Semiclassical wormholes

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Smooth-throat wormholes are treated as possessing quantum fluctuation energy with a scalar massive field as its source. The heat kernel coefficients of the Laplace operator are calculated in the background of the arbitrary-profile throat wormhole with the help of the zeta-function approach. Two specific profiles are considered. Some arguments are given that wormholes may exist. It serves as a solution of semiclassical Einstein equations in the range of specific values of the length, a certain radius of the wormhole's throat, and a constant of nonminimal connection.

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I. INTRODUCTION

Great interest in the space-time of wormholes dates back at least to 1916 [1]. Subsequent activity was initiated by both classical works of Einstein and Rosen in 1935 [2] in the context of black hole space-time structure and the later series of works by Wheeler in 1955 [3] with his excellent idea of creating everything from nothing. The more recent interest in the topic of wormholes has been rekindled by the works of Morris and Thorne [4] and Morris, Thorne, and Yurtsever [5] who made use of the concept of wormholes in a scientific discussion of "time machine." These authors constructed and investigated a class of objects they referred to as "traversable wormholes." Their work led to a flurry of activity in wormhole physics [6].

It is well known that the central problem of traversable wormholes is connected with the unavoidable violation of the null energy condition. This means that the matter which should be a source of this object has to possess some exotic properties. For this reason the traversable wormhole cannot be represented as a self-consistent solution of Einstein's equations with the usual classical matter as a source because the usual matter is sure to satisfy all energy conditions. One way out is to use quantum fields in the framework of semiclassical quantum gravity. The point is that the vacuum average value of the energy-momentum tensor of quantum fluctuations may violate energy conditions. Self-consistent wormholes in the framework of semiclassical quantum gravity have been studied in Ref. [7]. In our recent paper [8] we have considered the possibility of a self-consistent solution of semiclassical Einstein equations for a specific kind of wormhole—a short-throat flat-space wormhole. The model represents two identical copies of Minkowski space with spherical regions excised from each copy, and with boundaries of these regions to be identified. The space-time of this model is flat everywhere except a two-dimensional singular spherical surface. The vacuum average of the energy of quantum fluctuations of a massive scalar field with a nonminimal connection serves as a source for this space-time. Owing to the fact that this space-time is flat everywhere, a complete set of wave modes of the massive scalar field can be constructed and ground state energy can be calculated. In this paper we present a calculation of the full energy of the quantum fluctuations rather than the energy density and we use the Einstein's equations with the quantum source only, without a classical contribution. We found that the energy of fluctuations as a function of the radius of throat a may possess a minimum if the nonminimal connection constant $\xi > 0.123$. Utilization of the Einstein equations at the minimum gives the stable configurations of the wormhole. For instance, in the case of a conformal connection, $\xi = 1/6$, we found a relation between the radius a of the wormhole and mass m of the scalar field: $am \approx 0.16$. The Einstein equations say that the wormhole has a radius of throat $a \approx 0.0141 l_{Pl}$ and the mass of scalar field $m \approx 11.35 m_{Pl}$. Therefore, this kind of wormhole, if it exists, may possess a sub-Planckian radius of throat and it may be created by a massive scalar field with super-Planckian mass. Obviously, the validity of the results obtained are restricted by the model taken−short-throat flat-space wormhole.

The goal of this paper is to consider the wormholes with more real geometry of the throat and the energy of quantum fluctuations of a massive scalar field as a source of this background. The main problem in this case has a rather mathematical character. Even for the simple profile of a throat it becomes impossible to obtain a full set of solutions of a radial equation in order to find the energy density of quantum fluctuations in close form. Nevertheless, it is possible to make some predictions about the existence of the wormholes by considering the heat kernel coefficients [8]. In fact, the crucial point is the existence of the negative minimum of the zero point energy. The sufficient condition for the zero point energy to have negative minimum is that the heat kernel coefficients B_2 and B_3 be positive [8]. This gives a condition for the parameters of the model. More precisely, if a background is described by a parameter τ with a dimension of length and the domain where the space-time is "mainly" curved is defined by this parameter, then for the small size of the curved domain, $\tau \to 0$, the zero point energy shows the following behavior:

$$E^{ren} \approx -\frac{B_2 \ln(\tau m)^2}{32\pi^2},$$

and in opposite limit $\tau \to \infty$ we have

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$$E^{ren} \approx -\frac{B_3}{32\pi^2 m^2}.$$

If both these conditions are satisfied one can expect that the system will stay in a minimum of energy which is characterized by specific values of parameters of wormholes and a constant of the nonminimal connection ξ. The next step is the utilization of Einstein's equations with the energy-momentum tensor of quantum fluctuations as a source. The integration over the volume of the $t-t$ component of this equation gives an additional relation between the parameters of the wormhole and zero point energy, using which we obtain the size of a wormhole and the mass of a scalar field in terms of the Planck length and Planck mass correspondingly. At the beginning we may expect [8] that the size of the wormhole and the mass of field will be in the Planck scale. For this reason we are interested only in finding the domain of the wormhole's parameters and the constant nonminimal connection ξ for different models of the wormhole's profile.

The manifest expression for coefficient B_2 exists for an arbitrary background, but this is not the case for coefficient B_3. For this reason we adopt here the zeta-regularization approach (see Sec. III), in the frame of which it is possible to calculate the heat kernel coefficients and zero point energy itself. We pursue here another goal—to evolve the zeta-function approach for situations where it is impossible to find the full set of solutions of the radial equation in closed form. We find a method to calculate the heat kernel coefficients in the background of a wormhole with an arbitrary profile of the throat by using the WKB approach. Moreover, we obtain expressions for an arbitrary heat kernel coefficients and we reproduce them in manifest form up to B_3 for an arbitrary profile of a wormhole's throat.

The organization of the paper is as follows. In Sec. II we consider the geometry of a wormhole with a smooth throat. In Sec. III we discuss the method of the zeta function for the calculation of zero-point energy. The WKB approach for the scalar massive field is considered in Sec. IV. The heat kernel coefficients are obtained in Sec. V. We calculate them in manifest form for an arbitrary profile of throat. The specific profiles of throat are investigated in Secs. VI and VII. In Sec. VIII we discuss the results obtained. The Appendix contains some technical formulas which are too complicated to reproduce them in the text.

We use units $\hbar=c=G=1$. The signature of the space-time, the sign of the Riemann and Ricci tensors, are the same as in the book by Hawking and Ellis [9].

II. A TRAVERSABLE WORMHOLE WITH A SMOOTH THROAT

The metric of a space-time of wormhole which is under consideration has the form

$$ds^2=-dt^2+d\rho^2+r^2(\rho)(d\theta^2+\sin^2\theta d\varphi^2). \qquad (1)$$

The radial variable ρ changes from $-\infty$ to $+\infty$. In the paper we restrict ourselves to wormholes with a symmetric throat, which means that $r(-\rho)=r(+\rho)$. The radius a of a throat is defined as follows: $a=r(0)$. We suppose that far from the wormhole's throat the space-time becomes Minkowskian, that is

$$\lim_{\rho\to\pm\infty}\frac{r^2(\rho)}{\rho^2}=1.$$

The nonzero components of the Ricci tensor and the scalar curvature have the following form:

$$\mathcal{R}_\rho^\rho=-\frac{2r''}{r},$$

$$\mathcal{R}_\theta^\theta=\mathcal{R}_\varphi^\varphi=-\frac{-1+r'^2+rr''}{r^2},$$

$$\mathcal{R}=-\frac{2(-1+r'^2+2rr'')}{r^2}.$$

The energy-momentum tensor corresponding to this metric has a diagonal form from which we observe that the source of this metric possesses the following energy density and pressure:

$$\varepsilon=-\frac{-1+r'^2+2rr''}{8\pi r^2},$$

$$p_\rho=\frac{-1+r'^2}{8\pi r^2},$$

$$p_\theta=p_\varphi=\frac{r''}{8\pi r}.$$

In the paper we obtain general formulas for space-time (1) with an arbitrary symmetric function $r(\rho)$ obeying the above Minkowskian condition. Two specific kinds of throat's profile will be considered. In the first model the profile of the throat has the following form:

$$r(\rho)=\sqrt{\rho^2+a^2}, \qquad (2)$$

where a is the radius of a throat which characterizes the wormhole's size. The embedding into the three-dimensional Euclidean space of the section of the space-time by surface $t=\text{const}, \theta=\pi/2$ is plotted in Fig. 1(I) for two different values of the radius of the throat. In Euclidean space with cylindrical coordinates (r,φ,z) this surface may be found in parametric form from relations $r=r(\rho)$, $z'(\rho)=\sqrt{1-r'^2}$. In this background there is the only nonzero component of the Ricci tensor which reads

$$\mathcal{R}_\rho^\rho=-\frac{2a^2}{(\rho^2+a^2)^2}.$$

The second model has been considered in Ref. [10] and it is characterized by the following profile of a throat:

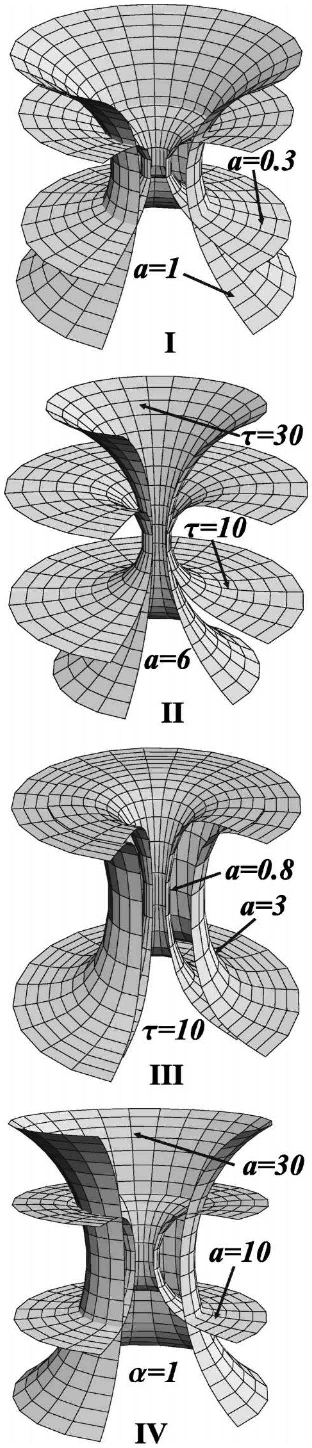

FIG. 1. (I) represents the section $t=\text{const}, \theta=\pi/2$ of the wormhole's space-time with the profile function $r(\rho)=\sqrt{\rho^2+a^2}$ for two different values of the radius of the throat. The next three figures illustrate the wormhole with the profile of the throat $r(\rho)=\rho\coth(\rho/\tau)-\tau+a$. (II) and (III) illustrate that the a and τ are the radius and the length of the throat, accordingly. In the last figure two wormholes with different a but with the same ratio of the radius and the length of the throat are depicted. It is seen that the parameter a characterizes the ''size'' of the wormhole and α describes the ''form'' of the wormhole.

$$r(\rho)=\rho\coth\left(\frac{\rho}{\tau}\right)-\tau+a. \tag{3}$$

This model possesses a more interesting structure. There are two parameters τ and a. The latter parameter is the radius of the throat. In this model we may introduce another parameter which may be called the length of the throat. The point is that the function $r(\rho)$ turns into a linear function of ρ starting from distance $\rho>\tau/2$ and the space-time becomes approximately Minkowskian. Therefore, the length of the throat $l=\tau$. Using new variables $y=\rho/a$, $\alpha=\tau/a$, one rewrites the function r in the form

$$r(y)=a\left[y\coth\left(\frac{y}{\alpha}\right)-\alpha+1\right].$$

The parameter α is the ratio of the length and the radius of the throat. This parameter will play the main role in our analysis. It allows us to consider wormholes of different forms, which are with different ratios of the radius and length of the throat.

In Fig. 1(II–IV) the sections $t=\text{const}$, $\theta=\pi/2$ of this wormhole space-time are shown for different values of a and τ. Namely in Fig. 1(II) we represent two wormholes with the same radius of the throat but with different lengths, and vice versa in Fig. 1(II), where we depict two wormholes with the same length of the throat but with different radii of throat. In last picture Fig. 1(IV) two wormholes with the same ratio of length and the radius of the throat, but with different values of the throats' radii, are depicted. Therefore, the size of the wormhole with the same ratio of the length and the radius of the throat is managed by parameter a. The parameter α describes the wormhole's form.

III. ZERO POINT ENERGY: ZETA-FUNCTION APPROACH

We exploit the zeta function regularization approach [12,11] developed in Ref. [13] and calculate the zero point energy of the massive scalar field in this background. Let us repeat some main formulas from those papers. In the framework of this approach the zero point energy,

$$E(s)=\frac{1}{2}\mu^{2s}\sum_j\sum_{(n)}(\lambda_{(n),j}^2+m^2)^{1/2-s}=\frac{1}{2}\mu^{2s}\zeta_{\mathcal{L}}\left(s-\frac{1}{2}\right), \tag{4}$$

of the scalar massive field Φ is expressed in terms of the zeta function

$$\zeta_{\mathcal{L}}\left(s-\frac{1}{2}\right)=\sum_j\sum_{(n)}(\lambda_{(n),j}^2+m^2)^{1/2-s} \tag{5}$$

of the Laplace operator $\mathcal{L}=-\triangle+m^2+\xi\mathcal{R}$. Here $\triangle=g^{kl}\nabla_k\nabla_l$ is the three-dimensional operator. The eigenvalues $\lambda_{(n),j}+m^2$ of operator $\mathcal{L}$ are found from the boundary condition which looks as follows:

$$\Psi_{(n)}(\lambda,R)=0, \tag{6}$$

where R denotes some boundary parameter. The solutions $\lambda=\lambda_{(n),j}$ of this equation depend on the numbers (n), and additionally they have the index $j=1,2,\ldots,$ which numerates the solutions of the boundary equation. Therefore, the zeta function is a sum of expressions which depend on zeros of function $\Psi_{(n)}$. Next, according to Ref. [13] we convert the series over j in the zeta function to the integral and arrive at the formula

$$E(s)=-\frac{1}{2}\mu^{2s}\sum_{(n)}\frac{\cos\pi s}{\pi}\times\int_m^\infty dk(k^2-m^2)^{1/2-s}\frac{\partial}{\partial k}\ln\Psi_{(n)}(ik,R), \tag{7}$$

where the function $\Psi_{(n)}$ in imaginary axes appears.

Expression (7) is divergent in the limit $s\to 0$ we are interested in. For renormalization we subtract from $E(s)$ all terms $E^{div}(s)$ which will survive in the limit $m\to\infty$:

$$E^{div}(s)=\lim_{m\to\infty}E(s)$$

and we define the renormalized energy as follows:

$$E^{ren}=\lim_{s\to 0}[E(s)-E^{div}(s)]. \tag{8}$$

Because the pole structure of the zeta function does not depend on the value of the parameters, it is obvious that in the limit $m\to\infty$ the divergent part will have the structure of the DeWitt-Schwinger expansion, which has the following form:

$$E^{div}(s)=\frac{1}{2}\left(\frac{\mu}{m}\right)^{2s}\frac{1}{(4\pi)^{3/2}\Gamma\left(s-\frac{1}{2}\right)}\left\{B_0m^4\Gamma(s-2)+B_{1/2}m^3\Gamma\left(s-\frac{3}{2}\right)+B_1m^2\Gamma(s-1)+B_{3/2}m\Gamma\left(s-\frac{1}{2}\right)+B_2\Gamma(s)\right\}, \tag{9}$$

where B_α are the heat kernel coefficients. In order to extract the divergent part of the energy we use the following procedure [13]. We subtract from and add to the integrand the uniform expansion of $\ln\Psi$ up to m^0. We denote this expansion as $(\ln\Psi_{(n)})^{as}$. Therefore, according to this, we represent the energy as the sum

$$E(s)=E_{fin}(s)+E_{as}(s) \tag{10}$$

of the finite (in the limit $s\to 0$) part

$$E_{fin}(s)=-\frac{1}{2}\mu^{2s}\sum_{(n)}\frac{\cos\pi s}{\pi}\times\int_m^\infty dk(k^2-m^2)^{1/2-s}\frac{\partial}{\partial k}[(\ln\Psi_{(n)}(ik,R)-(\ln\Psi_{(n)}(ik,R))^{as}], \tag{11}$$

and the remains, which will be obtained from the uniform expansion part

$$E_{as}(s)=-\frac{1}{2}\mu^{2s}\sum_{(n)}\frac{\cos\pi s}{\pi}\times\int_m^\infty dk(k^2-m^2)^{1/2-s}\frac{\partial}{\partial k}[\ln\Psi_{(n)}(ik,R)]^{as}.$$

The last expression contains all terms which will survive in the limit $m\to\infty$.

Taking into account the obtained expressions in Eq. (8) we arrive at the formula

$$E^{ren}=E_{fin}+E_{as}^{fin}, \tag{12a}$$

where

$$E_{fin}=E_{fin}(0)=-\frac{1}{2\pi}\sum_{(n)}\int_m^\infty dk\sqrt{k^2-m^2}\frac{\partial}{\partial k}\times\{\ln\Psi_{(n)}(ik,R)-[\ln\Psi_{(n)}(ik,R)]^{as}\}, \tag{12b}$$

$$E_{as}^{fin}=\lim_{s\to 0}[E_{as}(s)-E^{div}(s)]. \tag{12c}$$

The divergent part E^{div} is given by Eq. (9).

The finite part E_{fin} is calculated numerically. The second part, in practice, is found in the following way. By using the uniform expansion $(\ln\Psi_{(n)})^{as}$ we calculate in manifest form the $E_{as}(s)$ and after that we take the limit $m\to\infty$ in the expression obtained (the pole structure does not change). All terms which will survive in this limit constitute the DeWitt-Schwinger expansion (9) which we have to subtract in Eq. (12c). This way of calculation is more preferable because we may obtain the heat kernel coefficients in the manifest form. The calculations of heat kernel coefficients in framework of this approach shows that the approach is suitable for both a smooth background and for manifolds with singular surfaces of codimensions one [8] and two [14], the general formulas which were obtained in Refs. [15] and [16].

In consideration of the above we may find the zero-point energy for the large and small sizes of wormhole [8]. Let the parameter a characterize the size of the wormhole. In this case the E^{ren}/m is a dimensionless function and it depends on the parameter ma and some additional dimensionless parameters which characterize the form of wormhole. For example, in the first model (2) there is only the parameter a, which is the radius of the wormhole's throat, and it characterizes at the same time the size of the wormhole as a whole. Therefore in this model the E^{ren}/m depends on ma and there are no additional parameters. In the second model (3) there is an additional parameter $\alpha=\tau/a$ except parameter ma. For this reason the dependence of the zero point energy E^{ren}/m on the mass is the same as parameter a. Because for renormalization we subtracted all terms of the asymptotic over mass expansion up to B_2 the asymptotic $ma\to\infty$ is the following:

$$\frac{E^{ren}}{m}\approx-\frac{B_3}{32\pi^2 m^3}=-\frac{b_3}{32\pi^2(ma)^3}. \tag{13}$$

In the opposite case, $ma\rightarrow 0$, the behavior of the energy is defined by coefficient B_2 [see also Eq. (41)]:

$$\frac{E^{ren}}{m}\approx-\frac{\ln(ma)^2}{32\pi^2 m}B_2=-\frac{\ln(ma)^2}{32\pi^2(ma)}b_2. \tag{14}$$

Here b_3 and b_2 are dimensionless heat kernel coefficients which may depend on the additional parameters. Therefore from these expressions we obtain the following sufficient condition that the zero point energy has a minimum: both B_2 and B_3 have to be positive. An additional condition may be obtained from Einstein's equations (see Secs. VI and VII).

IV. MASSIVE SCALAR FIELD IN WORMHOLES BACKGROUND: THE WKB APPROACH

We consider the massive scalar quantum field in this background as a source for this space-time. In the framework of the approach used one has to find the spectrum of the Laplace operator $\mathcal{L}$:

$$(-\triangle+\xi\mathcal{R})\Phi=\lambda^2\Phi.$$

Taking into account the spherical symmetry of the problem we represent the equation in the following form:

$$\Phi=Y_l^m(\theta,\varphi)\phi,$$

where $Y_l^m(\theta,\varphi)$ are the spherical harmonics, $l=0,1,2,\ldots$ and $m\in[-l,l]$. The radial part of the wave function is the subject for the equation

$$\left(\partial_\rho^2+\frac{2r'}{r}\partial_\rho-\frac{l(l+1)}{r^2}-\xi\mathcal{R}\right)\phi=-\lambda^2\phi. \tag{15}$$

To find the spectrum λ we have to impose some appropriate boundary conditions. It does not matter what kind of boundary condition will be imposed because at the end of the calculation we will tend this boundary to infinity. We use the Dirichlet boundary condition in the spheres with radii R: $\rho=\pm R$. For simplifying formulas we will work here with the function $\zeta(s)=m^{2s}\zeta_{\mathcal{L}}(s)$. With this notations the regularized ground state energy reads

$$E(s)=\frac{1}{2}\left(\frac{\mu}{m}\right)^{2s}\zeta\left(s-\frac{1}{2}\right).$$

Because we need the solution for the imaginary energy only [see Eq. (7)], we change the integrand variable in the radial equation (15) to an imaginary axis, $\lambda\rightarrow i\nu k$, and rescale for simplicity the radial variable, $\rho k\rightarrow x$. Therefore we arrive at the following equation ($\nu=l+1/2$):

$$\ddot{\phi}+2\frac{\dot{r}_k}{r_k}\dot{\phi}-\nu^2\left(1+\frac{1}{r_k^2}\right)\phi+\left(\frac{1}{4r_k^2}-\xi\mathcal{R}_k\right)\phi=0, \tag{16}$$

where the dot is the derivative with respect to x; $r_k^2=r^2k^2$ and $\mathcal{R}_k=\mathcal{R}/k^2$.

A general solution of the radial equation (16) is the superposition of two linearly independent solutions

$$\bar{\Psi}(i\nu k,x)=C_1\phi_1(x)+C_2\phi_2(x). \tag{17}$$

The first function ϕ_1 tends to infinity far from the throat, for $\rho\rightarrow\infty$, and the second one tends to zero. We consider the behavior of the functions only for one part of space-time, namely, with $\rho>0$. The behavior of the solutions in the second part of space-time with negative ρ is found as a continuation of the solutions from the positive part of space-time. Now we impose the Dirichlet boundary condition at spheres $\rho=\pm R$:

$$\bar{\Psi}(i\nu k,+R)=C_1\phi_1(+R)+C_2\phi_2(+R)=0,$$

$$\bar{\Psi}(i\nu k,-R)=C_1\phi_1(-R)+C_2\phi_2(-R)=0.$$

The solution of this system exists if and only if the following condition is satisfied:

$$\Psi_l(i\nu k,R)=\phi_1(+R)\phi_2(-R)-\phi_1(-R)\phi_2(+R)=0. \tag{18}$$

The contribution from the second term in the equation above is exponentially small compared to the first one in the limit $R\rightarrow\infty$. In order to see this let us find the uniform expansion of solutions ϕ_1 and ϕ_2. Moreover, we need this expansion for the renormalization and the calculation of the heat kernel coefficients. Let us represent a solution ϕ in the exponential form

$$\phi(x)=\frac{1}{\sqrt{2a\nu}}e^{S(x)}, \tag{19}$$

where $a=r(0)$, and substitute it in the radial equation (16). One obtains a nonlinear equation

$$\ddot{S}+\dot{S}^2+2\frac{\dot{r}_k}{r_k}\dot{S}-\nu^2\left(1+\frac{1}{r_k^2}\right)+\left(\frac{1}{4r_k^2}-\xi\mathcal{R}_k\right)=0.$$

We represent now the solution in the WKB expansion form

$$S=\sum_{n=-1}^{\infty}\nu^{-n}S_n,$$

and substitute it in the equation above. This gives the following chain of equations:

$$\dot{S}_{-1}=\pm\sqrt{1+\frac{1}{r_k^2}}, \tag{20a}$$

$$\dot{S}_0=-\frac{1}{2}\frac{\ddot{S}_{-1}}{\dot{S}_{-1}}-\frac{\dot{r}_k}{r_k}, \tag{20b}$$

$$\dot{S}_1 = -\frac{1}{2\dot{S}_{-1}}\left[\ddot{S}_0 + \dot{S}_0^2 + 2\frac{\dot{r}_k}{r_k}\dot{S}_0 + \frac{1}{4r_k^2} - \xi\mathcal{R}_k\right], \quad (20c)$$

$$\dot{S}_{n+1} = -\frac{1}{2\dot{S}_{-1}}\left[\ddot{S}_n + \sum_{k=0}^{n}\dot{S}_k\dot{S}_{n-k} + 2\frac{\dot{r}_k}{r_k}\dot{S}_n\right],$$

$$n = 1,2,\ldots . \quad (20d)$$

There are two solutions to this chain that correspond to the sign in the first equation. The plus sign gives the growing (for positive coordinate ρ) solution which we mark "$+$" and the minus sign gives solution which tends to zero at infinity which we mark by the sign "$-$." Therefore

$$\phi_1(+R)\phi_2(-R) = \frac{1}{2a\nu}e^{S^+(+R)+S^-(-R)}.$$

To find an expansion for the sum $S^+(+R)+S^-(-R)$ we need the following properties of function $S^{\pm}(x)$:

$$\dot{S}_{2n-1}^-(x) = -\dot{S}_{2n-1}^+(x),$$

$$\dot{S}_{2n}^-(x) = +\dot{S}_{2n}^+(x),$$

and

$$\dot{S}_{2n-1}^{\pm}(x) = +\dot{S}_{2n-1}^{\pm}(-x),$$

$$\dot{S}_{2n}^{\pm}(x) = -\dot{S}_{2n}^{\pm}(-x),$$

where $n=0,1,\ldots$. The first two equations are the consequence of the structure of the chain and the last two equations are due to the symmetry of the metric function $r_k(x) = r_k(-x)$.

Taking into account these properties of symmetry we have

$$S^+(+x)+S^-(-x) = \sum_{n=0}^{\infty}\nu^{1-2n}\left[C_{2n-1}^+ + C_{2n-1}^- + \int_{-x}^{+x}\dot{S}_{2n-1}^+dx\right] + \sum_{n=0}^{\infty}\nu^{-2n} \times\left[C_{2n}^+ + C_{2n}^- + 2\int_{x_0}^{+x}\dot{S}_{2n}^+dx\right], \quad (21)$$

$$S^+(+x)+S^-(+x) = \sum_{n=0}^{\infty}\nu^{1-2n}[C_{2n-1}^+ + C_{2n-1}^-] + \sum_{n=0}^{\infty}\nu^{-2n}\left[C_{2n}^+ + C_{2n}^- + 2\int_{x_0}^{+x}\dot{S}_{2n}^+dx\right], \quad (22)$$

Here the C_n are the constant of the integration of system (20).

Therefore we may express the combination we need (21) in terms of Eq. (22):

$$S^+(+x)+S^-(-x) = S^+(+x)+S^-(+x) + \sum_{n=0}^{\infty}\nu^{1-2n}\int_{-x}^{+x}\dot{S}_{2n-1}^+dx.$$

To find the combination $S^+(+x)+S^-(+x)$ we use the Wronskian condition. Because these solutions are independent, they obey the equation ($a_k = ak$)

$$W(\phi_1(x),\phi_2(x)) = \frac{k}{r_k^2}.$$

The origin of this relation is the following. Suppose we try to find the scalar Green function of the Klein-Gordon equation:

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu - m^2 - \xi\mathcal{R})G(x,x') = \frac{\delta^4(x,x')}{\sqrt{-g(x)}} \quad (23)$$

in background (1). It is very easy to extract the time and the angular dependence of the Green function

$$G(x,x') = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_l^m(\theta,\varphi)Y_l^{-m} \times(\theta',\varphi')e^{-i\omega(t-t')}\phi(\rho,\rho'),$$

and we arrive at the equation for the radial part of the Green function which reads ($\lambda^2 = \omega^2 - m^2$)

$$\left\{\partial_\rho^2 + \frac{2r'}{r}\partial_\rho + \lambda^2 - \frac{l(l+1)}{r^2} - \xi\mathcal{R}\right\}\phi(\rho,\rho') = \frac{\delta(\rho-\rho')}{r^2}$$

or in dimensionless variables ($\lambda \to i\nu k$, $k\rho \to x$)

$$\left\{\partial_x^2 + 2\frac{\dot{r}_k}{r_k}\partial_x - \nu^2\left(1+\frac{1}{r_k^2}\right) + \left(\frac{1}{4r_k^2} - \xi\mathcal{R}_k\right)\right\}\phi(x,x') = \frac{k\delta(x-x)}{r_k^2}.$$

As usual we represent the radial Green function in standard form:

$$\phi(x,x') = \theta(x'-x)\phi_1(x)\phi_2(x') + \theta(x-x')\phi_2(x)\phi_1(x'),$$

where ϕ_1 and ϕ_2 are two linearly independent solutions of the homogenous equation and ϕ_1 tends to infinity for $\rho\to\infty$ and ϕ_2 tends to 0. The Wronskian condition appears if we substitute the radial Green function to the radial equation above:

$$W(\phi_1(x),\phi_2(x')) = \frac{k}{r_k^2}.$$

Therefore, if two functions ϕ_1 and ϕ_2 describe the system, they have to obey this Wronskian condition.

For the solution in exponential form (19) this condition gives

$$e^{S^+(x)+S^-(x)}=\frac{2\nu a_k}{r_k^2}\frac{1}{\dot{S}^+(x)-\dot{S}^-(x)}.$$

The denominator on the right-hand side (rhs) has the following form:

$$\dot{S}^+(x)-\dot{S}^-(x)=2\sum_{n=0}^{\infty}\nu^{1-2n}\dot{S}^+_{2n-1}.$$

Taking into account these two expressions above we arrive at the formula

$$\begin{aligned}S^+(x)+S^-(-x)&=\ln(a_k)-\frac{1}{2}\ln(\dot{S}^2_{-1}r_k^4)\\&\quad+\sum_{n=0}^{\infty}\nu^{1-2n}\int_{-x}^{+x}\dot{S}^+_{2n-1}dx\\&\quad-\ln\left\{1+\sum_{n=1}^{\infty}\nu^{-2n}\frac{\dot{S}^+_{2n-1}}{\dot{S}^+_{-1}}\right\}.\end{aligned}\tag{24}$$

The main achievements and peculiarities of this expression are as follows: (i) the rhs is expressed in terms of derivative of functions S_n^+, we do not need to find the constants of integration in the chain of equations (20); (ii) the odd and even power of ν are separated, which leads to the separation of the contribution to heat kernel coefficients with integer and half-integer indices; (iii) the rhs is expressed in terms of functions S_n with odd indices only. The first three functions $\dot{S}_{2n-1}$ are listed in the Appendix, formula (A1a). We would like to note that this formula is valid for an arbitrary, but symmetric, $r(\rho)=r(-\rho)$, metric coefficient.

From this expression we may conclude that the contribution from the second term in condition (18) is exponentially small compared with the first one. Indeed, the main WKB term in Eq. (24) gives the following contribution:

$$\phi_1(+R)\phi_2(-R)\approx\frac{k}{2\nu}\frac{1}{\dot{S}^+_{-1}r_k^2}\exp\left\{+\nu\int_{-kR}^{+kR}\dot{S}^+_{-1}dx\right\},$$

$$\phi_1(-R)\phi_2(+R)\approx\frac{k}{2\nu}\frac{1}{\dot{S}^+_{-1}r_k^2}\exp\left\{-\nu\int_{-kR}^{+kR}\dot{S}^+_{-1}dx\right\}.$$

Because the function $\dot{S}^+_{-1}$ is positive for arbitrary R we observe that the second expression gives an exponentially small (for $R\rightarrow\infty$) contribution compared with the first one and we will omit it in what follows.

V. HEAT KERNEL COEFFICIENTS

Let us now proceed to an evaluation of the heat kernel coefficients (HKC). The formula (24) allows us to find HKC in general form for arbitrary indices. Taking into account the above discussions we have the following expression for the zeta function:

$$\begin{aligned}\zeta\left(s-\frac{1}{2}\right)&=-m^{2s}\frac{2\cos\pi s}{\pi}\sum_{l=0}^{\infty}\nu^{2-2s}\int_{m/\nu}^{\infty}dk\\&\quad\times\left(k^2-\frac{m^2}{\nu^2}\right)^{1/2-s}\frac{\partial}{\partial k}\{S^+(+R)+S^-(-R)\}.\end{aligned}\tag{25}$$

To find the heat kernel coefficients we use the uniform expansion given by Eq. (24). As it will be clear later, the odd powers of ν will give a contribution to HKC with integer indices and even powers of ν produce the contribution to HKC with half-integer indices. The well-known asymptotic expansion of the zeta function in three dimensions has the form

$$\begin{aligned}\zeta_{as}\left(s-\frac{1}{2}\right)&=\frac{1}{(4\pi)^{3/2}}\frac{1}{\Gamma\left(s-\frac{1}{2}\right)}\sum_{l=0}^{\infty}\left\{m^{4-2l}B_l\Gamma(s+l-2)\right.\\&\quad\left.+m^{3-2l}B_{l+1/2}\Gamma\left(s+l-\frac{3}{2}\right)\right\}.\end{aligned}\tag{26}$$

For simplicity we introduce the density of HKC with integer indices $\overline{B_l}$ by the relation

$$B_l=\int_{-R}^{+R}d\rho\overline{B_l}(\rho)$$

and first of all we will obtain formulas for this density.

Let us consider the part of Eq. (24) with an odd degree of ν. The contribution to the zeta function is the following:

$$\begin{aligned}\zeta^{odd}_{as}\left(s-\frac{1}{2}\right)&=-m^{2s}\frac{2\cos\pi s}{\pi}\sum_{l=0}^{\infty}\nu^{2-2s}\\&\quad\times\int_{\frac{m}{\nu}}^{\infty}dk\left(k^2-\frac{m^2}{\nu^2}\right)^{1/2-s}\frac{\partial}{\partial k}\\&\quad\times\left\{\sum_{p=0}^{\infty}\nu^{1-2p}\int_{-kR}^{+kR}\dot{S}^+_{2p-1}(x)dx\right\}.\end{aligned}$$

We change now the variable of integration $x=k\rho$ and take the derivative with respect to k:

$$\begin{aligned}\zeta^{odd}_{as}\left(s-\frac{1}{2}\right)&=-m^{2s}\frac{2\cos\pi s}{\pi}\sum_{l=0}^{\infty}\nu^{2-2s}\\&\quad\times\int_{\frac{m}{\nu}}^{\infty}dkk\left(k^2-\frac{m^2}{\nu^2}\right)^{1/2-s}\\&\quad\times\sum_{p=0}^{\infty}\nu^{1-2p}\int_{-R}^{+R}s_{2p-1}(k,\rho)d\rho.\end{aligned}$$

The first four functions s_{2p-1} are listed in the Appendix, formula (A2a). The general structure of these functions is the following:

$$s_{2p-1}=\sum_{n=0}^{2p} \alpha_{2p-1,n} z^{-p-n-(1/2)},$$

where $\alpha_{2p-1,n}$ are the functions of $r(\rho)$, and $z=1+k^2r^2(\rho)$.

Next, we integrate over k using the formula

$$\int_{m/a}^{\infty} dk k\left(k^2-\frac{m^2}{\nu^2}\right)^{(1/2)-s}(1+k^2r^2)^{-q} =\frac{1}{2}r^{2s-3}\nu^{-3+2s+2q}(\nu^2+m^2r^2)^{3/2-s-q} \times\frac{\Gamma\left(\frac{3}{2}-s\right)\Gamma\left(q-\frac{3}{2}+s\right)}{\Gamma(q)}$$

and obtain the following expression for the odd part of the zeta function:

$$\zeta_{as}^{odd}\left(s-\frac{1}{2}\right)=\frac{m^{2s}}{\Gamma\left(s-\frac{1}{2}\right)}\int_{-R}^{+R}d\rho \times\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\sum_{n=0}^{2p}\alpha_{2p-1,n}r^{2s-3} \times\frac{\Gamma(s+p+n-1)}{\Gamma\left(p+n+\frac{1}{2}\right)}\frac{\nu^{2n+1}}{(\nu^2+m^2r^2)^{s+p+n-1}}. \tag{27}$$

By using the binomial of Newton we reduce the power of ν in the denominator

$$\sum_{l=0}^{\infty}\frac{\nu^{2n+1}}{(\nu^2+m^2r^2)^{s+p+n-1}} =\frac{1}{2}\sum_{q=0}^{n}(-m^2r^2)^{n-q}\frac{n!}{q!(n-q)!}\frac{\mathcal{Z}(s+p+n-1-q)}{\Gamma(s+p+n-1-q)}, \tag{28}$$

where

$$\mathcal{Z}(s)=2\Gamma(s)\sum_{l=0}^{\infty}\frac{\nu}{(\nu^2+m^2r^2)^s}.$$

To obtain the HKC we need asymptotic (over mass m) expansion of the zeta function. The asymptotic expansion of the function $\mathcal{Z}(s)$ was obtained in Ref. [17] and it has the form below

$$\mathcal{Z}(s)=(mr)^{-2s}\sum_{l=-1}^{\infty}A_l(s)(mr)^{-2l}, \tag{29a}$$

$$A_{-1}(s)=\Gamma(s-1), \tag{29b}$$

$$A_l(s)=2\frac{(-1)^l}{l!}\Gamma(l+s)\zeta_H\left(-1-2l,\frac{1}{2}\right), \tag{29c}$$

where the $\zeta_H(a,b)$ is the Hurwitz zeta function.

Taking into account formulas (28) and (29) one has the following asymptotic series for an odd part of the zeta function:

$$\zeta_{as}^{odd}\left(s-\frac{1}{2}\right) =\frac{1}{2\Gamma\left(s-\frac{1}{2}\right)}\int_{-R}^{+R}d\rho\sum_{l=0}^{\infty}\sum_{p=0}^{l}\sum_{n=0}^{2p}\sum_{q=0}^{n}\alpha_{2p-1,n} \times m^{4-2l}r^{-2l+1}\frac{\Gamma(p+n-1+s)}{\Gamma\left(p+n+\frac{1}{2}\right)}\frac{n!}{q!(n-q)!} \times(-1)^{n-q}\frac{A_{l-p-1}(s+p+n-1-q)}{\Gamma(s+p+n-1-q)}.$$

As was expected at the beginning this is a series over even degrees of mass and it gives a contribution to the HKC with integer indices. Comparing the above expression with the general equation (26) we obtain the general formula for the arbitrary HKC coefficient with an integer index

$$\overline{B_l}(\rho) =\frac{4\pi^{3/2}}{\Gamma(s+l-2)}\sum_{p=0}^{l}\sum_{n=0}^{2p}\sum_{q=0}^{n}\alpha_{2p-1,n}r^{-2l+1} \times\frac{\Gamma(p+n-1+s)}{\Gamma\left(p+n+\frac{1}{2}\right)}\frac{n!(-1)^{n-q}}{q!(n-q)!} \times\frac{A_{l-p-1}(s+p+n-1-q)}{\Gamma(s+p+n-1-q)}. \tag{30}$$

Therefore, to obtain the HKC with index l we have to take into account expansion up to ν^{1-2l}.

Let us now proceed to the HKC with half-integer indices. To find them we have to take into account the part of Eq. (24) with even powers of ν in an expression for the zeta function (25). The general form of the even part is

$$[S^{+}(x)+S^{-}(-x)]^{even} = \ln(a_k)-\frac{1}{2}\ln(\dot{S}_{-1}^{2}r_k^{4})-\ln\left\{1+\sum_{n=1}^{\infty}\nu^{-2n}\frac{\dot{S}_{2n-1}^{+}}{\dot{S}_{-1}^{+}}\right\} = \sum_{p=0}^{\infty}\nu^{-2p}E_{2p}, \tag{31}$$

where the first four functions E_{2p} are listed in the Appendix, formula (A1b).

We substitute now expansion (31) in the expression for the zeta function:

$$\zeta_{as}^{even}\left(s-\frac{1}{2}\right)=-m^{2s}\frac{2\cos\pi s}{\pi}\sum_{l=0}^{\infty}\nu^{2-2s}\times\int_{\frac{m}{\nu}}^{\infty}dk\left(k^2-\frac{m^2}{\nu^2}\right)^{1/2-s}\frac{\partial}{\partial k}\sum_{p=0}^{\infty}\nu^{-2p}E_{2p},$$

and take the derivative with respect to the k:

$$\zeta_{as}^{even}\left(s-\frac{1}{2}\right)=-m^{2s}\frac{2\cos\pi s}{\pi}\sum_{l=0}^{\infty}\nu^{2-2s}\times\int_{\frac{m}{\nu}}^{\infty}dk k\left(k^2-\frac{m^2}{\nu^2}\right)^{1/2-s}\sum_{p=0}^{\infty}\nu^{-2p}s_{2p}. \tag{32}$$

The functions s_{2p} have the following structure:

$$s_{2p}=\sum_{n=0}^{2p}\alpha_{2p,n}z^{-p-n-1},$$

where $z=1+k^2r^2(R)$. The first three coefficients are listed in manifest form in the Appendix [see Eq. (A2b)]. The coefficients $\alpha_{2p,n}$ depend on the parameters R and a and do not depend on the variable of integration k. Going the same way as we did for the HKC with integer indices we obtain the following asymptotic expression for the even part of the zeta function:

$$\zeta_{as}^{even}\left(s-\frac{1}{2}\right)=\frac{1}{2\Gamma\left(s-\frac{1}{2}\right)}\sum_{l=0}^{\infty}\sum_{p=0}^{l}\sum_{n=0}^{2p}\sum_{q=0}^{n}\alpha_{2p,n}m^{3-2l}r^{-2l}\times\frac{\Gamma\left(p+n-\frac{1}{2}+s\right)}{\Gamma(p+n+1)}\frac{n!}{q!(n-q)!}(-1)^{n-q}\times\frac{A_{l-p-1}\left(s+p+n-\frac{1}{2}-q\right)}{\Gamma\left(s+p+n-\frac{1}{2}-q\right)}.$$

As was expected the even part of the zeta function is the series over odd powers of mass and, therefore, it gives contributions to HKC with half-integer indices. Comparing this expression with the general asymptotic series for the zeta function we obtain the following formula for the HKC with half-integer indices:

$$B_{l+1/2}=\frac{4\pi^{3/2}}{\Gamma\left(s+l-\frac{3}{2}\right)}\sum_{p=0}^{l}\sum_{n=0}^{2p}\sum_{q=0}^{n}\alpha_{2p,n}r^{-2l}\times\frac{\Gamma\left(p+n-\frac{1}{2}+s\right)}{\Gamma(p+n+1)}\frac{n!(-1)^{n-q}}{q!(n-q)!}\times\frac{A_{l-p-1}\left(s+p+n-\frac{1}{2}-q\right)}{\Gamma\left(s+p+n-\frac{1}{2}-q\right)}. \tag{33}$$

We would like to note that the right-hand side of formulas (30) and (33) does not depend, in fact, on the s, which is confirmed by straightforward calculations.

These formulas look very complicated but calculation may be done easily using a simple program in the package MATHEMATICA. Indeed, the functions $\dot{S}_2^{+}(x)$ and E_{2n} may be found by using formulas (20) and (31). The functions s_n are obtained from the following relations:

$$s_{2n-1}(k,\rho)=\frac{1}{k}\frac{\partial}{\partial k}[k\dot{S}_{2n-1}^{+}(x)|_{x=k\rho}],$$

$$s_{2n}(k,R)=\frac{1}{k}\frac{\partial}{\partial k}[E_{2n}(x)|_{x=kR}].$$

The first four HKC coefficient (density) with integer indices are listed below

$$\overline{B}_0 = 4\pi r^2, \tag{34a}$$

$$\overline{B}_1 = \frac{8\pi}{3}[r'^2 + rr''] + 8\pi\left(\xi - \frac{1}{6}\right)[-1 + r'^2 + 2rr''], \tag{34b}$$

$$\overline{B}_2 = \frac{8\pi\xi^2}{r^2}[-1 + r'^2 + 2rr'']^2 + \frac{8\pi\xi}{3r^2}[(-1 + r'^2) - r(-5 + 7r'^2)r'' + 7r^2 r' r^{(3)} + 3r^3 r^{(4)}]$$
$$- \frac{2\pi}{315 r^2}[2(-21 + 17r'^4) - 6r(-35 + 59r'^2)r'' + 21r^2(7r'^2 + 24r' r^{(3)}) + 210 r^3 r^{(4)}], \tag{34c}$$

$$\overline{B}_3 = \frac{16\pi\xi^3}{3r^4}[-1 + r'^2 + 2rr'']^3 + \frac{8\pi\xi^2}{3r^4}[(-1 + r'^2)^2(1 + 9r'^2) - 2r(-1 + r'^2)(-5 + 9r'^2)r'' - 2r^2(-8r''^2 + 16r'^2 r''^2$$
$$- 3r' r^{(3)} + 3r'^3 r^{(3)}) + 2r^3(14 r' r'' r^{(3)} - 3r^{(4)} + 3r'^2 r^{(4)}) + 2r^4(5r^{(3)\,2} + 6r'' r^{(4)})]$$
$$- \frac{4\pi\xi}{315 r^4}[-42(-1 + r'^2)(1 + 15r'^2) + 2r(-252 + 105 r'^2 + 859 r'^4)r'' - 2r^2(-525 r''^2 + 2517 r'^2 r''^2 - 420 r' r^{(3)}$$
$$+ 808 r'^3 r^{(3)}) + 3r^3(308 r''^3 + 1354 r' r'' r^{(3)} - 175 r^{(4)} + 271 r'^2 r^{(4)}) + 21 r^4(27 r^{(3)\,2} + 27 r'' r^{(4)} - 13 r' r^{(5)}) - 105 r^5 r^{(6)}]$$
$$- \frac{\pi}{45045 r^4}[4(-572 - 9009 r'^2 + 9341 r'^6) - 4r(-6006 - 15015 r'^2 + 62039 r'^4)r'' + 13r^2(-4620 r''^2 + 32943 r'^2 r''^2$$
$$- 4620 r' r^{(3)} + 11564 r'^3 r^{(3)}) - 286 r^3(308 r''^3 + 1139 r' r'' r^{(3)} - 105 r^{(4)} + 223 r'^2 r^{(4)}) - 429 r^4(47 r^{(3)\,2} + 24 r'' r^{(4)}$$
$$- 74 r' r^{(5)}) + 12012 r^5 r^{(6)}]. \tag{34d}$$

In the above formulas the function r depends on the radial coordinate ρ whereas the heat kernel coefficients with half-integer indices,

$$B_{1/2} = -4\pi^{3/2} r^2, \tag{35a}$$

$$B_{3/2} = -8\pi^{3/2}\xi[-1 + r'^2 + 2rr''] + \frac{\pi^{3/2}}{3}[-4 + 3r'^2 + 6rr''], \tag{35b}$$

$$B_{5/2} = \frac{-8\pi^{3/2}\xi^2}{r^2}[-1 + r'^2 + 2rr'']^2 - \frac{2\pi^{3/2}\xi}{3r^2}[4(-1 + r'^2)$$
$$- 10r(-2 + 3r'^2)r'' - 3r^2(4r''^2 - 3r' r^{(3)}) + 6r^3 r^{(4)}]$$
$$+ \frac{\pi^{3/2}}{120 r^2}[2(-16 + 15 r'^4) - 5r(-32 + 63 r'^2)r''$$
$$- 10 r^2(5r''^2 - 14 r' r^{(3)}) + 90 r^3 r^{(4)}], \tag{35c}$$

depend on the radial function r at boundary $r = r(R)$. From Eqs. (34) and (35) we observe that the HKC B_l and $B_{l+1/2}$ are polynomial in ξ with degree l.

It is well known [11] that the heat kernel coefficients with integer indices consist of two parts. The first part is an integral over the volume and another one is an integral over the boundary. We obtained a slightly different representation for this coefficient as an integral over ρ. But it is easy to see that they are in agreement. Indeed, let us consider, for example, coefficient B_1. According to the standard formula we have

$$B_1 = \left(\frac{1}{6} - \xi\right)\int_V \mathcal{R} dV + \frac{1}{3}\int_S \operatorname{tr} K dS\bigg|_{\rho = +R} + \frac{1}{3}\int_S \operatorname{tr} K dS\bigg|_{\rho = -R}.$$

The volume contribution is exactly the same as we have already obtained (34b). Surface contribution from above formula is

$$\frac{1}{3}\int_S \operatorname{tr} K dS_{\rho = +R} + \frac{1}{3}\int_S \operatorname{tr} K dS_{\rho = -R} = \frac{16\pi}{3} r' r\bigg|_{\rho = R}.$$

From our result (34b) we get the same expression

$$\frac{8\pi}{3}\int_{-R}^{+R}[r'^2 + rr'']d\rho = \frac{8\pi}{3}\int_{-R}^{+R}[rr']' d\rho = \frac{16\pi}{3} rr'\bigg|_{\rho = R}.$$

It is not so difficult to verify that the heat kernel coefficients up to B_2 are in agreement with general expressions. There is no general expressions for higher coefficients.

According to Ref. [8] the condition sufficient for the existence of the self-consistent wormholes may be formulated in terms of two heat kernel coefficients

$$B_2=\int_{-\infty}^{+\infty}d\rho\overline{B}_2=h_{2,2}\xi^2+h_{2,1}\xi+h_{2,0},$$

$$B_3=\int_{-\infty}^{+\infty}d\rho\overline{B}_3=h_{3,3}\xi^3+h_{3,2}\xi^2+h_{3,1}\xi+h_{3,0}.$$

Namely, both B_2 and B_3 have to be positive [18]. The coefficients $h_{k,l}$ of the polynomials depend on the structure of the wormhole. Therefore the problem reduces to an analysis of the polynomial in ξ of second and third degrees, the coefficients of which depend on the structure of the wormhole's space-time. Wormholes with different forms may exist for different values of nonminimal connection ξ. For some ξ the above polynomials will be positive for specific forms of wormholes.

VI. MODEL OF THE THROAT: $R(\rho)=\sqrt{\rho^2+a^2}$

In this section we consider in detail the specific model of the wormhole with the following profile of the throat $r(\rho)=\sqrt{\rho^2+a^2}$. From the general expressions (34) we obtain the density of heat kernel coefficients with integer indices which are

$$\overline{B}_0=4\pi r^2,$$

$$\overline{B}_1=\frac{8\pi a^2}{r^2}\left(\xi-\frac{1}{6}\right)+\frac{8\pi}{3},$$

$$\overline{B}_2=\frac{2\pi}{315r^6}(1103a^4-796a^2r^2+8r^4)-\frac{8a^2\pi}{3r^6}(17a^2-12r^2)\times\xi+\frac{8\pi a^4}{r^5}\xi^2,$$

$$\overline{B}_3=-\frac{2\pi}{45045r^{10}}(2583561a^6-3157438a^4r^2+751820a^2r^4-480r^6)+\frac{4a^2\pi}{315r^{10}}(47263a^4-57464a^2r^2+13540r^4)\xi-\frac{8a^4\pi}{3r^{10}}(73a^2-62r^2)\xi^2+\frac{16\pi a^6}{3r^{10}}\xi^3.$$

Integrating over ρ from $-R$ to $+R$ we obtain the HKC. Here we reproduce their expansions in the limit $R\to\infty$ up to terms $1/R$,

$$B_0=\frac{8\pi R^3}{3}+8\pi a^2R, \tag{36a}$$

$$B_1\approx\frac{16\pi R}{3}+8\pi^2a\left(\xi-\frac{1}{6}\right)-\frac{16\pi a^2}{R}\left(\xi-\frac{1}{6}\right), \tag{36b}$$

$$B_2\approx\frac{\pi^2}{20a}(60\xi^2-20\xi+3)-\frac{32\pi}{315R}, \tag{36c}$$

$$B_3\approx\frac{\pi^2}{4032a^3}(5880\xi^3-6300\xi^2+2226\xi-257). \tag{36d}$$

The formulas for the first three coefficients with half integer indices may be found from the general expression (35). Here we have listed them with their expansions for large value of R,

$$B_{1/2}=-4\pi^{3/2}r^2=-4\pi^{3/2}(R^2+a^2),$$

$$B_{3/2}\approx-\frac{\pi^{3/2}}{3}-\frac{\pi^{3/2}a^2(8\xi-1)}{60R^2},$$

$$B_{5/2}\approx-\frac{\pi^{3/2}}{60R^2}.$$

Let us now proceed to the renormalization and calculation of the zero point energy. As noted in Sec. III [see Eq. (12)] we have to subtract all terms which will survive in the limit $m\to\infty$. According to the general asymptotic structure of the zeta function given by Eq. (26), in this limit the first five terms survive, namely the HKC up to B_2. Because the zero point energy is proportional to the zeta function we may speak about renormalization of the zeta function. According to Eq. (12) we take the asymptotic expansion for the zeta function up to ν^{-3} [in the limit $m\to\infty$ these terms give the asymptotic (over m) expansion (26) up to the heat kernel coefficient B_2] and subtract its expansion over m up to B_2 from it. After taking the limit $s\to 0$ we observe that this difference will give E_{as}^{fin} (12c). First of all we consider this part and later we will simplify the finite part (12b).

We should like to make a comment. In the problem under consideration we have two different scales: R and a which give us two dimensionless parameters mR and ma. To extract terms for renormalization we turn mass to infinity, which means the Compone wavelength of a scalar boson turns to zero and becomes smaller than all scales of the model. In other words, it means that we turn to infinity both parameters mR and ma. After renormalization we will turn mR to infinity separately in order to obtain the part which does not depend on the boundary.

Let us consider separately two parts of the asymptotic expansion of the zeta function according to the odd and even powers of ν. First of all we consider the odd part which gives the HKC with integer indices. All singularities are contained in the first three terms in Eq. (26) with B_0,B_1,B_2. After subtracting these singularities, $s\to 0$ and we obtain some infinite power series over parameters $m\rho$ and ma. Next, we have to integrate over ρ and $mR\to\infty$. For this reason we have to obtain some expression instead of a series to take this limit easily. It is impossible to take this limit directly in a series. We will use the Abel-Plana formula to extract the main contribution from a series in this limit. The rest will be a good expression for numerical calculations. Moreover, from this remaining part we will extract terms which will be divergent in the limit $ma\to 0$ to analyze our formulas.

Our starting formula for the odd part of the zeta function is Eq. (27) which, cut up to $p=2$, is

$$\zeta_{as,2}^{odd}\left(s-\frac{1}{2}\right)=\frac{m^{2s}}{\Gamma\left(s-\frac{1}{2}\right)}\int_{-R}^{+R}d\rho\sum_{l=0}^{\infty}\sum_{p=0}^{2}\sum_{k=0}^{2p}\alpha_{2p-1,k}\times r^{2s-3}\frac{\Gamma(s+p+k-1)}{\Gamma\left(p+k+\frac{1}{2}\right)}\frac{\nu^{2k+1}}{(\nu^2+m^2r^2)^{s+p+k-1}}. \tag{37}$$

Expanding the denominator with the help of the formula

$$(1+x^2)^{-s}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\Gamma(n+s)}{\Gamma(s)}x^{2n},$$

we represent Eq. (37) in the following forms:

$$\zeta_{as,2}^{odd}\left(s-\frac{1}{2}\right)=\frac{1}{(4\pi)^{3/2}}\frac{1}{\Gamma\left(s-\frac{1}{2}\right)}\times\int_{-R}^{+R}d\rho m^{2s}\sum_{n=0}^{\infty}\sum_{p=0}^{2}m^{2n}f_{n,p}(s), \tag{38}$$

where

$$f_{n,p}(s)=8\pi^{3/2}\frac{(-1)^n}{n!}r^{2n-3}\sum_{k=0}^{2p}\alpha_{2p-1,k}\times\frac{\Gamma(n+p+k+s-1)}{\Gamma\left(p+k+\frac{1}{2}\right)}\zeta_H\left(2n+2p+2s-3,\frac{1}{2}\right).$$

In order to make formulas more readable we make everything dimensionless but save the same notations. At any moment we may repair dimensional parameters by changing $R\to mR$ an $a\to ma$. In this case we rewrite the expression for the zeta function (37) in the following form:

$$\zeta_{as,2}^{odd}\left(s-\frac{1}{2}\right)=\frac{m}{(4\pi)^{3/2}}\frac{m^{2s}}{\Gamma\left(s-\frac{1}{2}\right)}\int_{-R}^{+R}d\rho\sum_{n=0}^{\infty}\sum_{p=0}^{2}f_{n,p}(s). \tag{39}$$

From this expression we observe that for $p=0$ the first three terms are divergent with $n=0,1,2$; for $p=1$ the first two terms with $n=0,1$ and at last for $p=2$ the only term is divergent with $n=0$. We recall that

$$\Gamma(s-n)_{s\to 0}=\frac{(-1)^n}{n!}\left(\frac{1}{s}+\Psi(n+1)\right)+O(s),$$

$$\zeta_H(s+1,q)_{s\to 0}=\frac{1}{s}-\Psi(q)+O(s).$$

For this reason we represent the zeta function (37) in the following form (for $s\to 0$):

$$\zeta_{as,2}^{odd}\left(s-\frac{1}{2}\right)=\frac{m}{(4\pi)^{3/2}}\frac{1}{\Gamma\left(s-\frac{1}{2}\right)}\times\int_{-R}^{+R}d\rho\left\{r^{2s}\sum_{n=0}^{2}f_{n,0}(s)+\sum_{n=3}^{\infty}f_{n,0}(0)\right\} \tag{40a}$$

$$+\frac{m}{(4\pi)^{3/2}}\frac{1}{\Gamma\left(s-\frac{1}{2}\right)}\times\int_{-R}^{+R}d\rho\left\{r^{2s}\sum_{n=0}^{1}f_{n,1}(s)+\sum_{n=2}^{\infty}f_{n,1}(0)\right\} \tag{40b}$$

$$+\frac{m}{(4\pi)^{3/2}}\frac{1}{\Gamma\left(s-\frac{1}{2}\right)}\times\int_{-R}^{+R}d\rho\left\{r^{2s}f_{0,2}(s)+\sum_{n=1}^{\infty}f_{n,2}(0)\right\} \tag{40c}$$

and we will analyze each part separately.

To illustrate the calculations we consider in details the first part (40a). First of all it is not difficult to find the manifest form of a singular part in the limit $s\to 0$:

$$\sum_{n=0}^{2}f_{n,0}(s)=4\pi r^2\Gamma(s-2)+\frac{\pi}{3}\Gamma(s-1)+\frac{7\pi}{120r^2}\Gamma(s)+\pi r^2[-3+4\gamma+8\ln(2)]+\frac{\pi}{3}[1+2\ln(2)+24\zeta_R'(-1)]-\frac{\pi}{120r^2}[-7+2\ln(2)-1680\zeta_R'(-3)].$$

We observe that this term gives a contribution to B_0, B_1, and B_2 according to the gamma functions. For renormalization we have to subtract from this expression the first three terms according to our scheme.

There is one important moment which is crucial for our analysis. The above formula contains all terms which survive in the limit $s\to 0$ for an arbitrary mass of field. For renormalization we have to subtract asymptotic expansion in the form (26). There is a difference in factor r^{2s}. For this reason after the renormalization factor

$$(r^{2s}-1)\left(4\pi r^2\Gamma(s-2)+\frac{\pi}{3}\Gamma(s-1)+\frac{7\pi}{120r^2}\Gamma(s)\right)_{s\to 0}$$
$$=\left[2\pi r^2-\frac{\pi}{3}+\frac{7\pi}{120r^2}\right]\ln r^2$$

appears. If we take into account all terms in Eq. (40) we obtain the following contribution:

$$\ln(r^2)\left(\frac{1}{2}\overline{B}_0-\overline{B}_1+\overline{B}_2\right). \tag{41}$$

Exactly the same structure was observed before in Refs. [17,8]. This term defines the behavior of energy for a small wormhole because it is maximally divergent for a small wormhole.

Therefore the renormalized contribution is

$$\sum_{n=0}^{2} f_{n,0}^{ren}(s)=\left[2\pi r^2-\frac{\pi}{3}+\frac{7\pi}{120r^2}\right]\ln r^2+\pi r^2[-3+4\gamma$$
$$+8\ln(2)]+\frac{\pi}{3}[1+2\ln(2)+24\zeta_R'(-1)]$$
$$-\frac{\pi}{120r^2}[-7+2\ln(2)-1680\zeta_R'(-3)].$$

We represent the finite part in the following form:

$$\sum_{n=3}^{\infty} f_{n,0}(0)=\frac{8\pi}{r^2}\sum_{l=0}^{\infty}\nu^3\left\{\ln\left(1+\frac{r^2}{\nu^2}\right)-\frac{r^2}{\nu^2}+\frac{1}{2}\frac{r^4}{\nu^4}\right.$$
$$\left.+\frac{r^2}{\nu^2}\left[\ln\left(1+\frac{r^2}{\nu^2}\right)-\frac{r^2}{\nu^2}\right]\right\}$$

by using a standard series representation for the Hurwitz zeta function. To find a more suitable form for these series we use the Abel-Plana formula and obtain

$$\sum_{l=0}^{\infty}\nu^2\left[\ln\left(1+\frac{r^2}{\nu^2}\right)-\frac{r^2}{\nu^2}\right]$$
$$=-\frac{1}{2}r^2\ln(r^2)-r^2\left[2\ln(2)+\gamma-\frac{1}{2}\right]$$
$$+2\int_0^\infty\frac{d\nu\,\nu}{e^{2\pi\nu}+1}\ln\left|1-\frac{r^2}{\nu^2}\right|,$$

$$\sum_{l=0}^{\infty}\nu^3\left[\ln\left(1+\frac{r^2}{\nu^2}\right)-\frac{r^2}{\nu^2}+\frac{1}{2}\frac{r^4}{\nu^4}\right]$$
$$=\frac{1}{4}r^4\ln(r^2)-\frac{1}{24}r^2+r^4\left[2\ln(2)+\gamma-\frac{1}{8}\right]$$
$$-2\int_0^\infty\frac{d\nu\,\nu^3}{e^{2\pi\nu}+1}\ln\left|1-\frac{r^2}{\nu^2}\right|.$$

Taking into account these formulas we have

$$\sum_{n=0}^{2} f_{n,0}(0)+\sum_{n=3}^{\infty} f_{n,0}^{ren}(0)$$
$$=16\pi\int_0^\infty\frac{d\nu\,\nu^3}{e^{2\pi\nu}+1}\left\{\ln\left|1-\frac{\nu^2}{r^2}\right|+\frac{\nu^2}{r^2}-\frac{\nu^2}{r^2}\ln\left|1-\frac{\nu^2}{r^2}\right|\right\}.$$

We now integrate this formula over ρ from $-R$ to $+R$ according to Eq. (40a) and take the limit $R\to\infty$. After this we arrive at the expression

$$(40a)-\frac{m}{16\pi^2}f_a=-\frac{m}{16\pi^2}\{f_a^{sing}+\omega_a\},$$

where

$$f_a^{sing}=\frac{7\pi^2}{60a}\ln(a)+\frac{\pi^2}{a}\left(\frac{7}{120}+\frac{1}{10}\ln(2)+14\zeta_R'(-3)\right).$$

The manifest form of the regular contribution is written out in the Appendix [see Eq. (A3)]. We extracted all terms with a logarithm and that which is singular for $a\to 0$ and collected them in f_a^{sing}. The remaining part, ω_a, is a regular contribution.

Using the same procedure for the second and third parts we obtain the following expressions:

$$(40b)=-\frac{m}{16\pi^2}f_b=-\frac{m}{16\pi^2}\{f_b^{sing}+\omega_b\},$$

where

$$f_b^{sing}=\pi^2\left[-16a\left(\xi-\frac{1}{6}\right)+\frac{1}{a}\left(\frac{2}{3}\xi-\frac{1}{6}\right)\right]\ln(a)$$
$$+\frac{1}{a}\left[-\frac{1}{3}[1+24\zeta_R'(-1)]\xi+\frac{1}{9}[1+18\zeta_R'(-1)]\right].$$

$$(40c)=-\frac{m}{16\pi^2}f_c=-\frac{m}{16\pi^2}\{f_c^{sing}+\omega_c\},$$

where

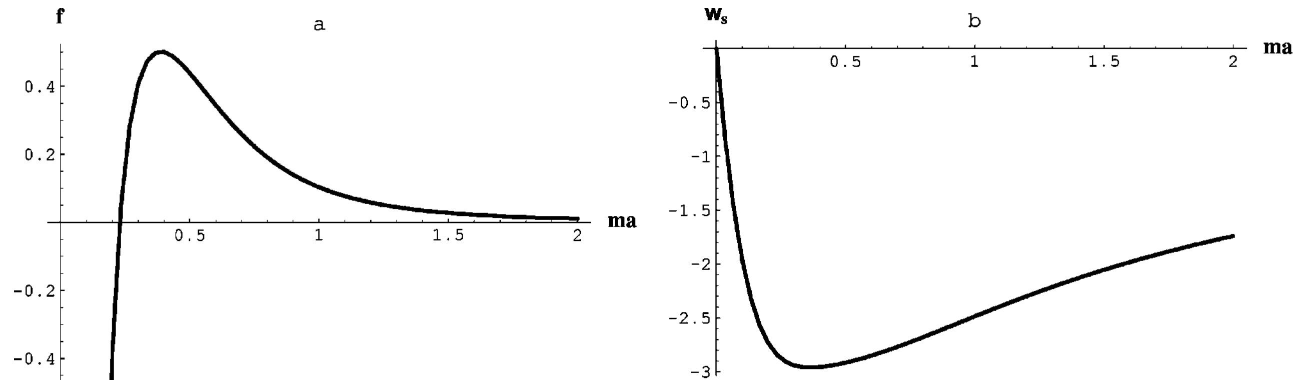

FIG. 2. The plot of the summary contributions: $f=f_a+f_b+f_c$ and $\omega_s=\omega_a+\omega_b+\omega_c$ for $\xi=\frac{1}{6}$: (a) summary contribution f and (b) regular part ω_s.

$$f_c^{sing}=\frac{\pi^2}{a}\left[6\xi^2-\frac{8}{3}\xi+\frac{7}{20}\right]\ln(a)+\frac{\pi^2}{a}\left[\frac{1}{2}(-7+12\gamma+36\ln 2)\xi^2+\frac{1}{6}(15-16\gamma-48\ln 2)\xi+\frac{1}{360}(-143+126\gamma+378\ln 2)\right].$$

The manifest form of the regular contributions ω_b and ω_c are out in the Appendix [Eq. (A3)].

Putting together all the contributions in Eq. (40) we obtain

$$\zeta_{odd}^{ren}=-\frac{m}{16\pi^2}\left(\ln(a^2)\pi^2\left\{\frac{1}{a}\left[3\xi^2-2\xi+\frac{3}{20}\right]+8a\left(\xi-\frac{1}{6}\right)\right\}+\omega\right),$$

where

$$\omega=\omega_a+\omega_b+\omega_c+\frac{\pi^2}{a}\left[\frac{1}{2}(-7+12\gamma+36\ln 2)\xi^2-\frac{1}{6}(-13+48\zeta_R'(-1)+16\gamma+48\ln 2)+\frac{1}{180}\times(-41+360\zeta_R'(-1)+2520\zeta_R'(-3)+63\gamma+207\ln 2)\right].$$

In Fig. 2 we reproduce a plot of the sum of all three contributions: $f=f_a+f_b+f_c$, $\omega_s=\omega_a+\omega_b+\omega_c$ for $\xi=\frac{1}{6}$.

Let us now proceed to the contribution from an even part of the zeta function. We start from Eq. (32) and do not take the limit of great mass. Integrating over k we obtain the following expression for this even part:

$$\zeta_{as}^{even}\left(s-\frac{1}{2}\right)=\frac{m^{2s}}{\Gamma\left(s-\frac{1}{2}\right)}\sum_{l=0}^{\infty}\sum_{p=0}^{1}\sum_{k=0}^{2p}\alpha_{2p,k}r^{2s-3}\times\frac{\Gamma\left(s+p+k-\frac{1}{2}\right)}{\Gamma(p+k+1)}\frac{\nu^{2k+1}}{(\nu^2+m^2r^2)^{s+p+k-1/2}}. \tag{42}$$

Here r is taken at the point R. Now we pass to dimensionless variables (with the same notations) and have

$$\zeta_{as}^{even}\left(s-\frac{1}{2}\right)=\frac{m}{(4\pi)^{3/2}\Gamma\left(s-\frac{1}{2}\right)}\sum_{p=0}^{1}\sum_{k=0}^{2p}\alpha_{2p,k}\times\frac{(4\pi)^{3/2}}{r^3}\frac{\Gamma\left(s+p+k-\frac{1}{2}\right)}{\Gamma(p+k+1)}\phi_{p,k}, \tag{43}$$

where

$$\phi_{p,k}=\sum_{l=0}^{\infty}\frac{\nu^{2k+1}}{(\nu^2+m^2r^2)^{s+p+k-1/2}}.$$

Analytical continuation $s\to 0$ in this series may be easily done by using the Abel-Plana formula:

$$\phi_{0,0}=-\frac{1}{3}r^3+\frac{1}{24}r-2r\int_r^{\infty}d\nu\,\nu Ex(\nu)-\frac{2}{r}\int_0^r\frac{\nu^3 d\nu}{\sqrt{1-\frac{\nu^2}{r^2}}}Ex(\nu),$$

$$\phi_{1,0}=-r+\frac{2}{r}\int_0^r d\nu\nu[3Ex(\nu)+\nu Ex'(\nu)]\sqrt{1-\frac{\nu^2}{r^2}},$$

$$\phi_{1,1}=-2r+\frac{2}{r}\int_0^r d\nu\nu[6Ex(\nu)+6\nu Ex'(\nu)+\nu^2 Ex''(\nu)]\sqrt{1-\frac{\nu^2}{r^2}},$$

$$\phi_{1,2}=-\frac{5}{3}r+\frac{2}{r}\int_0^r d\nu\nu\left[8Ex(\nu)+12\nu Ex'(\nu)+4\nu^2 Ex''(\nu)+\frac{1}{3}\nu^3 Ex'''(\nu)\right]\sqrt{1-\frac{\nu^2}{r^2}},$$

where

$$Ex(\nu)=\frac{1}{e^{2\pi\nu}+1}.$$

Now we substitute these formulas in Eq. (43) and take the limit $R\to\infty$. In this limit all integrals in expressions for $\phi_{p,k}$ are smaller than the $1/r$. Taking into account that $\alpha_{0,0}\sim r$ and $\alpha_{1,k}\sim r^2$ we obtain in this limit (we repair dimensional variables)

$$\zeta_{as}^{even}\left(s-\frac{1}{2}\right)=\frac{m}{(4\pi)^{3/2}\Gamma\left(s-\frac{1}{2}\right)}\left\{m^3B_{1/2}\Gamma\left(s-\frac{3}{2}\right)+mB_{3/2}\Gamma\left(s-\frac{1}{2}\right)+O\left(\frac{1}{R+a}\right)\right\}.$$

Therefore after renormalization (subtracting these two terms) we take the limit $R\to\infty$ and obtain zero contribution from this even part

$$\zeta_{ren}^{even}\left(s-\frac{1}{2}\right)=0.$$

Thus there is the only contribution from the odd part. Collecting all terms together we arrive at the following expression for zero point energy:

$$E^{ren}=-\frac{m}{32\pi^2}\left(\ln(ma)^2\pi^2\left\{\frac{1}{ma}\left[3\xi^2-\xi+\frac{3}{20}\right]+8ma\left(\xi-\frac{1}{6}\right)\right\}+\Omega\right), \tag{44}$$

where

$$\Omega=\omega+32\pi\sum_{l=0}^{\infty}\nu^2\int_{1/\nu}^{\infty}dy\left(y^2-\frac{1}{\nu^2}\right)^{1/2}\frac{\partial}{\partial y}\{S^+(+mR)+S^-(-mR)-[S^+(+mR)+S^-(-mR)]_3^{uni.exp.}\}_{R\to\infty}, \tag{45}$$

and $k=my$.

The main problem now is the calculation of the last term in the expression for Ω. Let us simplify the expression and show that in the limit $R\to\infty$ the divergent parts are cancelled. Indeed, let us consider the first five terms of uniform expansion

$$[S^+(R)+S^-(-R)]_3^{uni.exp.}=\sum_{n=0}^{2}\nu^{1-2n}\int_{-kR}^{+kR}\dot{S}_{2n-1}^+dx+\sum_{n=0}^{1}\nu^{-2n}E_{2n}.$$

It is very easy to take the limit in the part with an even power of ν by using the manifest form of E_{2n} listed in the Appendix. The only term which gives the nonzero contribution is E_0,

$$E_0|_{R\to\infty}=-2\ln(kR)+\ln(ka)+O(R^{-2}),$$

$$E_2|_{R\to\infty}=O(R^{-2}).$$

The part with an odd power of ν in the uniform expansion brings the single linear on R divergent contribution coming from $\dot{S}_{-1}^+$:

$$2(kR)\nu.$$

Therefore in the limit $R\to\infty$ the uniform expansion gives the following divergent contribution:

$$2(kR)\nu-2\ln(kR)+\ln(ka).$$

Because later we have to take the derivative with respect to k we may rewrite this expression in the following way:

$$2(kR)\nu-\ln(ka).$$

To take the limit of a large box in $S^+(+R)+S^-(-R)$ let us reduce the radial equation to a standard form of the scattering problem by changing the form of the radial function $\phi\to\psi/r(\rho)$. In this case the equation reads

$$\left[-\frac{d^2}{d\rho^2}+\frac{l(l+1)}{\rho^2+a^2}+\frac{a^2(1-2\xi)}{(\rho^2+a^2)^2}\right]\psi=\lambda^2\psi. \tag{46}$$

This equation looks similar to the equation of a scattering problem in one dimension [do not forget that $\rho\in(-\infty,+\infty)$] with a non-singular symmetric potential

$$V_l(\rho,a)=\frac{l(l+1)}{\rho^2+a^2}+\frac{a^2(1-2\xi)}{(\rho^2+a^2)^2}. \tag{47}$$

From a standard theory of one-dimensional scattering we know that there are two independent solutions which have the following properties [as opposite to traditional notations $\phi(x)$ we use the solutions $\phi(-x)$ to make coincidence with the functions we use]

$$\phi_1(\rho)\approx\begin{cases}s_{11}(\lambda)e^{-i\lambda\rho}, & \rho\to+\infty,\\ e^{-i\lambda\rho}+s_{12}(\lambda)e^{i\lambda\rho}, & \rho\to-\infty,\end{cases}$$

$$\phi_2(\rho)\approx\begin{cases} s_{21}(\lambda)e^{-i\lambda\rho}+e^{i\lambda\rho}, & \rho\to+\infty,\\ s_{22}(\lambda)e^{i\lambda\rho}, & \rho\to-\infty,\end{cases}$$

where $s_{\alpha\alpha}(\lambda)$ constitute the s matrix of the scattering problem. Due to the symmetry of the potential the components of the matrix obey the relation $s_{22}=s_{11}$.

Now we change energy to an imaginary axis: $\lambda\to ik\nu$ and obtain

$$\phi_1(+R)\phi_2(-R)_{R\to\infty}=s_{11}^2(ik\nu)e^{2k\nu R}.$$

Therefore

$$\begin{aligned}[S^+(+R)+S^-(-R)]_{R\to\infty}&=\ln[\phi_1(+R)\phi_2(-R)]_{R\to\infty}\\ &=\ln[s_{11}^2(ik\nu)]+2kR\nu,\end{aligned}$$

and the divergent parts in Eq. (45) are cancelled. Thus we arrive at the following expression for Ω:

$$\Omega=\omega+32\pi\sum_{l=0}^{\infty}\nu^2\int_{\frac{1}{\nu}}^{\infty}dy\left(y^2-\frac{1}{\nu^2}\right)^{1/2}\frac{\partial}{\partial y}\left\{\ln[ys_{11}^2(iy\nu m)]-\sum_{n=0}^{2}\nu^{1-2n}\int_{-\infty}^{+\infty}[\dot{S}_{2n-1}^{+}-\delta_{0,n}]dx\right\}. \quad (48)$$

Thus we express the finite part of the zero point energy in terms of the s matrix of the scattering problem, namely in terms of the transmission coefficient of the barrier in an imaginary axis. A similar relation was found by Bordag in Ref. [13]. The potential $V_l(\rho,a)$ of the scattering problem has the following properties:

$$V_l(0,a)=\frac{l(l+1)+1-2\xi}{a^2}$$

$-$the height of the barrier,

$$\int_{-\infty}^{+\infty}V_l(\rho,a)d\rho=\frac{\pi[2l(l+1)+1-2\xi]}{2a}$$

$-$the work against the potential barrier. (49)

Therefore the zero point energy has the form (44), where the function Ω is given by expression (48). We note that according to [17,8] the factor before the logarithm term in Eq. (44) is $(B_2-B_1)_{R\to\infty}$. The origin of this structure has been already noted in Eq. (41).

Now we analyze qualitatively without exact numerical calculations the behavior of energy for the small and large radii of the throat. According to Eqs. (14) and (13) the zero point energy in $3+1$ dimensions has the following behavior for small and large values of the throat:

$$E^{ren}\approx-\frac{B_2\ln(am)^2}{32\pi^2}, \quad a\to0,$$

$$E^{ren}\approx-\frac{B_3}{32\pi^2m^2}, \quad a\to\infty,$$

or in manifest form

$$E^{ren}\approx-\frac{m}{32}\frac{\ln(ma)^2}{ma}\left[3\xi^2-\xi+\frac{3}{20}\right], \quad a\to0, \quad (50)$$

$$E^{ren}\approx-\frac{m}{32}\frac{1}{(ma)^3}\frac{1}{4032}\times[5880\xi^3-6300\xi^2+2226\xi-257], \quad a\to\infty. \quad (51)$$

It is easy to verify that the coefficient after the logarithm in Eq. (50), which is the contribution from B_2 in the limit $R\to\infty$, is never to be zero or negative. It is always positive. For this reason the zero point energy is positive for a small radius of the throat for an arbitrary constant of the nonconformal coupling ξ. In the domain of the large radius of the throat the expression in the square brackets in Eq. (51), which is the contribution from B_3 in the limit $R\to\infty$, may change its sign. It is positive for $\xi>0.266$ (energy negative) and negative (energy positive) in the opposite case. Therefore we conclude that there is a minimum of ground state energy if constant $\xi>0.266$. The situation is opposite that which appeared in our last paper [8], where the energy for a large value of the throat (which was defined by $B_{5/2}$) was always positive, but for a small radius of the throat it could change its sign.

Let us now consider the semiclassical Einstein equations:

$$G_{\mu\nu}=\frac{8\pi G}{c^4}\langle T_{\mu\nu}\rangle^{\rm ren}, \quad (52)$$

where $G_{\mu\nu}$ is the Einstein tensor, and $\langle T_{\mu\nu}\rangle$ is the renormalized vacuum expectation values of the stress-energy tensor of the scalar field. The total energy in a static space-time is given by

$$E=\int_V\varepsilon\sqrt{g^{(3)}}d^3x,$$

where $\varepsilon=-\langle T_t^t\rangle^{\rm ren}=-G_t^tc^4/8\pi G$ is the energy density, and the integral is calculated over the whole space. In the spherically symmetric metric (1) we obtain

$$E=-\frac{c^4}{2G}\int_{-\infty}^{\infty}G_t^tr^2(\rho)d\rho=-\frac{c^4\pi a}{2G}. \quad (53)$$

The zero point energy has the following form:

$$E^{ren}=-\frac{\hbar c}{a}f(ma,\xi). \quad (54)$$

In the self-consistent case the total energy must coincide with the ground state energy of the scalar field. Equating Eqs. (54) and (53) gives

$$\frac{c^4a}{2G}=\frac{\hbar c}{a}f(ma,\xi),$$

or

$$a = l_P\sqrt{2f(ma,\xi)}.$$

Considering this equation at the minimum of the zero point energy we obtain some value of the wormhole's radius. The concrete value of the radius may be found from an exact numerical calculation of the zero point energy as a function of ma. But without this calculation we conclude that the wormholes with the throat's profile (2) may exist for $\xi > 0.266$.

VII. THE MODEL OF THE THROAT: $r(\rho)=\rho\coth(\rho/\tau)-\tau+a$

We will not reproduce here the density for heat kernel coefficients in manifest form due to their complexity. They may be found from general formulas (34). There are two parameters in this model a and $\alpha=\tau/a$. The dimensional parameter a characterizes this kind of wormhole as a whole. A small value of this parameter indicates the small size of a wormhole. The dimensionless parameter α characterizes the form of wormhole—its ratio of the length and the radius of the throat. By changing the integration variable $\rho = xa$ we observe that coefficients B_2 and B_3 have the following structure which is clear from dimensional consideration:

$$B_2=\int_{-\infty}^{+\infty} d\rho \overline{B}_2 = \frac{1}{a}[b_{2,2}\xi^2+b_{2,1}\xi+b_{2,0}],$$

$$B_3=\int_{-\infty}^{+\infty} d\rho \overline{B}_3 = \frac{1}{a^3}[b_{3,3}\xi^3+b_{3,2}\xi^2+b_{3,1}\xi+b_{3,0}],$$

where $b_{k,l}$ depend on the $\alpha=\tau/a$ only. We note that $b_{2,2} > 0$ as it is seen from Eq. (34c). Therefore we may analyze the zero point energy for different values of the parameter α. From the general point of view we have the following behavior of the zero point energy for the small size of wormhole that is for the small value of $a\to 0$:

$$E^{ren} \approx -\frac{\ln(ma)^2}{32\pi^2}B_2 = -\frac{\ln(ma)^2}{32\pi^2 a}[b_{2,2}\xi^2+b_{2,1}\xi+b_{2,0}]. \tag{55}$$

Using the general expression for coefficient B_2 it is possible to find in manifest form the polynomial in ξ in Eq. (55) for a great value of $\alpha\to\infty$ (a small radius of the wormhole throat compared with its length):

$$E^{ren} \approx -\frac{\sqrt{3\alpha}\ln(ma)^2}{240a}[30\xi^2-10\xi+1], \tag{56}$$

and for small value of $\alpha\to 0$ (a small length of the wormhole throat compared with its radius):

$$E^{ren} \approx -\frac{(15-\pi^2)\ln(ma)^2}{1350\pi a\alpha}[240\xi^2-80\xi+7]. \tag{57}$$

The numerical calculations of the discriminant $\Delta = b_{2,1}^2 - 4b_{2,2}b_{2,0}$ of the polynomial in ξ as a function of α is

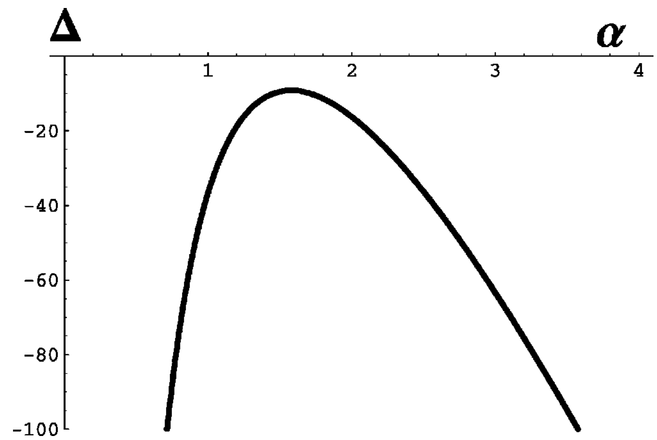

FIG. 3. The discriminant $\Delta = b_{2,1}^2 - 4b_{2,2}b_{2,0}$ of the polynomial in ξ as function of α. It is always negative for all values of α. It means that the zero point energy is always positive for a small value of the radius of the throat.

shown in Fig. 3. From this figure and Eqs. (56) and (57) we conclude that the discriminant is always negative for arbitrary values of α. It means that the zero point energy is always positive for a small wormhole for an arbitrary constant of the nonminimal coupling ξ and an arbitrary ratio of the length of throat and its radius.

The behavior of zero point energy for a large size of the wormhole ($a\to\infty$) has the following form:

$$E^{ren} \approx -\frac{B_3}{32\pi^2 m^2} = -\frac{1}{32\pi^2 m^2 a^3}[b_{3,3}\xi^3+b_{3,2}\xi^2+b_{3,1}\xi+b_{3,0}]. \tag{58}$$

The zero point energy will get a minimum for some value of a if the above expression will be negative. Let us consider the polynomial

$$P = b_{3,3}\xi^3+b_{3,2}\xi^2+b_{3,1}\xi+b_{3,0} \tag{59}$$

for different values of α starting from a small value of it. The zero point energy will get a minimum if this polynomial is positive. In the limit $\alpha\to 0$ we have, approximately,

$$\alpha^3 P \approx \frac{512\pi(45-4\pi^2)}{135}\alpha\xi^3 - \left(\frac{256\pi(21-2\pi^2)}{189} + \frac{256\pi(45-4\pi^2)}{135}\alpha\right)\xi^2 + \left(\frac{512\pi(21-2\pi^2)}{945} + \frac{224\pi(45-4\pi^2)}{675}\alpha\right)\xi - \frac{368\pi(21-2\pi^2)}{6615} + \frac{44\pi(45-4\pi^2)}{2025}\alpha.$$

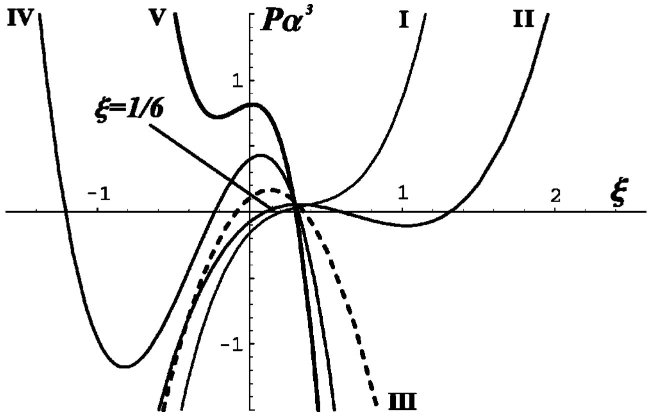

FIG. 4. The plots of the polynomial $P\alpha^3$ for different values of the ratio $\alpha=\tau/a$.

In this expression we saved terms up to α^{-3}. This polynomial in the limit $\alpha\to 0$ has two complex roots and one is real,

$$\xi\approx\frac{5(21-2\pi^2)}{14(45-4\pi^2)}\frac{1}{\alpha}.$$

Because the coefficient with ξ^3 is positive the polynomials will be positive for all

$$\xi>\frac{5(21-2\pi^2)}{14(45-4\pi^2)}\frac{1}{\alpha}.$$

Therefore, for small values of $\alpha=\tau/a$ we have a *low* boundary for parameter ξ where the wormhole may exist [see Fig. 4(I)]. The greater α the smaller the low boundary of ξ. For $\alpha>1.136$ the conformal connection $\xi=1/6$ will be greater than the low boundary. At the point $\alpha=1.26$ two domains appear where the polynomial is positive. The first domain is $0.188<\xi<0.841$ and the second is $\xi>0.841$ [see Fig. 4(II) for $\alpha>1.26$]. The low boundary of the second domain will increase for greater α and it disappears for $\alpha=1.65$. At this point the coefficients $b_{3,3}=0$ and the polynomial turns out to be a parabola [see Fig. 4(III)] with a positive part in the domain: $-0.088<\xi<0.358$. For greater α we obtain the *upper* boundary of ξ where the polynomial is positive because the coefficient with ξ^3 is negative. Starting from $\alpha=1.65$ we have two domains where the polynomial is positive [see Fig. 4(IV)]. First one closes to $-0.088<\xi<0.358$ and another one is smaller than some negative value of ξ. For $\alpha=2.08$ the high boundary of the second domain will coincide with the low boundary of the first domain and we get the only domain where the polynomial is positive $\xi<0.309$. For a greater value of α this high boundary of ξ tends to be constant [see Fig. 4(V)]. Indeed, in the limit $\alpha\to\infty$ we have

$$P\approx\pi^2\sqrt{\frac{\alpha}{3}}\left[-5\xi^3+\frac{5}{2}\xi^2-\frac{1}{2}\xi+\frac{1}{21}\right] \tag{60}$$

and it is positive for all $\xi<0.254$. We would like to note that for $\alpha>1.136$ the polynomial is positive for $\xi=1/6$.

Let us consider now what condition gives the Einstein's equations. The energy corresponding for this configuration is

$$E=-\frac{c^4}{2G}\int_{-\infty}^{\infty}G_t^t r^2(\rho)d\rho=-\frac{2c^4}{G}\left[1-\frac{15-\pi^2}{18}\alpha\right]a. \tag{61}$$

Equating this energy with zero point energy

$$E^{\rm ren}=-\frac{\hbar c}{a}f(am,\alpha,\xi),$$

we obtain the relation

$$\left[1-\frac{15-\pi^2}{18}\alpha\right]a^2=l_P^2 f(am,\alpha,\xi).$$

To find parameters of stable wormholes of this kind we have to consider this equation at the minimum of the function $f(am,\alpha,\xi)$. Because the function $f(am,\alpha,\xi)$ at the minimum is positive, we conclude that the stable wormhole may exist for $\alpha<18/(15-\pi^2)=3.5$. For $\alpha=3.5$ the polynomial is equal to zero for $\xi=0.278$. Therefore the stable wormholes with this profile of the throat may exist only for $\tau/a<3.5$. This upper boundary depends on the model of the throat. For example, the wormhole with the profile of the throat,

$$r(\rho)=\rho\tan\frac{\rho}{\tau}+a,$$

gives another boundary, namely, $\tau/a<36/(\pi^2-6)=9.3$. Specific values of τ,a and a region of ξ may be found by numerical calculation of the function $f(am,\alpha,\xi)$.

VIII. CONCLUSION

In the paper we analyzed the possibility of the existence of the semiclassical wormholes with the metric (1) and the throat's profile given by Eqs. (2),(3). Our approach consists of considering two heat kernel coefficients B_2 and B_3. We developed a method for the calculation of the heat kernel coefficient and obtained a general expression for an arbitrary coefficient in the background (1). The first seven coefficients in manifest form for an arbitrary profile of the throat are given by Eqs. (34) and (35). The sufficient condition for the existence of a wormhole is positivity of both B_2 and B_3. Some additional conditions may follow from the $t-t$ component of the Einstein's equations.

The common property of both models is that the zero point energy for a small size of wormhole is always positive for arbitrary constant ξ. This statement is opposite that obtained for the zero length throat model in Ref. [8]. The behavior of the zero point energy for a large wormhole crucially depends on the nonminimal coupling ξ and parameters of the model. We show that the wormholes with the first profile of the throat may be a self-consistent solution of semiclassical Einstein's equations if the constant of nonminimal connection $\xi>0.266$. This type of wormhole is characterized by the only parameter a, which is the radius of the wormhole's throat. The space outside of the throat polynomially tends to be Minkowkian and there is no way to define the length of the throat. We would like to note that the minimal connection $\xi=0$ and the conformal connection $\xi=1/6$ do not obey this condition.

The second model of the wormhole's throat (3) is characterized by two parameters τ and a. The latter is the radius of the wormhole's throat and the first is the length of the throat. It is possible to introduce the length of the throat because the space outside the throat becomes Minkowskian exponentially fast. A suitable illustration for this statement is Fig. 1(III). The existence of this kind of wormhole crucially depends on the parameter ξ and the ratio of the length and the radius of the throat: $\alpha=\tau/a$. The general condition for α follows from Einstein's equations, namely $\alpha<3.5$. The wormhole with a very small parameter α may be self-consistent, considered by a scalar massive field with a large value of $\xi\sim 1/\alpha$. The scalar field with conformal connection $\xi=1/6$ may self-consistently describe wormholes with $\tau/a \in (1.136,3.5)$. For $\xi=0$ we obtain another interval $\tau/a \in (1.473,3.5)$.

We would like to note that in the limit of zero length of the throat $\alpha=\tau/a\to 0$ there is no connection with the results of our last paper [8], where we considered a wormhole with zero length of throat. The point is that the model considered in that paper was singular at the beginning. The scalar curvature was singular at the throat and there was a singular surface with codimension one. For this case in Ref. [15] the general formulas for heat kernel coefficients were obtained which cannot be found to be a limited case of expression for the smooth background [11]. The reason for this lies in the following. The heat kernel coefficients are defined as an expansion of the heat kernel over some dimension parameter which must be smaller than the characteristic scale of the background. For a smooth background we may make this ratio small by taking the appropriate value of the expansion parameter, but a singular background has at the beginning the zero value of the background's scale. This leads to a new form of heat kernel coefficients. Furthermore, in the limit of large box $R\to\infty$ in this background the coefficient $B_{5/2}$ survives and it defines the behavior of energy for large wormhole.

Another interesting achievement of the paper is developing the zeta-function approach [13]. The radial equation in this background (15) cannot be solved in close form even for the simple profile of the throat (2). We obtain the general formula for the asymptotic expansion of solutions (24), using which we found the heat kernel coefficients (34), (35) in general form. After renormalization the zero point energy may be expressed in terms of the S matrix of the scattering problem (44), (48). More precisely, we need only the transmission coefficient s_{11} of the barrier (47), (49). The point is that the radial equation for the massive scalar field in the background (1) looks like a one-dimensional Schrödinger equation (46) for a particle with potential (47). This potential depends on both the orbital momentum l of a particle and a nonminimal coupling constant ξ, as well as on the radius of the throat a of the wormhole.

In the first model the domain of ξ for which the energy may possess a minimum is limited from below. The reason for this is connected with the fact that the effective mass

$$m_{eff}^2=m^2+\xi\mathcal{R}=m^2-\frac{2\xi a^2}{(\rho^2+a^2)^2}$$

may change its sign for some ξ limited from below. The same situation occurs in the short-throat flat-space wormhole [8] where the scalar curvature is negative, too. This is not the case for the second model. The scalar curvature in this model

$$\mathcal{R}=-\frac{6y^2+4\alpha-5\alpha^2+[4y^2+\alpha(-4+5\alpha)]\cosh(2y/\alpha)-2y(-2+5\alpha)\sinh(2y/\alpha)}{\tau^2\sinh^4(y/\alpha)[1-\alpha+y\coth(y/\alpha)]^2}$$

may change its sign, depending on the parameters of the model. For small α it is negative but starting from $\alpha=4/3$ the domain around $y=0$ appears, where the curvature becomes positive. This domain becomes larger for larger values of α and for α great enough the curvature is, in fact, positive. It is in qualitative agreement with the above consideration. Indeed for small values of α (negative scalar curvature) we obtained a *low* boundary for ξ and vice versa for large values of α (positive scalar curvature) we obtained an *upper* boundary for ξ.

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APPENDIX

In this appendix we reproduce in manifest form some expressions which are rather long to reproduce in the text. First of all let us consider the first five terms of uniform expansion:

$$\begin{aligned}S^+(x)+S^-(-x)&=\ln(a_k)-\frac{1}{2}\ln(\dot{S}_{-1}^2 r_k^4)\\&\quad+\sum_{k=0}^{\infty}\nu^{1-2k}\int_{-x}^{+x}\dot{S}_{2k-1}^+ dx\\&\quad-\ln\left\{1+\sum_{k=1}^{\infty}\nu^{-2k}\frac{\dot{S}_{2k-1}^+}{\dot{S}_{-1}^+}\right\}\\&=\sum_{k=0}^{\infty}\nu^{1-2k}\int_{-x}^{+x}\dot{S}_{2k-1}^+ dx+\sum_{k=0}^{\infty}\nu^{-2k}E_{2k}.\end{aligned}$$

The coefficients with odd powers of ν read (we give the integrands only)

$$[\nu^1]:\dot{S}^+_{-1}=\sqrt{1+\frac{1}{r_k^2}},$$

$$[\nu^{-1}]:\dot{S}^+_1=-\frac{\xi(-1+r_k'^2+2r_kr_k'')}{r_k\sqrt{1+r_k^2}}+\frac{(-1+r_k'^2)+2r_kr_k''+2r_k^2(-1+3r_k'^2)+6r_k^3r_k''-r_k^4+4r_k^5r_k''}{8r_k(1+r_k^2)^{5/2}},$$

$$\begin{aligned}[\nu^{-3}]:\dot{S}^+_3=&-\frac{\xi^2(-1+r_k'^2+2r_kr_k'')^2}{2r_k(1+r_k^2)^{3/2}}-\frac{\xi}{8r_k(1+r_k^2)^{7/2}}\{-(-1+r_k'^2)^2+4r_k(1+r_k'^2)r_k''-2r_k^2(1-8r_k'^2+7r_k'^4-2r_k''^2-8r_k'r_k^{(3)})\\&+2r_k^3[(7-31r_k'^2)r_k''+2r_k^{(4)}]+r_k^4(-1-11r_k'^2+12r_k'^4-4r_k''^2+12r_k'r_k^{(3)})+2r_k^5[(5-8r_k'^2)r_k''+4r_k^{(4)}]-4r_k^6(2r_k''^2\\&+r_k'r_k^{(3)})+4r_k^7r_k^{(4)}\}+\frac{1}{128r_k(1+r_k^2)^{11/2}}\{-(-1+r_k'^2)^2+4r_k(1+3r_k'^2)r_k''+4r_k^2(-1+8r_k'^2+9r_k'^4+3r_k''^2+8r_k'r_k^{(3)})\\&+4r_k^3[(7+43r_k'^2)r_k''+2r_k^{(4)}]+2r_k^4(-3+17r_k'^2-294r_k'^4+40r_k''^2+88r_k'r_k^{(3)})\\&+r_k^5[4(15-136r_k'^2)r_k''+40r_k^{(4)}]+4r_k^6(-1-5r_k'^2+120r_k'^4+23r_k''^2+56r_k'r_k^{(3)})+4r_k^7[(13-168r_k'^2)r_k''+18r_k^{(4)}]\\&-r_k^8(1+24r_k'^2+8r_k''^2-48r_k'r_k^{(3)})+8r_k^9[2(1+2r_k'^2)r_k''+7r_k^{(4)}]-32r_k^{10}(r_k''^2+r_k'r_k^{(3)})+16r_k^{11}r_k^{(4)}\},\end{aligned}\tag{A1a}$$

and the coefficients with even powers of ν are

$$[\nu^0]:E_0=\ln(a_k)-\frac{1}{2}\ln(\dot{S}^2_{-1}r_k^4),$$

$$[\nu^{-2}]:E_2=-\frac{\dot{S}^+_1}{\dot{S}^+_{-1}}.\tag{A1b}$$

The functions s_p are defined by the relation

$$\begin{aligned}\frac{\partial}{\partial k}[S^+(x)+S^-(-x)]&=\frac{\partial}{\partial k}\left\{\sum_{p=0}^{\infty}\nu^{1-2p}\int_{-kR}^{+kR}\dot{S}^+_{2p-1}(x)dx+\sum_{k=0}^{\infty}\nu^{-2k}E_{2k}\right\}\\&=\sum_{p=0}^{\infty}\nu^{1-2p}\int_{-R}^{+R}s_{2p-1}(k\rho)d\rho+\sum_{p=0}^{\infty}\nu^{-2p}s_{2p}.\end{aligned}$$

Here is a list of the first four functions s_p with odd indices [here $r=r(\rho)$ and $z=1+k^2r^2(\rho)$]

$$s_{-1}=rz^{-1/2},$$

$$s_1=z^{-3/2}r\left[\xi(-1+r'^2+2rr'')+\frac{1}{8}\{1-4rr''\}\right]+z^{-5/2}\frac{3r}{4}[-3r'^2+rr'']+z^{-7/2}\left[\frac{25}{8}rr'^2\right],$$

$$\begin{aligned}s_3=&z^{-5/2}\frac{3r}{2}\left[\xi^2(-1+r'^2+2rr'')^2+\xi\frac{1}{4}\{(-1+r'^2)(1+12r'^2)-2r(-5+8r'^2)r''-4r^2(2r''^2+r'r^{(3)})+4r^3r^{(4)}\}\right.\\&\left.+\frac{1}{64}\{1+24r'^2-16r(1+2r'^2)r''+32r^2(r''^2+r'r^{(3)})-16r^3r^{(4)}\}\right]+z^{-7/2}\frac{5r}{4}\left[\xi\{-19(-1+r'^2)r'^2\right.\\&\left.-3r(1+5r'^2)r''+2r^2(3r''^2+5r'r^{(3)})\}+\frac{1}{8}\{-r'^2(19+120r'^2)+r(3+200r'^2)r''-2r^2(15r''^2+22r'r^{(3)})+2r^3r^{(4)}\}\right]\end{aligned}$$

$$
\begin{aligned}
&+z^{-9/2}\frac{175r}{8}\left[\xi r'^2(-1+r'^2+2rr'')+\frac{1}{200}\{1014r'^4+38r^2r''^2+r'^2(25-832rr'')+56r^2r'r^{(3)}\}\right]\\
&+z^{-11/2}\left[\frac{1989rr'^2}{32}(-3r'^2+rr'')\right]+z^{-13/2}\left[\frac{12155rr'^4}{128}\right],
\end{aligned}
$$

$$
\begin{aligned}
s_5=&z^{-7/2}\frac{5r}{2}\left[\xi^3(-1+r'^2+2rr'')^3+\frac{\xi^2}{8}\{(-1+r'^2)^2(3+112r'^2)+12r(-1+r')(1+r')(4+5r'^2)r''-4r^2(-27r''^2\right.\\
&+48r'^2r''^2-26r'r^{(3)}+26r'^3r^{(3)})+24r^3(-2r''^3-2r'r''r^{(3)}-r^{(4)}+r'^2r^{(4)})+8r^4(5r^{(3)\,2}+6r''r^{(4)})\}+\frac{\xi}{64}\{(-1+r'^2)\\
&\times(3+224r'^2+960r'^4)-2r(-39-860r'^2+1424r'^4)r''+8r^2(-54r''^2+186r'^2r''^2-69r'r^{(3)}+80r'^3r^{(3)})\\
&+8r^3(12r''^3+32r'r''r^{(3)}+11r^{(4)})-16r^4(11r^{(3)\,2}+15r''r^{(4)}+3r'r^{(5)})+16r^5r^{(6)}\}+\frac{1}{512}\{1+112r'^2+960r'^4-4r(9\\
&+424r'^2+192r'^4)r''+16r^2(21r''^2+106r'^2r''^2+28r'r^{(3)}+48r'^3r^{(3)})-64r^3(7r''^3+27r'r''r^{(3)}+r^{(4)}+6r'^2r^{(4)})\\
&\left.+32r^4(9r^{(3)\,2}+13r''r^{(4)}+4r'r^{(5)})-32r^5r^{(6)}\}\right]+z^{-9/2}\frac{7r}{8}\left[5\xi^2(-1+r'^2+2rr'')[-29r'^4+r'^2(29-15rr'')\right.\\
&+3rr''(-1+2rr'')+20r^2r'r^{(3)}]+\frac{\xi}{4}\{-(-1+r'^2)r'^2(145+2628r'^2)+r(-15-2839r'^2+2592r'^4)r''+2r^2(128r''^2\\
&+643r'^2r''^2+191r'r^{(3)}+284r'^3r^{(3)})-2r^3(150r''^3+638r'r''r^{(3)}+5r^{(4)}+165r'^2r^{(4)})+8r^4(7r^{(3)\,2}+12r''r^{(4)}\\
&+7r'r^{(5)})\}+\frac{1}{64}\{-r'^2(145+5256r'^2+20160r'^4)+3r(5+1776r'^2+17952r'^4)r''-4r^2(113r''^2+8440r'^2r''^2\\
&+166r'r^{(3)}+3864r'^3r^{(3)})+4r^3(676r''^3+2816r'r''r^{(3)}+5r^{(4)}+756r'^2r^{(4)})-8r^4(58r^{(3)\,2}+93r''r^{(4)}+46r'r^{(5)})\\
&\left.+8r^5r^{(6)}\}\right]+z^{-11/2}\frac{9r}{16}\left[175r'^2\xi^2(-1+r'^2+2rr'')^2+\frac{\xi}{4}\{(-1+r'^2)r'^2(175+15054r'^2)+2rr'^2(4413+1976r'^2)r''\right.\\
&-14r^2(19r''^2+813r'^2r''^2+28r'r^{(3)}+414r'^3r^{(3)})+4r^3(133r''^3+638r'r''r^{(3)}+221r'^2r^{(4)})\}+\frac{1}{64}\{r'^2(175+30108r'^2\\
&+393408r'^4)-8rr'^2(2119+88298r'^2)r''+4r^2(133r''^2+65998r'^2r''^2+196r'r^{(3)}+31264r'^3r^{(3)})-16r^3(628r''^3\\
&\left.+2729r'r''r^{(3)}+791r'^2r^{(4)})+8r^4(69r^{(3)\,2}+110r''r^{(4)}+54r'r^{(5)})\}\right]+z^{-13/2}\frac{11r}{32}\left[221\xi r'^2\{-37(-1+r'^2)r'^2\right.\\
&-9r(1+5r'^2)r''+2r^2(9r''^2+5r'r^{(3)})\}+\frac{1}{8}\{-r'^4(8177+332178r'^2)+rr'^2(1989+396718r'^2)r''-4r^2r'^2(21015r''^2\\
&\left.+10168r'r^{(3)})+2r^3(631r''^3+2782r'r''r^{(3)}+815r'^2r^{(4)})\}\right]+z^{-15/2}\frac{13rr'^2}{128}\left[12155\xi r'^2(-1+r'^2+2rr'')\right.\\
&\left.+\frac{1}{8}\{r'^2(12155+2052348r'^2)-1484372rr'^2r''+4r^2(34503r''^2+16880r'r^{(3)})\}\right]+z^{-17/2}\left[\frac{3727125rr'^4}{512}(-3r'^2+rr'')\right]\\
&+z^{-19/2}\left[\frac{7040125rr'^6}{1024}\right],
\end{aligned}
\tag{A2a}
$$

and here is a list of the first three functions s_p with even indices [$r=r(R)$ and $z=1+k^2r^2(R)$]

$$s_0=-r^2z^{-1},$$

$$s_2=-z^{-2}2r^2\left[\frac{1}{8}(1-4rr'')+\xi(-1+r'^2+2rr'')\right]-z^{-3}r^2[-3r'^2+rr'']-z^{-4}\left[\frac{15r^2r'^2}{4}\right],$$

$$s_4 = -z^{-3}r^2\Big[4\xi^2(-1+r'^2+2rr'')^2+\xi\{(-1+r'^2)(1+6r'^2)-2r(-4+5r'^2)r''-2r^2(4r''^2+r'r^{(3)})+2r^3r^{(4)}\}$$
$$+\frac{1}{16}\{1+12r'^2-4r(3+4r'^2)r''+8r^2(3r''^2+2r'r^{(3)})-8r^3r^{(4)}\}\Big]-z^{-4}3r^2\Big[\xi[-11(-1+r'^2)r'^2-2r(1+5r'^2)r''$$
$$+r^2(4r''^2+5r'r^{(3)})]+\frac{1}{8}\{-(1+4r'^2)(-1+15r'^2)+2r(1+53r'^2)r''+r^2(-17r''^2-22r'r^{(3)})+r^3r^{(4)}\}\Big]$$
$$-z^{-5}r^2\Big[30\xi r'^2(-1+r'^2+2rr'')+\frac{1}{4}\{3r'^2(5+172r'^2)-432rr'^2r''+4r^2(5r''^2+7r'r^{(3)})\}\Big]$$
$$-z^{-6}\Big[\frac{565r^2r'^2}{8}(-3r'^2+rr'')\Big]-z^{-7}\Big[\frac{1695r^2r'^4}{16}\Big]. \tag{A2b}$$

Here are the functions ω_a, ω_b and ω_c with definitions of the corresponding integrals:

$$\omega_a=\pi^2\left[-\frac{2}{3}a+32a^3\int_0^1\frac{d\nu\,\nu}{e^{2\pi\nu a}+1}\left(\sqrt{1-\nu^2}+\nu^2\ln\left[\frac{\nu}{1+\sqrt{1-\nu^2}}\right]\right)\right],$$
$$\omega_b=-\frac{5}{3}\pi^4a^2+a\pi^2\left[-16(\gamma+2\ln 2)\xi+\frac{4}{3}(3+2\gamma)\right]-8\pi^2a\ln(2a)[1-\tanh(\pi a)](1-2\xi)-32\pi^2a\xi V_1$$
$$+\frac{4}{3}\pi^2a[V_1+6V_2-4\pi aV_3+2\pi aV_4-5\pi^2a^2V_5],$$
$$\omega_c=U_1+U_2+U_3+U_4+U_5+U_6. \tag{A3}$$

Here we introduced five functions for ω_b

$$V_1=\frac{\pi a}{2}\int_0^1\frac{\ln(2a\nu)}{\cosh^2(\pi a\nu)}d\nu-\int_0^1\frac{\nu d\nu}{e^{2\pi a\nu}+1}\left[\ln\left(\frac{\nu}{1+\sqrt{1-\nu^2}}\right)+\frac{1}{1+\sqrt{1-\nu^2}}\right],$$
$$V_2=\int_0^1\frac{\nu d\nu}{e^{2\pi a\nu}+1}\ln\left(\frac{\nu}{1+\sqrt{1-\nu^2}}\right),$$
$$V_3=\int_0^1\frac{d\nu}{\cosh^2(\pi a\nu)}\ln\left(\frac{\nu}{1+\sqrt{1-\nu^2}}\right),$$
$$V_4=\int_0^1\frac{d\nu\sqrt{1-\nu^2}}{\cosh^2(\pi a\nu)},$$
$$V_5=\int_0^1\frac{\sinh(\pi a\nu)d\nu}{\cosh^2(\pi a\nu)}[\nu\sqrt{1-\nu^2}-\arccos\nu],$$

and six for ω_c

$$U_1=-\frac{229\pi^4}{2520}-\pi a\int_0^1 f_1(a\nu)\arccos\nu\, d\nu,$$
$$U_2=\pi a\int_1^\infty f_3(a\nu)\frac{d\nu}{\nu}+\pi a\int_0^1\frac{f_3(a\nu)\nu d\nu}{1+\sqrt{1-\nu^2}},$$
$$U_3=\pi a\int_1^\infty f_5(a\nu)\frac{d\nu}{\nu}\left(\frac{1}{2}+\frac{1}{3\nu^2}\right)+\frac{\pi}{3}a\int_0^1 f_5(a\nu)\nu d\nu\left[\frac{2}{1+\sqrt{1-\nu^2}}+\frac{1}{2(1+\sqrt{1-\nu^2})^2}\right],$$
$$U_4=\frac{\pi^2}{2}\int_0^1\frac{f_2 d\nu}{\cosh^2(\pi a\nu)}\ln\left(\frac{\nu}{1+\sqrt{1-\nu^2}}\right),$$
$$U_5=\frac{\pi^2}{4}\int_1^\infty\frac{f_4 d\nu}{\nu^2\cosh^2(\pi a\nu)}+\frac{\pi^2}{4}\int_0^1\frac{f_4 d\nu}{\cosh^2(\pi a\nu)}\left[\ln\left(\frac{\nu}{1+\sqrt{1-\nu^2}}\right)+\frac{\nu}{1+\sqrt{1-\nu^2}}\right],$$

$$U_5=\frac{\pi^2}{8}\int_1^\infty \frac{f_6(\nu^2+1)d\nu}{\nu^4\cosh^2(\pi a\nu)}+\frac{3\pi^2}{16}\int_0^1\frac{f_6 d\nu}{\cosh^2(\pi a\nu)}\left[\ln\left(\frac{\nu}{1+\sqrt{1-\nu^2}}\right)+\frac{\nu}{1+\sqrt{1-\nu^2}}+\frac{1}{3}\frac{\nu}{(1+\sqrt{1-\nu^2})^2}\right],$$

where

$$f_1(\nu)=\frac{4\pi}{315}[14595\Pi_2(\nu)-43638\Pi_3(\nu)+47736\Pi_4(\nu)-17680\Pi_5(\nu)],$$

$$f_3(\nu)=\frac{32\pi\xi}{3}[29\Pi_2(\nu)-10\Pi_3(\nu)]-\frac{8\pi}{315}[32445\Pi_2(\nu)-64113\Pi_3(\nu)+55692\Pi_4(\nu)-17680\Pi_5(\nu)],$$

$$f_5(\nu)=-\frac{64\pi\xi}{3}[16\Pi_2(\nu)-5\Pi_3(\nu)]+\frac{32\pi}{315}[6720\Pi_2(\nu)-10773\Pi_3(\nu)+7956\Pi_4(\nu)-2210\Pi_5(\nu)],$$

$$f_2(\nu)=-\frac{229\pi}{630},$$

$$f_4(\nu)=\frac{376\pi}{3}\xi-\frac{5948\pi}{315},$$

$$f_6(\nu)=32\pi\xi^2-\frac{544\pi}{3}\xi+\frac{8824\pi}{315},$$

and

$$\Pi_2(\nu)=\frac{\pi^2}{2}Sc(\nu),$$

$$\Pi_3(\nu)=\pi^2\left[Sc(\nu)+\frac{1}{8}\nu Sc'(\nu)\right],$$

$$\Pi_4(\nu)=\pi^2\left(\frac{3}{2}Sc(\nu)+\frac{17}{48}\nu Sc'(\nu)+\frac{1}{48}\nu^2 Sc''(\nu)\right),$$

$$\Pi_5(\nu)=\pi^2\left(2Sc(\nu)+\frac{259}{384}\nu Sc'(\nu)+\frac{29}{384}\nu^2 Sc''(\nu)+\frac{1}{384}\nu^3 Sc'''(\nu)\right),$$

$$Sc(\nu)=\frac{\sinh(\pi a\nu)}{\cosh^3(\pi a\nu)}.$$

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