k-essence and the coincidence problem

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(Received 16 April 2003; published 16 July 2003)

k-essence has been proposed as a possible means of explaining the coincidence problem of the Universe beginning to accelerate only at the present epoch. We carry out a comprehensive dynamical systems analysis of the *k*-essence models given so far in the literature. We numerically study the basin of attraction of the tracker solutions and we highlight the behavior of the field close to sound speed divergences. We find that, when written in terms of parameters with a simple dynamical interpretation, the basins of attraction represent only a small region of the phase space.

DOI: 10.1103/PhysRevD.68.023512 PACS number(s): 98.80.Cq

I. INTRODUCTION

The search for an explanation for the observed acceleration of the Universe [1] is one of the most important challenges in contemporary cosmology. One usually assumes the existence of a dark energy component which breaks the strong energy condition, and the simplest model that one can build is that of introducing a cosmological constant. Yet, in order to obtain its domination today it has to be precisely set to an extremely small and so far unexplained value, one manifestation of the cosmological constant problem [2]. More generally, the fact that the dark energy overtakes dark matter at a recent epoch ($z<1$) when we cosmologists are able to observe it is known as the coincidence problem.

Allowing the dark energy to be dynamical may help to solve this issue. In this respect, a light scalar field, known as quintessence [3,4], has been proposed. The field is meant to slow-roll down its potential, with its potential energy acting analogously to that of early Universe inflation. A wide class of tracker models [4] features an attractor solution which roughly mimics the behavior of the dominant component of the Universe, rendering the evolution of the field fairly independent of its initial conditions. Unfortunately, in order to obtain quintessence domination today, the parameters of the potentials so far discussed also need a fine-tuning, and so as yet those models have not led to a compelling resolution of the coincidence problem.

More recently, models based on scalar fields with noncanonical kinetic energy [5], dubbed *k*-essence [6–9], have emerged. A subclass of models [7,8] feature a tracker behavior during radiation domination, and a cosmological-constant-like behavior shortly after the transition to matter domination. As long as this transition seems to occur generically for purely dynamical reasons, these models are claimed to solve the coincidence problem without fine-tuning.

In this paper we will analyze the models given in the literature so far [7,8]. We will study the size of the basin of attraction of their tracker solutions and comment on the fine-tuning of the parameters. We will also look at the behavior of a general *k*-essence field close to singularities corresponding to a diverging sound speed.

Throughout this article a prime denotes a derivative with respect to the argument of the function to which it is applied, and a dot denotes a derivative with respect to proper time. We assume $3/8\pi G=1$.

II. *k*-ESSENCE FORMALISM

In general *k*-essence is defined as a scalar field with noncanonical kinetic energy, but usually the models are restricted to the Lagrangian

$$\mathcal{L}_{\rm k}\equiv K(\phi)\tilde{p}(X), \tag{1}$$

where $K(\phi)>0$ and $X=\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$. We note that the definition includes quintessence models (in this paper meaning scalar fields with a canonical kinetic term). Using the perfect fluid analogy, the pressure and the energy density are given by

$$p_{\rm k}(\phi,X)=K(\phi)\tilde{p}(X), \tag{2}$$

$$\varepsilon_{\rm k}(\phi,X)=K(\phi)\tilde{\varepsilon}(X), \tag{3}$$

where

$$\tilde{\varepsilon}(X)=2X\tilde{p}'(X)-\tilde{p}(X). \tag{4}$$

Following Refs. [7,8], we set $K(\phi)\equiv 1/\phi^2$, define a new variable $y\equiv 1/\sqrt{X}$ and reexpress $\tilde{p}(X)$ as $\tilde{p}[X(y)]\equiv g(y)/y$. In this case the pressure and the energy density become

$$p_{\rm k}(\phi,y)=\frac{g(y)}{\phi^2 y}, \tag{5}$$

$$\varepsilon_k(\phi,y) = -\frac{g'(y)}{\phi^2}. \tag{6}$$

We assume $\varepsilon_k(\phi,y)>0$, hence $g'(y)<0$. The equation of state parameter and the effective sound speed are given by

$$w_k(y) = -\frac{g(y)}{yg'(y)}, \tag{7}$$

$$c_{sk}^2(y) = \frac{g(y)-yg'(y)}{y^2 g''(y)}. \tag{8}$$

We also assume $w_k(y)>-1$ and $c_{sk}^2(y)>0$ which implies $g''(y)>0$. As a result, $g(y)$ must be a convex and decreasing function of y.

From now on, we consider a flat Robertson-Walker universe defined by the metric

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2. \tag{9}$$

In this case, the Euler-Lagrange equation for the *k*-essence field is

$$\tilde{\varepsilon}'(X)\ddot{\phi} + 3H\tilde{p}'(X)\dot{\phi} + \frac{K'(\phi)}{K(\phi)}\tilde{\varepsilon}(X) = 0, \tag{10}$$

where $H=\dot{a}/a$. Then, if the Universe is filled with another fluid with energy density ε_f and equation of state parameter w_f constant one can find the following system of equations in terms of the independent variables y and Ω_k:

$$\frac{dy}{dN} = \frac{\sqrt{-8g'(y)}[r(y)-\sqrt{\Omega_k}]}{yg''(y)}, \tag{11}$$

$$\frac{d\Omega_k}{dN} = 3\Omega_k(1-\Omega_k)[w_f - w_k(y)], \tag{12}$$

where $N\equiv\ln(a/a_0)$, $\Omega_k\equiv\varepsilon_k/(\varepsilon_k+\varepsilon_f)$ and

$$r(y)\equiv\frac{3[g(y)-yg'(y)]}{\sqrt{-8g'(y)}}>0. \tag{13}$$

Here, we have assumed that $\dot{\phi}>0$. Therefore, for $0<y$ and $0<\Omega_k<1$, the dynamics is completely described by trajectories in the y-Ω_k plane. As long as $g''(y)\neq 0$ the system is well defined. As we can see, (y_s,Ω_{ks}) is a stationary point if $w_k(y_s)=w_f$ and $r^2(y_s)=\Omega_{ks}$. As shown in Ref. [8] this is a stable point—and therefore corresponds to a perfect tracking $(w_k=w_f)$ of the dominant fluid—if $c_{sk}^2(y_s)>w_k(y_s)$.

III. SOUND SPEED DIVERGENCE

As we will see, for the particular class of models we will analyze, we have $g''(y_c)=0$ for some y_c, which implies that $\tilde{\varepsilon}'(y_c)=0$ and that the sound speed diverges at y_c. In that case, from Eq. (11) we see that at $y=y_c$ there is a unique possible value $\Omega_k=\Omega_{kc}$ given by the constraint equation

$$\Omega_{kc} = r^2(y_c). \tag{14}$$

This means that the phase space y-Ω_k is cut into (at least) two parts separated by the line $y=y_c$ which is not allowed by the model except at $\mathbf{S_0}\equiv(y_c,\Omega_{kc})$. To study the dynamics close to this line we use the expansion $y=y_c+\delta y$ with $\delta y/y_c\ll 1$, compute Eq. (11) to $\mathcal{O}(\delta y^2)$ and find the equation

$$\frac{d\delta y}{dN} \simeq \frac{C_1}{\delta y} + C_2\delta y, \tag{15}$$

where

$$C_1 = \frac{\sqrt{-8g'(y_c)}[r(y_c)-\sqrt{\Omega_k}]}{y_c g'''(y_c)} \tag{16}$$

$$C_2 = \frac{3[w_k(y_c)-1]}{4} - \frac{2[r(y_c)-\sqrt{\Omega_k}]}{y_c\sqrt{-8g'(y_c)}}. \tag{17}$$

If $g'''(y_c)\neq 0$, Eq. (15) allows us to study the behavior of the field close to the line $y=y_c$, and unless $\Omega_k=r^2(y_c)$ the first term dominates. For the particular class of models we will analyze we have $g'''(y_c)>0$, and therefore if $\Omega_k<r^2(y_c)$, hence $C_1>0$, the solution moves away from the line, whereas if $\Omega_k>r^2(y_c)$ the solution ceases to exist (in that it hits the singularity y_c) within a finite time $\Delta N\simeq\delta y^2/2|C_1|$ as it approaches the line (this has been checked numerically). We define $\mathbf{S}_+\equiv\{(y_c,\Omega_k)|\Omega_k<r^2(y_c)\}$ as the segment of the singularity from which some trajectories spontaneously emerge and $\mathbf{S}_-\equiv\{(y_c,\Omega_k)|\Omega_k>r^2(y_c)\}$ as the segment of the singularity on which some trajectories abruptly end. We can also use Eq. (15) to determine the nature of the perturbation δy at the regular point $\mathbf{S_0}=(y_c,\Omega_{kc})$. In that case the equation simplifies to give

$$\frac{d\delta y}{dN} \simeq \frac{3}{4}[w_k(y_c)-1]\delta y. \tag{18}$$

Depending on the value of $w(y_c)$ this solution either grows or decreases exponentially fast. In the cases we will study here, we have $w_k(y_c)<1$ and therefore δy decreases.

The existence of this singular behavior means that a diverging sound speed leads to serious problems. In some situations it may well be possible to argue that the theory is valid up to a certain cutoff which excludes the singularity, but as we will show this is not the case for the cosmologically realistic models proposed so far [7,8]. Instead, in these situations we must deal with the singular regions as we meet them.

IV. ANALYSIS OF TWO MODELS

k-essence models can possess many different attractor solutions [6–8], especially trackers which perfectly mimic the dominant component of the Universe and attractors with negative equation of state leading to domination of the field. By choosing an appropriate function $g(y)$, it is possible to build a model with a certain number of attractor solutions which can feature some interesting dynamical properties. For

TABLE I. This table shows the sign of some variables as a function of X for the two models analyzed in Sec. IV. For each model the values of X_c, X_w and $X_\star$ are given in the text. The symbol "∞" stands for a diverging value.

	$(0,X_c)$	X_c	(X_c,X_w)	X_w	$(X_w,X_\star)$	$X_\star$	$X_\star<X$
ε_k	+	+	+	0	−	−	−
w_k+1	+	+	+	∞	−	0	+
c_{sk}^2	+	∞	−	−	−	0	+

instance, a stable tracker solution during radiation domination (**R**) could render the late-time evolution of the field fairly independent of its initial conditions. Then, the lack of such a solution during matter domination would force the field to reach another pseudo-attractor with $w_k \approx -1$ leading to the *k*-essence domination attractor (**K**). This possibility was discussed in Refs. [7,8] and in that sense it is possible to find a model which could solve the coincidence problem: we cosmologists would observe the acceleration of the Universe today because we happen to appear soon after the onset of matter domination which leads to the formation of structures—and human beings. In this section we analyze the models proposed so far which are built in order to fulfill this principle [7,8]. We study their sensitivity to initial conditions and comment on the fine-tuning involved. Throughout the discussion we will refer to and make use of the parameters first introduced in Refs. [7,8].

A. Model 1 (Ref. [7])

Following the classification scheme introduced in Ref. [8], this first model is of type (B_r). It is defined by

$$\widetilde{p}(X) \equiv -2.01 + 2\sqrt{1+X} + 3\times 10^{-17} X^3 - 10^{-24} X^4. \qquad (19)$$

As we will see, it appears to contain a number of problematic issues which on the face of it contradict what is stated in Ref. [7]. For example for some ranges of X we see that it is possible to have $\varepsilon_k<0$, $w_k<-1$ and $c_{sk}^2<0$. Moreover we find that c_{sk}^2 and w_k diverge at $X=X_c\approx 1.6\times 10^7$ and $X=X_w\approx 2.1\times 10^7$ respectively. These properties are summarized in Table I, where we have also introduced $X_\star \approx 2.3\times 10^7$ as the value of X for which the sound speed vanishes. From Eqs. (7) and (8), this also corresponds to the value where the equation of state parameter $w_k=-1$.

In what follows we only analyze the dynamics for $X<X_c$, which corresponds to $y>y_c\approx 2.4\times 10^{-4}$, since in this case $\varepsilon_k>0$, $w_k>-1$ and $c_{sk}^2>0$ and also the solution which "solves" the coincidence problem is in this region of the phase diagram.

Since our concern over the dependency on initial conditions really only relates to the period of radiation domination, we assume that the Universe is filled with radiation $(w_f=1/3)$. We have run simulations in order to determine the size of the basin of attraction of the tracker **R**. In Fig. 1 we have plotted the phase diagram for the *k*-essence field during radiation domination. The solid and long-dashed lines are the limiting solutions which separate the different types of trajectories. The vertical dashed line is the singularity $y=y_c$. As explained in Sec. II, it can be divided in two parts: $\mathbf{S}_+$ (below $\mathbf{S_0}$) and $\mathbf{S}_-$ (above $\mathbf{S_0}$). All the trajectories contained within the long-dashed line originate from the same point **0** and the rest originate from segment $\mathbf{S}_+$. The solid lines separate the trajectories ending on $\mathbf{S}_-$, the ones reaching the tracker solution **R** and the ones reaching the *k*-essence domination attractor **K**. (Having a look at the trajectories plotted in Fig. 2 can be useful for understanding the dynamics, although this figure describes the model we study next.) As we can see, the basin of attraction of **R** (shaded region) is very small and most of the solutions reach **K**. However as this is still during radiation domination, they reach the *k*-essence domination attractor too early [typically after about 12 *e*-foldings, i.e. after an increase of the scale factor $a(t)$ by about a factor 10^5] to be associated with the onset of matter-radiation equality and to be a candidate for dark energy. We also note that in a small region of the phase diagram the solutions cease to exist after a finite time as they reach $\mathbf{S}_-$, the part of singularity $y=y_c$ above $\mathbf{S_0}$.

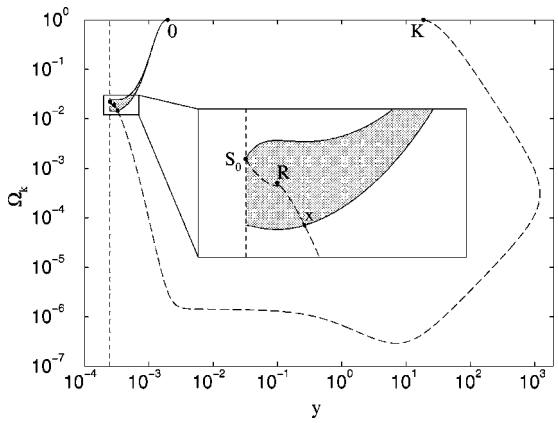

FIG. 1. Basin of attraction (shaded region) of the tracker solution **R** during radiation domination for the first model analyzed in Sec. IV. The points **K** and **x** denote the *k*-essence domination attractor and the saddle point respectively, defined in Ref. [8]. The solid lines are the limiting solutions which demarcate the basin of attraction and the long-dashed line separates the trajectories originating from **0** and the ones originating from the singularity. The vertical dashed line is the singularity $y=y_c$. $\mathbf{S_0}$ denotes the point (y_c,Ω_{kc}).

Therefore, when analyzed in detail, this first model exhibits a rather different phase diagram structure from that sketched in Fig. 3 of Ref. [8] to generically describe models of class (B_r). Unfortunately a complete comparison is not possible as the precise equations used to generate that figure are not disclosed, so that we do not know to which *k*-essence model the figure corresponds (nor whether the model has an explicit Lagrangian description or is only of a more phenomenological nature).

B. Model 2 (Ref. [8])

This second model is defined by

$$\widetilde{p}(X) \equiv -2.05 + 2\sqrt{1+f(X)}, \qquad (20)$$

where

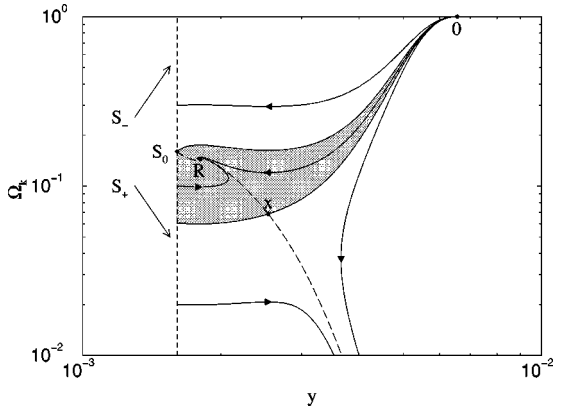

FIG. 2. Basin of attraction (shaded region) of the tracker solution **R** during radiation domination for the second model analyzed in Sec. IV. The explanations are the same as for Fig. 1. $\mathbf{S}_+$ and $\mathbf{S}_-$ denote the parts of the singularity from where some trajectories originate and end respectively. A few trajectories have been plotted.

$$f(X) \equiv X - 10^{-8}X^2 + 10^{-12}X^3 - 10^{-16}X^4 + 10^{-20}X^5 - 10^{-24}X^6/2^6. \quad (21)$$

As in the previous example, this model is not always well defined. For $X > X_{\max} \simeq 6.3 \times 10^5$ we have $1 + f(X) < 0$ and therefore $\tilde{p}(X)$ becomes ill-defined. Moreover, for $X < X_{\max}$, this model has the same problems as those of the model we have analyzed above. As before, its properties are summarized in Table I, where this time $X_c \approx 3.9 \times 10^5$, $X_w \approx 5.0 \times 10^5$ and $X_\star \approx 5.3 \times 10^5$.

As in the first example, this is a model of type (B_r) and for $y > y_c \approx 1.6 \times 10^{-3}$ its phase space has a similar structure to that shown in Fig. 1. In Fig. 2 we have plotted the basin of attraction of the tracker solution **R** during radiation domination for this second model. In order to help in understanding the dynamics a few trajectories have been added. Again, we see that for most of the initial conditions the field does not reach the tracker, but instead it reaches either an early *k*-essence domination solution **K** or the singularity on $\mathbf{S}_-$. Therefore, this second model suffers the same fine-tuning as that of the first example.

V. DISCUSSION

As introduced in Refs. [7,8], the idea of *k*-essence seems appealing. Current models of quintessence suffer in general because of the fine-tuning of the potential parameters to account for the fact that the field has only recently started dominating. *k*-essence was introduced as an extension of quintessence models by taking into account noncanonical kinetic terms. In a subclass of these models [7,8] a tracker behavior occurs during radiation domination, and a cosmological-constant-like behavior shortly after the transition to matter domination. Since the *k*-essence field seems to change its behavior generically for purely dynamical reasons, these models could be claimed to solve the coincidence problem without fine-tuning of the initial conditions. In that sense they are more natural than quintessence models which rely on a different, nonobvious, scale for the transition to occur.

In this short paper, we have addressed in a bit more detail the question over the nature of this attractor solution during radiation domination which is meant to avoid a fine-tuning of the initial conditions. We have numerically solved the evolution equations for the *k*-essence fields and written the results in terms of the physically motivated parameters y and Ω_k, following Refs. [7,8]. The key result we have found is that the basin of attraction for the tracker solution appears to be very small compared to the basin of attraction for the *k*-essence domination solution and therefore it cannot be seen as equivalent to that of quintessence models. In other words, for almost all initial conditions, the system would evolve rapidly into *k*-essence domination. It would have done so way before matter-radiation equality and in that sense, the required behavior of *k*-essence can only hold for a specific subset of initial conditions. If these turn out to be an important set of conditions then *k*-essence can be thought of as providing an elegant way of obtaining the acceleration we see today. However, if there is no particular reason for choosing such initial conditions, then we believe that *k*-essence suffers from the same fine-tuning issues that plague quintessence models.

With regard to this issue, in Refs. [7,8] the authors argue that the basin of attraction of the radiation attractor of the two models studied in this paper is compatible with equipartition. This particular initial condition may be the case should the *k*-essence or quintessence field be associated with one of the many fields produced at the end of a period of inflation. However, it need not be the case, and if it was not, equipartition would not help in choosing initial conditions. Indeed, equipartition is generally applied to systems where particle production occurs after the decay of the inflaton, whereas in the case of vacuum energy the initial condition may well have been set by early Universe physics (see for example Ref. [10]). In any case, both basins of attraction are so small that their main parts do not overlap and therefore there remains a fine-tuning issue even when assuming equipartition.

We have also shown that the models proposed so far feature a singularity associated with a diverging sound speed. This leads to problems like the sudden disappearance of some field trajectories. The presence of this singularity could be avoided by adding an extra term to the functions given by Eqs. (19) and (21), but our attempts to do so have led to the creation of a second radiation attractor which goes against the goal of having a model for which the late-time behavior is independent of the initial conditions. Another way of solving this problem would be to argue that the theory is valid up to a certain cutoff which excludes the singularity, but as long as the radiation tracker is very close to the singularity this argument cannot be applied.

In this paper we have dealt with the specific Lagrangians given in Refs. [7,8] at face value. Those Lagrangians have a complicated form which is not appealing, and it is disappointing that so far it has not proved possible to produce much simpler models, as one would expect to be able to if the desired triggering behavior really is generic to *k*-essence. The particular nature of the models makes it hard to assess

how difficult it is to build a working example, for instance quite how much fine-tuning is required to establish the time needed between matter-radiation equality and *k*-essence domination, and how easily one can avoid the presence of many different attractors during radiation domination. Unfortunately, we are not aware of any particular particle physics motivated models which would deliver the Lagrangians of the two models, and, as with quintessence, we believe that a realistic model of *k*-essence remains a challenge which has to be met.

ACKNOWLEDGMENTS

M.M. was supported by the Fondation Barbour, the Fondation Wilsdorf, the Janggen-Pöhn-Stiftung and the ORS, and A.R.L. in part by the Leverhulme Trust. We thank Mark Trodden for many discussions relating to *k*-essence.

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