Casimir effect, Achucarro-Ortiz black hole, and the cosmological constant

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We treat the two-dimensional Achucarro-Ortiz black hole (also known as a $1+1$ dilatonic black hole) as a Casimir-type system. The stress tensor of a massless scalar field satisfying Dirichlet boundary conditions on two one-dimensional "walls" ("Dirichlet walls") is explicitly calculated in three different vacua. Without employing known regularization techniques, the expression in each vacuum for the stress tensor is reached by using Wald's axioms. Finally, within this asymptotically nonflat gravitational background, it is shown that the equilibrium of the configurations, obtained by setting the Casimir force to zero, is controlled by the cosmological constant.

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I. INTRODUCTION

In the framework of quantum field theory in curved spacetime, there is no natural definition of particles. Unfortunately, only in exceptional cases does the particle concept in curved spacetime correspond to the intuitive picture of subatomic physics [1]. Therefore, we are led to study other observables that are not globally defined, which is obviously part of the problem with the particle definition. One of the most interesting objects, if not the very most, is the stress (or energy-momentum) tensor $T_{\mu\nu}(x)$. Furthermore, the interest in explicitly calculating the stress tensor is augmented by the presence of a gravitational background. The main reason is that the role of the stress tensor is now twofold. It describes the physical character of the quantum field at a spacetime point x, and it is also the source of gravity in this gravitational background. There are a plethora of field theoretical procedures [1–5], known as regularization techniques, for computing a finite and renormalized $\langle T_{\mu\nu}\rangle_{reg}$, such as the dimensional regularization [6–8], Green's function method [9,10], heat kernel method [11,12], zeta function regularization [13], point-splitting method [14–16], and Pauli-Villars regularization [17]. In this article, we are going to derive the exact form of the stress tensor of a massless scalar field by implementing some general properties of the renormalized stress tensor known as Wald's axioms [19,20], avoiding in this way employing any of the above-mentioned techniques.

In 1948, Casimir [21] was trying at first to calculate the van der Waals force between two polarized atoms. In the end, he was led to the problem of two parallel conducting plates. He evaluated the attractive force between the two plates and the electromagnetic energy that was localized between the two conducting plates. The Casimir effect, i.e., the disturbance to the electromagnetic vacuum induced by the presence of two parallel conducting plates, is in contact with laboratory physics [22,23]. Nowadays, Casimir-type systems [24,23] are viewed as tractable field theoretical models in which the general curved spacetime formalism can be applied and sensible results can be reached [25–27].

The scenario to be considered in our semiclassical analysis is as follows. (a) The gravitational background is the two-dimensional Achucarro-Ortiz black hole [28,29] which is asymptotically an AdS_2 spacetime, (b) two one-dimensional "walls," separated by a distance L, are placed in the aforementioned gravitational background, and (c) the quantum field whose stress tensor we are going to evaluate is a massless scalar one satisfying Dirichlet boundary conditions on the one-dimensional "walls" ("Dirichlet walls"). It is obvious that the Achucarro-Ortiz black hole will be treated as a Casimir-type system [30–33].

The paper is organized as follows. The next section is devoted to the presentation of Wald's axioms. In Secs. III and IV we describe the Achucarro-Ortiz and AdS_2 black hole geometries and calculate some of their geometrical quantities which are useful in the subsequent analysis. In Sec. V the vacuum expectation value of the stress tensor of the massless scalar field in the Achucarro-Ortiz black hole geometry is explicitly evaluated, respectively, in the Boulware vacuum (labeled by η) [34], the Hartle-Hawking vacuum (labeled by υ) [35–37], and the Unruh vacuum (labeled by ξ) [38]. The energy density, pressure, energy, and corresponding force between the two Dirichlet walls are specified. In Sec. VI, requiring the configurations to be in equilibrium, the distance between the Dirichlet walls is seemed to be determined by the two-dimensional cosmological constant. Finally, Sec. VII closes with conclusions and prospects for future work.

II. WALD'S AXIOMATIC ANALYSIS

In the mid-1970s there was a variety of techniques using complicated mathematical devices for computing the stress tensors. There was still the question of how to define a unique renormalized stress tensor $\langle T_{\mu\nu}\rangle$ purely by imposing physical requirements. Wald proffered five "axioms" to be satisfied by the stress tensors [19,20]. The axioms, called from now on Wald's axioms, are as follows: (1) The expectation values of the energy-momentum tensor are covariantly conserved; (2) causality holds; (3) in Minkowski spacetime, standard results should be obtained; (4) standard results for the off-diagonal elements should also be obtained; (5) the energy-momentum tensor is a local functional of the metric;

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i.e., it depends only on the metric and its derivatives which appear through the Riemann curvature tensor. It should be remarked that recently there was a significant generalization of the above-mentioned framework by Hollands and Wald [39].

Additionally, it must be noted that in a classical theory with a conformally invariant Lagrangian the trace vanishes. However, in the corresponding quantized theory the stress tensor may acquire a nonvanishing trace through renormalization (this is called conformal or trace anomaly) [6,7]. In two dimensions, the trace T^{α}_{α} can only be proportional to the Ricci scalar R of the theory [8,40]. This is in agreement with Wald's axioms.

III. ACHUCARRO-ORTIZ BLACK HOLE

The black hole solutions of Bañados, Teitelboim, and Zanelli (BTZ) in $2+1$ spacetime dimensions are derived from a three-dimensional theory of gravity [41]

$$S=\int dx^3\sqrt{-g}\,({}^{(3)}R+2\Lambda) \tag{1}$$

with a negative cosmological constant $(\Lambda=1/l^2>0)$. The corresponding line element is

$$ds^2=-\left(-M+\Lambda r^2+\frac{J^2}{4r^2}\right)dt^2+\frac{dr^2}{(-M+\Lambda r^2+J^2/4r^2)} + r^2\left(d\theta-\frac{J}{2r^2}dt\right)^2. \tag{2}$$

There are many ways to reduce the three-dimensional BTZ black hole solutions to the two-dimensional charged and uncharged dilatonic black holes [28]. The Kaluza-Klein reduction of the metric of the $(2+1)$-dimensional BTZ black hole (2) yields the two-dimensional line element

$$ds^2=-g(r)dt^2+g(r)^{-1}dr^2, \tag{3}$$

where

$$g(r)=\left(-M+\Lambda r^2+\frac{J^2}{4r^2}\right) \tag{4}$$

with M the Arnowitt-Deser-Misner (ADM) mass, J the charge of the two-dimensional charged black hole, a U(1) gauge field

$$A_t=-\frac{J}{2r^2}, \tag{5}$$

and a dilaton field

$$\Phi=r. \tag{6}$$

For the positive mass black hole spectrum with charge ($J\neq 0$), the line element (3) has two horizons

$$r_{\pm}^2=\frac{M\pm\sqrt{M^2-\Lambda J^2}}{2\Lambda} \tag{7}$$

with r_+,r_- the outer and inner horizon, respectively.

The Hawking temperature T_H of the event (outer) horizon is [42]

$$T_H=\frac{\sqrt{2\Lambda}}{2\pi}\frac{\sqrt{M^2-\Lambda J^2}}{(M+\sqrt{M^2-\Lambda J^2})^{1/2}} =\frac{\Lambda}{2\pi}\left(\frac{r_+^2-r_-^2}{r_+}\right). \tag{8}$$

The analytical formulas for the nonvanishing Christoffel symbols are

$$\Gamma^r_{tt}=\frac{1}{2}\left(-M+\Lambda r^2+\frac{J^2}{4r^2}\right)\left(2\Lambda r-\frac{J^2}{2r^3}\right) \tag{9}$$

$$\Gamma^r_{rr}=-\frac{1}{2}\frac{(2\Lambda r-J^2/2r^3)}{(-M+\Lambda r^2+J^2/4r^2)} \tag{10}$$

$$\Gamma^t_{rt}=\frac{1}{2}\frac{(2\Lambda r-J^2/2r^3)}{(-M+\Lambda r^2+J^2/4r^2)}. \tag{11}$$

The Ricci scalar is given by

$$R(r)=-\left[2\Lambda+\frac{3J^2}{2r^4}\right], \tag{12}$$

and therefore the nonzero trace of the stress tensor corresponding to the Achucarro-Ortiz black hole takes the form

$$T^{\alpha}_{\alpha}(r)=-\left[\frac{\Lambda}{12\pi}+\frac{J^2}{16\pi r^4}\right], \tag{13}$$

where we have used the expression for the trace of a stress tensor in two dimensions [8,40]:

$$T^{\alpha}_{\alpha}(r)=\frac{R(r)}{24\pi}. \tag{14}$$

IV. AdS_2 SPACE

The two-dimensional anti–de-Sitter geometry (AdS_2) can be derived either by restricting the Achucarro-Ortiz black hole to its spinless sector $J=0$ or by fixing the value of the dilaton field that appears in the above-mentioned reduced theory [28]. We adopt the first option and the resulting AdS_2 metric takes the form

$$ds^2=-g_{\text{AdS}}(r)dt^2+g_{\text{AdS}}(r)^{-1}dr^2 \tag{15}$$

where

$$g_{\text{AdS}}(r)=(-M+\Lambda r^2), \tag{16}$$

which has a horizon at

$$r_H = \sqrt{\frac{M}{\Lambda}}. \tag{17}$$

The temperature of the AdS_2 black hole is [43]

$$T_H^{\mathrm{AdS}} = \frac{\sqrt{\Lambda M}}{2\pi}. \tag{18}$$

The analytical formulas for the nonvanishing Christoffel symbols are

$$\Gamma^r_{tt} = \Lambda r(-M + \Lambda r^2), \tag{19}$$

$$\Gamma^r_{rr} = -\frac{\Lambda r}{(-M + \Lambda r^2)}, \tag{20}$$

$$\Gamma^t_{rt} = \frac{\Lambda r}{(-M + \Lambda r^2)}. \tag{21}$$

The Ricci scalar is given by

$$R(r) = -2\Lambda, \tag{22}$$

and therefore the nonzero trace of the Achucarro-Ortiz black hole takes the form

$$T^\alpha_\alpha(r) = -\frac{\Lambda}{12\pi}. \tag{23}$$

Using the formula

$$T^{\mathrm{AdS}}_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}_{\mathrm{grav}}}{\delta g^{\mu\nu}} \bigg|_{g^{\mu\nu} = g^{\mu\nu}_{\mathrm{AdS}}}, \tag{24}$$

the explicit expression for the stress tensor of the gravitational field of the AdS_2 space is easily calculated:

$$T^{\mathrm{AdS}}_{\mu\nu} = \begin{bmatrix} \frac{r^2}{2} & 0 \\ 0 & \frac{r^2}{2(-M + \Lambda r^2)^2} \end{bmatrix}. \tag{25}$$

V. CASIMIR EFFECT AND STRESS TENSOR

In this section a detailed expression for the renormalized stress tensor of the massless scalar is obtained by enforcing Wald's axioms and using its trace.

The starting point is Wald's first axiom, i.e., that the conservation equation must be satisfied by the renormalized expectation value of the stress tensor $\langle T^\mu{}_\nu \rangle_{\mathrm{reg}} \equiv T^\mu{}_\nu$,

$$T^\mu{}_{\nu;\mu} = 0, \tag{26}$$

which "splits" into two equations:

$$\frac{dT^r_t}{dr} + \Gamma^r_{rr} T^r_t - \Gamma^r_{tt} T^t_r = 0, \tag{27}$$

$$\frac{dT^r_r}{dr} + \Gamma^t_{tr} T^r_r - \Gamma^t_{rt} T^t_t = 0, \tag{28}$$

and since $T^t_r = -T^r_t$ and $T^t_t = T^\alpha_\alpha - T^r_r$, we get

$$\frac{dT^r_t}{dr} + [\Gamma^r_{rr} + \Gamma^r_{tt}] T^r_t = 0, \tag{29}$$

$$\frac{dT^r_r}{dr} + 2\Gamma^t_{rt} T^r_r = \Gamma^t_{rt} T^\alpha_\alpha. \tag{30}$$

Substituting the Christoffel symbols (9)–(11) into Eqs. (29), (30) and solving them, we get, respectively,

$$T^r_t(r) = \frac{1}{g(r)} \delta, \tag{31}$$

where

$$\delta = \alpha g^{3/2}(r) e^{-g^2(r)/4}, \tag{32}$$

and

$$T^r_r(r) = \frac{1}{g(r)} [\beta + H_2(r)], \tag{33}$$

where

$$H_2(r) = \frac{1}{2} \int_{r_+}^{r} \frac{dg(r')}{dr'} T^\alpha_\alpha(r') dr' \tag{34}$$

and the parameters α, β are constants of integration while the point r_+ is where the outer horizon is placed. It can be shown that $H_2(r)$ for the Achucarro-Ortiz black hole background (3), (4) becomes

$$H_2(r) = \frac{1}{96\pi} \left[2\Lambda r - \frac{J^2}{2r^3} \right]^2 - D, \tag{35}$$

where D is constant,

$$D = \frac{1}{96\pi} \left[2\Lambda r_+ - \frac{J^2}{2r_+^3} \right]^2. \tag{36}$$

Now the following limiting values of $H_2(r)$ are obtained from Eq. (35):

$$\text{if} \quad r \to r_+ \quad \text{then} \quad H_2(r) = 0,$$

$$\text{if} \quad r \to +\infty \quad \text{then} \quad H_2(r) = \frac{\Lambda^2}{24\pi} r^2 - D.$$

Therefore, using Eqs. (31) and (33), we have the most general expression for the regularized stress tensor in our gravitational background:

$$T^{\mu}_{\nu}=\begin{bmatrix} T^{\alpha}_{\alpha}(r)-g^{-1}(r)H_2(r) & 0 \\ 0 & g^{-1}(r)H_2(r)\end{bmatrix}+g^{-1}(r)\begin{bmatrix}-\beta & -\delta \\ \delta & \beta\end{bmatrix}, \tag{37}$$

or, substituting Eqs. (32) and (35), a more explicit expression is

$$T^{\mu}_{\nu}=\begin{bmatrix} \frac{\Lambda}{12\pi}+\frac{J^2}{16r^4}-\frac{1}{96\pi g(r)}\left[2\Lambda r-\frac{J^2}{2r^3}\right]^2+g^{-1}(r)D & 0 \\ 0 & \frac{1}{96\pi g(r)}\left[2\Lambda r-\frac{J^2}{2r^3}\right]^2-g^{-1}(r)D\end{bmatrix}+g^{-1}(r)$$
$$\times\begin{bmatrix}-\beta & -\alpha g^{3/2}(r)e^{-g^2(r)/4} \\ \alpha g^{3/2}(r)e^{-g^2(r)/4} & \beta\end{bmatrix}, \tag{38}$$

where the Achucarro-Ortiz black hole background (3), (4) and relations (13), (32), (35), and (36) have been used. In this expression, the only unknowns are the parameters α and β; we hope to determine them by imposing the third and fourth Wald axioms treating the Achucarro-Ortiz black hole as a Casimir system [1]. Two one-dimensional walls at a proper distance (between them) L are placed at points r_1 and r_2. The massless scalar field whose energy-momentum tensor we try to evaluate satisfies the Dirichlet boundary conditions on the walls, i.e., $\phi(r_1)=\phi(r_2)=0$.

We are now going to find the explicit form of the regularized stress tensor in the different vacua.

A. Boulware vacuum

In this vacuum there are no particles detected at infinity ($\mathcal{J}^+$) and the regularized stress tensor (38) should coincide at infinity with the sum of the standard Casimir stress tensor [1,2] in the Minkowski spacetime

$$T^{\mu}_{\nu}=\frac{\pi}{24L^2}\begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix} \tag{39}$$

and of the stress tensor of the gravitational field of the AdS_2 space

$$T^{\mu(\mathrm{AdS})}_{\nu}=\begin{bmatrix}-\frac{r^2}{2(-M+\Lambda r^2)} & 0 \\ 0 & \frac{r^2}{2(-M+\Lambda r^2)^2}\end{bmatrix}$$
$$=\frac{r^2}{2(-M+\Lambda r^2)}\begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix}, \tag{40}$$

since the Achucarro-Ortiz black hole is asymptotically an AdS_2 space.

The constants of integration α and β are evaluated by demanding that the regularized stress tensor given in Eq. (38) coincide at infinity, i.e., $r\rightarrow+\infty$, with the sum of the above-mentioned stress tensors (39) and (40).

Therefore we get

$$\alpha=0,\quad \beta=\left(\frac{\Lambda}{12\pi}+\frac{\pi}{24L^2}\right)g_{\mathrm{AdS}}(r)+\left(\frac{1}{2}-\frac{\Lambda^2}{24\pi}\right)r^2+D, \tag{41}$$

and the regularized stress tensor has been explicitly calculated. It can also be written as a direct sum:

$$T^{(\eta)\mu}_{\nu}=T^{\mu}_{\nu(\mathrm{gravitational})}+T^{\mu}_{\nu(\mathrm{boundary})}+T^{\mu}_{\nu(\mathrm{ANFG})}, \tag{42}$$

where η denotes that the regularized stress tensor has been calculated under the assumption that there are no particles (vacuum state) at infinity (Boulware vacuum). The first term denotes the contribution to the vacuum polarization due to the nontrivial topology in which the contribution of the trace anomaly is included, the second term denotes the contribution due to the presence of the two Dirichlet walls, and the third term denotes the contribution due to the asymptotically nonflat geometry (ANFG) of the Achucarro-Ortiz black hole.

The detected energy density, pressure, and energy at infinity ($r\rightarrow+\infty$) are given by

$$\rho=T^{(\eta)t}_{t}=-\frac{\pi}{24L^2}-\frac{1}{2\Lambda}, \tag{43}$$

$$p=-T^{(\eta)x}_{x}=-\frac{\pi}{24L^2}-\frac{1}{2\Lambda}-\frac{\Lambda}{12\pi}, \tag{44}$$

$$E(L)=\int_{r_1}^{r_2=r_1+L}\rho dr=-\frac{\pi}{24L}-\frac{1}{2\Lambda}L. \tag{45}$$

The corresponding Casimir force between the walls is not always attractive as expected:

$$F(L)=-\frac{\partial E(L)}{\partial L}=-\frac{\pi}{24L^2}+\frac{1}{2\Lambda}. \tag{46}$$

It is clear that the Casimir force is

(a) attractive,

$$L<\sqrt{\frac{\pi}{12}}\Lambda^{1/2}, \tag{47}$$

(b) zero,

$$L=\sqrt{\frac{\pi}{12}}\Lambda^{1/2}, \tag{48}$$

and (c) repulsive,

$$L>\sqrt{\frac{\pi}{12}}\Lambda^{1/2}. \tag{49}$$

B. Hartle-Hawking vacuum

In this vacuum the Achucarro-Ortiz black hole (3), (4) is in thermal equilibrium with an infinite reservoir of blackbody radiation at a temperature T which is equal to its Hawking temperature. The regularized stress tensor (38) should coincide with the following stress tensor:

$$T^{\mu}_{\nu}=\frac{\pi}{24L^2}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}+\frac{r^2}{2(-M+\Lambda r^2)}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}+\frac{\pi T^2}{6}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, \tag{50}$$

where the last term is the stress tensor for a two-dimensional black hole in thermal equilibrium at temperature T [18].

The constants of integration α and β are evaluated by demanding that the regularized stress tensor given in Eq. (38) coincide at infinity, i.e., $r\rightarrow+\infty$, with the sum of the above-mentioned stress tensors (39) and (40). Therefore we get

$$\alpha=0, \tag{51}$$

$$\beta=\left(\frac{\Lambda}{12\pi}+\frac{\pi}{24L^2}+\frac{\pi}{6}(T_H^{\mathrm{AdS}})^2\right)g_{\mathrm{AdS}}(r)+\left(\frac{1}{2}-\frac{\Lambda^2}{24\pi}\right)r^2+D \tag{52}$$

$$=\left(\frac{\Lambda}{12\pi}+\frac{\pi}{24L^2}+\frac{\Lambda}{24\pi}M\right)g_{\mathrm{AdS}}(r)+\left(\frac{1}{2}-\frac{\Lambda^2}{24\pi}\right)r^2+D, \tag{53}$$

and the regularized energy-momentum tensor has been explicitly calculated. It can also be written as a direct sum:

$$T^{(\upsilon)\mu}_{\nu}=T^{\mu}_{\nu(\mathrm{gravitational})}+T^{\mu}_{\nu(\mathrm{boundary})}+T^{\mu}_{\nu(\mathrm{ANFG})}+T^{\mu}_{\nu(\mathrm{bath})}, \tag{54}$$

where υ denotes that the regularized stress tensor has been calculated under the assumption that massless particles (blackbody radiation) are detected at infinity (toward $\mathcal{J}^{+}$) (Hartle-Hawking vacuum), and the fourth term in Eq. (54) denotes the contribution to the vacuum polarization due to the thermal bath at temperature T_H.

In this vacuum the asymptotically ($r\rightarrow+\infty$) detected energy density, pressure, and energy at infinity are given by

$$\rho=T_t^{(\eta)t}=-\frac{\pi}{24L^2}-\frac{1}{2\Lambda}-\frac{\Lambda}{24\pi}M, \tag{55}$$

$$p=-T_x^{(\eta)x}=-\frac{\pi}{24L^2}-\frac{1}{2\Lambda}-\frac{\Lambda}{24\pi}M-\frac{\Lambda}{12\pi}, \tag{56}$$

$$E(L)=\int_{r_1}^{r_2=r_1+L}\rho dr=-\frac{\pi}{24L}-\frac{1}{2\Lambda}L-\frac{\Lambda}{24\pi}ML. \tag{57}$$

The corresponding Casimir force between the walls is not always attractive as expected:

$$F(L)=-\frac{\partial E(L)}{\partial L}=-\frac{\pi}{24L^2}+\frac{1}{2\Lambda}+\frac{\Lambda}{24\pi}M. \tag{58}$$

It is clear that the Casimir force is

(a) attractive,

$$L<\pi\sqrt{\frac{\Lambda}{12\pi+\Lambda^2M}}, \tag{59}$$

(b) zero,

$$L=\pi\sqrt{\frac{\Lambda}{12\pi+\Lambda^2M}}, \tag{60}$$

and (c) repulsive,

$$L>\pi\sqrt{\frac{\Lambda}{12\pi+\Lambda^2M}}. \tag{61}$$

Thus, if the last condition is satisfied, the outer Dirichlet wall moves toward infinity. It can be studied as a moving mirror creating particles whose energy rate detected at infinity is given by the third term in Eq. (57):

$$\frac{dE}{dt}=\frac{\Lambda}{24\pi}ML=\frac{\pi L}{6}(T_H^{\mathrm{AdS}})^2. \tag{62}$$

This is the rate at which energy is radiated for the case of the massless two-dimensional field [44,45].

C. Unruh vacuum

In this vacuum an outward flux of radiation is detected at infinity. Thus, since the Achucarro-Ortiz black hole (3), (4) radiates and its spectrum distribution is thermal at the Hawking temperature T_H [46,47], the Unruh vacuum state is identified with the vacuum obtained after the Achucarro-Ortiz black hole has settled down to an ''equilibrium'' of temperature T_H. The regularized stress tensor (38) should now coincide at infinity with the following stress tensor:

$$T^{\mu}_{\nu}=\frac{\pi}{24L^2}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}+\frac{r^2}{2(-M+\Lambda r^2)}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}+\frac{\pi(T_H^{\mathrm{AdS}})^2}{12}\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}, \tag{63}$$

where the last term is the stress tensor for a radiating two-dimensional black hole which has settled down to an ''equilibrium'' of temperature T_H [18].

The constants of integration α and β are evaluated by demanding that the regularized stress tensor given in Eq. (38) coincide at infinity, i.e., $r\rightarrow+\infty$, with the sum of the above-mentioned stress tensors (39) and (40). Therefore we get

$$\alpha=\frac{\pi(T_H^{\mathrm{AdS}})^2}{12}g_{\mathrm{AdS}}^{-1/2}(r)e^{g^2_{\mathrm{AdS}}(r)/4}=\frac{\Lambda}{48\pi}Mg_{\mathrm{AdS}}^{-1/2}(r)e^{g^2_{\mathrm{AdS}}(r)/4}, \tag{64}$$

$$\begin{aligned}\beta&=\left(\frac{\Lambda}{12\pi}+\frac{\pi}{24L^2}+\frac{\pi}{12}(T_H^{\mathrm{AdS}})^2\right)g_{\mathrm{AdS}}(r)+\left(\frac{1}{2}-\frac{\Lambda^2}{24\pi}\right)r^2+D\\&=\left(\frac{\Lambda}{12\pi}+\frac{\pi}{24L^2}+\frac{\Lambda}{48\pi}M\right)g_{\mathrm{AdS}}(r)+\left(\frac{1}{2}-\frac{\Lambda^2}{24\pi}\right)r^2+D,\end{aligned} \tag{65}$$

and the regularized stress tensor has been explicitly calculated. It can also be written as a direct sum:

$$T^{(\xi)\mu}_{\nu}=T^{\mu}_{\nu(\mathrm{gravitational})}+T^{\mu}_{\nu(\mathrm{boundary})}+T^{\mu}_{\nu(\mathrm{ANFG})}+T^{\mu}_{\nu(\mathrm{radiation})}, \tag{66}$$

where ξ denotes that the regularized stress tensor has been calculated under the assumption that massless particles are detected at infinity due to the Hawking radiation of the Achucarro-Ortiz black hole (Unruuh vacuum), and the fourth term in Eq. (66) denotes the contribution to the vacuum polarization due to Hawking radiation at temperature T_H.

In this vacuum the asymptotically ($r\rightarrow+\infty$) detected energy density, pressure, and energy at infinity are given by

$$\rho=T^{(\eta)t}_{t}=-\frac{\pi}{24L^2}-\frac{1}{2\Lambda}-\frac{\Lambda}{48\pi}M, \tag{67}$$

$$p=-T^{(\eta)x}_{x}=-\frac{\pi}{24L^2}-\frac{1}{2\Lambda}-\frac{\Lambda}{48\pi}M-\frac{\Lambda}{12\pi}, \tag{68}$$

$$E(L)=\int_{r_1}^{r_2=r_1+L}\rho dr=-\frac{\pi}{24L}-\frac{1}{2\Lambda}L-\frac{\Lambda}{48\pi}ML. \tag{69}$$

The corresponding Casimir force between the walls is not always attractive as expected:

$$F(L)=-\frac{\partial E(L)}{\partial L}=-\frac{\pi}{24L^2}+\frac{1}{2\Lambda}+\frac{\Lambda}{48\pi}M. \tag{70}$$

It is clear that the Casimir force is

(a) attractive,

$$L<\pi\sqrt{\frac{2\Lambda}{24\pi+\Lambda^2M}}, \tag{71}$$

(b) zero,

$$L=\pi\sqrt{\frac{2\Lambda}{24\pi+\Lambda^2M}}, \tag{72}$$

and (c) repulsive,

$$L>\pi\sqrt{\frac{2\Lambda}{24\pi+\Lambda^2M}}. \tag{73}$$

As in the case of the Hartle-Hawking vacuum, if the last condition is satisfied the outer wall moves toward infinity. It can be studied as a moving mirror creating particles whose energy rate detected at infinity is given by the second term in Eq. (69):

$$\frac{dE}{dt}=\frac{\Lambda}{48\pi}ML=\frac{\pi L}{12}(T_H^{\mathrm{AdS}})^2. \tag{74}$$

This is the rate at which energy is radiated for the case of the massless two-dimensional field [44,45].

VI. EQUILIBRIUM AND COSMOLOGICAL CONSTANT

It is obvious that in the case that the net force which the Dirichlet walls experience turns out to be repulsive the system will be uninteresting since it will be decompactified as $L\rightarrow\infty$. On the other hand, if the net force exerted on the Dirichlet walls turns out to be attractive then the system inevitably will evolve in such a way that at some finite time the distance L will be of order of the Planck length, where the semiclassical analysis adopted here will no longer be valid. Therefore the case of a zero net force on the Dirichlet walls sounds the most interesting for our scenario.

The net force exerted on the Dirichlet walls can be evaluated using the Casimir force in any of the three vacua. It should be noted that for the cases of the Hartle-Hawking and Unruuh vacua the last term in Eqs. (58) and (70), respectively, should be removed. The reason is that in both vacua the forces acting on both sides of each Dirichlet wall due to the thermal bath or radiation, respectively, are the same, and thus their total contribution to the net force is zero. Therefore the net force that the Dirichlet walls experience is given by

$$F_{\mathrm{net}} = -\frac{\pi}{24L^2} + \frac{1}{2\Lambda}, \tag{75}$$

and on setting it equal to zero the distance L between the Dirichlet walls receives the value

$$L = \sqrt{\frac{\pi}{12}}\Lambda^{1/2}. \tag{76}$$

It is clear that the distance L between the Dirichlet walls is controlled by the value of the cosmological constant.

VII. CONCLUSIONS

In this paper we explicitly calculated in the Achucarro-Ortiz black hole background the regularized stress tensor of a massless scalar field satisfying Dirichlet boundary conditions on one-dimensional walls (Dirichlet walls). The regularized stress tensor is separately treated in the Boulware, Hartle-Hawking, and Unruh vacua. In all these vacua, expressions for the asymptotically detected energy, energy density, and pressure acting on the Dirichlet walls are obtained. The values of the above-mentioned quantities are all negative, exhibiting a violation of all energy conditions [48]. This ‘‘problem’’ is expected to take place in our scenario to some extent, since violations of some or all of the energy conditions appear as soon as scalar fields couple to gravity [49,50]. In the Hartle-Hawking and Unruuh vacua, the corresponding Casimir force is evaluated and proved, as expected, to be not always attractive: it can be attractive, repulsive, or zero according to the distance L between the Dirichlet walls. In contradistinction to what was known until now [30–33], in the Boulware vacuum, the Casimir force is also not always attractive. Additionally, we evaluated the net force exerted on the Dirichlet walls. It was easily demonstrated by imposing the condition of equilibrium on the Dirichlet walls, i.e., zero net force, that the distance between the one-dimensional walls is tuned by the cosmological constant.[1]

It would be very interesting for our scenario to be utilized in higher dimensions, and specifically in braneworlds. Of course, it is well known that the trace anomaly—which plays a key role in the technique presented here—is zero for odd-dimensional spacetimes. Therefore, only even-dimensional spacetimes should be considered. It should also be pointed out that our scenario is not directly applicable to higher even-dimensional spacetimes, since more conditions are required in order to completely fix the form of the renormalized stress tensor corresponding to the quantized scalar field. Indeed, there are a number of recent works that deal with the Casimir effect in different models of braneworlds [51–56]. These scenarios are more complicated than the one analyzed here; just to mention a few complications, the branes are located in the bulk space [57,58], not at points of the spacetime in which we live, and there exists a radion field which has to be stabilized [59].

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[1] The stability of the configurations can be checked by using Eq. (45) in the Boulware vacuum. It is easily derived that the configurations are unstable against small displacements. The same result can be derived by using Eqs. (57) and (69) in the Hartle-Hawking and Unruuh vacua, respectively, but the last term in these equations has to be dropped for the reason given in Sec. VI.

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