475Q_2^5Q_4^5+11891917800Q_2^2Q_3^7Q_5$$
$$-36105463800Q_2^3Q_3^5Q_4Q_5+27953667000Q_2^4Q_3^3Q_4^2Q_5-4457716200Q_2^5Q_3Q_4^3Q_5+6285855240Q_2^4Q_3^4Q_5^2$$
$$-6471756144Q_2^5Q_3^2Q_4Q_5^2+565259688Q_2^6Q_4^2Q_5^2+380939328Q_2^6Q_3Q_5^3-4375251160Q_2^3Q_3^6Q_6$$
$$+10317018600Q_2^4Q_3^4Q_4Q_6-5113813320Q_2^5Q_3^2Q_4^2Q_6+238888440Q_2^6Q_4^3Q_6-3203871552Q_2^5Q_3^3Q_5Q_6$$
$$+1758685824Q_2^6Q_3Q_4Q_5Q_6-88566912Q_2^7Q_5^2Q_6+335466432Q_2^6Q_3^2Q_6^2-55073088Q_2^7Q_4Q_6^2+1351294560Q_2^4Q_3^5Q_7$$
$$-2341442880Q_2^5Q_3^3Q_4Q_7+626542560Q_2^6Q_3Q_4^2Q_7+619520832Q_2^6Q_3^2Q_5Q_7-123524352Q_2^7Q_4Q_5Q_7$$
$$-96574464Q_2^7Q_3Q_6Q_7+4048704Q_2^8Q_7^2-341160120Q_2^5Q_3^4Q_8+386210160Q_2^6Q_3^2Q_4Q_8-30837240Q_2^7Q_4^2Q_8$$
$$-78073632Q_2^7Q_3Q_5Q_8+5848416Q_2^8Q_6Q_8+70415520Q_2^6Q_3^3Q_9-43424640Q_2^7Q_3Q_4Q_9$$
$$+5255712Q_2^8Q_5Q_9)-\frac{(n+1/2)^6 i}{300987187200Q_2^{12}\sqrt{2Q_2}}(-72576Q_{12}Q_2^9+1886976Q_{11}Q_2^8Q_3-22135680Q_{10}Q_2^7Q_3^2$$
$$+27463538375Q_3^{10}+2903040Q_{10}Q_2^8Q_4-141448688325Q_2Q_3^8Q_4+240655765350Q_2^2Q_3^6Q_4^2-152907158250Q_2^3Q_3^4Q_4^3$$
$$+28724479875Q_2^4Q_3^2Q_4^4-413669025Q_2^5Q_4^5+59058073200Q_2^2Q_3^7Q_5-164264209200Q_2^3Q_3^5Q_4Q_5$$
$$+113654696400Q_2^4Q_3^3Q_4^2Q_5-15166342800Q_2^5Q_3Q_4^3Q_5+26061194880Q_2^4Q_3^4Q_5^2-23876233920Q_2^5Q_3^2Q_4Q_5^2$$
$$+1767189312Q_2^6Q_4^2Q_5^2+1292433408Q_2^6Q_3Q_5^3-18902165520Q_2^3Q_3^6Q_6+40256773200Q_2^4Q_3^4Q_4Q_6$$
$$-17116974000Q_2^5Q_3^2Q_4^2Q_6+483582960Q_2^6Q_4^3Q_6-11384150400Q_2^5Q_3^3Q_5Q_6+5285056896Q_2^6Q_3Q_4Q_5Q_6$$
$$-246903552Q_2^7Q_5^2Q_6+992779200Q_2^6Q_3^2Q_6^2-101860416Q_2^7Q_4Q_6^2+4966859520Q_2^4Q_3^5Q_7-7661606400Q_2^5Q_3^3Q_4Q_7$$
$$+1683037440Q_2^6Q_3Q_4^2Q_7+1861574400Q_2^6Q_3^2Q_5Q_7316141056Q_2^7Q_4Q_5Q_7-235146240Q_2^7Q_3Q_6Q_7+8895744Q_2^8Q_7^2$$
$$-1042372800Q_2^5Q_3^4Q_8+1016789760Q_2^6Q_3^2Q_4Q_8-52436160Q_2^7Q_4^2Q_8-189060480Q_2^7Q_3Q_5Q_8$$
$$+9217152Q_2^8Q_6Q_8+175190400Q_2^6Q_3^3Q_9-87816960Q_2^7Q_3Q_4Q_9+10378368Q_2^8Q_5Q_9). \tag{A3}$$

All six WKB corrections printed in MATEMATICA are available from the author in electronic form upon request.

APPENDIX B: QNMS OF A FOUR-DIMENSIONAL SCHWARZSCHILD BLACK HOLE

‘‘The potential’’ $Q(x)$ in case of a Schwarzschild black hole has the form

$$Q(x)=\omega^2-\left(1-\frac{1}{r}\right)\left(\frac{l(l+1)}{r^2}+\frac{1-s^2}{r^3}\right), \tag{B1}$$

TABLE II. Schwarzschild QN frequencies for perturbations of different spin.

$s=0$	Numerical	Third order WKB	Sixth order WKB
$l=0,\ n=0$	$0.1105-0.1049i$	$0.1046-0.1152i$	$0.1105-0.1008i$
$l=1,\ n=0$	$0.2929-0.0977i$	$0.2911-0.0980i$	$0.2929-0.0978i$
$l=1,\ n=1$	$0.2645-0.3063i$	$0.2622-0.3074i$	$0.2645-0.3065i$
$l=2,\ n=0$	$0.4836-0.0968i$	$0.4832-0.0968i$	$0.4836-0.0968i$
$l=2,\ n=1$	$0.4639-0.2956i$	$0.4632-0.2958i$	$0.4638-0.2956i$
$l=2,\ n=2$	$0.4305-0.5086i$	$0.4317-0.5034i$	$0.4304-0.5087i$
$s=1/2$	Numerical	Third order WKB	Sixth order WKB
$l=1,\ n=0$		$0.2803-0.0969i$	$0.2822-0.0967i$
$l=1,\ n=1$		$0.2500-0.3049i$	$0.2525-0.3040i$
$l=2,\ n=0$		$0.4768-0.9639i$	$0.4772-0.0963i$
$l=2,\ n=1$		$0.4565-0.2947i$	$0.4571-0.2945i$
$l=2,\ n=2$		$0.4244-0.5016i$	$0.4231-0.5070i$
$l=3,\ n=0$		$0.6706-0.0963i$	$0.6708-0.0963i$
$l=3,\ n=1$		$0.6557-0.2917i$	$0.6560-0.2917i$
$l=3,\ n=2$		$0.6299-0.4931i$	$0.6286-0.4950i$
$l=3,\ n=3$		$0.5970-0.6997i$	$0.5932-0.7102i$
$s=1$	Numerical	Third order WKB	Sixth order WKB
$l=1,\ n=0$	$0.2483-0.0925i$	$0.2459-0.0931i$	$0.2482-0.0926i$
$l=1,\ n=1$	$0.2145-0.2937i$	$0.2113-0.2958i$	$0.2143-0.2941i$
$l=2,\ n=0$	$0.4576-0.0950i$	$0.4571-0.0951i$	$0.4576-0.0950i$
$l=2,\ n=1$	$0.4365-0.2907i$	$0.4358-0.2910i$	$0.4365-0.2907i$
$l=2,\ n=2$	$0.4012-0.5016i$	$0.4023-0.4959i$	$0.4009-0.5017i$
$l=3,\ n=0$	$0.6569-0.0956i$	$0.6567-0.0956i$	$0.6569-0.0956i$
$l=3,\ n=1$	$0.6417-0.2897i$	$0.6415-0.2898i$	$0.6417-0.2897i$
$l=3,\ n=2$	$0.6138-0.4921i$	$0.6151-0.4901i$	$0.6138-0.4921i$
$l=3,\ n=3$	$0.5779-0.7063i$	$0.5814-0.6955i$	$0.5775-0.7065i$
$s=3/2$	Numerical	Third order WKB	Sixth order WKB
$l=1,\ n=0$		$0.1817-0.0866i$	$0.1739-0.08357i$
$l=1,\ n=1$		$0.1354-0.2812i$	$0.1198-0.2813i$
$l=2,\ n=0$		$0.4231-0.926i$	$0.4236-0.0925i$
$l=2,\ n=1$		$0.4000-0.2842i$	$0.4007-0.2838i$
$l=2,\ n=2$		$0.3636-0.4853i$	$0.3618-0.4919i$
$l=3,\ n=0$		$0.6332-0.0945i$	$0.6333-0.0944i$
$l=3,\ n=1$		$0.6173-0.2864i$	$0.6175-0.2863i$
$l=3,\ n=2$		$0.5898-0.4846i$	$0.5884-0.4868i$
$l=3,\ n=3$		$0.5547-0.6882i$	$0.5505-0.7000i$
$s=2$	Numerical	Third order WKB	Sixth order WKB
$l=2,\ n=0$	$0.3737-0.0890i$	$0.3732-0.0892i$	$0.3736-0.0890i$
$l=2,\ n=1$	$0.3467-0.2739i$	$0.3460-0.2749i$	$0.3463-0.2735i$
$l=2,\ n=2$	$0.3011-0.4783i$	$0.3029-0.4711i$	$0.2985-0.4776i$
$l=3,\ n=0$	$0.5994-0.0927i$	$0.5993-0.0927i$	$0.5994-0.0927i$
$l=3,\ n=1$	$0.5826-0.2813i$	$0.5824-0.2814i$	$0.5826-0.2813i$
$l=3,\ n=2$	$0.5517-0.4791i$	$0.5532-0.4767i$	$0.5516-0.4790i$
$l=3,\ n=3$	$0.5120-0.6903i$	$0.5157-0.6774i$	$0.5111-0.6905i$
$l=4,\ n=0$	$0.8092-0.0942i$	$0.8091-0.0942i$	$0.8092-0.0942i$
$l=4,\ n=1$	$0.7966-0.2843i$	$0.7965-0.2844i$	$0.7966-0.2843i$
$l=4,\ n=2$	$0.7727-0.4799i$	$0.7736-0.4790i$	$0.7727-0.4799i$
$l=4,\ n=3$	$0.7398-0.6839i$	$0.7433-0.6783i$	$0.7397-0.6839i$
$l=4,\ n=4$	$0.7015-0.8982i$	$0.7072-0.8813i$	$0.7006-0.8985i$

where $s=0$ corresponds to scalar perturbations; $s=\frac{1}{2}$, neutrino perturbations; $s=1$, electromagnetic perturbations; $s=\frac{3}{2}$, gravitino perturbations; $s=2$, gravitational perturbations. The quasinormal frequencies at third and sixth WKB orders and in comparison with numerical results [35] are presented in Table II.

[1] S. Dimopoulos and S. Landsberg, Phys. Rev. Lett. **87**, 161602 (2002).
[2] V.P. Frolov and D. Stojkovic, gr-qc/0301016.
[3] V. Cardoso, O.J.C. Dias, and J.P.S. Lemos, Phys. Rev. D **67**, 064026 (2003).
[4] G.T. Horowitz and V. Hubeny, Phys. Rev. D **62**, 024027 (2000).
[5] J.S.F. Chan and R.B. Mann, Phys. Rev. D **59**, 064025 (1999).
[6] V. Cardoso and J.P.S. Lemos, Phys. Rev. D **63**, 124015 (2001).
[7] D. Birmingham, I. Sachs, and S.N. Solodukhin, Phys. Rev. Lett. **88**, 151301 (2002).
[8] D. Birmingham, I. Sachs, and S.N. Solodukhin, Phys. Rev. D **67**, 104026 (2003).
[9] R.A. Konoplya, Phys. Rev. D **66**, 084007 (2002).
[10] R.A. Konoplya, Phys. Rev. D **66**, 044009 (2002).
[11] A.O. Starinets, Phys. Rev. D **66**, 124013 (2002).
[12] D.T. Son and A.O. Starinets, J. High Energy Phys. **09**, 042 (2002).
[13] R. Aros, C. Martinez, R. Troncoso, and J. Zanelli, Phys. Rev. D **67**, 044014 (2003).
[14] M. Musiri and G. Siopsis, hep-th/0301081.
[15] I.G. Moss and J.P. Norman, Class. Quantum Grav. **19**, 2323 (2002).
[16] K. Kokkotas and B. Schmidt, Living Rev. Relativ. **2**, 2 (1999); H.P. Nollert, Class. Quantum Grav. **16**, R159 (2000).
[17] O. Dreyer, Phys. Rev. Lett. **90**, 081301 (2003).
[18] G. Kunstatter, gr-qc/0211076.
[19] V. Cardoso and J.P.S. Lemos, Phys. Rev. D **67**, 084020 (2003).
[20] L. Motl, gr-qc/0212096.
[21] W.T. Kim and J.J. Oh, Phys. Lett. B **514**, 155 (2001).
[22] R.A. Konoplya, Phys. Lett. B **550**, 117 (2002).
[23] L. Motl and A. Neitzke, hep-th/0301173.
[24] H.-J. Blome and B. Mashhoon, Phys. Lett. **100A**, 231 (1984).
[25] B.F. Schutz and C.M. Will, Astrophys. J. Lett. **291**, L33 (1985).
[26] S. Iyer and C.M. Will, Phys. Rev. D **35**, 3621 (1987).
[27] S. Iyer, Phys. Rev. D **35**, 3632 (1987).
[28] K. Kokkotas and B.F. Schutz, Phys. Rev. D **37**, 3378 (1988).
[29] K. Kokkotas, Nuovo Cimento Soc. Ital. Fis., B **108**, 991 (1993).
[30] L.E. Simone and C.M. Will, Class. Quantum Grav. **9**, 963 (1992).
[31] N. Andersson and H. Onozawa, Phys. Rev. D **54**, 7470 (1996); N. Andersson, *ibid.* **52**, 1808 (1995).
[32] V. Ferrari, M. Pauri, and F. Piazza, Phys. Rev. D **63**, 064009 (2001).
[33] H. Onozawa, T. Okamura, T. Mishima, and H. Ishihara, Phys. Rev. D **53**, 7033 (1996).
[34] R.A. Konoplya, Gen. Relativ. Gravit. **34**, 329 (2002).
[35] E. Leaver, Proc. R. Soc. London **A402**, 285 (1985).
[36] R.F. Stark and T. Piran, Phys. Rev. Lett. **55**, 891 (1985).
[37] O.B. Zaslavskii, Phys. Rev. D **43**, 605 (1991).
[38] V. Ferrari and B. Mashhoon, Phys. Rev. Lett. **52**, 1361 (1984).
[39] W.H. Press, Astrophys. J., Lett. Ed. **170**, L105 (1971).
[40] N. Andersson, Class. Quantum Grav. **11**, 3003 (1994); Y. Decanini, A. Folacci, and B. Jensen, Phys. Rev. D **67**, 124017 (2003).