de Sitter waves and the zero curvature limit

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We show that a particular set of global modes for the massive de Sitter scalar field (the de Sitter waves) allows us to manage the group representations and the Fourier transform in the flat (Minkowskian) limit. This is in opposition to the usual acceptance based on a previous result, suggesting the appearance of negative energy in the limit process. This method also confirms that the Euclidean vacuum, in de Sitter spacetime, is preferred as far as one wishes to recover ordinary quantum field theory in the flat limit.

I. INTRODUCTION

A major issue in the resurgence of de Sitter (dS) space physics motivated by inflation scenarios [1,2], astronomical observation [3], dS conformal field theory (CFT) correspondence [4–7], and the study of a simple maximally symmetric space with nonvanishing curvature, concerns the status of a "preferred" vacuum state for the associated quantum field theory (QFT). The absence of a global timelike Killing vector field in de Sitter space (nonstationary) excludes the "natural" choice characterized by the spectrum of a Hamiltonian operator unlike the Minkowski case. The presence of a maximal symmetry group does not get rid of this problem: there exists a family of inequivalent vacua which are all invariant under the dS group [8–10].

Nevertheless, thanks to this group, one can study the limit at vanishing curvature owing to the method of group contraction which allows us to follow the unitary irreducible representations (UIR) in that limit. It has been shown [11] that the representations of the de Sitter group associated with the massive scalar field, i.e., the principal series of SO(1,4), contract (in the zero curvature limit) toward the direct sum of two UIRs of the Poincaré group associated, respectively, with positive and negative frequencies massive scalar fields: namely,

$$D_\nu \rightarrow \mathcal{P}(+m) \oplus \mathcal{P}(-m). \qquad (1)$$

This result could appear as somewhat confusing since it suggests that the curvature is in some sense responsible for the emergence of negative frequency modes in QFT. This is all the more disturbing since a recent paper shows that these modes necessarily occur in the covariant quantization of the minimally coupled scalar field [12]. Since, on the level of two-point functions, the flat limit seems to work perfectly well, it has been argued that group representation contractions were not adapted for the study of QFT [13]. Attempts have been made in replacing SO(1,4) by the de Sitter

"causal semigroup" which contracts toward the Poincaré causal semigroup [14]. In view of the decisive role played by group theory in ordinary QFT and in defining de Sitter space objects as mass or spin, it is really frustrating that one cannot manage the group representation in the flat limit process. In this paper we propose to amend this drawback.

The Euclidean vacuum has been studied before and singled out by analyticity requirements [13,15], flat space behavior, or further reasons listed in [16]. Although the Euclidean vacuum seems to be favored, it remains sensible to use the whole vacua family, for instance, as tools in order to investigate the effects of trans-Planckian physics [1,2]. In this paper, we reconsider the flat limit through the modes. The flat limit for a mode is obtained by considering the latter on a domain which is small compared to the inverse of the curvature. This process can be applied of course at any point of spacetime with different results. The use of ambient space formalism allows us to show in a very simple way that the Euclidean vacuum is the only vacuum for which the flat limit yields, *in any point of spacetime*, positive frequency modes. Furthermore the use of the de Sitter waves shows that the whole free QFT tends toward the flat theory when the curvature vanishes, including the de Sitter Fourier transform which becomes the ordinary Fourier transform in the limit. Some of us will show in future works that these de Sitter waves are also very well adapted to group representations and spinorial computation.

Moreover, our procedure will allow us to reconsider the significance of the result on group contractions quoted before. In this paper we argue that although Eq. (1) *can* hold it does not represent the only possibility. Actually, we show that the principal series of SO(1,4) can contract toward the positive energy representation of the Poincaré group, a result which is, as far as we know, new.

The de Sitter waves and ambient space formalism are summarized in Sec. II. The flat limit is investigated in Sec. III. The problem of the contraction of group representations is tackled in Sec. IV. Section V is devoted to some concluding remarks.

II. THE DE SITTER WAVES

The de Sitter space is conveniently seen as a hyperboloid embedded in a five-dimensional time oriented Minkowski space E_5:

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$$M_H = \{X \in E_5 | X^2 = \eta_{\alpha\beta} X^\alpha X^\beta = -H^{-2}\},$$

where $\eta^{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1,-1)$. The (pseudo-) sphere M_H is obviously invariant under O(1,4). We only consider the connected component of the identity $SO_o(1,4)$, the so-called de Sitter group. We are in particular interested in the flat limit (i.e., $H \rightarrow 0$) of the massive scalar free quantum field and the behavior of the group representation in this limit.

The free massive scalar field on this spacetime is, in the Wigner sense, an *elementary system* whose associated unitary irreducible representation belongs to the principal series of representations of $SO_o(1,4)$. This UIR is characterized by the eigenvalue $\nu^2 + 9/4$ of the Casimir operator Q_0 which is linked to the Laplace-Beltrami operator on M_H through $-H^2 Q_0 \equiv \Box_H$ [17]. The contraction of that UIR has already been studied in a group theoretical context [11]. The result is usually written in the following way: the massive representations contract toward the direct sum of the positive energy and negative energy representations of the Poincaré group. We emphasize that this result has been achieved on a purely group theoretical level through an *ad hoc* process of contraction. Although it is from this point of view remarkable that the irreducible representation *can contract* toward a reducible representation, there is no uniqueness in this choice of contraction procedure. In the framework of QFT this result played a rather misleading role in order to understand field theory on de Sitter background from our Minkowskian point of view. Actually, we will see that the negative energy plane waves do not appear when the curvature vanishes as soon as the Euclidean vacuum has been chosen.

In [13,15], the authors use a set of global modes, the de Sitter waves, solutions of the de Sitter Klein-Gordon equation, which are the formal analogue of the plane waves in Minkowski spacetime. We will see that these modes reduce to the usual plane waves when the curvature tends to zero as far as their analyticity domain has been conveniently chosen.

Let $\mathcal{C}^+ = \{\xi \in E_5 ; \xi^2 = 0, \xi^0 > 0\}$ be the null upper cone of E_5. The multivalued functions defined on dS spacetime by

$$X \mapsto (HX \cdot \xi)^s, \ \xi \in \mathcal{C}^+, \ X \cdot \xi \neq 0, \ s \in \mathbb{C}, \tag{2}$$

are solutions of the de Sitter Klein-Gordon equation $(\Box_H + m^2 + 12H^2 \zeta)\phi = 0$, where ζ is a positive gravitational coupling with the de Sitter background and

$$s = -\frac{3}{2} - i\nu \text{ where } \nu = \frac{1}{2}\sqrt{4m^2 H^{-2} + 48\zeta - 9} \ \in \mathbb{R},$$

corresponds to the principal series of UIR (massive case). The expression defined by Eq. (2) is, *as a function of* ξ, homogeneous with degree s on $\mathcal{C}^+$ and thus is entirely determined by specifying its values on a well chosen three-dimensional submanifold (the so-called orbital basis) γ of $\mathcal{C}^+$. These dS waves, as functions on de Sitter spacetime, are only locally defined because they are singular on specific lower dimensional subsets of M_H and multivalued since $(HX \cdot \xi)$ can be negative. In order to get a single-valued global definition, they have to be viewed as distributions which

are boundary values of analytic continuations to suitable domains in the complexified de Sitter space $M_H^{(c)}$:

$$M_H^{(c)} = \{Z = X + iY \in E_5 + iE_5 ; \eta_{\alpha\beta} Z^\alpha Z^\beta = -H^{-2}\}.$$

The minimal domains of analyticity which yield single-valued functions on de Sitter spacetime are the forward and backward tubes of $M_H^{(c)}$: $\mathcal{T}^\pm = T^\pm \cap M_H^{(c)}$, where $T^\pm = E_5 - iV^\pm$ and $V^\pm = \{X \in E_5; \quad X^0 \gtrless \sqrt{\|\vec{X}\|^2 + (X^4)^2}\}$. Details are given in [13].

When Z varies in $\mathcal{T}^+$ and ξ lies in the positive cone $\mathcal{C}^+$, the functions given in Eq. (2) are globally well defined since the imaginary part of $(Z.\xi)$ is nonpositive. We define the de Sitter waves $\phi_\xi(X)$ as the boundary value of the analytic continuation to the future tube of Eq. (2):

$$\phi_\xi(X) \equiv c_\nu \mathrm{bv}(HZ \cdot \xi)^s$$
$$= c_\nu [\theta(HX \cdot \xi) + \theta(-HX \cdot \xi) e^{-i\pi s}] |HX \cdot \xi|^s, \tag{3}$$

where θ is the Heaviside function. The real valued constant c_ν is determined by imposing the Hadamard condition on the two-point function. This choice of modes corresponds to the Euclidean vacuum. In terms of de Sitter waves, the two-point function reads [15]:

$$W(z,z') = c_\nu^2 \int_\gamma (HZ \cdot \xi)^s (HZ' \cdot \xi)^{s*} d\sigma_\gamma(\xi),$$

where $Z \in \mathcal{T}^+$ and $Z' \in \mathcal{T}^-$. The measure $d\sigma_\gamma(\xi)$ on the orbital basis γ is chosen to be m^2 times the natural one induced from the $\mathbb{R}^5$ Lebesgue measure. The calculation, similar to that of [15] yields:

$$c_\nu = \sqrt{\frac{H^2(\nu^2 + 1/4)}{2(2\pi)^3(1 + e^{-2\pi\nu})m^2}}.$$

III. THE FLAT LIMIT OF DE SITTER WAVES

Hereafter, we investigate the behavior of the mode $\phi_\xi(X)$ under vanishing curvature. We consider a region around any point X_A in which all the distances are small compared to H^{-1}. With this assumption we will prove that

$$\lim_{H \rightarrow 0} \phi_\xi(X) = \frac{1}{\sqrt{2(2\pi)^3}} \exp(-ikx) \text{ for } X_A \cdot \xi > 0,$$

$$\lim_{H \rightarrow 0} \phi_\xi(X) = 0 \quad \text{for } X_A \cdot \xi < 0. \tag{4}$$

In other words, these modes do not generate negative frequency modes in the flat limit, whatever the point around which the limit is computed.

Due to the homogeneity of the de Sitter space under the de Sitter group action, one can choose a system of coordinates such that $X_A^4 = H^{-1}$ and $X_A^\mu = 0$. In the neighborhood of this point, for $H \rightarrow 0$, the de Sitter spacetime meets its tangent plane (the four-dimensional Minkowski spacetime), and the coordinates X of this neighborhood read:

$$X^\mu = x^\mu + o(H),$$

$$X^4 = H^{-1} + o(1). \tag{5}$$

For $s \sim -\frac{3}{2} - imH^{-1}$, $\exp(-i\pi s) \to 0$ and one obtains:

$$\lim_{H \to 0} \phi_\xi(X) = \lim_{H \to 0} c_\nu \, \theta(HX \cdot \xi) |HX \cdot \xi|^s.$$

The Heaviside function yields $\xi^4 < 0$ since $HX \cdot \xi \simeq -\xi^4$ and finally, for small H:

$$\phi_\xi(X) \simeq \frac{|\xi_4|^s}{\sqrt{2(2\pi)^3}} \left(1 + \frac{H\xi_\mu x^\mu}{|\xi_4|} \right)^{-3/2 - imH^{-1}} \theta(-\xi^4).$$

This limit exists only for $|\xi_4| = 1$. As a consequence, we use the orbital basis $\gamma = \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1, \mathcal{C}_2$ are defined by

$$\xi = \left(\frac{\omega_k}{m}, \frac{k}{m}, -1 \right) \in \mathcal{C}_1, \quad \xi = \left(\frac{\omega_k}{m}, \frac{k}{m}, 1 \right) \in \mathcal{C}_2,$$

with $\omega_k = \sqrt{k^2 + m^2}$. Note that the induced measure on γ is $dk/(m^2 \omega_k)$ and therefore $d\sigma_\gamma(\xi) = dk/\omega_k$. We finally obtain Eq. (4) according to whether ξ belongs to $\mathcal{C}_1$ or $\mathcal{C}_2$, i.e., $X_A \cdot \xi$ positive or negative.

Thus, due to the analyticity condition at the origin of the $\exp(-i\pi s)$ term, the negative energy modes are (exponentially) suppressed whereas the positive energy modes give the Minkowskian on-shell modes corresponding to a particle of mass m.

We insist on the fact that the result leads to positive frequency plane waves whatever the point X_A we choose. This choice of modes, which corresponds to the Euclidean vacuum, is the only one having this property. Any Bogoliubov transformation on these modes leads to the appearance of conjugate modes ϕ_ξ^* whose flat limit at some point X_B is a negative frequency mode as soon as B has been chosen in such a way that $X_B \cdot \xi < 0$.

As a consequence, any vacuum different from the Euclidean vacuum would lead to physically unacceptable Minkowskian QFT. The Euclidean vacuum has therefore to be preferred with respect to the flat limit criterion.

The de Sitter waves allow us to define a de Sitter Fourier transform which becomes the ordinary Fourier transform in the flat limit. In fact, one can realize the de Sitter one particle sector $\mathcal{H}_H$ as distributions on spacetime through this de Sitter Fourier transform: any $\psi \in \mathcal{H}_H$ can be written as

$$\psi(X) = \int_{\xi \in \gamma} \phi_\xi(X) \tilde{\psi}(\xi) d\sigma_\gamma(\xi), \quad \tilde{\psi} \in L^2(\gamma, d\sigma_\gamma), \tag{6}$$

see [13] for details. Let X_A be a point of de Sitter spacetime in the neighborhood of which we will proceed to the flat limit. The space $\mathcal{H}_H$ can then be decomposed into $\mathcal{H}_H = \mathcal{H}_H^1 \oplus \mathcal{H}_H^2$ using the decomposition of the orbital basis:

$$\psi(X) = \int_{\xi \in \mathcal{C}_1} \phi_\xi(X) \tilde{\psi}(\xi) d\sigma_\gamma(\xi)$$

$$+ \int_{\xi \in \mathcal{C}_2} \phi_\xi(X) \tilde{\psi}(\xi) d\sigma_\gamma(\xi). \tag{7}$$

In the limit of null curvature, the second integral of the above expression vanishes and only the positive frequency remains:

$$\lim_{H \to 0} \psi(x) = \int \frac{e^{-ikx}}{\sqrt{2(2\pi)^3}} \tilde{\psi}(k) \frac{dk}{\omega_k}.$$

As a consequence, the ordinary Fourier transform is the flat limit of the de Sitter Fourier transform. Once again, one can see the significance of de Sitter waves which play in de Sitter space the role of plane waves in Minkowski space, including a good behavior with respect to the de Sitter group: one can see easily using $\phi_\xi(g^{-1}X) = \phi_{g\xi}(X)$ that each space $\mathcal{H}_H^i$ is invariant under the subgroup generated by the M_{ab} with $0 \leq a < b \leq 3$ (see the Appendix). This subgroup, isomorphic to $SO_o(1,3)$, is the stabilizer of X_A.

IV. GROUP CONTRACTION

The Minkowski spacetime is the flat limit of the de Sitter spacetime with respect to all the objects of QFT. In order to emphasize this fact and clarify the link between our approach and that of [11] we will present the concept of contractions in a slightly different manner from the usual presentation.

Let us consider a family of representations U^H of a group G into some spaces $\mathcal{H}_H$ and a representation U of a group G' into a space $\mathcal{H}$. One wants to give a precise meaning to the assertion $U^H \to U$ for $H \to 0$ (one says that the representations U^H contract toward U).

First, we must have a bijection $G \to_i G'$ (which is not a homomorphism) conveying the "similarity" between the two groups. Second we need a space, equipped with a topology, in which all the representations are written. This is obtained by writing an injective map A_H from $\mathcal{H}_H$ to $E \supset \mathcal{H}$ where E is a topological space containing $\mathcal{H}$ in such a way that for any $\phi \in \mathcal{H}_H$ the limit $\lim_{H \to 0} A_H \phi = h$ exists in E and belongs to $\mathcal{H}$.

We say that the representations U^H contract toward U if

$$\forall \psi \in \mathcal{H}_H, \quad \lim_{H \to 0} A_H U_g^H \psi = U_{g'} h = U_{g'} \lim_{H \to 0} A_H \psi, \tag{8}$$

where g' is the element of G' identified with $g \in G$ by means of i.

Let us now return to de Sitter context. For x in Minkowski space we define X in de Sitter space through Eq. (5). We then can define A_H: for the de Sitter waves ϕ_ξ we define $A_H \phi_\xi$ as a function on Minkowski spacetime through

$$(A_H \phi_\xi)(x) = \phi_\xi(X).$$

This definition extends linearly to $\mathcal{H}_H$ through Eq. (7). Then in view of Eq. (4) we have, at least in a weak sense, for any ψ in $\mathcal{H}_H$:

$$\lim_{H \to 0} A_H \psi = f,$$

where f is a positive frequency wave packet on Minkowski spacetime.

We now turn to the representations. The de Sitter and Poincaré groups are identified as explained in the Appendix. Consider

$$g = \exp\left(\sum_{ab} \alpha_{ab} M_{ab} \right) \overset{i}{\mapsto} g'$$

$$= \exp\left(\sum_j \alpha_{0j} B_j + \sum_{ij} \alpha_{ij} R_{ij} + \sum_\mu \alpha_{\mu 4} T_\mu \right).$$

The representation of the de Sitter group is defined by

$$U_g^H \psi(X) = \psi(g_H^{-1} X),$$

where g_H is the 5×5 matrix defined by

$$g_H = \exp\left(\sum_{a < b < 4} \alpha_{ab} M_{ab} + H \sum_\mu \alpha_{\mu 4} M_{\mu 4} \right),$$

and the representation of the Poincaré group is defined by

$$U_{g'} \psi(x) = \psi(g'^{-1} x).$$

One can easily see that for $\xi \in \mathcal{C}_1$:

$$(H(g_H^{-1} X) \cdot \xi) = (H(g'^{-1} x) \cdot k + o(H) + 1). \qquad (9)$$

Then Eq. (8) follows and the principal series of representations of the de Sitter group contract toward the positive energy representation of the Poincaré group. Once again, no negative energy can appear in this process. Nevertheless this is not in contradiction with [11] for which this series can contract toward another representation. In fact, in our context, the result of [11] can be recovered by modifying A_H in the following way. One can define $\tilde{A}_H$ by $\tilde{A}_H = A_H$ on $\mathcal{H}_H^1$ and $\tilde{A}_H = \exp(+i\pi s) A_H$ on $\mathcal{H}_H^2$. With this operator, one obtains the result of [11] because the artificial exponential term cancels the natural one which is present in the definition of ϕ_ξ thanks to the property of analyticity.

V. CONCLUSIONS

Recently, several papers summarized the theory of irreducible unitary representations of the de Sitter group. A re-sult, commonly quoted in these summaries, suggests that the appearance of negative energies for a Minkowskian observer is an unavoidable consequence of group theory. For this reason some authors claimed that the contraction procedure of group representations was not suitable in order to investigate the flat limit of dS-QFT. We have shown that this is inexact. We also conclude that the Euclidean vacuum is preferred as far as one wishes to recover ordinary QFT in the flat limit. To that end we used the formalism of de Sitter waves which turned out to be a very convenient tool, possibly as useful in de Sitter space as the plane waves in Minkowski space.

APPENDIX: IDENTIFICATION OF DE SITTER AND POINCARE GROUPS

We begin with Lie algebras. Let $\Delta_{a,b}$ be the 5×5 matrices whose entries a_{nm} are defined by $a_{nm} = \delta_{na} \delta_{mb}$.

The following matrices are a basis of the Lie algebra so$(1,4)$ of the Lie group $SO_o(1,4)$:

$$M_{0b} = \Delta_{0b} + \Delta_{b0} \quad \text{for} \ b = 1,2,3,4,$$

$$M_{ab} = \Delta_{ab} - \Delta_{ba} \quad \text{for} \ 0 < a < b \leq 4.$$

The following matrices are a basis of the Lie algebra p$(1,3)$ of the Poincaré group:

$$B_j = \Delta_{0j} + \Delta_{j0} \quad \text{for} \ j = 1,2,3,$$

$$R_{ij} = \Delta_{ij} - \Delta_{ji} \quad \text{for} \ 0 < i < j \leq 3,$$

$$T_\mu = \Delta_{\mu 4} \quad \text{for} \ \mu = 0,1,2,3.$$

The identification between the two Lie algebras is obtained through

$$M_{0j} \simeq B_j \quad \text{for} \ j = 1,2,3,$$

$$M_{ij} \simeq R_{ij} \quad \text{for} \ 0 < i < j \leq 3,$$

$$M_{\mu 4} \simeq T_\mu \quad \text{for} \ \mu = 0,1,2,3.$$

The identification between the groups follows, using the exponential application.

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