Reconstruction of inhomogeneous metric perturbations and electromagnetic four-potential in Kerr spacetime

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We present a procedure that allows the construction of the metric perturbations and electromagnetic four-potential, for gravitational and electromagnetic perturbations produced by sources in Kerr spacetime. This may include, for example, the perturbations produced by a point particle or an extended object moving in orbit around a Kerr black hole. The construction is carried out in the frequency domain. Previously, Chrzanowski derived the vacuum metric perturbations and electromagnetic four-potential by applying a differential operator to a certain potential Ψ. Here we construct Ψ for inhomogeneous perturbations, thereby allowing the application of Chrzanowski's method. We address this problem in two stages: First, for vacuum perturbations (i.e. pure gravitational or electromagnetic waves), we construct the potential from the modes of the Weyl scalars ψ_0 or φ_0. Second, for perturbations produced by sources, we express Ψ in terms of the mode functions of the source, i.e. the energy-momentum tensor $T_{\alpha\beta}$ or the electromagnetic current vector J_α.

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I. INTRODUCTION

The gravitational perturbations of Kerr black holes (BHs) are fully described by the metric perturbation (MP) $h_{\alpha\beta}$. The latter satisfies the linearized Einstein equation, which is a set of coupled, linear, partial differential equations. Teukolsky [1,2] showed, however, that the curvature Weyl scalars ψ_0 and ψ_4 each satisfy a decoupled field equation, the "master equation." Furthermore, this decoupled equation may be separated, leading to ordinary differential equations for the radial and angular parts. This leads to a great simplification of the problem of determining the gravitational perturbations.

The problem of electromagnetic perturbations over a Kerr background has a similar status. The Maxwell equations form a set of coupled, linear, partial differential equations for the four-potential A_α (or for the Maxwell field $F_{\alpha\beta}$). In this case, too, Teukolsky [1,2] derived separable, decoupled, equations for the two Maxwell scalars φ_0 and φ_2.

For several problems, e.g. the calculation of energy and angular-momentum outflux to infinity, knowledge of the Teukolsky variables (e.g. ψ_4 or φ_2) is sufficient. However, there are problems for which one needs the full perturbation field (i.e. the MP $h_{\alpha\beta}$ in the gravitational case, and A_α—or alternatively the full tensor field $F_{\alpha\beta}$—in the electromagnetic case). This includes, for example, the calculation of gravitational or electromagnetic self-force acting on a pointlike particle orbiting a spinning BH.

In principle, each of the Weyl scalars ψ_0 and ψ_4 contains the full information on the gravitational perturbation in vacuum [3] (up to a few nonradiative degrees of freedom, e.g. infinitesimal changes in the BH's mass and spin). Chrzanowski [4] developed a procedure which allows the determination of the general homogeneous (i.e. vacuum) solution for the MP $h_{\alpha\beta}$, by applying a certain differential operator to the homogeneous solutions for ψ_0 or ψ_4. It was later shown [5], however, that this operator, when applied to a particular solution ψ_0 or ψ_4 yields a result $h_{\alpha\beta}$ which is a valid vacuum solution, but yet it represents a *physically different* gravitational perturbation. Let us rephrase this in a more explicit manner: Consider a vacuum gravitational perturbation characterized by a particular function ψ_4. Then, there exists a certain function Ψ, from which $h_{\alpha\beta}$ can be constructed by applying Chrzanowski's differential operator. This function Ψ satisfies the same Teukolsky equation as the function $\rho^{-4}\psi_4$ (where ρ is a certain quantity defined below), but yet Ψ does *not* coincide with the quantity $\rho^{-4}\psi_4$ of the gravitational perturbation under consideration.

The same situation occurs in the case of electromagnetic perturbations. Here, too, the full information about the (radiative part of the) electromagnetic perturbation is encoded in each of the Maxwell scalars φ_0 and φ_2 [3]. Chrzanowski's method [4] allows the determination of the general, homogeneous solution for A_α by applying a certain differential operator to the homogeneous solutions for φ_0 or φ_2. However, this procedure, when applied to a particular solution φ_0 or φ_2, yields a vacuum solution A_α which represents a physically different electromagnetic perturbation [5].

In view of the above, the problem of constructing the MP $h_{\alpha\beta}$ (the four-potential A_α) from ψ_0 or ψ_4 (φ_0 or φ_2) includes two stages: First, construct the potential Ψ from ψ_0 or ψ_4 (φ_0 or φ_2), and second, construct $h_{\alpha\beta}$ (A_α) from Ψ. The second part is well known—this is Chrzanowski's procedure [4]. The goal of the present paper is to address the first part, namely, the determination of Ψ from ψ_0 (or from φ_0 in the electromagnetic case).[1] This problem was recently addressed, for gravitational perturbations, by Lousto and Whiting (LW) [6] in the case of a Schwarzschild BH. Here we provide the solution to this problem in the Kerr case (in the frequency domain).

[1]We shall restrict our attention in this paper to the construction of $h_{\alpha\beta}$ or A_α in the *ingoing* radiation gauge from ψ_0 or φ_0, respectively. The analogous problem of constructing $h_{\alpha\beta}$ or A_α in the outgoing radiation gauge from ψ_4 or φ_2 may be treated in a similar way.

We shall consider here two different physical situations: (i) pure gravitational or electromagnetic waves (i.e. a vacuum perturbation in the entire spacetime), and (ii) gravitational (or electromagnetic) perturbations produced by a (charged) object orbiting the BH. The first problem is fairly simple, but the second one, that of perturbations with sources, is a bit more involved. The explicit solution for Ψ in this case of inhomogeneous perturbations, which is the primary goal of this paper, is summarized in Secs. VIII (gravitational case) and IX (electromagnetic case).

The equations relating the potential Ψ to the relevant Teukolsky variables were derived by Wald [5] for a general algebraically special, vacuum, background spacetime. The reduction of these equations to the Kerr case is given in Ref. [7] for the electromagnetic case and in Ref. [6] for the gravitational case. Our goal is to determine Ψ by solving these equations. For either the gravitational or electromagnetic case, there are two such differential equations relating Ψ to the Teukolsky variables: a *radial equation* (i.e. one including r derivatives), which relates Ψ to ψ_0 or φ_0, and an *angular equation* (i.e. one including θ derivatives), relating Ψ to ψ_4 or φ_2. In Ref. [6] LW elaborated on the angular equation [Eq. (2.7) therein], and constructed its solution in the Schwarzschild case (for gravitational perturbations). Here we shall elaborate on the radial equation [Eq. (2.6) therein, or its electromagnetic counterpart]. This in fact turns out to be a simple ordinary differential equation, which is not difficult to solve even in the Kerr case.

The MP $h_{\alpha\beta}$ and four-potential A_α constructed via Chrzanowski's method are given in the ingoing (or outgoing) radiation gauge [4]. Barack and Ori [8] recently investigated the local asymptotic behavior of the radiation gauge $h_{\alpha\beta}$ (either the ingoing or outgoing one) near a point particle, by locally integrating the equations defining this gauge. They found that in this gauge $h_{\alpha\beta}$ cannot be well defied all around the particle. Instead, there is a line of singularity that emerges from the particle to either the ingoing or outgoing radial direction, over which $h_{\alpha\beta}$ diverges. (This line forms a $1+1$ dimensional singularity set in spacetime.) One can choose to have a regular function $h_{\alpha\beta}$ at $r>r_{particle}$, where r is the radial coordinate, but this will inevitably lead to a line singularity at $r<r_{particle}$; and vice versa. (Barack and Ori demonstrated this in the simplest case, i.e. a static particle located at $r=r_{particle}$ in flat spacetime, but the same situation should occur also for moving particles in Kerr.) Although the analysis in Ref. [8] was restricted to the gravitational case, it is easily extended to the electromagnetic case as well. It shows that the radiation-gauge A_α also has a line singularity, either at $r>r_{particle}$ or at $r<r_{particle}$.

The solution constructed here provides an independent demonstration to this pathology of the radiation-gauge quantities $h_{\alpha\beta}$ and A_α near a point source. Throughout this paper we shall assume that the source is confined to a range $r_{\min}\leq r\leq r_{\max}$ (consider, e.g. a point mass moving on an elliptical or circular orbit). In both the electromagnetic and gravitational cases (and for either the ingoing or outgoing radiation gauge), one may choose to integrate the equations governing Ψ from $r>r_{\max}$ towards smaller r values. Then Ψ is perfectly regular at $r>r_{\max}$; but it turns out that at $r<r_{\max}$, Ψ is irregular on an outgoing null ray emerging form the particle inwardly (i.e. in the past direction). Alternatively one may integrate these equations from $r<r_{\min}$ towards larger r values, in which case Ψ is perfectly regular at $r<r_{\min}$ but at $r>r_{\min}$ it develops an irregularity on an outgoing null ray emerging form the particle. This is perfectly consistent with the above-mentioned irregularity of the radiation-gauge $h_{\alpha\beta}$ and A_α, found earlier (for $h_{\alpha\beta}$) by Barack and Ori [8]. Throughout this paper we shall mostly refer to the solution for Ψ which is regular at $r>r_{\max}$ but has a line singularity at $r<r_{\max}$, which we shall denote Ψ^+ for concreteness. The analogous solution Ψ^- (which is regular at $r<r_{\min}$ but has a line singularity at $r>r_{\min}$) may be constructed in a fully analogous manner, as briefly summarized at the end of Secs. VIII and IX. (In Sec. X we briefly discuss the possible implications of this line singularity to the self-force problem.)

The two solutions Ψ^+ and Ψ^- yield two different solutions for the MP $h_{\alpha\beta}$ or the four-potential A_α, both for the "same" (e.g. the ingoing) radiation-gauge condition, which we denote $h^+_{\alpha\beta},A^+_\alpha$ and $h^-_{\alpha\beta},A^-_\alpha$, correspondingly. To avoid confusion we emphasize that these two solutions (for either $h_{\alpha\beta}$ or A_α, and, say, in the ingoing gauge) essentially represent *the same physical perturbation*, and they differ by gauge. That is, the ingoing (or outgoing) radiation-gauge condition does not completely fix the gauge.[2]

Since there is full analogy between the gravitational and electromagnetic cases, the detailed calculations presented in the next seven sections will refer to the gravitational case only. The electromagnetic perturbations may be treated exactly in the same manner. Only in Sec. IX we shall return to the electromagnetic case and present the final procedure of constructing Ψ for electromagnetic perturbations.

In Sec. II we give the basic field equations and establish some notation. Section III presents the basic calculation of Ψ in the case of pure gravitational waves, expressing it in terms of the modes of ψ_0. This result is then further simplified in Sec. IV, by expressing both ψ_0 and Ψ in terms of basis solutions (of the relevant homogeneous Teukolsky equation) which admit a simple asymptotic behavior at either the large-r limit or at the event horizon (EH). In Sec. V we develop the general homogeneous solution for Eq. (11), the above-mentioned "radial equation" relating Ψ to ψ_0, which is a fourth-order differential equation. This general homogeneous solution is required later for the construction of the relevant inhomogeneous solution (for perturbations with sources).

In Sec. VI we address the physical situation which provides the main motivation for this paper, namely, gravitational perturbations produced by sources (e.g. a point particle, or an extended object, in orbit around a Kerr BH). We construct the solution for Ψ (more specifically, Ψ^+) in this case, using the general homogeneous solution constructed earlier in Sec. V. In the first stage the potential Ψ is expressed in terms of the inhomogeneous mode functions of ψ_0. Then we further simplify the solution, expressing Ψ di-

[2]It appears, though, that the "+" and "−" perturbations also differ by some (non-gauge) non-radiative component, i.e. the so-called "$l=0$" and/or "$l=1$" modes.

rectly in terms of the mode functions of the energy-momentum distribution in spacetime. (In both stages we also use the homogeneous basis modes of ψ_4 in our expression for Ψ.) Whereas our detailed construction refers to Ψ^+, the construction of Ψ^- proceeds in a fully analogous manner, and we provide the final result for Ψ^- as well. Some details of the calculations are given in Appendixes A and B.

In Sec. VII we study the domain of validity of the solution Ψ^+ (and similarly of Ψ^-). For a point particle we find that Ψ^+ is regular everywhere except at a $(1+1)$ surface spanned by outgoing principal null geodesics emanating from the particle's worldline in the small-r (i.e. past) direction. We denote this surface Σ^+. For an extended object, the solution Ψ^+ (which is typically regular throughout) is valid everywhere except in a domain denoted (again) Σ^+, which is now a four-dimensional set (the definition of which is provided therein). In a fully analogous manner, the other solution Ψ^- is, for an extended object, valid everywhere except in a four-dimensional set Σ^-; and at the point-like limit Σ^- degenerates to a $(1+1)$ surface spanned by outgoing principal null geodesics emanating from the particle's worldline in the large-r (i.e. future) direction, with Ψ^- becoming irregular on Σ^-.

Section VIII provides a summary of the construction of Ψ, in the gravitational case, for the benefit of the reader who wishes to implement this method in practical calculations. Then, in Sec. IX we return to the electromagnetic problem and summarize the procedure of constructing Ψ in this case, leaving many details of the derivation to Appendix C. Finally in Sec. X we give some concluding remarks.

II. PRELIMINARIES

Consider the spacetime of a Kerr BH with mass M and specific angular momentum a. We shall use the standard Boyer-Lindquist coordinates (t,r,θ,φ), and following Teukolsky [1] we denote $\Delta\equiv r^2-2Mr+a^2$, $\Sigma\equiv r^2+a^2\cos^2\theta$, and $\rho\equiv-1/(r-ia\cos\theta)$. The Newman-Penrose Weyl scalars ψ_0 and ψ_4 (corresponding to $s=+2$ and $s=-2$, respectively) satisfy two decoupled master equations. Defining

$$\psi_{+2}\equiv\psi_0,\quad \psi_{-2}\equiv\rho^{-4}\psi_4,$$

we may formally write the two master equations as

$$W_{\pm2}[\psi_{\pm2}]=4\pi\Sigma T_{\pm2},\tag{1}$$

where $W_{\pm2}$ is the second-order partial differential operator

$$\begin{aligned}W_s\equiv&-\Delta^{-s}\partial_r[\Delta^{s+1}\partial_r]+[(r^2+a^2)^2/\Delta-a^2\sin^2\theta]\partial_{tt}\\&+4aMr\Delta^{-1}\partial_{\varphi t}+[a^2/\Delta-\sin^{-2}\theta]\partial_{\varphi\varphi}\\&-\sin^{-1}\theta\partial_\theta[\sin\theta\partial_\theta]-2s[a(r-M)/\Delta\\&+i\cos\theta\sin^{-2}\theta]\partial_\varphi-2s[M(r^2-a^2)/\Delta-r\\&-ia\cos\theta]\partial_t+(s^2\cot^2\theta-s),\end{aligned}\tag{2}$$

and $T_{\pm2}$ is the corresponding energy-momentum source term given explicitly in Refs. [1,2]. Teukolsky further showed that these two Weyl scalars may be decomposed as

$$\psi_{\pm2}=\sum_{\lambda m\omega}R^{\lambda m\omega}_{\pm2}(r)S^{\lambda m\omega}_{\pm2}(\theta)e^{i(m\varphi-\omega t)},\tag{3}$$

where $R^{\lambda m\omega}_{\pm2}$ and $S^{\lambda m\omega}_{\pm2}$ are, respectively, solutions of the radial and angular Teukolsky equations (given below).[3] The source terms are expanded in a similar manner:

$$4\pi\Sigma T_{\pm2}=\sum_{\lambda m\omega}T^{\lambda m\omega}_{\pm2}(r)S^{\lambda m\omega}_{\pm2}(\theta)e^{i(m\varphi-\omega t)}.\tag{4}$$

The radial and angular ordinary differential equations take the form

$$P^{\lambda m\omega}_{\pm2}[R^{\lambda m\omega}_{\pm2}(r)]=T^{\lambda m\omega}_{\pm2}(r)\tag{5}$$

and

$$\Theta^{\lambda m\omega}_{\pm2}[S^{\lambda m\omega}_{\pm2}(\theta)]=0.\tag{6}$$

Here $P^{\lambda m\omega}_{\pm2}$ and $\Theta^{\lambda m\omega}_{\pm2}$ are second-order linear differential operators, given by

$$P^{\lambda m\omega}_s\equiv\Delta^{-s}\partial_r[\Delta^{s+1}\partial_r]+V_r(r)\tag{7}$$

and

$$\Theta^{\lambda m\omega}_s\equiv\sin^{-1}\theta\partial_\theta[\sin\theta\partial_\theta]+V_\theta(\theta),$$

where the potentials are

$$\begin{aligned}V_r(r)=&[(r^2+a^2)^2\omega^2-4aM\omega mr+a^2m^2+2iams(r-M)\\&-2iM\omega s(r^2-a^2)]\Delta^{-1}+2i\omega sr-a^2\omega^2-\tilde{A}\end{aligned}$$

and

$$\begin{aligned}V_\theta(\theta)=&a^2\omega^2\cos^2\theta-m^2/\sin^2\theta-2a\omega s\cos\theta\\&-2ms\cos\theta/\sin^2\theta-s^2\cot^2\theta+s+\tilde{A}.\end{aligned}$$

The parameter $\tilde{A}$ is Teukolsky's [1] separation constant, which we write as

$$\tilde{A}=\lambda-s-|s|,$$

where λ is the separation constant used by Chandrasekhar [9] (often denoted $\lambda\!\!\!^{-}$ there). The parameter λ runs over all eigenvalues of the angular Teukolsky equation (6). Throughout this paper we prefer to use the separation constant λ rather than $\tilde{A}$ because the angular equations for $s=2$ and $s=-2$ have the same set of eigenvalues λ [9] (which is not the case for $\tilde{A}$).[4] Also, this will allow an easier connection with Chandrasekhar's formalism.

Our goal is to construct the MP. In a vacuum spacetime (i.e. $T_{\pm2}=0$), the MP in the ingoing radiation gauge can be

[3] In case the spectrum is continuous, the sum over ω should be replaced by an integral.

[4] In the special case $a\omega=0$, the separation constant λ becomes $l(l+1)-s^2+|s|$; hence, common λ also means common l.

derived from a potential Ψ_{IRG} [4], by applying to the latter a certain second-order differential operator [10]. We shall formally denote this differential operator by Π_{IRG}, namely,

$$h_{IRG}=\Pi_{IRG}[\Psi_{IRG}], \tag{8}$$

where h_{IRG} denotes the MP in the ingoing radiation gauge (for brevity we shall often omit the spacetime indices of the MP). Similarly, the MP in the outgoing radiation gauge can be obtained from another potential Ψ_{OUT}, through another differential operator Π_{OUT} [10], namely,

$$h_{ORG}=\Pi_{ORG}[\Psi_{ORG}].$$

In this paper we shall only consider the case of ingoing radiation gauge, but the potential Ψ_{ORG} (and hence the MP in the outgoing radiation gauge) may be constructed in a fully analogous manner. For brevity we shall use here the notation $\Psi\equiv\Psi_{IRG}$ (this variable is denoted ψ_G in Ref. [5]), $\Pi\equiv\Pi_{IRG}$, and $h_{\alpha\beta}\equiv h_{IRG}$, hence

$$h_{\alpha\beta}=\Pi[\Psi]. \tag{9}$$

The function Ψ has to be a solution of the vacuum Teukolsky equation for ψ_{-2} [5], namely,

$$W_{-2}[\Psi]=0. \tag{10}$$

In addition, it must satisfy the following differential equation [5,6]:

$$\psi_0=D^4[\bar{\Psi}], \tag{11}$$

where throughout this paper an overbar denotes complex conjugation. Here D is the differential operator

$$D=l^\mu\partial_\mu=\frac{r^2+a^2}{\Delta}\partial_t+\partial_r+(a/\Delta)\partial_\varphi,$$

where l^μ is the standard outgoing Kinnersley's tetrad vector (see, e.g. Ref. [1]). We use here the abbreviated notation $D^4\equiv DDDD$, and the same for other operators used below.

Our goal in this paper is to construct the function Ψ that satisfies Eqs. (10) and (11). This will allow the construction of $h_{\alpha\beta}$, the MP in the ingoing radiation gauge, through Eq. (9). We shall first consider the case of pure vacuum gravitational waves in the entire spacetime. In this case we assume that ψ_0 is known (it encodes the information on the gravitational waves). Subsequently we shall consider the case of gravitational perturbations produced by a point particle (or any other matter source) moving in the Kerr spacetime. In this case we shall assume that T_{+2}, the $s=+2$ energy-momentum source term, is known.

III. PURE GRAVITATIONAL WAVES

We now consider the case of pure vacuum gravitational waves, namely, $T_{\pm2}=0$ in the entire spacetime. The information about the gravitational waves is given by means of the Weyl scalar ψ_0. Since we are dealing here with linear perturbations, it will be sufficient to consider a particular mode $\lambda m\omega$ of ψ_0; the entire perturbation is then obtained by a superposition. Thus, we now assume that ψ_0 takes the decomposed form

$$\psi_0=R_{+2}^{\lambda m\omega}(r)S_{+2}^{\lambda m\omega}(\theta)e^{i(m\varphi-\omega t)}, \tag{12}$$

and the radial function satisfies the vacuum radial equation:

$$P_{+2}^{\lambda m\omega}[R_{+2}^{\lambda m\omega}(r)]=0. \tag{13}$$

We shall now use the Teukolsky-Starobinsky relations to show that the desired solution of Eqs. (10),(11) is

$$\bar{\Psi}=p^{-1}\Delta^2(D^\dagger)^4[\Delta^2\psi_0], \tag{14}$$

where p is a constant to be determined later, and $D^\dagger$ is the differential operator

$$D^\dagger=-\frac{r^2+a^2}{\Delta}\partial_t+\partial_r-(a/\Delta)\partial_\varphi.$$

For a single $\lambda m\omega$ mode we define the "reduced" operators

$$D_{m\omega}=\partial_r+iK/\Delta,\quad D_{m\omega}^\dagger=\partial_r-iK/\Delta,$$

where

$$K\equiv am-(r^2+a^2)\omega,$$

such that for any functions $f(r)$ and $g(\theta)$,

$$D[f(r)g(\theta)e^{i(m\varphi-\omega t)}]=g(\theta)e^{i(m\varphi-\omega t)}D_{m\omega}[f(r)],$$

and the same relation holds between the operators $D^\dagger,D^\dagger_{m\omega}$. (Note that $D_{m\omega}$ and $D^\dagger_{m\omega}$ are the same as the operators "$\mathcal{D}_0$" and "$\mathcal{D}_0^\dagger$," respectively, in Chandrasekhar's notation [9].) Using the decomposition

$$\bar{\Psi}=\hat{R}_{-2}^{\lambda m\omega}S_{+2}^{\lambda m\omega}(\theta)e^{i(m\varphi-\omega t)}, \tag{15}$$

Eq. (14) becomes

$$\hat{R}_{-2}^{\lambda m\omega}=p^{-1}\Delta^2(D^\dagger_{m\omega})^4[\Delta^2R_{+2}^{\lambda m\omega}(r)], \tag{16}$$

and Eq. (11) now reduces to

$$R_{+2}^{\lambda m\omega}(r)=(D_{m\omega})^4[\hat{R}_{-2}^{\lambda m\omega}]. \tag{17}$$

Let us first verify that Ψ [the complex conjugate of Eq. (14)] satisfies Eq. (10). The Teukolsky-Starobinsky relations (see, e.g. [9]) imply that the radial function $\hat{R}_{-2}^{\lambda m\omega}(r)$ constructed in Eq. (16) is a solution of the $s=-2$ radial vacuum equation, namely,

$$P_{-2}^{\lambda m\omega}[\hat{R}_{-2}^{\lambda m\omega}(r)]=0. \tag{18}$$

The complex conjugate of the radial function $\hat{R}_{-2}^{\lambda m\omega}$ is a radial vacuum solution with the same λ and $s=-2$, but with negative sign for m and ω, which we denote $\hat{R}_{-2}^{\lambda,-m,-\omega}$. The angular Teukolsky equation is real, and (holding λ fixed) is invariant under the simultaneous change of signs of s,m,ω.

Hence, $S_{+2}^{\lambda m\omega}(\theta)$ is a real function which is also a solution to the angular Teukolsky equation (6) with $-m,-\omega$, and $s=-2$; and correspondingly we may write it as $cS_{-2}^{\lambda,-m,-\omega}$, where c is a constant (whose value is unimportant to us). We find that

$$\Psi = c\hat{R}_{-2}^{\lambda,-m,-\omega}(r)S_{-2}^{\lambda,-m,-\omega}(\theta)e^{-i(m\varphi-\omega t)}. \tag{19}$$

Thus, Ψ is indeed a solution to the $s=-2$ vacuum Teukolsky equation (10)—a solution characterized by the set of indices $\lambda,-m,-\omega$.

We still need to check that $\hat{R}_{-2}^{\lambda m\omega}$ constructed in Eq. (16) satisfies Eq. (17) [this would in turn imply that the expression (14) satisfies Eq. (11)], and to determine the constant p. In fact all we need to show is that

$$(D_{m\omega})^4\{\Delta^2(D_{m\omega}^\dagger)^4[\Delta^2 R_{+2}^{\lambda m\omega}(r)]\}$$

is a constant multiple of $R_{+2}^{\lambda m\omega}$. This follows immediately by applying the two parts of Theorem 1 in Chap. 9 of Ref. [9]. Consequently, there exists a constant p such that

$$R_{+2}^{\lambda m\omega}(r) = p^{-1}(D_{m\omega})^4\{\Delta^2(D_{m\omega}^\dagger)^4[\Delta^2 R_{+2}^{\lambda m\omega}(r)]\}. \tag{20}$$

From the analysis therein it becomes obvious[5] that p is the real constant

$$p=\lambda^2(\lambda+2)^2-8\omega^2\lambda[\alpha^2(5\lambda+6)-12a^2]+144\omega^4\alpha^4 + 144\omega^2M^2, \tag{21}$$

where $\alpha^2\equiv a^2-am/\omega$. Note that the coefficient p depends on the mode.[6]

As was mentioned in Sec. I, the potential Ψ must also satisfy an angular differential equation, i.e. Eq. (2.7) in Ref. [6]. The compliance of the above-constructed vacuum solution with this additional equation is guaranteed by virtue of the following considerations: (i) According to the analysis in Ref. [5], there must exist a solution to the three simultaneous equations [i.e. Eqs. (10) and (11), and the angular equation], and (ii) the solution (15),(16) is the *unique* solution to the simultaneous equations (10) and (11). For any nontrivial solution to the homogeneous part of Eq. (11) will violate Eq. (10) [one can easily verify this, based on the general homogeneous solution to Eq. (11), constructed in Sec. V below].

IV. FURTHER SIMPLIFICATION OF THE VACUUM SOLUTION

Equations (15),(16) provide the full solution for $\overline{\Psi}$. It is possible, however, to construct a simpler and more explicit expression for the radial function $\hat{R}_{-2}^{\lambda m\omega}$. The latter satisfies Eq. (18), which is the vacuum Teukolsky equation for the $s=-2$ radial function $R_{-2}^{\lambda m\omega}$. Since this is a second-order ordinary differential equation, its general solution may be spanned by any pair of independent solutions. Let $R_{\pm2}^{\lambda m\omega(a)}$ and $R_{\pm2}^{\lambda m\omega(b)}$ be two such pairs of independent solutions (one pair for $s=+2$ and one for $s=-2$). Let H denote the operator which maps a vacuum $s=+2$ radial function $R_{+2}^{\lambda m\omega}$ into the corresponding function $\hat{R}_{-2}^{\lambda m\omega}$ of Eq. (16), namely,

$$H\equiv p^{-1}\Delta^2(D_{m\omega}^\dagger)^4\Delta^2.$$

For each mode $\lambda m\omega$ there must exist a constant 2×2 matrix C_{ij} such that

$$H[R_{+2}^{\lambda m\omega(i)}(r)] = C_{ij}R_{-2}^{\lambda m\omega(j)}(r), \tag{22}$$

where i,j run over the two basis states a and b. The problem thus reduces to the determination of the four constants C_{ij}.

There are two preferred bases, however, for which this matrix becomes diagonal and especially easy to calculate. One such basis is the pair of solutions characterized by positive and negative exponents of r_* at large r. Here r_* is a function of r, defined by

$$dr_*/dr=(r^2+a^2)/\Delta \tag{23}$$

(and given explicitly below). Note that $r_*\to\infty$ as $r\to\infty$. The other basis is that of positive and negative exponents of r_* at the EH (where $r_*\to-\infty$). These two bases are also preferable for the physical interpretation of the solution, and for its construction via a Green function (described in Sec. VI). The asymptotic behavior of the vacuum radial Teukolsky functions are given in e.g., Ref. [3] for all values of s, both at the limit of large r and at the EH.

In what follows we shall describe the application of Eq. (22), and the determination of the required coefficients, for these two special bases.

A. Large-*r* asymptotic behavior

Considering the large-r asymptotic behavior of the vacuum radial functions $R_{+2}^{\lambda m\omega}(r)$ and $R_{-2}^{\lambda m\omega}(r)$, we may take the two basic solutions (for each s) to be those of positive and negative exponents of r_*. These two solutions take the asymptotic form

$$R_{+2}^{\lambda m\omega(in)}\cong e^{-i\omega r_*}/r, \quad R_{+2}^{\lambda m\omega(out)}\cong e^{i\omega r_*}/r^5 \tag{24}$$

and

$$R_{-2}^{\lambda m\omega(in)}\cong e^{-i\omega r_*}/r, \quad R_{-2}^{\lambda m\omega(out)}\cong e^{i\omega r_*}r^3 \tag{25}$$

(see [3], and recall the factor $\rho^{-4}\propto r^4$ in the above definition of ψ_{-2}). To avoid confusion we emphasize that the basis solutions $R_{\pm2}^{\lambda m\omega(in,out)}$ are defined to be the *exact* solutions of the corresponding radial equations, which satisfy the asymptotic form (24),(25) at the leading order in $1/r$ (the same remark applies to the event-horizon basis functions defined below).

[5]This may easily be deduced by applying the operator $\Delta^2(D_{m\omega}^\dagger)^4\Delta^2$ to both sides of Eq. (20), and then using Chandrasekhar's Theorem 1, as well as his Eq. (43) (both in Chap. 9 of Ref. [9]).

[6]It is assumed that p is finite and non-vanishing for all real ω.

One can easily verify that the operators $D_{m\omega}, D^{\dagger}_{m\omega}$ do not mix positive and negative exponents or r_*. Therefore, $H[R^{\lambda m\omega(in)}_{+2}]$ and $H[R^{\lambda m\omega(out)}_{+2}]$ must take the simple forms

$$H[R^{\lambda m\omega(in)}_{+2}]=C^{(in)}R^{\lambda m\omega(in)}_{-2} \tag{26}$$

and

$$H[R^{\lambda m\omega(out)}_{+2}]=C^{(out)}R^{\lambda m\omega(out)}_{-2}, \tag{27}$$

and the problem reduces to the determination of the two constants $C^{(in)}$ and $C^{(out)}$. These constants may be determined from the large-r asymptotic form of Eqs. (16) or (17). Ignoring terms of higher order in $1/r$, we have

$$D_{m\omega}\cong\partial_r - i\omega, \quad D^{\dagger}_{m\omega}\cong\partial_r + i\omega,$$

and $\Delta\cong r^2$.

In principle, both Eqs. (16) and (17) may be used for the determination of each of the coefficients $C^{(in)}, C^{(out)}$. Notice, however, that when $D_{m\omega}$ acts on $R^{\lambda m\omega(out)}_{\pm 2}$ and $D^{\dagger}_{m\omega}$ on $\Delta^2 R^{\lambda m\omega(in)}_{\pm 2}$, the leading-order term proportional to ω cancels out. Therefore, in these cases the operator effectively decreases the powers of r (by 1 at least), as ∂_r differentiates this power of r. This leads to a complication, because then we cannot ignore the higher-order terms (in $1/r$) in the operators $D_{m\omega}, D^{\dagger}_{m\omega}$ and in Δ, and also the higher-order terms in the basis solutions $R^{\lambda m\omega(in,out)}_{\pm 2}$. On the other hand, no such cancelation of the leading order term occurs when $D_{m\omega}$ acts on $R^{\lambda m\omega(in)}_{\pm 2}$ and $D^{\dagger}_{m\omega}$ on $R^{\lambda m\omega(out)}_{\pm 2}$. Instead, we get

$$D_{m\omega}[R^{\lambda m\omega(in)}_{\pm 2}]\cong -2i\omega R^{\lambda m\omega(in)}_{\pm 2}$$

and

$$D^{\dagger}_{m\omega}[R^{\lambda m\omega(out)}_{\pm 2}]\cong 2i\omega R^{\lambda m\omega(out)}_{\pm 2}.$$

It will therefore be convenient to calculate $C^{(in)}$ from Eq. (17) and $C^{(out)}$ from Eq. (16), by substituting in these equations

$$R^{\lambda m\omega}_{+2}=R^{\lambda m\omega(a)}_{+2}, \quad \hat{R}^{\lambda m\omega}_{-2}=C^{(a)}R^{\lambda m\omega(a)}_{-2} \tag{28}$$

(with "a" standing for either "in" or "out," as appropriate). Equation (17) then becomes

$$R^{\lambda m\omega(in)}_{+2}\cong 16\omega^4 C^{(in)}R^{\lambda m\omega(in)}_{-2},$$

and Eq. (16) reads

$$C^{(out)}R^{\lambda m\omega(out)}_{-2}\cong 16\omega^4 p^{-1} r^8 R^{\lambda m\omega(out)}_{+2}.$$

Since Eqs. (24),(25) imply

$$R^{\lambda m\omega(in)}_{-2}\cong R^{\lambda m\omega(in)}_{+2}, \quad R^{\lambda m\omega(out)}_{-2}\cong r^8 R^{\lambda m\omega(out)}_{+2},$$

we obtain

$$C^{(in)}=1/(16\omega^4), \quad C^{(out)}=16\omega^4/p. \tag{29}$$

B. Event-horizon asymptotic behavior

In a completely analogous manner, we can use for the expansion of $R^{\lambda m\omega}_{+2}$ and $\hat{R}^{\lambda m\omega}_{-2}$ basis solutions characterized by either negative or positive exponents of r_* at the EH (the latter corresponds to $r_*\to-\infty$). These basis solutions take the asymptotic form

$$R^{\lambda m\omega(down)}_{+2}\cong\Delta^{-2}e^{-ikr_*}, \quad R^{\lambda m\omega(up)}_{+2}\cong e^{ikr_*} \tag{30}$$

and

$$R^{\lambda m\omega(down)}_{-2}\cong\Delta^{2}e^{-ikr_*}, \quad R^{\lambda m\omega(up)}_{-2}\cong e^{ikr_*} \tag{31}$$

(see [3]). Here $k\equiv\omega-ma/(2Mr_+)$, where r_+ is the r value at the event horizon, given by

$$r_\pm = M\pm(M^2-a^2)^{1/2}.$$

In this case, again, one can verify that the operators $D_{m\omega}, D^{\dagger}_{m\omega}$ do not mix positive and negative exponents of r_*. Therefore,

$$H[R^{\lambda m\omega(down)}_{+2}]=C^{(down)}R^{\lambda m\omega(down)}_{-2} \tag{32}$$

and

$$H[R^{\lambda m\omega(up)}_{+2}]=C^{(up)}R^{\lambda m\omega(up)}_{-2}, \tag{33}$$

and the problem reduces to the determination of the two constants $C^{(down)}$ and $C^{(up)}$.

The leading-order form of $\Delta=(r-r_+)(r-r_-)$ is

$$\Delta\cong q\,\delta r,$$

where $\delta r\equiv r-r_+$ and

$$q=r_+-r_-=2(M^2-a^2)^{1/2}.$$

Correspondingly, the leading-order forms of $D_{m\omega}$ and $D^{\dagger}_{m\omega}$ near the EH are

$$D_{m\omega}\cong\frac{2Mr_+}{\Delta}(\partial_{r_*}-ik), \quad D^{\dagger}_{m\omega}\cong\frac{2Mr_+}{\Delta}(\partial_{r_*}+ik),$$

where we have used $r_+^2+a^2=2Mr_+$. However, when applying the operators $D_{m\omega}, D^{\dagger}_{m\omega}$ to the above basis solutions, it is most convenient to view r and r_* in Eqs. (30),(31) as two independent variables. In this context we have

$$D_{m\omega}\cong\partial_r+\frac{2Mr_+}{\Delta}(\partial_{r_*}-ik),$$

and

$$D^{\dagger}_{m\omega}\cong\partial_r+\frac{2Mr_+}{\Delta}(\partial_{r_*}+ik).$$

The above basis functions all take the general form $F(r)e^{\pm ikr_*}$. For such functions we have

$$D_{m\omega}[Fe^{ikr_*}]=F'e^{ikr_*},$$

$$D_{m\omega}[Fe^{-ikr_*}]=\left[F'-\frac{iw}{\Delta}\right]e^{-ikr_*},$$

$$D^{\dagger}_{m\omega}[Fe^{ikr_*}]=\left[F'+\frac{iw}{\Delta}\right]e^{ikr_*},$$

$$D^{\dagger}_{m\omega}[Fe^{-ikr_*}]=F'e^{-ikr_*},$$

where $w=4kMr_+$ and a prime denotes d/dr. Note that when $D_{m\omega}$ acts on $R^{\lambda m\omega(up)}_{\pm 2}$ and $D^{\dagger}_{m\omega}$ on $\Delta^2 R^{\lambda m\omega(down)}_{+2}$, the leading-order term proportional to k cancels out, and we are left with higher-order terms that take the lead, which is an inconvenient situation. For this reason we shall calculate $C^{(down)}$ from Eq. (17) and $C^{(up)}$ from Eq. (16). Using again the substitution (28) (this time with "a" standing for either "down" or "up"), Eqs. (17) and (16) become, respectively,

$$R^{\lambda m\omega(down)}_{+2}=C^{(down)}(D_{m\omega})^4[R^{\lambda m\omega(down)}_{-2}] \tag{34}$$

and

$$C^{(up)}R^{\lambda m\omega(up)}_{-2}=p^{-1}\Delta^2(D^{\dagger}_{m\omega})^4[\Delta^2 R^{\lambda m\omega(up)}_{+2}].$$

For the first equation we need to calculate the quantity

$$(D_{m\omega})^4[\Delta^2 e^{-ikr_*}]. \tag{35}$$

A straightforward calculation yields (at the leading order)

$$(D_{m\omega})^4[\Delta^2 e^{-ikr_*}]\cong Q\Delta^{-2}e^{-ikr_*},$$

where

$$Q=(w+2iq)(w+iq)w(w-iq).$$

We thus find

$$C^{(down)}=1/Q. \tag{36}$$

For the second equation we need to calculate the quantity

$$(D^{\dagger}_{m\omega})^4[\Delta^2 e^{ikr_*}].$$

This is just the complex conjugate of expression (35), and we find

$$(D^{\dagger}_{m\omega})^4[\Delta^2 e^{ikr_*}]\cong \bar{Q}\Delta^{-2}e^{ikr_*},$$

which leads to

$$C^{(up)}=\bar{Q}/p. \tag{37}$$

V. THE GENERAL HOMOGENEOUS SOLUTION TO THE FOURTH-ORDER EQUATION

The equation (11) that determines the potential $\bar{\Psi}$ is an inhomogeneous fourth-order linear differential equation. In the preceding section we constructed the relevant inhomogeneous solution of this equation in the case of pure vacuum perturbations, for each mode of the source term ψ_0 [by "relevant" we refer here to the solution that also solves the vacuum Teukolsky equation (10)]. Later we shall also need the general solution of this fourth-order equation in order to construct the inhomogeneous solutions relevant to nonvacuum perturbations. To this end, we shall now construct the *general homogeneous solution* to Eq. (11).

This equation is in fact a trivial ordinary differential equation. Let us denote by ξ the null geodesics whose tangent is the null tetrad vector l^{μ} (namely, ξ are the members of the outgoing principal null congruence). Let γ be an affine parameter along the geodesics ξ, namely,

$$l^{\mu}=\frac{dx^{\mu}}{d\gamma}(\xi).$$

Then for any function $f(t,r,\theta,\varphi)$,

$$D[f]\equiv l^{\mu}\partial_{\mu}f=\frac{df}{d\gamma}(\xi).$$

The differential equation (11) thus reads

$$\frac{d^4\bar{\Psi}}{d\gamma^4}(\xi)=\psi_0. \tag{38}$$

Its general homogeneous solution is a third-order polynomial in γ, whose four coefficients may be taken to be arbitrary functions of ξ:

$$\bar{\Psi}=\sum_{i=0}^{3} b_i(\xi)\gamma^i \quad \text{(homogeneous)}. \tag{39}$$

We wish, however, to rewrite this homogeneous solution more explicitly as a function of the four spacetime coordinates. To this end we need to explicitly parametrize the null geodesics ξ. From the definition of l^{μ}, along each null geodesic ξ we have

$$\frac{dr}{d\gamma}=1,\quad \frac{d\theta}{d\gamma}=0,\quad \frac{dt}{d\gamma}=\frac{r^2+a^2}{\Delta},\quad \frac{d\varphi}{d\gamma}=a/\Delta.$$

We choose the origin of γ such that $\gamma=r$ along the geodesic. Then t,θ,φ along the geodesic are given by

$$\theta=\theta_0,\quad t=t_0+r_*(r),\quad \varphi=\varphi_0+u(r), \tag{40}$$

where θ_0,t_0,φ_0 are arbitrary constants, and $r_*(r)$ and $u(r)$ are given by the two integrals

$$r_*(r)=\int\frac{r^2+a^2}{\Delta}dr,\quad u(r)=\int\frac{a}{\Delta}dr.$$

Specifically we take

$$r_*(r)=r+\frac{r_+^2+a^2}{q}\ln(r-r_+)-\frac{r_-^2+a^2}{q}\ln(r-r_-) \tag{41}$$

and

$$u(r)=(a/q)\ln[(r-r_+)/(r-r_-)]. \tag{42}$$

The null geodesics ξ are thus parametrized by the three parameters $\theta_0\equiv\theta$, $t_0\equiv t-r_*(r)$, and $\varphi_0\equiv\varphi-u(r)$, and the above general solution takes the explicit form

$$\bar{\Psi}=\sum_{i=0}^{3} b_i(\theta_0,t_0,\varphi_0)r^i \quad \text{(homogeneous)}, \tag{43}$$

where b_i are arbitrary functions of their arguments.

Later we shall also need the form of this general homogeneous solution in the frequency domain. In order to comply with the decomposed form (15), for a particular mode $\lambda m\omega$ the arbitrary functions b_i must take the form

$$b_i=B_i S_{+2}^{\lambda m\omega}(\theta)e^{im(\varphi-u)}e^{-i\omega(t-r_*)},$$

where B_i are four arbitrary constants (for each mode). Correspondingly the (homogeneous-solution) radial function $\hat{R}_{-2}^{\lambda m\omega}$ is given by

$$\hat{R}_{-2}^{\lambda m\omega}(r)=e^{-i(mu-\omega r_*)}\sum_{i=0}^{3} B_i r^i \quad \text{(homogeneous)}. \tag{44}$$

One can easily verify that this solution indeed satisfies the homogeneous part of Eq. (17), namely,

$$(D_{m\omega})^4[\hat{R}_{-2}^{\lambda m\omega}]=0, \quad \text{(homogeneous)}.$$

To this end, it is sufficient to note that for any function $f(r)$,

$$D_{m\omega}[f(r)e^{-i(mu-\omega r_*)}]=\frac{df}{dr}e^{-i(mu-\omega r_*)}. \tag{45}$$

VI. GRAVITATIONAL PERTURBATIONS PRODUCED BY SOURCES

Consider now gravitational waves produced by a point-like particle that moves freely in a Kerr spacetime. For concreteness let us assume that the orbit is confined to the range $r_{\min}\leqslant r\leqslant r_{\max}$ (but this assumption may be relaxed, at least partially, as we discuss in Sec. X). The orbit needs not be equatorial. Of special importance is the case of a circular orbit, $r_{\min}=r_{\max}\equiv r_0$. Alternatively, we may assume that the gravitational waves are produced by a finite-size mater distribution. In this case, too, we shall assume that the matter is confined to the range $r_{\min}\leqslant r\leqslant r_{\max}$.

In the formalism used here the single function Ψ is required to satisfy two differential equations—Eq. (10) and the inhomogeneous equation (11). These equations are mutually consistent in the case the source term for Eq. (11) is a *vacuum* gravitational field ψ_0, but otherwise we should expect to have an over-determination. Therefore, in a spacetime with a matter source (either a finite-size or a point-like source), we cannot expect to have a solution to both Eqs. (10) and (11) in the entire spacetime, or even in the entire vacuum part of spacetime.

The nonexistence of a global solution Ψ can be demonstrated from another point of view. For a point particle in an otherwise-vacuum spacetime, Barack and Ori [8] showed there is no global radiation-gauge solution $h_{\alpha\beta}$ around the particle. As was discussed in Sec. I, one can construct a solution $h_{\alpha\beta}^+$ which is entirely regular at $r>r_{particle}$, but this solution will necessarily have a singularity in the range $r<r_{particle}$, along a line emanating from the particle. Alternatively one may construct a solution $h_{\alpha\beta}^-$ which is regular in the entire domain $r<r_{particle}$, but this solution will have a line singularity in the range $r>r_{particle}$.

Although it is not possible to construct a solution Ψ valid in the entire vacuum region, it is possible (for a point source) to construct a solution which is valid everywhere throughout the vacuum region, except in a set of zero measure. (In fact there are two such solutions, those denoted Ψ^+ and Ψ^- in Sec. I.) We shall now proceed to construct such a solution. Specifically we shall describe the construction of the solution Ψ^+ (but the other solution Ψ^- may be constructed in a fully analogous manner). This construction is applicable in both cases of a point source and an extended source (though in the latter case the domain in which Ψ^+ violates the required equations is no longer of zero measure, as we discuss in the next section).

As in the previous sections, we shall consider here the function $\bar{\Psi}^+$ sourced by a particular mode $\lambda m\omega$ of ψ_0. This function takes the decomposed form (15), and we need to construct the radial function $\hat{R}_{-2}^{\lambda m\omega}$. In the vacuum region $r>r_{\max}$, $\hat{R}_{-2}^{\lambda m\omega}$ is just the solution described in Sec. IV. This was shown to be a valid solution of both Eqs. (17) and (18) (and this is the *only* valid solution). Note that in this external region ψ_0 is made of outgoing modes only,

$$R_{+2}^{\lambda m\omega}(r)=A^{(out)}R_{+2}^+(r) \quad (r>r_{\max}), \tag{46}$$

where hereafter we denote the relevant basis functions for brevity as

$$R_{\pm2}^+\equiv R_{\pm2}^{\lambda m\omega(out)}, \quad R_{\pm2}^-\equiv R_{\pm2}^{\lambda m\omega(down)},$$

and similarly we use $C^+\equiv C^{(out)}$, $C^-\equiv C^{(down)}$. (Only $R_{\pm2}^{\lambda m\omega(out)}$ and $R_{\pm2}^{\lambda m\omega(down)}$ will be relevant here, because these are the two homogeneous basis functions involved in the construction of the retarded Green's functions for the Teukolsky variables $\psi_{\pm2}$; see below.) Therefore, in $r>r_{\max}$ the radial function of $\bar{\Psi}^+$ (for a particular mode $\lambda m\omega$) is simply given by

$$\hat{R}_{-2}^{\lambda m\omega}(r)=C^+A^{(out)}R_{-2}^+(r) \quad (r>r_{\max}). \tag{47}$$

Consider next the extension of this solution into the range $r<r_{\max}$. Here $R_{+2}^{\lambda m\omega}(r)$ is not a vacuum solution (it fails to be a vacuum solution everywhere in $r_{\min}<r<r_{\max}$), and we can no longer require $\hat{R}_{-2}^{\lambda m\omega}$ to satisfy both Eqs. (17) and (18).[7] We therefore choose to extend $\hat{R}_{-2}^{\lambda m\omega}$ into $r<r_{\max}$ as a solution of Eq. (17), and, for the time being, forget about Eq.

[7]For extending $\hat{R}_{-2}^{\lambda m\omega}$ into $r<r_{\max}$ as a solution of Eq. (18) would automatically yield the external solution (47) in this range too. But then $(D_{m\omega})^4[\hat{R}_{-2}^{\lambda m\omega}]$ would necessarily be the analytically-extended vacuum function $R_{+2}^{\lambda m\omega}$, which does not conform with the actual, non-vacuum, function $R_{+2}^{\lambda m\omega}$ in $r_{\min}<r<r_{\max}$.

(18). Nevertheless, in the next section we shall show that this procedure yields a valid solution Ψ^+, which solves both required (time-domain) equations (10) and (11), even at $r < r_{\max}$—except in the domain Σ^+ (which is of zero measure for a point source).

To construct the solution of Eq. (17) we proceed as follows. The source term in this linear differential equation is $R_{+2}^{\lambda m\omega}(r)$, the radial function of ψ_0. This function is in turn sourced by the energy-momentum distribution in spacetime. It can thus be expressed by means of the energy-momentum source term $T_{+2}^{\lambda m\omega}(r)$ via the Green's-function method:

$$R_{+2}^{\lambda m\omega}(r) = \int_{r_{\min}}^{r_{\max}} T_{+2}^{\lambda m\omega}(r')G(r,r')dr'. \qquad (48)$$

The Green's function $G(r,r')$ is constructed from the two vacuum solutions admitting the desired asymptotic behavior, namely, outgoing waves at large r and down-going waves at the EH:

$$G(r,r') = A^+(r')R_{+2}^+(r)\theta(r-r') + A^-(r')R_{+2}^-(r)\theta(r'-r), \qquad (49)$$

where θ denotes the standard step function, namely $\theta(x) = 0$ for $x<0$ and $\theta(x)=1$ for $x>0$. The functions $A^+(r')$ and $A^-(r')$ are given by

$$A^{\pm} = R_{+2}^{\mp}/(\Delta W[R_{+2}^-, R_{+2}^+]) \qquad (50)$$

(all quantities evaluated at r'), where $W[R_{+2}^-, R_{+2}^+]$ denotes the Wronskian of the two homogeneous solutions $R_{+2}^{\pm}$. Note that G, viewed as a function of r, is continuous at $r=r'$, namely,

$$A^+(r')R_{+2}^+(r') = A^-(r')R_{+2}^-(r'). \qquad (51)$$

The coefficient $A^{(out)}$ in the external solution (46) is thus given by

$$A^{(out)} = \int_{r_{\min}}^{r_{\max}} T_{+2}^{\lambda m\omega}(r')A^+(r')dr'. \qquad (52)$$

We now wish to construct a "Green-like function" $H(r,r')$ such that the function $\hat{R}_{-2}^{\lambda m\omega}$ will be given by

$$\hat{R}_{-2}^{\lambda m\omega}(r) = \int_{r_{\min}}^{r_{\max}} T_{+2}^{\lambda m\omega}(r')H(r,r')dr', \qquad (53)$$

in analogy with Eq. (48). This will be a solution to Eq. (17) if $H(r,r')$ satisfies the equation

$$(D_{m\omega})^4[H(r,r')] = G(r,r') \qquad (54)$$

(in which the operator $D_{m\omega}$ differentiates with respect to r, not r'). Motivated by the above form of $G(r,r')$, we assume a function $H(r,r')$ of a similar form,

$$H(r,r') = H^+(r,r')\theta(r-r') + H^-(r,r')\theta(r'-r), \qquad (55)$$

where $H^+(r,r')$ and $H^-(r,r')$ are smooth functions of their arguments. Equation (54) is then satisfied if the following two conditions hold: (i) the two functions $H^{\pm}(r,r')$ satisfy

$$(D_{m\omega})^4[H^{\pm}(r,r')] = A^{\pm}(r')R_{+2}^{\pm}(r), \qquad (56)$$

and (ii) $H(r,r')$ is continuous and differentiable four times (with respect to r) at $r=r'$; In other words, the function

$$y(r,r') \equiv H^+(r,r') - H^-(r,r'),$$

and its derivatives with respect to r up to fourth order, vanish at $r=r'$:

$$\frac{\partial^n y}{\partial r^n}(r=r') = 0, \quad n=0,\ldots,4 \qquad (57)$$

(in which $\partial^0 y/\partial r^0 \equiv y$ is to be understood). Condition (i) guarantees the validity of Eq. (54) at $r>r'$ and $r<r'$ [the "+" and "−" cases in Eq. (56), respectively]. Condition (ii) is required for the validity of Eq. (54) at $r=r'$. [To see this, rewrite Eq. (55) as

$$H(r,r') = y(r,r')\theta(r-r') + H^-(r,r'),$$

and recall the continuity of G at $r=r'$.]

The form of the general solution $H^{\pm}(r,r')$ to Eq. (56) is obvious from the analysis in Secs. IV and V. We have a specific inhomogeneous solution $C^{\pm}A^{\pm}(r')R_{-2}^{\pm}(r)$, and the general homogeneous solution (44) (in which the coefficients B_i are now allowed to be arbitrary functions of r'); Hence the most general solution is

$$H^{\pm}(r,r') = C^{\pm}A^{\pm}(r')R_{-2}^{\pm}(r) + e^{-i(mu-\omega r_*)}\sum_{i=0}^{3} B_i^{\pm}(r')r^i. \qquad (58)$$

It should be noted that since this is the most general solution to Eq. (17), this form must be satisfied by both Ψ^+ and Ψ^-. The difference between these two solutions should emerge from the choice of the free functions $B_i^{\pm}$, which are to be determined by the boundary conditions. As we are considering here the solution Ψ^+, the radial function $\hat{R}_{-2}^{\lambda m\omega}(r)$ must satisfy Eq. (47) at $r>r_{\max}$. This is achieved by simply choosing $B_i^+(r')\equiv 0$ for all i, namely,

$$H^+(r,r') = C^+A^+(r')R_{-2}^+(r). \qquad (59)$$

The internal part $H^-(r,r')$ has a more complicated form,

$$H^-(r,r') = C^-A^-(r')R_{-2}^-(r) + e^{-i(mu-\omega r_*)}\sum_{i=0}^{3} B_i^-(r')r^i, \qquad (60)$$

in which the four arbitrary functions $B_i^-(r')$ are to be determined by condition (ii) above, i.e. by matching to $H^+(r,r')$ at $r=r'$. The function $y(r,r')$ is given by

$$y(r,r')=C^+A^+(r')R^+_{-2}(r)-C^-A^-(r')R^-_{-2}(r) - e^{-i(mu-\omega r_*)}\sum_{i=0}^{3} B_i^-(r')r^i,$$

and we must impose Eq. (57). This might look problematic at first glance, because apparently the latter equation imposes five requirements on the four arbitrary functions $B_i^-(r')$. However, one of these requirements is automatically satisfied. To see this, it will be convenient to rewrite Eq. (57) as

$$(D_{m\omega})^n[y]=0,\quad n=0,\ldots,4\quad (r=r')$$

(where $(D_{m\omega})^0[y]\equiv y$ is to be understood). Considering the case $n=4$, the operator $(D_{m\omega})^4$ annihilates the last term in the above expression for $y(r,r')$, and we have

$$(D_{m\omega})^4[y]=A^+(r')R^+_{+2}(r)-A^-(r')R^-_{+2}(r),$$

which vanishes by virtue of Eq. (51). We can therefore reexpress condition (ii) as

$$(D_{m\omega})^n[y]=0,\quad n=0,\ldots,3\quad (r=r'). \tag{61}$$

The four arbitrary functions $B_i^-(r')$ should thus be determined from the four conditions involved in this equation. In Appendix A we show that these four functions may be expressed as

$$B_i^-(r')=C^+A^+(r')\left[f_i(r')R^+_{-2}(r')+g_i(r')\frac{d}{dr'}R^+_{-2}(r')\right] - C^-A^-(r')\left[f_i(r')R^-_{-2}(r') + g_i(r')\frac{d}{dr'}R^-_{-2}(r')\right]$$

where $f_i(r')$ and $g_i(r')$ are certain functions of r' explicitly specified therein.

Further simplification of the solution

The procedure described so far for the construction of $\hat{R}^{\lambda m\omega}_{-2}(r)$ requires the two $s=-2$ homogeneous basis functions $R^{\pm}_{-2}$, and also the two $s=+2$ homogeneous functions $R^{\pm}_{+2}$. The latter functions are required for the determination of $A^{\pm}(r')$ (and their derivatives are involved in the Wronskian $W[R^-_{+2},R^+_{+2}]$). However, in principle ψ_0 can be determined from ψ_4 (and vice versa), and this implies that $R^{\pm}_{+2}$ may be determined from $R^{\pm}_{-2}$. In Appendix B we undertake this goal and reexpress $A^{\pm}(r')$ in terms of the functions $R^{\pm}_{-2}$ and their first-order derivatives. We find

$$A^{\pm}(r')=(pC^{\pm}W[R^-_{-2},R^+_{-2}])^{-1}\left[\bar{A}(r')\frac{d}{dr'}R^{\mp}_{-2}(r') + \bar{B}(r')R^{\mp}_{-2}(r')\right], \tag{62}$$

where p is the parameter defined in Eq. (21), $\bar{A}$ and $\bar{B}$ are functions specified in Eq. (B7), and

$$W[R^-_{-2},R^+_{-2}]=\text{const}\times\Delta(r') \tag{63}$$

is the Wronskian of the two basis solutions $R^{\pm}_{-2}$ (evaluated at r').

In the above construction of $H(r,r')$, $A^{\pm}$ and $C^{\pm}$ only appear through their products A^+C^+ and A^-C^-. We therefore define

$$a^{\pm}(r')\equiv C^{\pm}A^{\pm}(r'),$$

and obtain

$$a^{\pm}(r')=(pW[R^-_{-2},R^+_{-2}])^{-1}\left[\bar{A}(r')\frac{d}{dr'}R^{\mp}_{-2}(r') + \bar{B}(r')R^{\mp}_{-2}(r')\right]. \tag{64}$$

The functions $H^{\pm}(r,r')$ and $B_n^-(r')$ can now be reexpressed as

$$H^+(r,r')=a^+(r')R^+_{-2}(r), \tag{65}$$

$$H^-(r,r')=a^-(r')R^-_{-2}(r)+e^{-i(mu-\omega r_*)}\sum_{i=0}^{3} B_i^-(r')r^i, \tag{66}$$

and

$$B_i^-(r')=a^+(r')\left[f_i(r')R^+_{-2}(r')+g_i(r')\frac{d}{dr'}R^+_{-2}(r')\right] - a^-(r')\left[f_i(r')R^-_{-2}(r')+g_i(r')\frac{d}{dr'}R^-_{-2}(r')\right].$$

Note that when expressed in this form the function $H(r,r')$—and hence also $\hat{R}^{\lambda m\omega}_{-2}(r)$—is invariant to a rescaling of R^+_{-2} or R^-_{-2} by constants. Therefore there is no need to require here a specific normalization for these basis solutions.

VII. DOMAIN OF VALIDITY OF THE CONSTRUCTED SOLUTION

In the case of perturbations produced by matter sources, we must carefully examine in what parts of spacetime the construction of Ψ and $h_{\alpha\beta}$ is valid. First, the Chrzanowski's construction requires that the potential Ψ satisfies both equations (10) and (11). These equations are mutually consistent in vacuum, but are generally inconsistent in the presence of matter. The matter source therefore leads to the violation of at least one of these equations. This violation occurs not only in the region occupied by the matter, but also in certain vacuum parts of the spacetime.[8]

At this stage it will be conceptually simpler to assume that the massive object that creates the perturbation has a finite size and a regular energy-momentum distribution $T_{\mu\nu}(x^\alpha)$. (The case of a point mass will then follow in a trivial manner.) Let us define Σ^+ to be the collection of all points P in spacetime which have the following property: The null geodesic ξ (a member of the outgoing principal null congruence) passing through P intersects matter (more precisely, nonvanishing source term T_{+2}) on its approach from P towards future null infinity. The collection of all other points of (the part $r>r_+$ of) spacetime is denoted $\hat{\Sigma}^+$. In an analogous manner, we define Σ^- to be the collection of all points P for which ξ intersects matter on its approach from P towards the EH, and $\hat{\Sigma}^-$ is the rest of (the part $r>r_+$ of) spacetime. In a more pictorial language, imagine that light rays propagate along all geodesics ξ of the outgoing principal null congruence. Then Σ^- is the portion of spacetime "shadowed" by the matter-energy distribution $T_{+2}(x^\alpha)$, and $\hat{\Sigma}^-$ is the non-shadowed part. Similarly, Σ^+ is the "past-shadowed" part, i.e. the part of spacetime that would be shadowed if the light rays were propagating along the null congruence from future to past (and from large r towards the EH), and $\hat{\Sigma}^+$ is the rest of $r>r_+$. By definition, $\hat{\Sigma}^+$ and $\hat{\Sigma}^-$ are pure vacuum domains. Note that $\hat{\Sigma}^+$ contains the entirely-vacuum domain $r>r_{\max}$, and generally it also extends through $r<r_{\max}$ into the other entirely-vacuum domain $r<r_{\min}$—though the latter domain is not entirely contained in $\hat{\Sigma}^+$. (Similarly, $\hat{\Sigma}^-$ contains the entire domain $r<r_{\min}$, and generally extends through $r>r_{\min}$ into $r>r_{\max}$.) Also, at the point-like limit Σ^+ and Σ^- each degenerates to a $(1+1)$-dimensional surface that emerges out of the particle's worldline in either the past or future (i.e. inward or outward) direction of ξ. Hence in the point-like limit $\hat{\Sigma}^+$ or $\hat{\Sigma}^-$ cover the entire spacetime except a set of measure zero. On the other hand, when the object is extended, Σ^+ and Σ^- are four-dimensional sets.

We shall now argue that the above-constructed potential Ψ^+—and the MP $h^+_{\alpha\beta}$ constructed from it via Eq. (9)—are valid in the entire domain $\hat{\Sigma}^+$. We shall first establish this for the potential Ψ^+, namely, we shall show that Eqs. (10) and (11) are satisfied throughout $\hat{\Sigma}^+$. Then we shall show that the MP solution $h^+_{\alpha\beta}$ constructed from Ψ^+ is valid throughout $\hat{\Sigma}^+$ too. (The same arguments apply to the validity of Ψ^- and $h^-_{\alpha\beta}$ throughout $\hat{\Sigma}^-$.)

A. Validity of the constructed potential Ψ^+

In the range $r>r_{\max}$, the above construction of the "Green-like function" $H(r,r')$ ensures that Eq. (47) is satisfied by $\hat{R}^{\lambda m\omega}_{-2}(r)$, and hence also Eqs. (17) and (18). In the time domain this implies that Ψ^+ satisfies Eqs. (10) and (11) throughout $r>r_{\max}$.

In the range $r<r_{\max}$ we have constructed $\hat{R}^{\lambda m\omega}_{-2}(r)$ such that it satisfies Eq. (17) for all modes, hence in the time domain compliance with Eq. (11) is guaranteed. But Eq. (18) is violated throughout $r<r_{\max}$ [note that the homogeneous solution (44) violates Eq. (18)]. We shall now employ analyticity considerations to demonstrate that Ψ^+ does satisfy the *time-domain* Teukolsky equation (10) throughout $\hat{\Sigma}^+$.

Each mode $\lambda m\omega$ of $\bar{\Psi}^+$ is analytic throughout $r>r_{\max}$ by construction [cf. Eq. (47)]. Assuming convergence of the mode sum at $r>r_{\max}$, we may assume that $\bar{\Psi}^+$ itself is analytic at $r>r_{\max}$ too.[9] Now, in the above construction we extended $\bar{\Psi}^+$ into the entire domain $r<r_{\max}$ as a solution of Eq. (11). This implies that $\bar{\Psi}^+$ is analytic throughout $\hat{\Sigma}^+$, as we now show.

Equation (11) is an ordinary differential equation along the null geodesics ξ, which we may write as

$$\frac{d^4\bar{\Psi}^+(\gamma;\xi)}{d\gamma^4}=\psi_0(\gamma;\xi). \tag{67}$$

We write its general solution explicitly (in a recursive manner) as $\bar{\Psi}^+(\gamma;\xi)\equiv\Phi_1(\gamma;\xi)$, with

$$\Phi_n(\gamma;\xi)=c_n(\xi)+\int_{r_0}^{\gamma}\Phi_{n+1}(\gamma';\xi)d\gamma', \quad n=1\ \ldots 4, \tag{68}$$

where $\Phi_5\equiv\psi_0$. Here $c_i(\xi)$ $(i=1,\ldots,4)$ are four arbitrary functions of the three variables θ_0,t_0,φ_0. [Recall that the geodesics ξ are parametrized by the three quantities θ_0,t_0,φ_0, defined through Eq. (40), that take constant values along the geodesic. Also recall that we have set $\gamma=r$.] We take the lower integration limit to be, say, $r_0=2r_{\max}$. The transformation from the coordinates (t,r,θ,φ) to $(\gamma,\theta_0,t_0,\varphi_0)$ can be read off Eq. (40), and it is manifestly analytic everywhere in $r>r_+$. Therefore, the analyticity of $\bar{\Psi}^+$ in the domain $r>r_{\max}$ implies it is analytic in $(\gamma,\theta_0,t_0,\varphi_0)$ as well. This in turn implies that all four functions $c_i(\xi)\equiv c_i(\theta_0,t_0,\varphi_0)$ are analytic in (θ_0,t_0,φ_0). Now, the function ψ_0 is presumably analytic everywhere in the

[8]To verify this, note that in the above construction of Ψ^+, the individual-mode radial function $\hat{R}^{\lambda m\omega}_{-2}(r)$ violates the radial Teukolsky equation in the entire domain $r<r_{\max}$—which in particular includes the vacuum domain $r<r_{\min}$. This violation follows from the non-vanishing of the coefficients B_i^-. This does not necessarily mean that the time-domain Teukolsky equation (10) is violated everywhere in $r<r_{\max}$ (in fact it does not, as we show below); but it does indicate the existence of a domain of violation that extends at any r value in $r<r_{\max}$.

[9]For our argument to hold it is sufficient that the mode sum converges throughout some range $r>r'_{\max}\geqslant r_{\max}$, or even throughout some open interval of r values located somewhere at $r>r_{\max}$.

vacuum region.[10] When the solution (68) is restricted to the domain $\hat{\Sigma}^+$, we observe that only vacuum points $(\gamma';\xi)$ are encountered in the integration, hence $\psi_0(\gamma';\xi)$ is analytic. This immediately implies that $\bar{\Psi}^+$ given in Eq. (68) is analytic in $(\gamma;\theta_0,t_0,\varphi_0)$ throughout $\hat{\Sigma}^+$, and hence also in (t,r,θ,φ).

From the analyticity of $\bar{\Psi}^+$ (which implies the analyticity of Ψ^+) it follows that $W_{-2}[\Psi^+]$ is analytic too. The vanishing of the latter at $r>r_{\max}$ therefore implies its vanishing throughout $\hat{\Sigma}^+$. We have thus established the compliance of Ψ^+ with Eqs. (10) and (11) throughout $\hat{\Sigma}^+$.

It is easy to see why this argument fails at Σ^+: The function ψ_0 fails to be analytic at the point particle, or—in the case of an extended object—at the boundary of the matter distribution. As a consequence, along each null geodesic ξ intersecting the source, $\bar{\Psi}^+$ will be analytic only up to the intersection point. Then $\bar{\Psi}^+$ will usually be non-analytic at the boundary of Σ^+. Therefore we cannot expect Ψ^+ to satisfy Eq. (10) in Σ^+. The violation of the corresponding frequency-domain equation (18) throughout $r<r_{\max}$ indicates that Eq. (10) is indeed violated somewhere in Σ^+ (and for any value of r in this range).

Finally we note that the compliance of Ψ^+ with the "angular equation" (i.e. Eq. (2.7) in Ref. [6]) throughout $\hat{\Sigma}^+$ may be deduced by exactly the same analyticity argument.

B. Validity of the constructed metric perturbation

Our goal here is to establish the validity of $h^+_{\alpha\beta}$ (constructed from Ψ^+ via the Chrzanowski's method) throughout $\hat{\Sigma}^+$, despite the presence of matter in spacetime. By "validity" we mean that (i) $h^+_{\alpha\beta}$ satisfies the linearized vacuum Einstein equations, and (ii) the $s=+2$ Weyl scalar constructed from it coincides with the original field ψ_0. To this end we use analyticity considerations, similar to those used above for analyzing the validity of Ψ^+. Here we shall briefly sketch these considerations [11].

Consider first the validity of $h^+_{\alpha\beta}$ in $r>r_{\max}$. To this end, expand Ψ^+ (and ψ_0) into modes. For a particular mode $\lambda m\omega$, extend the $r>r_{\max}$ vacuum solution analytically into $r>r_{\max}$. This extended solution represents a pure vacuum perturbation. Chrzanowski's construction may now be applied to it, yielding the MP solution $h^{+\lambda m\omega}_{\alpha\beta}=\Pi[\Psi^{+\lambda m\omega}]$ for the mode under consideration. Upon summation over the modes, we obtain a valid MP solution $h^+_{\alpha\beta}=\Pi[\Psi^+]$ in the range $r>r_{\max}$.

Next consider the validity of the solution $h^+_{\alpha\beta}\equiv\Pi[\Psi^+]$ in the part $r<r_{\max}$ of $\hat{\Sigma}^+$. From the analyticity of Ψ^+ throughout $\hat{\Sigma}^+$ (established above) it follows that $h^+_{\alpha\beta}$ is also analytic in this range. Recall also the analyticity of ψ_0 throughout $\hat{\Sigma}^+$. The above criteria (i),(ii) for the validity of a MP solution $h_{\alpha\beta}$ are both formulated in terms of analytic differential operators acting on $h_{\alpha\beta}$. From the validity of these criteria in the range $r>r_{\max}$ it now follows that they must hold throughout $\hat{\Sigma}^+$.

[10] In the point-like case ψ_0 is irregular at the particle's location. In the case of a smooth extended source, ψ_0 will fail to be analytic at the boundary of the region occupied by matter. But in both cases we may assume that ψ_0 is analytic throughout the vacuum region.

VIII. SUMMARY OF MAIN RESULTS: GRAVITATIONAL PERTURBATIONS

Here we briefly summarize our procedure for constructing the potential Ψ, for gravitational perturbations. We use the decomposition

$$\bar{\Psi}=\sum_{\lambda m\omega}\hat{R}^{\lambda m\omega}_{-2}S^{\lambda m\omega}_{+2}(\theta)e^{i(m\varphi-\omega t)},$$

and our goal is to construct the radial functions $\hat{R}^{\lambda m\omega}_{-2}$. We shall now summarize this construction in the two different cases: (i) pure gravitational waves, and (ii) perturbations with sources.

A. Pure gravitational waves

In this case we assume that ψ_0 is given. This field is decomposed into modes too,

$$\psi_0=\sum_{\lambda m\omega}R^{\lambda m\omega}_{+2}(r)S^{\lambda m\omega}_{+2}(\theta)e^{i(m\varphi-\omega t)}.$$

For each mode $\lambda m\omega$, $R^{\lambda m\omega}_{+2}(r)$ is a solution of the vacuum radial Teukolsky equation, and we assume this function is provided as a linear combination of two basis solutions. Two sets of convenient basis solutions are (i) the *large-r set*, in which the basis solutions for $R^{\lambda m\omega}_{+2}$ and $R^{\lambda m\omega}_{-2}$ are given in Eqs. (24),(25), and (ii) the *EH set*, in which the basis solutions for $R^{\lambda m\omega}_{\pm2}$ are given in Eqs. (30),(31).

Assume now that the information about ψ_0 is given in terms of any two of the above four $s=+2$ basis functions, namely,

$$R^{\lambda m\omega}_{+2}(r)=A^{(a)}R^{\lambda m\omega(a)}_{+2}(r)+A^{(b)}R^{\lambda m\omega(b)}_{+2}(r),$$

and the coefficients $A^{(a)}$ and $A^{(b)}$ are provided for each mode $\lambda m\omega$. (Here "a" and "b" denote either the large-r basis solutions, or the horizon basis solutions, or any combination of these two sets, e.g. "a"="out" and "b"="$down$.") Then, the corresponding radial functions $\hat{R}^{\lambda m\omega}_{-2}$ of $\bar{\Psi}$ are simply given by

$$\hat{R}^{\lambda m\omega}_{-2}(r)=C^{(a)}A^{(a)}R^{\lambda m\omega(a)}_{-2}(r)+C^{(b)}A^{(b)}R^{\lambda m\omega(b)}_{-2}(r).$$

The four coefficients $C^{(in)}$, $C^{(out)}$, $C^{(down)}$ and $C^{(up)}$ are specified in Eqs. (29),(36),(37).

B. Gravitational perturbations produced by sources

Here we consider the case in which the perturbation is produced by a distribution of matter-energy. This may be either a point-like particle, or an extended object. In both cases we assume that we are given the radial energy-momentum source function $T_{+2}^{\lambda m\omega}(r)$ for each mode [this is the source term in the $s=+2$ radial Teukolsky equation (5)]. For simplicity we assume here that the matter source is restricted to the range $r_{\min}\leqslant r\leqslant r_{\max}$ (but this assumption may be relaxed—at least partially—as we discuss in Sec. X).

Then $\hat{R}_{-2}^{\lambda m\omega}(r)$ is given by

$$\hat{R}_{-2}^{\lambda m\omega}(r)=\int_{r_{\min}}^{r_{\max}}T_{+2}^{\lambda m\omega}(r')H(r,r')dr', \tag{69}$$

where

$$H(r,r')=H^{+}(r,r')\,\theta(r-r')+H^{-}(r,r')\,\theta(r'-r), \tag{70}$$

and $H^{\pm}(r,r')$ are two smooth functions. We construct these functions from the two $s=-2$ homogeneous radial solutions $R_{-2}^{\lambda m\omega(out)}\equiv R_{-2}^{+}$ and $R_{-2}^{\lambda m\omega(down)}\equiv R_{-2}^{-}$, defined by their asymptotic behavior

$$R_{-2}^{+}(r)\propto r^{3}e^{i\omega r_{*}} \qquad (r\to\infty)$$

and

$$R_{-2}^{-}(r)\propto\Delta^{2}e^{-ikr_{*}} \qquad (r_{*}\to-\infty),$$

where $k=\omega-ma/(2Mr_{+})$. (The $s=+2$ basis solutions are not required here. Also we do not require here a specific normalization for R_{-2}^{+} and R_{-2}^{-}.) We find

$$H^{+}(r,r')=a^{+}(r')R_{-2}^{+}(r) \tag{71}$$

and

$$H^{-}(r,r')=a^{-}(r')R_{-2}^{-}(r)+e^{-i(mu-\omega r_{*})}\sum_{i=0}^{3}B_{i}^{-}(r')r^{i}, \tag{72}$$

where u and r_{*} are defined in Eqs. (42), (41), respectively, and

$$a^{\pm}(r')=(pW[R_{-2}^{-},R_{-2}^{+}])^{-1}\left[\overline{A}(r')\frac{d}{dr'}R_{-2}^{\mp}(r')+\overline{B}(r')R_{-2}^{\mp}(r')\right]. \tag{73}$$

Here p is a parameter given in Eq. (21), and $W[R_{-2}^{-},R_{-2}^{+}]$ is the Wronskian of the two basis functions (which is proportional to Δ), evaluated at r'. The functions $\overline{A}(r'),\overline{B}(r')$ are specified in Appendix B. The four functions $B_{i}^{-}(r')$ are given by

$$B_{i}^{-}(r')=a^{+}(r')\left[f_{i}(r')R_{-2}^{+}(r')+g_{i}(r')\frac{d}{dr'}R_{-2}^{+}(r')\right]-a^{-}(r')\left[f_{i}(r')R_{-2}^{-}(r')+g_{i}(r')\frac{d}{dr'}R_{-2}^{-}(r')\right],$$

where f_i and g_i are functions of r' specified in Appendix A.

The above construction yields the radial functions $\hat{R}_{-2}^{\lambda m\omega}(r)$ for the solution Ψ^{+} that is valid (and regular) through $\hat{\Sigma}^{+}$. This domain includes the entire range $r>r_{\max}$, but not all points of $r<r_{\max}$. The other solution Ψ^{-} that is valid through $\hat{\Sigma}^{-}$ (i.e. everywhere in $r<r_{\min}$ but not at all points of $r>r_{\min}$) may be constructed in a fully analogous manner. The only difference is in the functions $H^{\pm}(r,r')$, which now take the forms

$$H^{+}(r,r')=a^{+}(r')R_{-2}^{+}(r)-e^{-i(mu-\omega r_{*})}\sum_{i=0}^{3}B_{i}^{-}(r')r^{i} \qquad (\Psi^{-}),$$

$$H^{-}(r,r')=a^{-}(r')R_{-2}^{-}(r) \qquad (\Psi^{-}).$$

IX. SUMMARY OF MAIN RESULTS: ELECTROMAGNETIC PERTURBATIONS

The electromagnetic case is treated in full analogy with the gravitational case. Here, again, the four-potential A_{α} is constructed in Chrzanowski's method, by applying a certain differential operator Π_{EM} to a potential Ψ_{EM}:

$$A_{\alpha}=\Pi_{EM}[\Psi_{EM}]. \tag{74}$$

Throughout this section we shall denote the electromagnetic potential Ψ_{EM} as Ψ for brevity. This potential satisfies equations analogous to Eqs. (10) and (11):

$$W_{-1}[\Psi]=0 \tag{75}$$

and

$$\varphi_{0}=-D^{2}[\overline{\Psi}], \tag{76}$$

where φ_0 is the $s=+1$ Weyl scalar, and W_{-1} is the $s=-1$ case of the differential operator (2). (See, e.g. [5], in which Ψ is denoted "φ_{E}." The last equation is the reduction of Eq. (15) therein to the Kerr case.) We use the decomposition

$$\overline{\Psi}=\sum_{\lambda m\omega}\hat{R}_{-1}^{\lambda m\omega}S_{+1}^{\lambda m\omega}(\theta)e^{i(m\varphi-\omega t)},$$

and our goal is to construct the radial functions $\hat{R}_{-1}^{\lambda m\omega}$. Again, we shall construct this function first in the case of pure electromagnetic waves, and then for perturbations with sources. Here we shall summarize the results. The main steps in the derivations are presented in Appendix C.

A. Pure electromagnetic waves

In this case we assume that φ_0 is given. This field is decomposed into modes as

$$\varphi_{0}=\sum_{\lambda m\omega}R_{+1}^{\lambda m\omega}(r)S_{+1}^{\lambda m\omega}(\theta)e^{i(m\varphi-\omega t)}.$$

For each mode $\lambda m\omega$, $R_{+1}^{\lambda m\omega}(r)$ is a solution of the vacuum radial Teukolsky equation, and we assume this function is provided as a linear combination of two basis solutions. The two sets of convenient basis solutions are (i) the *large-r set*,

$$R_{+1}^{\lambda m\omega(in)} \cong e^{-i\omega r_*}/r, \qquad R_{+1}^{\lambda m\omega(out)} \cong e^{i\omega r_*}/r^3, \tag{77}$$

$$R_{-1}^{\lambda m\omega(in)} \cong e^{-i\omega r_*}/r, \qquad R_{-1}^{\lambda m\omega(out)} \cong r e^{i\omega r_*}, \tag{78}$$

and (ii) the *EH set*,

$$R_{+1}^{\lambda m\omega(down)} \cong \Delta^{-1} e^{-ikr_*}, \qquad R_{+1}^{\lambda m\omega(up)} \cong e^{ikr_*}, \tag{79}$$

$$R_{-1}^{\lambda m\omega(down)} \cong \Delta e^{-ikr_*}, \qquad R_{-1}^{\lambda m\omega(up)} \cong e^{ikr_*}, \tag{80}$$

where $k=\omega-ma/(2Mr_+)$.

Assume now that φ_0 is given in terms of any two of the above four basis functions for $R_{+1}^{\lambda m\omega}$, namely,

$$R_{+1}^{\lambda m\omega}(r)=A^{(a)}R_{+1}^{\lambda m\omega(a)}(r)+A^{(b)}R_{+1}^{\lambda m\omega(b)}(r), \tag{81}$$

and the coefficients $A^{(a)}$ and $A^{(b)}$ are provided for each mode $\lambda m\omega$. (Here, again, "a" and "b" denote two of the above four basis functions, e.g. "a" = "*out*" and "b" = "*down*.") Then, the corresponding radial functions $\hat{R}_{-1}^{\lambda m\omega}$ of $\overline{\Psi}$ are given by

$$\hat{R}_{-1}^{\lambda m\omega}(r)=C^{(a)}A^{(a)}R_{-1}^{\lambda m\omega(a)}(r)+C^{(b)}A^{(b)}R_{-1}^{\lambda m\omega(b)}(r). \tag{82}$$

The four coefficients $C^{(in)}$, $C^{(out)}$, $C^{(down)}$, $C^{(up)}$ now take the values

$$C^{(in)}=1/(4\omega^2), C^{(out)}=4\omega^2/p \qquad \text{(EM)}, \tag{83}$$

$$C^{(up)}=\overline{Q}/p, C^{(down)}=1/Q \qquad \text{(EM)}, \tag{84}$$

where in the electromagnetic case we have

$$p=\lambda^2-4\alpha^2\omega^2=\lambda^2+4a\omega(m-a\omega) \qquad \text{(EM)}$$

and

$$Q=w(w+iq) \qquad \text{(EM)}, \tag{85}$$

and, recall,

$$w=4kMr_+, \quad q=r_+-r_-=2(M^2-a^2)^{1/2}.$$

B. Electromagnetic perturbations produced by sources

Here we consider the case in which the perturbation is produced by charges and/or currents (e.g. a point charge or an extended charged object orbiting the BH). We assume that we are given the radial electromagnetic source function $T_{+1}^{\lambda m\omega}(r)$ for each mode [this is the source term in the $s=+1$ analogue of the radial Teukolsky equation (5)]. As before we assume for simplicity that the source is restricted to the range $r_{\min}\leqslant r\leqslant r_{\max}$.

The radial function $\hat{R}_{-1}^{\lambda m\omega}(r)$ then takes the form

$$\hat{R}_{-1}^{\lambda m\omega}(r)=\int_{r_{\min}}^{r_{\max}} T_{+1}^{\lambda m\omega}(r')H(r,r')dr', \tag{86}$$

where

$$H(r,r')=H^+(r,r')\,\theta(r-r')+H^-(r,r')\,\theta(r'-r), \tag{87}$$

and $H^\pm(r,r')$ are two smooth functions. We construct these functions from the two $s=-1$ homogeneous radial solutions $R_{-1}^{\lambda m\omega(out)}\equiv R_{-1}^{+}$ and $R_{-1}^{\lambda m\omega(down)}\equiv R_{-1}^{-}$, defined by their asymptotic behavior

$$R_{-1}^{+}(r)\propto r e^{i\omega r_*} \qquad (r\to\infty)$$

and

$$R_{-1}^{-}(r)\propto \Delta e^{-ikr_*} \qquad (r_*\to-\infty).$$

(Here, again, we do not require a specific normalization for $R_{-1}^{\pm}$.) We find

$$H^+(r,r')=a^+(r')R_{-1}^{+}(r) \tag{88}$$

and

$$H^-(r,r')=a^-(r')R_{-1}^{-}(r)+e^{-i(mu-\omega r_*)}\sum_{i=0}^{1} B_i^-(r')r^i, \tag{89}$$

where u and r_* are defined in Eqs. (42), (41), respectively, and

$$a^\pm(r')=-(pW[R_{-1}^{-},R_{-1}^{+}])^{-1}\left[\overline{A}(r')\frac{d}{dr'}R_{-1}^{\mp}(r') + \overline{B}(r')R_{-1}^{\mp}(r')\right]. \tag{90}$$

Here $W[R_{-1}^{-},R_{-1}^{+}]=$const is the Wronskian of the two basis functions $R_{-1}^{\pm}$, and

$$\overline{A}(r')=2iK, \ \ \overline{B}(r')=\lambda+2i\omega r'-2K^2/\Delta \qquad \text{(EM)}$$

[with all quantities evaluated at r', e.g. $K=am-(r'^2+a^2)$]. The two functions $B_i^-(r')$ are given by

$$B_i^-(r')=a^+(r')\left[f_i(r')R_{-1}^{+}(r')+g_i(r')\frac{d}{dr'}R_{-1}^{+}(r')\right] - a^-(r')\left[f_i(r')R_{-1}^{-}(r')+g_i(r')\frac{d}{dr'}R_{-1}^{-}(r')\right]$$

(for $i=0,1$), where the functions $f_i(r'),g_i(r')$ are

$$f_0(r')=(1-iKr'/\Delta)e^{i(mu-\omega r_*)},$$

$$g_0(r')=-r'e^{i(mu-\omega r_*)},$$

$$f_1(r')=(i/\Delta)Ke^{i(mu-\omega r_*)}$$

$$g_1(r')=e^{i(mu-\omega r_*)}$$

(again, with u,r_*,K,Δ all evaluated at r').

The above construction yields the radial functions $\hat{R}^{\lambda m\omega}_{-1}(r)$ for the solution Ψ^+ that is valid (and regular) throughout $\hat{\Sigma}^+$. The other solution Ψ^- that is valid throughout $\hat{\Sigma}^-$ is constructed in a fully analogous manner. The only difference is in the functions $H^{\pm}(r,r')$, which now take the forms

$$H^+(r,r')=a^+(r')R^+_{-1}(r) - e^{-i(mu-\omega r_*)}\sum_{i=0}^{1} B_i^-(r')r^i \quad (\Psi^-),$$

$$H^-(r,r')=a^-(r')R^-_{-1}(r) \qquad (\Psi^-).$$

X. DISCUSSION

Although in most of this paper we referred explicitly to gravitational perturbations, the same construction applies to the electromagnetic case as well, as outlined in Sec. IX. In particular, the domains of validity are the same in both cases: $\hat{\Sigma}^+$ for Ψ^+ (and for $h^+_{\alpha\beta}$ or A^+_α derived from the latter), and $\hat{\Sigma}^-$ for Ψ^- (and for $h^-_{\alpha\beta}$ or A^-_α).

Also, although we have explicitly considered the ingoing radiation gauge throughout this paper, an analogous construction may be applied to the *outgoing* radiation gauge. In this latter gauge, too, there are two solutions, Ψ^+_{ORG} and Ψ^-_{ORG} (and correspondingly h^+_{ORG},A^+_{ORG} and h^-_{ORG},A^-_{ORG}), which are valid in the domains $\hat{\Sigma}^+_{ORG}$ and $\hat{\Sigma}^-_{ORG}$ (but invalid in Σ^+_{ORG} or Σ^-_{ORG}), respectively. The two domains $\hat{\Sigma}^{\pm}_{ORG}$ are completely analogous to $\hat{\Sigma}^{\pm}\equiv\hat{\Sigma}^{\pm}_{IRG}$, except that they are defined with respect to the *ingoing* rather than outgoing principal null congruence. In the rest of this discussion, too, we shall refer explicitly to the ingoing gauge, but the same remarks will be applicable to the outgoing gauge as well.

Consider the case of a point particle. Our analysis shows there does not exist a single solution for the radiation-gauge $h_{\alpha\beta}$ or A_α that is regular in the entire off-worldline neighborhood of the particle. Instead, the solution Ψ^+ (and correspondingly $h^+_{\alpha\beta},A^+_\alpha$) has a line singularity along the outgoing null geodesic ξ emanating from the particle towards the past and smaller *r*. Similarly, Ψ^- (and correspondingly $h^-_{\alpha\beta},A^-_\alpha$) has a line singularity along the null geodesic ξ emanating from the particle towards the future and larger *r*. The inevitability of such a line singularity in the radiation-gauge MP was previously demonstrated in Ref. [8] based on independent arguments. (The existence of ingoing radiation-gauge solutions other than $h^{\pm}_{\alpha\beta},A^{\pm}_\alpha$, which admit a line singularity in a different direction, not tangent to ξ, has not been explored yet.)

The unavoidable occurrence of a line singularity in the radiation-gauge fields $h_{\alpha\beta}$, A_α is obviously an inconvenient property. Nevertheless, it does not pose a too serious obstacle (at least in some important applications). We must recall that this singularity is after all a gauge artifact, which may in principle be removed by an appropriate gauge transformation. Therefore, whenever the local values of $h_{\alpha\beta}$ or A_α are required for the calculation of any local gauge-invariant quantity, the solutions $h^+_{\alpha\beta}$, A^+_α and/or $h^-_{\alpha\beta}$, A^-_α may be used regardless of the line singularity.

An important application which requires the knowledge of $h_{\alpha\beta}$ or A_α is the radiation-reaction problem for a point mass or point charge. Generically the full analysis of this phenomenon requires the calculation of the local self force acting on the particle. The electromagnetic self force is gauge invariant. The situation in the gravitational problem is more delicate, because the gravitational self force is a gauge-dependent entity. Nevertheless, within the context of the adiabatic approximation, the orbit-integrated change (induced by the self force) in any of the orbit's constants of motion is gauge-invariant. One thus may use any gauge to calculate the self force, and hence the rate of change of the constants of motion. Consider the calculation of the self force according to the Mino-Sasaki-Tanaka [12] formulation. Then the self force is the limit of the ''tail-force'' field at the particle's location. This limit may be taken from any desired direction. Two especially convenient directions are the ingoing and outgoing radial directions (so far the mode-sum method [13] has been fully developed for these radial directions only). To this end, one may use the solution $h^+_{\alpha\beta}$ or A^+_α when calculating the self-force from the radial direction $r>r_{particle}$, and the solution $h^-_{\alpha\beta}$ or A^-_α for calculating the self-force from $r<r_{particle}$. In both cases the line singularity is not encountered.[11]

In the case of a smooth extended source, Σ^+ (or Σ^-) becomes a four-dimensional set. In this case Ψ^+ does not develop an irregularity at Σ^+; however, Eq. (10) is violated there. This *suggests* that the quantity $h^+_{\alpha\beta}$ (constructed from Ψ^+ by applying the differential operator Π) will not be valid at Σ^+, even in its vacuum part (namely, it will fail to satisfy the vacuum Einstein equation, and/or to reproduce the original Teukolsky field ψ_0); but this still needs to be verified.

In the above construction we have assumed that the particle's worldline or the extended source is restricted to a range $r_{\min}\leq r\leq r_{\max}$. This assumption was made primarily for conceptual clarity, as it allows us to discuss the behavior of, e.g. Ψ^+, in the two vacuum regions, $r>r_{\max}$ and $r<r_{\min}$, but it can be relaxed at least partially, as we now discuss.

Consider, first, the situation in which the source is re-

[11]Recall, however, that in the gravitational case there is another difficulty associated with the radiation gauge: The leading-order asymptotic behavior of the MP, on approaching the particle's location from a generic direction, differs from that of the harmonic-gauge MP, making this an ''irregular gauge'' in the terminology of Ref. [8]. This kind of irregularity (which is unrelated to the line singularity on $\Sigma^{\pm}$) also occurs in e.g. the Regge-Wheeler gauge in the Schwarzschild case. This difficulty may in principle be overcome by transforming to an ''intermediate gauge,'' as outlined in Ref. [8].

stricted to the range $r \leqslant r_{\max}$ with no minimal value $r_{\min}$.[12] Then the construction of Ψ^{+} follows just as prescribed above, without any difficulties. The construction of Ψ^{-} in this case may formally be carried out as above; However, the proof given in Sec. VII for the validity of Ψ^{-} throughout $\hat{\Sigma}^{-}$ fails in this case: This proof (when applied to Ψ^{-} rather than Ψ^{+}) starts from the trivial observation that (provided that the source is restricted to $r \geqslant r_{\min}$) Eq. (10) is satisfied by Ψ^{-} throughout $r < r_{\min}$. Then this feature is analytically extended to the entire domain $\hat{\Sigma}^{-}$. In the present case (i.e. no $r_{\min}$) this proof is inapplicable even at its starting point. It therefore still needs be verified whether in this case the so constructed solution Ψ^{-} is valid in $\hat{\Sigma}^{-}$.

In the analogous case, in which the source extends from infinity to some $r_{\min}$, the situation is basically similar, though technically it is slightly more involved. Consider for example an unbounded orbit that arrives from infinity and scatters off the BH back to infinity. Here, the solution Ψ^{-} can in principle be constructed as above, but the solution Ψ^{+} is not guaranteed to hold (for the reason explained just above). In this case, however, due to the slow decay at large r of the potential term in the radial Teukolsky equation, the standard integral solution (48), (49) for ψ_0 diverges. One then has to use another Green's function [14] for the construction of ψ_0, and this may modify the function $H(r,r')$. We shall not elaborate on this case here.

Finally we note that there are a few types of special modes which require special treatment. First, for the stationary modes $\omega = 0$, the large-r basis solutions $R_{\pm 2}^{\lambda m \omega (in,out)}$ constructed in Sec. IV must be replaced by some other ones, and the same holds for the corresponding constants $C^{(in)}$ and $C^{(out)}$. Second, for "marginally superradiant" modes $k=0$, the EH basis solutions $R_{\pm 2}^{\lambda m \omega (up,down)}$ and the corresponding constants $C^{(up,down)}$ are to be modified. It appears likely, though, that in both cases the inhomogeneous solution described in, e.g. Sec. VIII, remains valid, provided that one substitutes the appropriate basis functions $R_{-2}^{\pm}(r)$ (i.e. those satisfying the correct boundary conditions at large r or at the EH). Other cases which require special attention are the so-called "$l=0,1$ modes" (the "$l=0$ mode" in the electromagnetic case). These are the perturbation modes for which the Teukolsky variables ψ_0 and ψ_4 vanish identically, while ψ_2 is nonvanishing. The extension of this construction to include the $l=0,1$ modes, as well as all other $\omega=0$ modes, is now underway.

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[12] In the case of a point particle, this situation may be realized by a "fine-tuned" geodesic that asymptotes to an unstable circular orbit in the far past, but falls into the BH in the future. We prefer not to consider here bound geodesics emerging out of the white-hole horizon, to avoid the conceptual complications associated with the latter's causal properties.

APPENDIX A

We rewrite Eq. (61) as

$$(D_{m\omega})^n \left[e^{-i(mu-\omega r_*)} \sum_{i=0}^{3} B_i^{-}(r') r^i \right] = C^{+}A^{+}(r')(D_{m\omega})^n [R_{-2}^{+}(r)] - C^{-}A^{-}(r') \times (D_{m\omega})^n [R_{-2}^{-}(r)] \tag{A1}$$

(applied at $r=r'$ and for $n=0,\ldots,3$). It will be convenient to rewrite the polynomial $\Sigma_i B_i^{-}(r')r^i$ as $\Sigma_i \hat{B}_i(r')(r-r')^i$. By virtue of Eq. (45), the left-hand side of the last equation reads

$$e^{-i(mu-\omega r_*)} \sum_{i=0}^{3} \hat{B}_i(r') \frac{d^n}{dr^n}(r-r')^i = e^{-i(mu-\omega r_*)} n! \hat{B}_n(r').$$

Evaluating Eq. (A1) at $r=r'$ then implies (for $n=0,\ldots,3$)

$$e^{-i(mu-\omega r_*)} n! \hat{B}_n(r') = C^{+}A^{+}(r')(D_{m\omega})^n [R_{-2}^{+}] - C^{-}A^{-}(r')(D_{m\omega})^n [R_{-2}^{-}] \tag{A2}$$

(where $D_{m\omega}^n[R_{-2}^{\pm}]$ is to be evaluated at $r=r'$). The operator $D_{m\omega}$ is given by

$$D_{m\omega} = \frac{d}{dr} + (i/\Delta)K.$$

Using the radial Teukolsky equation (13) we can express any derivative of an $s=-2$ vacuum radial function $R_{-2}^{\lambda m \omega}$ as a linear combination of $R_{-2}^{\lambda m \omega}$ and $(d/dr)R_{-2}^{\lambda m \omega}$ (and consequently we can express any power of $D_{m\omega}$ as a linear combination of $R_{-2}^{\lambda m \omega}$ and $D_{m\omega}[R_{-2}^{\lambda m \omega}]$). As a consequence, when applied to any homogeneous solutions $R_{-2}^{\lambda m \omega}$ (and in particular $R_{-2}^{\pm}$), we have the following operator identities [15]:

$$(D_{m\omega})^2 = (2/\Delta)(iK + r - M) D_{m\omega} + \frac{\lambda + 6i\omega r}{\Delta} = (2/\Delta)(iK + r - M)\frac{d}{dr} + \Delta^{-2}[2iK(iK + r - M) + \Delta(\lambda + 6i\omega r)]$$

and

$$(D_{m\omega})^3=\Delta^{-2}[4iK(iK+r-M)+(\lambda+2-2i\omega r)\Delta]D_{m\omega}+\Delta^{-2}[2iK(\lambda+6i\omega r)+6i\omega\Delta]$$
$$=\Delta^{-2}[4iK(iK+r-M)+(\lambda+2-2i\omega r)\Delta]\frac{d}{dr}+\Delta^{-3}\{iK[4iK(iK+r-M)+(\lambda+2-2i\omega r)\Delta]+\Delta[2iK(\lambda+6i\omega r)+6i\omega\Delta]\}.$$

Thus, we may write Eq. (A2) as

$$\hat{B}_n(r')=C^+A^+(r')\left[\hat{f}_n(r')R^+_{-2}(r')+\hat{g}_n(r')\frac{d}{dr'}R^+_{-2}(r')\right]-C^-A^-(r')\left[\hat{f}_n(r')R^-_{-2}(r')+\hat{g}_n(r')\frac{d}{dr'}R^-_{-2}(r')\right]$$

(for $n=0,\ldots,3$), where the functions $\hat{f}_n,\hat{g}_n$ are given by

$$\hat{f}_0(r)=e^{i(mu-\omega r_*)},\quad \hat{g}_0(r)=0,$$

$$\hat{f}_1(r)=(i/\Delta)Ke^{i(mu-\omega r_*)},\quad \hat{g}_1(r)=e^{i(mu-\omega r_*)},$$

$$\hat{f}_2(r)=(2\Delta^2)^{-1}[2iK(iK+r-M)+\Delta(\lambda+6i\omega r)]e^{i(mu-\omega r_*)},$$

$$\hat{g}_2(r)=\Delta^{-1}(iK+r-M)e^{i(mu-\omega r_*)},$$

$$\hat{f}_3(r)=(6\Delta^3)^{-1}\{iK[4iK(iK+r-M)+(\lambda+2-2i\omega r)\Delta]+\Delta[2iK(\lambda+6i\omega r)+6i\omega\Delta]\}e^{i(mu-\omega r_*)},$$

$$\hat{g}_3(r)=(6\Delta^2)^{-1}[4iK(iK+r-M)+(\lambda+2-2i\omega r)\Delta]e^{i(mu-\omega r_*)}.$$

Once the coefficients $\hat{B}_i(r')$ are determined, the original coefficients $B^-_n(r')$ may be constructed through

$$\sum_{i=0}^{3}B^-_i(r')r^i=\sum_{i=0}^{3}\hat{B}_i(r')(r-r')^i,$$

which yields

$$B^-_0=\hat{B}_0-\hat{B}_1r'+\hat{B}_2r'^2-\hat{B}_3r'^3,$$

$$B^-_1=\hat{B}_1-2\hat{B}_2r'+3\hat{B}_3r'^2,$$

$$B^-_2=\hat{B}_2-3\hat{B}_3r',\qquad B^-_3=\hat{B}_3.$$

This allows us to express the functions $B^-_n(r')$ as

$$B^-_n(r')=C^+A^+(r')\left[f_n(r')R^+_{-2}(r')+g_n(r')\frac{d}{dr'}R^+_{-2}(r')\right]-C^-A^-(r')\left[f_n(r')R^-_{-2}(r')+g_n(r')\frac{d}{dr'}R^-_{-2}(r')\right],$$

with the functions $f_n(r'),g_n(r')$ given by

$$f_0=\hat{f}_0-\hat{f}_1r'+\hat{f}_2r'^2-\hat{f}_3r'^3;\quad f_1=\hat{f}_1-2\hat{f}_2r'+3\hat{f}_3r'^2;$$

$$f_2=\hat{f}_2-3\hat{f}_3r',\quad f_3=\hat{f}_3,$$

and similarly

$$g_0=\hat{g}_0-\hat{g}_1r'+\hat{g}_2r'^2-\hat{g}_3r'^3;\quad g_1=\hat{g}_1-2\hat{g}_2r'+3\hat{g}_3r'^2;$$

$$g_2=\hat{g}_2-3\hat{g}_3r',\quad g_3=\hat{g}_3$$

(with all functions $\hat{g}_n$, $\hat{f}_n$ evaluated at r' rather than r).

APPENDIX B

Our goal is to construct the basis functions $R^{\pm}_{+2}$ from the corresponding functions $R^{\pm}_{-2}$. To simplify the notation, throughout this appendix we shall view $R^{\pm}_{\pm 2}$ and all other "radial" variables as functions or r, not r'. When implementing the result (B6) back in Sec. VI, one should simply substitute $r\rightarrow r'$.

The analysis in Sec. IV, Eqs. (27), (32), implies

$$H[R^{\pm}_{+2}]=C^{\pm}R^{\pm}_{-2},$$

which [since H is the inverse of the operator $(D_{m\omega})^4$] is equivalent to

$$R^{\pm}_{+2}=C^{\pm}(D_{m\omega})^4[R^{\pm}_{-2}].$$

With the aid of the radial Teukolsky equation, the operator $(D_{m\omega})^4$ acting on any vacuum solution $R^{\lambda m\omega}_{-2}$ may be expressed in terms of $R^{\lambda m\omega}_{-2}$ and its first-order derivative. Chandrasekhar derived the formula (see Eq. (49CH), where throughout this appendix "CH" refers to equations in Chap. 9 of Ref. [9])

$$(D_{m\omega})^4[R^{\lambda m\omega}_{-2}]=(A_0/\Delta^3)D_{m\omega}[R^{\lambda m\omega}_{-2}]+(B_0/\Delta^3)R^{\lambda m\omega}_{-2},\tag{B1}$$

where

$$A_0=-8iK[K^2+(r-M)^2]+[4iK(\lambda+2)-8i\omega r(r-m)]\Delta+8i\omega\Delta^2,\tag{B2}$$

$$B_0=[(\lambda+2-2i\omega r)(\lambda+6i\omega r)+12i\omega(iK-r+M)]\Delta+4iK(iK-r+M)(\lambda+6i\omega r).\tag{B3}$$

This yields

$$R^{\pm}_{+2}=(C^{\pm}/\Delta^3)(A_0D_{m\omega}[R^{\pm}_{-2}]+B_0R^{\pm}_{-2}). \tag{B4}$$

We also need to express the determinant $W[R^-_{+2},R^+_{+2}]$ in terms of $R^{\pm}_{-2}$. We find it useful to express this determinant as

$$W[R^-_{+2},R^+_{+2}]\equiv R^-_{+2}R^+_{+2,r}-R^+_{+2}R^-_{+2,r} = R^-_{+2}\mathcal{D}^{\dagger}_{-1}[R^+_{+2,r}]-R^+_{+2,r}\mathcal{D}^{\dagger}_{-1}[R^-_{+2}],$$

where $\mathcal{D}^{\dagger}_{-1}\equiv\partial_r-(iK/\Delta)-2(r-M)/\Delta$, which allows us to make use of Eq. (50CH). Writing

$$\mathcal{D}^{\dagger}_{-1}(A_0D_{m\omega}+B_0)=A_1D_{m\omega}+B_1$$

[with A_1,B_1 specified in Eq. (51CH)], we obtain

$$W[R^-_{+2},R^+_{+2}]=C^+C^-\Delta^{-6}(B_0A_1-A_0B_1)W[R^-_{-2},R^+_{-2}].$$

A straightforward calculation yields

$$B_0A_1-A_0B_1=p\Delta^2,$$

leading to

$$W[R^-_{+2},R^+_{+2}]=C^+C^-p\Delta^{-4}W[R^-_{-2},R^+_{-2}]. \tag{B5}$$

(Note the consistency of this result with the general expression for the Wronskian of the Teukolsky equation: For any s, and any pair of independent solutions R^a_s,R^b_s,

$$W[R^a_s,R^b_s]=\text{const}\times\Delta^{-s-1}.$$

Hence $W[R^-_{+2},R^+_{+2}]=\text{const}\times\Delta^{-3}$ and $W[R^-_{-2},R^+_{-2}]=\text{const}\times\Delta$, in agreement with Eq. (B5).) Combining this result with Eqs. (B4) and (50), we obtain

$$A^{\pm}=(C^{\pm}pW[R^-_{-2},R^+_{-2}])^{-1}(A_0D_{m\omega}[R^{\mp}_{-2}]+B_0R^{\mp}_{-2}).$$

This yields

$$C^{\pm}A^{\pm}=(pW[R^-_{-2},R^+_{-2}])^{-1}\left[\overline{A}\frac{d}{dr}R^{\mp}_{-2}+\overline{B}R^{\mp}_{-2}\right], \tag{B6}$$

with

$$\overline{A}=A_0,\quad \overline{B}=B_0+(iK/\Delta)A_0. \tag{B7}$$

APPENDIX C

For a vacuum mode $\lambda m\omega$ of electromagnetic perturbations, the radial function $\hat{R}^{\lambda m\omega}_{-1}$ must satisfy the two equations

$$P^{\lambda m\omega}_{-1}[\hat{R}^{\lambda m\omega}_{-1}(r)]=0 \tag{C1}$$

and

$$R^{\lambda m\omega}_{+1}(r)=-(D_{m\omega})^2[\hat{R}^{\lambda m\omega}_{-1}(r)]. \tag{C2}$$

The general solution for these two equations is

$$\hat{R}^{\lambda m\omega}_{-1}\equiv-p^{-1}\Delta(D^{\dagger}_{m\omega})^2\Delta[R^{\lambda m\omega}_{+1}(r)], \tag{C3}$$

where

$$p=\lambda^2-4\alpha^2\omega^2=\lambda^2+4a\omega(m-a\omega)\quad(\text{EM}).$$

Considering the four asymptotic basis solutions $R^{\lambda m\omega(in,out,up,down)}_{\pm1}$ specified in Sec. IX, the corresponding parameters $C^{(in)}$, $C^{(out)}$, $C^{(down)}$, $C^{(up)}$ are easily calculated just as in the gravitational case. One finds

$$C^{(in)}=1/(4\omega^2),\quad C^{(out)}=4\omega^2/p\quad(\text{EM})$$

$$C^{(up)}=Q/p,\quad C^{(down)}=1/Q\quad(\text{EM}).$$

The general solution to the homogeneous part of Eq. (C2), namely $(D_{m\omega})^2[\hat{R}^{\lambda m\omega}_{-1}]=0$, is easily constructed:

$$\hat{R}^{\lambda m\omega}_{-1}(r)=e^{-i(mu-\omega r_*)}\sum_{i=0}^{1}B_ir^i. \tag{C4}$$

Consider next the case of inhomogeneous electromagnetic perturbations. The general solution for the radial function of φ_0 may be expressed as

$$R^{\lambda m\omega}_{+1}(r)=\int_{r_{\min}}^{r_{\max}}T^{\lambda m\omega}_{+1}(r')G(r,r')dr'. \tag{C5}$$

The Green's function is

$$G(r,r')=A^+(r')R^+_{+1}(r)\theta(r-r')+A^-(r')\times R^-_{+1}(r)\theta(r'-r),$$

where

$$A^{\pm}=R^{\mp}_{+1}/(\Delta W[R^-_{+1},R^+_{+1}]) \tag{C6}$$

(evaluated at r'). Then $\hat{R}^{\lambda m\omega}_{-1}(r)$ is given by

$$\hat{R}^{\lambda m\omega}_{-1}(r)=\int_{r_{\min}}^{r_{\max}}T^{\lambda m\omega}_{+1}(r')H(r,r')dr', \tag{C7}$$

where $H(r,r')$ satisfies

$$(D_{m\omega})^2[H(r,r')]=-G(r,r') \tag{C8}$$

(and the appropriate boundary conditions at $r>r_{\max}$). We find $H(r,r')$ to be of the form

$$H(r,r')=H^+(r,r')\theta(r-r')+H^-(r,r')\theta(r'-r), \tag{C9}$$

with

$$H^+(r,r')=C^+A^+(r')R^+_{-1}(r) \tag{C10}$$

and

$$H^-(r,r')=C^-A^-(r')R^-_{-1}(r)+e^{-i(mu-\omega r_*)}\sum_{i=0}^{1} B_i^-(r')r^i. \tag{C11}$$

The two functions $B_i^-(r')$ are determined by regularity conditions at $r=r'$, which yield

$$B_i^-(r')=C^+A^+(r')\left[f_i(r')R^+_{-1}(r')+g_i(r')\frac{d}{dr'}R^+_{-1}(r')\right] -C^-A^-(r')\left[f_i(r')R^-_{-1}(r') +g_i(r')\frac{d}{dr'}R^-_{-1}(r')\right].$$

In full analogy with the gravitational case (see Appendix A) we find

$$f_0=\hat{f}_0-\hat{f}_1r',\quad f_1=\hat{f}_1,\quad g_0=\hat{g}_0-\hat{g}_1r',\quad g_1=\hat{g}_1,$$

and

$$\hat{f}_0(r)=e^{i(mu-\omega r_*)},\quad \hat{g}_0(r)=0,$$

$$\hat{f}_1(r)=(i/\Delta)Ke^{i(mu-\omega r_*)},\quad \hat{g}_1(r)=e^{i(mu-\omega r_*)},$$

yielding the functions f_i,g_i specified in Sec. IX.

Next we express $R^{\mp}_{+1}$ in terms of $R^{\mp}_{-1}$, using

$$R^{\pm}_{+1}=-C^{\pm}(D_{m\omega})^2[R^{\pm}_{-1}]. \tag{C12}$$

The vacuum radial Teukolsky equation for the $s=-1$ radial function may be expressed as [9]

$$D^{\dagger}_{m\omega}D_{m\omega}=(\lambda+2i\omega r)/\Delta.$$

Using this equation to reduce the order of differentiation, and recalling $D_{m\omega}=D^{\dagger}_{m\omega}+2iK/\Delta$, we obtain

$$(D_{m\omega})^2[R^{\pm}_{-1}]=(A_0/\Delta)D_{m\omega}[R^{\pm}_{-1}]+(B_0/\Delta)R^{\pm}_{-1}, \tag{C13}$$

where

$$A_0=2iK,\quad B_0=\lambda+2i\omega r.$$

We may rewrite Eqs. (C12),(C13) as

$$R^{\pm}_{+1}=-(C^{\pm}/\Delta)\left(\bar{A}\frac{d}{dr}R^{\pm}_{-1}+\bar{B}R^{\pm}_{-1}\right), \tag{C14}$$

where

$$\bar{A}=A_0=2iK \tag{C15}$$

and

$$\bar{B}=B_0+(iK/\Delta)A_0=\lambda+2i\omega r-2K^2/\Delta.$$

We now calculate the determinant $W[R^-_{+1},R^+_{+1}]$, using

$$W[R^-_{+1},R^+_{+1}]=R^-_{+1}D_{m\omega}[R^+_{+1}]-R^+_{+1}D_{m\omega}[R^-_{+1}],$$

along with Eqs. (C12),(C13). The calculation yields

$$W[R^-_{+1},R^+_{+1}]=C^+C^-p\Delta^{-2}W[R^-_{-1},R^+_{-1}].$$

Substituting this and Eq. (C14) into Eq. (C6) we obtain

$$a^{\pm}(r)\equiv C^{\pm}A^{\pm}(r) =-(pW[R^-_{-1},R^+_{-1}])^{-1}\left(\bar{A}\frac{d}{dr}R^{\mp}_{-1}+\bar{B}R^{\mp}_{-1}\right).$$

Finally, substituting this in the above equations for $B_i^-(r')$ and $H^{\pm}(r,r')$ (with the substitution $r\to r'$), we obtain the expressions for these quantities as specified in Sec. IX.

[1] S.A. Teukolsky, Phys. Rev. Lett. **29**, 1114 (1972).
[2] S.A. Teukolsky, Astrophys. J. **185**, 635 (1973).
[3] S.A. Teukolsky and W.H. Press, Astrophys. J. **193**, 443 (1974).
[4] P.L. Chrzanowski, Phys. Rev. D **11**, 2042 (1975).
[5] R.M. Wald, Phys. Rev. Lett. **41**, 203 (1978).
[6] C.O. Lousto and B.F. Whiting, Phys. Rev. D **66**, 024026 (2002).
[7] J.M. Cohen and L.S. Kegeles, Phys. Rev. D **10**, 1070 (1974).
[8] L. Barack and A. Ori, Phys. Rev. D **64**, 124003 (2001).
[9] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).
[10] See Table I in Ref. [4]. The differential operators Π_{IRG} and Π_{ORG} are also given in Eqs. (2.3) and (2.5) of Ref. [6].
[11] It should be pointed out that Wald's derivation in Ref. [5] is applicable even on a local basis [R.M. Wald (private communication)]. This implies that, since Ψ^+ satisfies all the required equations everywhere in $\hat{\Sigma}^+$, $h^+_{\alpha\beta}\equiv\Pi[\Psi^+]$ is a valid MP solution throughout this domain. We preferred, however, to demonstrate here the validity of $h^+_{\alpha\beta}$ without referring to these local considerations.
[12] Y. Mino, M. Sasaki, and T. Tanaka, Phys. Rev. D **55**, 3457 (1997).
[13] L. Barack, Y. Mino, H. Nakano, A. Ori, and M. Sasaki, Phys. Rev. Lett. **88**, 091101 (2002).
[14] E. Poisson, Phys. Rev. D **55**, 639 (1997).
[15] See Eqs. (47), (48) in Chap. 9 of Ref. [9].