Extensive entropy bounds

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It is shown that, for systems in which the entropy is an extensive function of the energy and volume, the Bekenstein and the holographic entropy bounds predict new results. More explicitly, the Bekenstein entropy bound leads to the entropy of thermal radiation (the Unruh-Wald bound) and the spherical entropy bound implies the "causal entropy bound." Surprisingly, the first bound shows a close relationship between black hole physics and the Stephan-Boltzmann law (for the energy and entropy flux densities of the radiation emitted by a hot blackbody). Furthermore, we find that the number of different species of massless fields is bounded by $\sim 10^4$.

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According to classical general relativity, ordinary matter that crosses the event horizon will disappear into the spacetime singularity of the black hole. In 1971, Wheeler raised the question: what happens to the entropy initially present in the matter? It seems that there is no gain of ordinary entropy in the universe, to compensate for the loss of entropy of the matter that has been absorbed by the black hole. Therefore, the second law of thermodynamics is violated in this process.

Bekenstein [1] found a way out of this paradox by introducing the notion of black hole entropy. He assigned an entropy S_{BH} that is proportional to the horizon surface area of the black hole. After the discovery of Hawking radiation, this entropy has been elevated to the status of a physical theory. Furthermore, Bekenstein proposed to replace the ordinary second law of thermodynamics by the *generalized second law* (GSL): The generalized entropy, $S_g \equiv S + S_{BH}$, of a system consisting of a black hole and ordinary matter (with entropy S) never decreases with time (for an excellent review on the thermodynamics of black holes and the validity of the GSL, see Wald [2]).

The GSL not only resolves the difficulty emphasized by Wheeler but also imposes entropy bounds to hold for arbitrary systems. The first entropy bound was proposed by Bekenstein [3] more than 20 years ago. He considered a *gedanken-experiment* such that one lowers adiabatically a spherical box of radius R toward a black hole (Geroch process). The box is lowered from infinity where the total energy of the box plus matter contents is E. It was shown [3] that the entropy S of the box must obey (throughout the paper $c = k_B = 1$)

$$S \leqslant \frac{2\pi RE}{\hbar}, \tag{1}$$

in order to preserve the GSL.

The derivation of Eq. (1) in [3] was criticized by Unruh and Wald [4,5] who have argued that, since the process of lowering the box is a quasistatic one (and therefore can be considered as a sequence of static-accelerating boxes), the box should experience a buoyant force due to the Unruh radiation [6]. Describing the acceleration radiation as a fluid, they have shown that this buoyant force alters the work done by the box such that no entropy bound in the form of Eq. (1) is necessary for the validity of the GSL. A few years ago, Pelath and Wald [7] gave further arguments in favor of this result.

Bekenstein [8,9], on the other hand, argued that, only for very flat systems, the Unruh-Wald effect may be important. Later on, he has shown [10] that, if the box is not almost at the horizon, the typical wavelengths in the radiation are larger than the size of the box and, as a result, the derivation of the buoyant force from a fluid picture is incorrect. The question of whether the Bekenstein bound follows from the GSL via the *Geroch process* remains controversial (see [2,11–13]). However, as it was shown by Bousso [14] (see the following paragraphs), there is another *link* connecting the GSL with the Bekenstein bound.

Susskind [15] has shown, by considering the conversion of a system to a black hole, that the GSL implies a spherical entropy bound

$$S \leqslant \frac{1}{4l_p^2} A, \tag{2}$$

where S is the entropy of a system that can be enclosed by a sphere with area A. A few years later, Bousso [16,17] had found an elegant way to generalize Eq. (2) and write it in a covariant form. He proposed the *covariant entropy bound*: "the entropy on any light-sheet $L(B)$ of a surface B will not exceed the area of B." That is,

$$S[L(B)] \leqslant \frac{A(B)}{4l_p^2}, \tag{3}$$

where the light-sheet $L[B]$ is constructed by the light rays that emanate from the surface B and are not expanding (for an excellent review see Bousso [17]).

When a matter system with initial entropy S falls into a black hole, the horizon surface area increases at least by $4l_p^2 S$ due to the GSL. This motivated Flanagan, Marolf and Wald [18] to generalize Eq. (3) into the following form:

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$$S[L(B,B')]\leq\frac{A(B)-A(B')}{4l_p^2},\qquad(4)$$

where $L(B,B')$ is a light-sheet which starts at the cross section B and cuts off at the cross-section B' before it reaches a caustic.

Unlike the controversial issues regarding the relationship between the GSL and Eq. (1), the entropy bounds in Eqs. (2), (3), (4) are closely related to the GSL. However, very recently, Bousso [17] has shown that the Bekenstein entropy bound follows from Eq. (4) for any isolated, weakly gravitating system. Hence, even though it is not clear whether quantum effects should be taken into consideration in the derivation of Eq. (1) (via the Geroch process), there is a strong link between the GSL and the Bekenstein bound.

In the following, we provide another link connecting the bound (1) with the entropy of thermal radiation and the Stephan-Boltzmann law. In our derivation, we consider systems in which the entropy density is a function of the energy density. Later on, we show that for such systems, the spherical entropy bound (2) yields the *causal entropy bound* proposed by Brustein and Veneziano [19] and independently by Sasakura [20]. We conclude that our results provide universal upper bounds for extensive systems.

Consider an isolated spherical box[1] of size R and volume $V=(4\pi/3)R^3$. Let us denote by $S(E,V)$ the *maximum* entropy of the box under the condition that S is an extensive function of E and V. Bekenstein's bound in terms of E and V is given by

$$S(E,V)<\eta\frac{EV^{1/3}}{\hbar},\qquad(5)$$

where $\eta=(6\pi^2)^{1/3}$.

Since the maximum entropy, S(E,V), will preserve the extensivity property of entropy, it can be written as follows:

$$S(E,V)=VF\left(\frac{E}{V}\right),\qquad(6)$$

where F is some function of the energy density, $\varepsilon\equiv E/V$. Equation (6) is equivalent to Euler's theorem on homogeneous functions[2]:

$$E\frac{\partial S}{\partial E}+V\frac{\partial S}{\partial V}=S.\qquad(7)$$

Now, Eq. (1) and Eq. (6) imply a bound on F:

$$F(E/V)<\frac{\eta E}{V^{2/3}\hbar}.\qquad(8)$$

[1]Throughout the paper we shall assume a spherical symmetry even though it is not always necessary.

[2]We assume the case where there are no other thermodynamic functions such as an electric or chemical potential.

In order to compare between dimensionless quantities, let us multiple Eq. (3) by l_p^3 and define the following dimensionless quantities:

$$x\equiv\frac{l_p^3E}{\epsilon_pV},\quad y\equiv\frac{V^{1/3}}{l_p},\quad\text{and}\quad f(x)\equiv l_p^3F\left(\frac{E}{V}\right),\qquad(9)$$

where ϵ_p and l_p are the Planck energy and the Planck length, respectively. In these notations, Eq. (3) can be written as

$$f(x)<\eta xy.\qquad(10)$$

However, x and y can be considered as two independent parameters. Therefore, let us fix x and take y to its minimal value. Reducing y implies reducing *both* R and E since x is kept constant (actually E decreases faster than R). Hence, the minimal value of y can be obtained by the requirement $R>\hbar/E$ (otherwise, the energy will leak out of the box). In terms of x and y, it means $y>x^{-1/4}$. Thus, taking $y\sim x^{-1/4}$, we find that

$$f(x)<x^{3/4},\qquad(11)$$

where from this point we will stress functional dependence, while ignoring numerical factors.

By substituting this condition in Eq. (6), we obtain the following "extensive entropy bound":

$$S(E,V)<\frac{E^{3/4}V^{1/4}}{\hbar^{3/4}}=\left(\frac{ER}{\hbar}\right)^{3/4}.\qquad(12)$$

The above result, by itself, is not surprising. For example, consider a gas of radiation at temperature T that is confined in the box. The energy and the entropy are given by the Stephan-Boltzmann law (neglecting corrections due to the discreteness of modes)

$$E\sim n_sR^3T^4\quad\text{and}\quad S\sim n_sR^3T^3,\qquad(13)$$

where n_s is the number of different (non-interacting) species of particles in the gas. Hence, in terms of E and R, the entropy is proportional to

$$S\sim n_s^{1/4}\left(\frac{ER}{\hbar}\right)^{3/4}.\qquad(14)$$

That is, the entropy of thermal radiation saturates Eq. (12). It is a good guess that no other system has more entropy, because the rest mass of ordinary particles only enhances gravitational instability without contributing to the entropy. Thus, the bound (12) is understandable.

However, there are three points about Eq. (12) that are very interesting and somewhat surprising: First, one does *not* have to define unconstrained thermal radiation to be the maximum entropy system (as did, for example, Unruh and Wald [4,5] and Pelath and Wald [7]). It comes out that the entropy of extensive systems is no higher if one assumes Bekenstein's bound.

Second, the GSL leads to Bekenstein's bound and extensivity leads to a bound proportional to the thermal radiation entropy. That is, the GSL implies the Stephan-Boltzmann law. Boltzmann and other physicists in his time would have never imagined that one would be able to obtain the thermal radiation entropy from black hole physics.[3] Let us take this moment to mention that both the black hole entropy formula, $S \propto A$ (A is the horizon surface area), and the Stephan-Boltzmann formula, $u \propto T^3$ and $s \propto T^4$ (u and s are the energy and entropy flux densities of the radiation emitted by a hot blackbody), can be derived purely by classical thermodynamics [21]. This shows another similarity between the physics of black holes and blackbodies.

Third, the species problem: one of the objections to *all* kinds of entropy bounds is that one can take n_s in Eq. (14) to be arbitrarily large. Consider, for example, the spherical entropy bound. In order for S in Eq. (14) to become greater than $A/4l_p^2$, one has to take $n_s > A/l_p^2$. Of course, we have no evidence (experimental or string theoretical) that n_s can run into such a high number, as would be required to violate the bound. That is, one can always hold the position that the bound is telling us about the world as it is, not as it might be in the imagination of a physicist who needs counterexamples. Furthermore, if the number of species grows, one can raise the question whether interactions will not nullify the assumption of ''free particles.''

However, the species problem manifests in a much more conspicuous form in the extensive bound (12). This bound implies that $n_s^{1/4}$ must be of order unity. That is, n_s cannot be much greater than 10^4. This number is much smaller than A/l_p^2 and it raises the question whether there are more realistic bounds on the number of species in nature. Since the arguments that lead to Eq. (12) include the assumption that the minimum value of R is approximately the Compton wavelength $\hbar/E$, we could not obtain the exact dimensionless numerical factor that should be added to Eq. (12). This numerical factor would have provided an exact bound on n_s.

In the above considerations, it was assumed that the system does not exceed Bekenstein's entropy bound. Let us now instead consider the the relationship between the spherical entropy bound (2) and the extensivity property of the entropy function $S(E,V)$. As we will see in the following, this relationship yields the ''causal entropy bound'' [19] which scales as $\sqrt{EV}$. We will first obtain this result by a simple heuristic argument, and then we will prove it rigorously.

Consider a box of size R (volume R^3) with energy E. According to the holographic bound (2), the entropy of the box cannot exceed $\sim R^2/l_p^2$. Now, consider N^3 (N is an integer) identical boxes arranged in a much bigger box of size NR (volume N^3R^3). If the interactions between the boxes are negligible and N is not too big (i.e. the big box is not a black hole), the entropy of the big box is N^3S, where S is the entropy of a single box. However, by applying the holographic bound for the big box, we get

$$N^3 S < \frac{(NR)^2}{l_p^2} \tag{15}$$

or equivalently

$$S < \frac{R^2}{N l_p^2}. \tag{16}$$

As it was expected, the holographic bound implies a tighter bound for ''non-gravitating'' systems.

Now, the total mass (energy) of the big box, N^3E, should be smaller than the size of the box NR (the big box is not a black hole). Therefore, the maximum possible value of N is of the order $\sim(\epsilon_p R/l_p E)^{1/2}$. By supplementing this in Eq. (16) we get

$$S \lesssim \frac{R^2}{\hbar} \times \left(\frac{l_p E}{\epsilon_p R}\right)^{1/2} \propto \sqrt{EV}, \tag{17}$$

where V is the volume of the small box.

The above arguments clarify why for extensive systems the holographic principle predicts the bound (17). We shall now prove this bound in a more formal way.

Theorem: Denote by $S(E,V)$ the maximum entropy that an isolated spherical system with energy E and volume V can have, under the condition that the entropy is distributed uniformly. The spherical entropy bound then implies that $S(E,V) \lesssim \sqrt{EV}$.

The entropy is distributed uniformly if and only if it can be written in the form given in Eq. (6). On the other hand, the spherical entropy bound implies

$$S(E,V) < \frac{V^{2/3}}{l_p^2}. \tag{18}$$

Therefore, the bound (2) leads to the result

$$F\left(\frac{E}{V}\right) < \frac{1}{l_p^2 V^{1/3}}, \tag{19}$$

where F is defined in Eq. (6). In terms of the dimensionless quantities which are defined in Eq. (9), the above inequality can be written in the form

$$f(x) < y^{-1}. \tag{20}$$

However, the two dimensionless parameters x and y^{-1} can be considered as independent. Therefore, one can keep x constant and take y^{-1} to its minimum value. The minimum value of y^{-1} occurs when the size of the system $V^{1/3}$ becomes comparable with its energy. That is, when $x = y^{-2}$. Hence,

$$f(x) < \sqrt{x}. \tag{21}$$

Equation (21) provides a proof for the theorem above. It shows a close relationship between the holographic principle and the causal entropy bound obtained by Brustein and Veneziano [19] and independently by Sasakura [20]. In [19] the causal entropy bound is defined covariantly and, hence, it is much more general then our derivation. Note, however, that for weakly gravitating systems, the Bekenstein bound is

[3]In some way, it also gives further evidence of Bekenstein's identification of black hole entropy with the horizon area.

tighter than the causal bound so that the later may be useful only for strongly gravitating systems. It may seem that the causal bound does not imply that the number of fundamental degrees of freedom is related to the area surfaces in space-time. However, from our derivation of the causal bound (based on the spherical bound), we learn that the causal bound can be incorporated into a holographic world.

In conclusion, in this paper we considered the applications of the entropy bounds (1), (2) into extensive systems. It was shown that extensivity provides links between different entropy bounds. One of the main results was the derivation of a bound proportional to the entropy of thermal radiation from black hole physics. In the future, we hope to generalize the results to charged and rotating systems.

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