Parity violating spin-two gauge theories

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Nonlinear covariant parity-violating deformations of free spin-two gauge theory are studied in $n \geqslant 3$ spacetime dimensions, using a linearized frame and spin-connection formalism, for a set of massless spin-two fields. It is shown that the only such deformations yielding field equations with a second order quasilinear form are the novel algebra-valued types in $n=3$ and $n=5$ dimensions already found in some recent related work concentrating on lowest order deformations. The complete form of the deformation to all orders in $n=5$ dimensions is worked out here and some features of the resulting new algebra-valued spin-two gauge theory are discussed. In particular, the internal algebra underlying this theory on five-dimensional Minkowski space is shown to cause the energy for the spin-two fields to be of indefinite sign. Finally, a Kaluza-Klein reduction to $n=4$ dimensions is derived, giving a parity-violating nonlinear gauge theory of a coupled set of spin-two, spin-one, and spin-zero massless fields.

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I. INTRODUCTION

It is widely believed that the only allowed type of gauge symmetry for a nonlinear massless spin-two field is a diffeomorphism symmetry, corresponding to a gravitational self-interaction of the field. Recent work [1,2] has addressed this question more generally for a set of any number of coupled massless spin-two fields by the approach of deformations of linear Abelian spin-two gauge theory. A deformation, here, means adding linear and higher power terms to the Abelian spin-two gauge symmetry while also adding quadratic and higher power terms to the linear spin-two field equation, such that a gauge invariant action principle exists which is not equivalent to the undeformed linear theory by nonlinear field redefinitions. The condition of gauge invariance has various formulations [3–5] that yield determining equations to solve for the allowed form of the deformation terms, added order by order, in powers of the fields. This approach, in contrast with earlier efforts in the literature, makes no assumptions on the possible structure for the commutator algebra of the deformed gauge symmetries and takes advantage of the necessary requirement that these gauge symmetries should be realized by a nonlinear theory given by a deformation of the Lagrangian of the linear theory.

The results of the deformation analysis in Ref. [1] show that if all deformation terms are required to involve no more derivatives of the spin-two field and gauge symmetry parameter than appear in the free theory, then in four dimensions the unique possible deformation for a set of one or more spin-two fields is given by an algebra-valued generalization of the Einstein gravity theory [6,7] based on a commutative, associative, invariant-normed algebra. As such a set of fields is mathematically equivalent to a single algebra-valued spin-two field [8], the only type of gauge symmetry indeed allowed is a diffeomorphism symmetry (in an algebra-valued setting). An extension of this result to all higher spacetime dimensions is given in Ref. [2] by using a BRST cohomological formulation [4,5] of the deformation determining equations.

Very interestingly, in Ref. [9] a deformation different than the Einstein field equation and diffeomorphism gauge symmetry for a single spin-two field in three-dimensional Minkowski space is constructed by deforming the Abelian gauge symmetry by a linear term that contains first derivatives of both the spin-two field and the gauge symmetry parameter. Gauge invariance is maintained at lowest order by also deforming the free Lagrangian by a term that is cubic in first derivatives of the spin-two field. The resulting quadratic terms in the field equation contain one more derivative than in the free theory but are still second order in derivatives. These deformation terms, moreover, also involve the Minkowski volume tensor and hence possess the interesting feature of being parity noninvariant. Of course, in three dimensions there are no local dynamical degrees of freedom for a free spin-two field. Indeed the full deformation to all orders is found to be related through field redefinitions to certain topological three-dimensional gravity theories [10] in a scaling limit in which the gravitational interaction is turned off. Intriguingly, parity noninvariant deformation terms of a similar form to the three-dimensional ones also exist at lowest order in five dimensions for an algebra-valued spin-two field using an anticommutative algebra; but it was left open in Ref. [9] whether a full deformation actually exists to all orders. The resulting nonlinear spin-two gauge theory, if it were to exist, would be of obvious potential physical and mathematical interest to investigate. It would lead, for instance, to an exotic four-dimensional gauge theory of a nonlinearly coupled set of spin-two and spin-one fields obtained via a Kaluza-Klein reduction of the five-dimensional theory.

The main purpose of this paper is to show that, in fact, the novel first order deformation in five dimensions explored in Ref. [9] exists to all orders only if the anticommutative algebra is nilpotent (and invariant-normed), due to an integrability condition that arises in solving the deformation determining equations at second order. It will also be shown that there is a striking connection between this algebraic structure and the sign of the energy at lowest order in the deformation.

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These results will follow from a more general classification theorem that will be proven here in $n \geqslant 3$ dimensions for nonlinear spin-two gauge theories with more derivatives than contained in the linear theory, but which preserve the number of local dynamical degrees of freedom. For this analysis, it turns out to be most convenient to employ a linearized spin-connection/frame formalism for free spin-two fields, in terms of which the deformation obtained in $n=3$ dimensions in Ref. [9] takes its simplest form (indeed, no other complete formulation for that deformation has yet been derived).

II. DEFORMATION ANALYSIS AND MAIN RESULTS

We introduce the set of $N \geqslant 1$ fields $h_{a\mu}{}^A$ (viewed as linearized frames), together with a set of auxiliary fields (with the role of linearized spin-connections)

$$\overset{(1)}{\omega}_{a\mu\nu}{}^A = \partial_a h_{[bc]}{}^A \sigma^b{}_\mu \sigma^c{}_\nu - 2\sigma^b{}_{[\nu}\partial_{\mu]} h_{(ab)}{}^A, \qquad (2.1)$$

in terms of $h_{ab}{}^A = \sigma_b{}^\mu h_{a\mu}{}^A$, $A = 1,\dots,N$, where $\sigma_b{}^\mu$ denotes any fixed orthonormal frame for the Minkowski metric $\eta_{ab} = \sigma_a{}^\mu \sigma_b{}^\nu \eta_{\mu\nu}$ [with $\eta^{\mu\nu} = \mathrm{diag}(-1,+1,\dots,+1)$] while $\sigma^b{}_\mu$ is the inverse frame [11]. The free spin-two theory for these fields on n-dimensional Minkowski space $(\mathbb{R}^n, \eta_{ab})$ is given by the linear field equations

$$\overset{(1)}{E}_{a\mu}{}^A = \overset{(1)}{R}_{ab\mu\nu}{}^A \sigma^{b\nu} = 0, \quad \overset{(1)}{R}_{ab\mu\nu}{}^A = \partial_{[a} \overset{(1)}{\omega}_{b]\mu\nu}{}^A, \qquad (2.2)$$

and the Abelian gauge symmetries

$$\overset{(0)}{\delta}_\xi h_{a\mu}{}^A = \partial_a \xi_\mu{}^A, \quad \overset{(0)}{\delta}_\chi h_{a\mu}{}^A = \sigma_a{}^\nu \chi_{\nu\mu}{}^A \qquad (2.3)$$

involving parameters $\xi_\mu{}^A$, $\chi_{\nu\mu}{}^A = \chi_{[\nu\mu]}{}^A$ which are arbitrary functions of the spacetime coordinates. Here $\overset{(1)}{R}_{ab\mu\nu}{}^A$ is a gauge invariant field strength, as is seen due to

$$\overset{(0)}{\delta}_\xi \overset{(1)}{\omega}_{a\mu\nu}{}^A = 0, \quad \overset{(0)}{\delta}_\chi \overset{(1)}{\omega}_{a\mu\nu}{}^A = \partial_a \chi_{\nu\mu}{}^A. \qquad (2.4)$$

This free theory comes from the gauge-invariant Lagrangian

$$\overset{(2)}{L} = -\tfrac{1}{2}\big(3 h_{[a}{}^{\mu A} \partial_b \overset{(1)}{\omega}_{c]}{}^{\nu\rho B} \sigma^c{}_\rho + \overset{(1)}{\omega}_{[a}{}^{\mu\rho A} \overset{(1)}{\omega}_{b]\rho}{}^{\nu B}\big) \sigma^a{}_\mu \sigma^b{}_\nu \delta_{AB}, \qquad (2.5)$$

where δ_{AB} is a fixed symmetric matrix. A variation of $h_{a\mu}{}^A$ yields the field equations (2.2) to within irrelevant trace terms,

$$\delta^{AB}\left(\delta_a{}^b \delta_\mu{}^\nu + \frac{1}{2-n}\sigma^{b\nu}\sigma_{a\mu}\right)\delta \overset{(2)}{L} / \delta h_{b\nu}{}^B = \overset{(1)}{E}_{a\mu}{}^A, \qquad (2.6)$$

while an independent variation of $\overset{(1)}{\omega}_{a\mu\nu}{}^A$ gives a total divergence as a consequence of the auxiliary equations (2.1). Hence it is necessary only to vary $h_{a\mu}{}^A$ alone in considering variations of the Lagrangian (2.5) (analogous to a "1.5 formalism" [12]). To see how the formalism here is related to the familiar free theory of spin-two fields, observe that if the gauge condition $h_{[ab]}{}^A = 0$ is imposed using the gauge freedom $h_{ab}{}^A \rightarrow h_{ab}{}^A + \chi_{ab}{}^A$ involving the skew-tensor functions $\chi_{ab}{}^A = \sigma_a{}^\nu \sigma_b{}^\rho \chi_{\nu\rho}{}^A$, then the field equations (2.2) and gauge symmetries (2.3) reduce to the ordinary Fierz-Pauli spin-two equations

$$\eta^{cd} \partial_{[a|} \partial_{[b} \gamma_{c|d]}{}^A = 0 \qquad (2.7)$$

and spin-two gauge invariance

$$\gamma_{cd}{}^A \rightarrow \gamma_{cd}{}^A + \partial_{(c} \xi_{d)}{}^A \qquad (2.8)$$

in terms of the symmetric tensor fields $\gamma_{cd}{}^A = h_{(cd)}{}^A$ and covector functions $\xi_b{}^A = \sigma_b{}^\mu \xi_\mu{}^A$. Furthermore, note these spin-two fields will have positive energy as obtained from the conserved stress-energy tensor of the free Lagrangian (2.5) if (and only if) $\delta_{AB} = \mathrm{diag}(+1,\dots,+1)$ is a positive definite matrix.

We now consider deformations of the gauge symmetries and field equations

$$\delta_\xi h_{a\mu}{}^A = \overset{(0)}{\delta}_\xi h_{a\mu}{}^A + \overset{(1)}{\delta}_\xi h_{a\mu}{}^A + \cdots,$$

$$\delta_\chi h_{a\mu}{}^A = \overset{(0)}{\delta}_\chi h_{a\mu}{}^A + \overset{(1)}{\delta}_\chi h_{a\mu}{}^A + \cdots, \qquad (2.9)$$

$$E_{a\mu}{}^A = \overset{(1)}{E}_{a\mu}{}^A + \overset{(2)}{E}_{a\mu}{}^A + \cdots = \mathrm{E}_h(L)_{a\mu}{}^A,$$

$$L = \overset{(2)}{L} + \overset{(3)}{L} + \cdots, \qquad (2.10)$$

satisfying gauge invariance of the Lagrangian L, so that $\delta_\xi L$ and $\delta_\chi L$ are total divergences. This determining condition has the following direct formulation (see Refs. [1, 3]):

$$\mathrm{E}_h(\delta_\xi h_{b\nu}{}^B E^{b\nu}{}_B)_{a\mu}{}^A = \mathrm{E}_h(\delta_\chi h_{b\nu}{}^B E^{b\nu}{}_B)_{a\mu}{}^A = 0, \qquad (2.11)$$

where $\mathrm{E}_h(\cdot)_{a\mu}{}^A$ is the Euler-Lagrange operator with respect to $h_{a\mu}{}^A$ [modified by trace projection (2.6)]. Two deformations will be regarded as equivalent if they are related by field redefinitions

$$h_{a\mu}{}^A \rightarrow h'{}_{a\mu}{}^A = h_{a\mu}{}^A + \overset{(2)}{h}{}'{}_{a\mu}{}^A + \cdots \qquad (2.12)$$

or by parameter redefinitions

$$\xi_\mu{}^A \rightarrow \xi'{}_\mu{}^A = \xi_\mu{}^A + \overset{(1)}{\xi}{}'{}_\mu{}^A + \cdots,$$

$$\chi_{\mu\nu}{}^A \rightarrow \chi'{}_{\mu\nu}{}^A = \chi_{\mu\nu}{}^A + \overset{(1)}{\chi}{}'{}_{\mu\nu}{}^A + \cdots. \qquad (2.13)$$

The deformation terms are taken to be locally constructed from the Minkowski frame, metric and volume tensors, spin-two fields, gauge symmetry parameters, and derivatives of the fields and parameters. As the main assumption, the derivatives appearing in the deformation terms will be restricted so that the number of local dynamical degrees of freedom of the spin-two fields in the linear theory is preserved order by order in a nonlinear deformation. With this requirement the most general possible form for nonlinear spin-two field equations is that of a quasilinear second order system of partial differential equations (PDEs), namely, highest derivative terms are of second order, while the coefficient of these terms depends on at most first order derivatives of the spin-two fields. In turn, due to a general relation known to hold [3,13] between the form of lowest order deformation terms in the field equations and Noether currents of rigid symmetries associated with the lowest order deformation terms in the gauge symmetries, the most general possible form for spin-two gauge symmetries is required to be at most first order in derivatives. It is worth noting that stronger assumptions have been made in all systematic analyses to date and essentially lead to nonlinear spin-two field equations being restricted to the form of a second order system of semilinear PDEs, where the coefficient of the second order derivative terms involves *no* derivatives of the spin-two field but is allowed to depend on the spin-two field itself. Such a form arises directly if the deformation terms in the Lagrangian are restricted to contain at most two derivatives, i.e., second order derivatives appear linearly, or more generally, first order derivatives appear quadratically. In contrast, the weaker assumptions made here allow these deformation terms to have a general polynomial dependence on first derivatives (with no higher order derivatives appearing), which is compatible with a quasilinear form for the spin-two field equations as occurs for the deformations investigated in Ref. [9].

In addition, corresponding to the role of the auxiliary field (2.1) in the free theory, all derivatives of the spin-two fields $h_{a\mu}{}^{A}$ in a nonlinear deformation will be assumed to appear only through $\overset{(1)}{\omega}{}_{a\mu\nu}{}^{A}$. Consequently, it follows that at lowest order the deformation of the gauge symmetries and field equations is required to take the form (indices suppressed)

$$\overset{(1)}{\delta}_{\chi}h = Ah\chi + Bw\chi + Ch\partial\chi, \quad \overset{(1)}{\delta}_{\xi}h = Fh\xi + Gw\xi + Hh\partial\xi, \tag{2.14}$$

and

$$\overset{(2)}{E} = Iw\partial w + Jh\partial w + Kww + Mhw + Nhh, \tag{2.15}$$

where the coefficients are constant tensors $A,\ldots,N$ locally constructed just from $\sigma_a{}^{\mu}$, η_{ab}, $\epsilon_{a_1\cdots a_n}$. Such deformations will be called "quasilinear covariant."

A useful field-theoretic formulation of gauge invariance (2.11) is given by the following necessary and sufficient Lie derivative equations (indices suppressed):

$$\mathcal{L}_{\delta_\xi}E = 0, \quad \mathcal{L}_{\delta_\chi}E = 0, \tag{2.16}$$

$$\mathcal{L}_{[\delta_{\xi_1},\delta_{\xi_2}]}E = 0, \quad \mathcal{L}_{[\delta_{\chi_1},\delta_{\chi_2}]}E = 0, \quad \mathcal{L}_{[\delta_{\xi_1},\delta_{\chi_1}]}E = 0, \tag{2.17}$$

where $\mathcal{L}_\delta$ denotes the Lie derivative operator as defined with respect to field variations δh regarded formally as tangent vector fields on the space of field configurations h (see Ref. [8]). An expansion of Eqs. (2.16) and (2.17) in powers of h gives a system of determining equations for all allowed deformation terms order by order in the field equations and gauge symmetries.

The first order deformation terms (2.14) and (2.15) can be determined by solving the expanded determining equations (2.17) to zeroth order and (2.16) to first order using the methods of Ref. [1]. This leads to the following classification result.

Theorem 1. All first order quasilinear covariant deformations (2.14) *and* (2.15) *of the free spin-two gauge theory* (2.1)–(2.5) *in* $n \geqslant 3$ *dimensions are equivalent to a combination of the types*

$$\overset{(1)}{\delta}_{\xi}h_{a\mu}{}^{A} = a^{A}{}_{BC}\overset{(1)}{\omega}{}_{a\mu\nu}{}^{B}\xi^{\nu C}, \quad \overset{(1)}{\delta}_{\chi}h_{a\mu}{}^{A} = a^{A}{}_{BC}h_{a}{}^{\nu B}\chi_{\nu\mu}{}^{C}, \tag{2.18a}$$

$$\begin{aligned}\overset{(3)}{L} = -(a_{ABC}(3\overset{(1)}{R}{}_{[ab}{}^{\alpha\beta A}h_{c}{}^{\nu B}h_{d]}{}^{\rho C}\sigma^{d}{}_{\rho} \\ + \tfrac{3}{2}\overset{(2)}{R}{}_{[ab}{}^{\alpha\beta A}h_{c]}{}^{\nu B})\sigma^{c}{}_{\nu} \\ + \delta_{AB}\tfrac{1}{2}\overset{(2)}{\omega}{}_{[a}{}^{\alpha\rho A}\overset{(1)}{\omega}{}_{b]\rho}{}^{\beta B})\sigma^{a}{}_{\alpha}\sigma^{b}{}_{\beta},\end{aligned} \tag{2.18b}$$

$$a_{ABC} = a_{(ABC)}, \tag{2.18c}$$

where

$$\overset{(2)}{R}{}_{ab}{}^{\nu\rho A} = \partial_{[a}\overset{(2)}{\omega}{}_{b]}{}^{\nu\rho A} + a^{A}{}_{BC}\overset{(1)}{\omega}{}_{[a}{}^{\nu\mu B}\overset{(1)}{\omega}{}_{b]\mu}{}^{\rho C}, \tag{2.19}$$

$$\begin{aligned}\overset{(2)}{\omega}{}_{a}{}^{\nu\rho A} = a^{A}{}_{BC}((2h^{b[\nu B}\sigma^{\rho]c}\sigma_{a\mu} \\ - h_{a\mu}{}^{B}\sigma^{b\nu}\sigma^{c\rho})\partial_{[b}h_{c]}{}^{\mu C} - 2h^{b[\nu B}\partial_{[a}h_{b]}{}^{\rho]C})\end{aligned} \tag{2.20}$$

(corresponding to an algebra-valued gravitational interaction), or if $n=3$

$$\overset{(1)}{\delta}_{\xi}h_{a\mu}{}^{A} = 0, \quad \overset{(1)}{\delta}_{\chi}h_{a\mu}{}^{A} = b^{A}{}_{BC}\epsilon^{\nu\rho\alpha}\overset{(1)}{\omega}{}_{a\nu\rho}{}^{B}\chi_{\alpha\mu}{}^{C}, \tag{2.21a}$$

$$\overset{(3)}{L} = \tfrac{1}{2}b_{ABC}\epsilon^{abc}\overset{(1)}{\omega}{}_{a}{}^{\alpha\beta A}\overset{(1)}{\omega}{}_{b\alpha}{}^{\nu B}\overset{(1)}{\omega}{}_{c\nu\beta}{}^{C}, \tag{2.21b}$$

$$b_{ABC} = b_{(ABC)} \tag{2.21c}$$

(corresponding to the parity violating part of an algebra-valued topological gravity interaction), or if $n=5$

$$\overset{(1)}{\delta}_{\xi}h_{a\mu}{}^{A} = 0, \quad \overset{(1)}{\delta}_{\chi}h_{a\mu}{}^{A} = c^{A}{}_{BC}\epsilon_{\mu\nu\rho\alpha\beta}\overset{(1)}{\omega}{}_{a}{}^{\nu\rho B}\chi^{\alpha\beta C}, \tag{2.22a}$$

$$\begin{aligned}\overset{(3)}{L} = -\tfrac{1}{2}c_{ABC}\epsilon^{abc\nu\rho}\overset{(1)}{\omega}{}_{a}{}^{\alpha\beta A}(\overset{(1)}{\omega}{}_{b\alpha\beta}{}^{B}\overset{(1)}{\omega}{}_{c\nu\rho}{}^{C} \\ - 4\overset{(1)}{\omega}{}_{b\alpha\nu}{}^{B}\overset{(1)}{\omega}{}_{c\beta\rho}{}^{C}),\end{aligned} \tag{2.22b}$$

$$c_{ABC} = c_{[ABC]} \tag{2.22c}$$

(corresponding to a parity violating exotic interaction). The structure on the internal vector space $\mathbb{R}^N$ associated with the set of $N \geqslant 1$ spin-two fields $h_{a\mu}{}^{A}$ is a commutative, invariant-normed algebra in the first two types of deformations and an anticommutative, invariant-normed algebra in the third type of deformation.

These results determine the commutator structure of the deformed gauge symmetries to lowest order, $[\overset{(0)}{\delta}_1, \delta_2] = \overset{(0)}{\delta}_3$. For type (2.18), the nonvanishing commutators are given by

$$[\overset{(0)}{\delta}_{\chi_1}, \delta_{\chi_2}] = \overset{(0)}{\delta}_{\chi_3} \tag{2.23}$$

with

$$\chi_{3_{\mu\nu}}{}^{A} = 2\chi_{1_{[\mu}}{}^{\rho B}\chi_{2_{\nu]\rho}}{}^{C} a^{A}{}_{BC}, \tag{2.24}$$

and

$$[\overset{(0)}{\delta}_{\chi_1}, \delta_{\xi_1}] = \overset{(0)}{\delta}_{\xi_3} \tag{2.25}$$

with

$$\xi_{3_{\mu}}{}^{A} = \chi_{1_{\mu\nu}}{}^{B}\xi_1{}^{\nu C} a^{A}{}_{BC}. \tag{2.26}$$

In contrast, the only nonvanishing commutator for types (2.21) and (2.22) is given by

$$[\overset{(0)}{\delta}_{\chi_1}, \delta_{\chi_2}] = \overset{(0)}{\delta}_{\xi_3} \tag{2.27}$$

with, respectively,

$$\xi_{3_{\mu}}{}^{A} = \epsilon^{\nu\rho\alpha}\chi_{1_{\nu\rho}}{}^{B}\chi_{2_{\alpha\mu}}{}^{C} b^{A}{}_{BC} \quad \text{when} \quad n=3 \tag{2.28}$$

and

$$\xi_{3_{\mu}}{}^{A} = \epsilon_{\mu\nu\rho\alpha\beta}\chi_1{}^{\nu\rho B}\chi_2{}^{\alpha\beta C} c^{A}{}_{BC} \quad \text{when} \quad n=5. \tag{2.29}$$

An analysis of the determining equation (2.17) by the methods of Ref. [1] shows that the same commutator structure holds at next order

$$[\overset{(1)}{\delta}_1, \delta_2]|_{\overset{(1)}{E}=0} = \overset{(1)}{\delta}_3|_{\overset{(1)}{E}=0} \tag{2.30}$$

when $h_{a\mu}{}^{A}$ satisfies the linear field equation (2.2).

Higher order deformation terms in the gauge symmetries and field equations can be derived by continuing to solve the determining equations (2.16) and (2.17) at successively higher orders. However, an integrability condition on the first order deformation terms (2.18)–(2.22) arises from the closure result for the gauge symmetry commutator structure at first order in Eq. (2.30) if we consider the terms that involve second derivatives of $h_{a\mu}{}^{A}$. For type (2.18), all such terms come from the commutator $[\overset{(1)}{\delta}_{\xi_1}, \overset{(1)}{\delta}_{\xi_2}]\gamma_{ab}{}^{A}$, which yields

$$a^{A}{}_{B[C}a^{B}{}_{D]E}\xi_1{}^{eD}\xi_2{}^{dC}\overset{(1)}{R}_{a(ed)b}{}^{E}|_{\overset{(1)}{E}=0}. \tag{2.31}$$

These terms must vanish to within a symmetrized derivative by Eq. (2.30). Since $\partial_{[g|}\partial_{[f}\overset{(1)}{R}_{a](de)|b]}{}^{E} \neq 0$, we consequently obtain the integrability condition

$$a^{A}{}_{B[C}a^{B}{}_{D]E} = 0. \tag{2.32}$$

Next, for type (2.21) we find the terms that come from $[\overset{(1)}{\delta}_{\chi_1}, \overset{(1)}{\delta}_{\chi_2}]\gamma_{ab}{}^{A}$ with second derivatives of $h_{a\mu}{}^{A}$ are given by

$$b^{A}{}_{B[C}b^{B}{}_{D]E}\tilde{\chi}_1{}^{eD}\tilde{\chi}_2{}^{dC}\overset{(1)}{R}_{a(ed)b}{}^{E}|_{\overset{(1)}{E}=0}, \tag{2.33}$$

where $\tilde{\chi}^{\nu A} = \epsilon^{\nu\alpha\beta}\chi_{\alpha\beta}{}^{A}$. But Eq. (2.33) vanishes since, in $n=3$ dimensions, the linear spin-two field equation (2.2) is well known to imply $\overset{(1)}{R}_{adeb}{}^{E}=0$. Thus, due to the absence of local dynamical degrees of freedom, no integrability condition arises from Eq. (2.30) for deformation (2.21). However, if deformations (2.18) and (2.21) are combined in $n=3$ dimensions, then we obtain an integrability condition

$$a^{A}{}_{BC}b^{B}{}_{DE} = b^{A}{}_{BE}a^{B}{}_{DC}. \tag{2.34}$$

Finally, for type (2.22) the same commutator now yields

$$c^{A}{}_{BC}c^{B}{}_{DE}(\chi_1{}^{deC}\chi_{2_{(a}}{}^{cD} - \chi_{1_{(a}}{}^{cD}\chi_2{}^{deC})\overset{(1)}{R}_{b)cde}{}^{E}|_{\overset{(1)}{E}=0} + \eta_{ab} \ \text{terms}, \tag{2.35}$$

which does not vanish to within a symmetrized derivative. Hence, it follows from equation (2.30) that these terms (2.35) must be canceled by suitable quadratic deformation terms of the form (indices suppressed)

$$\overset{(2)}{\delta}_{\chi}h = bb\overset{(1)}{R}h\chi + (\text{lower derivative terms}). \tag{2.36}$$

In turn, a similar analysis for the resulting commutator $[\overset{(1)}{\delta}_{\xi_1}, \overset{(1)}{\delta}_{\chi_1}]\gamma_{ab}{}^{A}$ leads to further quadratic deformation terms

$$\overset{(2)}{\delta}_{\xi}h = bb\overset{(1)}{R}h\partial\xi + (\text{lower derivative terms}). \tag{2.37}$$

Then we find that the commutator $[\overset{(1)}{\delta}_{\xi_1}, \overset{(1)}{\delta}_{\xi_2}]\gamma_{ab}{}^{A}$ produces second derivative terms of the same form as Eq. (2.35) where $\chi_{1_{de}}{}^{D}$ and $\chi_{2_{ac}}{}^{C}$ are replaced by $\partial_d\xi_{1_e}{}^{D}$ and $\partial_a\xi_{1_c}{}^{C}$. Since the resulting terms do not vanish to within a symmetrized derivative, we thus derive an integrability condition

$$c^{A}{}_{BC}c^{B}{}_{DE} = 0. \tag{2.38}$$

Similar integrability conditions occur if deformations (2.22) and (2.18) are combined in $n=5$ dimensions,

$$a^{A}{}_{BC}c^{B}{}_{DE} = c^{A}{}_{BC}a^{B}{}_{DE} = 0. \tag{2.39}$$

Integrability conditions (2.32) and (2.38) assert that the underlying internal algebras associated with the spin-two fields in the deformations (2.18) and (2.22) are, respectively, associative and nilpotent of degree three. By the results in Refs. [2, 9], there are no further integrability conditions on the construction of deformations (2.18) and (2.21) to all higher orders in solving the determining equations. On the other hand, deformation (2.22) can be shown to satisfy the determining equations to all orders itself, since any higher order deformation terms necessarily vanish as a consequence of nilpotency (2.38) of the algebra. Thus, this deformation

$$L=\overset{(2)}{L}+\overset{(3)}{L},\quad \delta_\xi h_{a\mu}{}^{A}=\overset{(0)}{\delta}_\xi h_{a\mu}{}^{A},$$

$$\delta_\chi h_{a\mu}{}^{A}=\overset{(0)}{\delta}_\chi h_{a\mu}{}^{A}+\overset{(1)}{\delta}_\chi h_{a\mu}{}^{A} \tag{2.40}$$

given by Eqs. (2.3), (2.5), (2.22), and (2.32) yields a full, nonlinear spin-two gauge theory. Hence we arrive at the following main classification result.

Theorem 2. The nonlinear spin-two gauge theories in $n>2$ *dimensions determined by the respective first-order deformations* (2.18), (2.21), (2.22) *are equivalent to an algebra-valued Einstein gravity theory for* $n\geqslant 3$ *with a commutative, associative, invariant-normed algebra, or if* $n=3$, *a novel nonlinear theory related to a scaling limit of algebra-valued topological gravity theory with a commutative, invariant-normed algebra, or if* $n=5$ *a new algebra-valued nonlinear theory with an anticommutative, nilpotent, invariant-normed algebra. Additional nonlinear spin-two gauge theories arise from the gravity deformation* (2.18) *combined with either of the other two deformations* (2.21) *or* (2.22), *describing exotic (parity violating) generalizations of algebra-valued Einstein gravity theory in* $n=3,5$ *dimensions* [*with the algebras restricted by conditions* (2.34) *and* (2.39)]. *There are no other nonlinear spin-two gauge theories of quasilinear covariant type*.

The five-dimensional nonlinear theory without gravitational interactions has the following features. Its field equations (to within trace terms)

$$E_{a\mu}{}^{A}=R_{a\mu}{}^{A}=\partial_{[a}\omega_{b]\mu\nu}{}^{A}\sigma^{b\nu}=0 \tag{2.41}$$

are given by the quadratic spin-connection

$$\omega_{a\mu\nu}{}^{A}=\overset{(1)}{\omega}_{a\mu\nu}{}^{A}+\Omega_{a\mu\nu}{}^{A}-2\sigma_a{}^{\rho}\sigma^{b}{}_{[\mu|}\Omega_{b|\nu]\rho}{}^{A}+\tfrac{4}{3}\sigma^{b\rho}\sigma_{a[\mu|}\Omega_{b\rho|\nu]}{}^{A} \tag{2.42}$$

with

$$\Omega^{a\alpha\beta A}=c^{A}{}_{BC}[2\epsilon^{abc\nu\rho}(2\overset{(1)}{\omega}_{b\nu}{}^{\alpha B}\overset{(1)}{\omega}_{c\rho}{}^{\beta C}-\overset{(1)}{\omega}_{b}{}^{\alpha\beta B}\overset{(1)}{\omega}_{c\nu\rho}{}^{C})-\epsilon^{abc\alpha\beta}\overset{(1)}{\omega}_{b}{}^{\nu\rho B}\overset{(1)}{\omega}_{c\nu\rho}{}^{C}+2\epsilon^{abc\rho[\alpha}\overset{(1)}{\omega}_{b}{}^{\beta]\mu B}\overset{(1)}{\omega}_{c\mu\rho}{}^{C}], \tag{2.43}$$

where $\Omega_a{}^{\alpha\beta A}-\frac{2}{3}\Omega_c{}^{\rho[\beta A}\sigma_a{}^{\alpha]}\sigma^{c}{}_{\rho}$ plays the role of a contorsion tensor. Exponentiation of its gauge symmetries generates a group of finite gauge transformations $h_{a\mu}{}^{A}\rightarrow h'_{a\mu}{}^{A}$ given by

$$h'_{a\mu}{}^{A}=h_{a\mu}{}^{A}+\partial_a\xi_\mu{}^{A}+\sigma_a{}^{\nu}\chi_{\nu\mu}{}^{A}+c^{A}{}_{BC}\epsilon_{\mu\nu\rho\alpha\beta}(\overset{(1)}{\omega}_a{}^{\nu\rho B}+\tfrac{1}{2}\partial_a\chi^{\nu\rho A})\chi^{\alpha\beta C}. \tag{2.44}$$

Using this gauge freedom in the theory, we can make a nonlinear change of field variable to $\gamma'_{ab}{}^{A}=\sigma_{(a}{}^{\mu}h'_{b)\mu}{}^{A}$ with $\chi_{\nu\mu}{}^{A}$ determined in terms of $h_{a\mu}{}^{A}$ by $\sigma_{[a}{}^{\mu}h'_{b]\mu}{}^{A}=0$. The gauge symmetries then consist of (after a parameter redefinition)

$$\delta_\xi\gamma'_{ab}{}^{A}=\partial_{(a}\xi_{b)}{}^{A}+c^{A}{}_{BC}\epsilon_{(a|}{}^{cdef}\partial_c\gamma'_{d|b)}{}^{B}\partial_e\xi_f{}^{C}, \tag{2.45}$$

while the field equations become

$$E'_{ab}{}^{A}=-\eta^{cd}(2\partial_{[a|}\partial_{[b}\gamma'_{c]|d]}{}^{A}+\partial_c(\Omega'_{(ab)d}{}^{A}(\gamma')+\tfrac{1}{3}\Omega'_{ed}{}^{eA}(\gamma')\eta_{ab}))=0, \tag{2.46}$$

which are of quasilinear second order form, where

$$\Omega'_{amn}{}^{A}(\gamma')=4c^{A}{}_{BC}(\epsilon_{abcde}\Gamma_m{}^{dbB}\Gamma_n{}^{ecC}-\epsilon_{abcmn}\Gamma^{bdeB}\Gamma^{c}{}_{de}{}^{C}-\epsilon_{abcd[m}\Gamma^{b}{}_{n]e}{}^{B}\Gamma^{edcC}) \tag{2.47}$$

with $\Gamma_{amn}{}^{A}=\partial_{[m}\gamma'_{n]a}{}^{A}$.

Since the algebra $(c^{A}{}_{BC},\mathbb{R}^N)$ associated with the spin-two fields in this theory is anticommutative and nilpotent of degree three, it satisfies the Jacobi identity and hence is equivalently characterized as being a solvable Lie algebra of length two [14], with an invariant norm. Note that the existence of an invariant norm puts some restriction on the Lie bracket structure in the algebra. The simplest example of such an algebra $(c^{A}{}_{BC},\mathbb{R}^N)$ is given by $c_{ABC}=u_{[A}v_Bw_{C]}$, where u_A, v_B, w_C are mutually orthogonal null vectors in a $(N=6)$-dimensional vector space with norm $\delta_{AB}=\mathrm{diag}(+1,+1,+1,-1,-1,-1)$.

However, if we impose a physically natural requirement that the individual spin-two fields should have positive energy (or more precisely that the weak energy condition [15] holds), using the conserved stress-energy tensor derived from the Lagrangian (2.40), this severely restricts the allowed non-Abelian structure of the algebra. Specifically, as already noted for the free theory, positivity of energy forces the norm on the algebra to be positive definite. As a consequence, from nilpotency (2.38) combined with norm-invariance (2.22c), we have $c^{A}{}_{BC}c_{ADE}=0$, which implies $c^{A}{}_{BC}=0$ due to positive definiteness of the norm. Hence, in this situation, every anticommutative, nilpotent, invariant-normed algebra $(c^{A}{}_{BC},\mathbb{R}^N)$ is Abelian. In a similar manner, as shown in Ref. [2], any commutative, associative, invariant-normed algebra $(a^{A}{}_{BC},\mathbb{R}^N)$ with a positive definite norm is the direct sum of one-dimensional unit algebras $\mathbb{R}$. Thus we obtain the following no-go result.

Theorem 3. The only quasilinear covariant deformations in $n\geqslant 4$ *dimensions for spin-two gauge theories with positive energy are semilinear, in particular, equivalent to Einstein gravity theory with no interaction between different spin-two fields*.

This strengthens the no-go theorem in Ref. [2] to the more general class of quasilinear covariant deformations considered here. In $n=3$ dimensions, positivity of energy is compatible with a nontrivial algebra structure $(b^{A}{}_{BC},\mathbb{R}^N)$, as emphasized in Ref. [9]. The resulting three-dimensional nonlinear theory, using the formalism here, without gravitational interactions is given by the gauge symmetries

$$\delta_\xi h_{a\mu}{}^{A}=\partial_a \xi_\mu{}^{A},\quad \delta_\chi h_{a\mu}{}^{A}=\epsilon_{a\mu\nu}\tilde{\chi}^{\nu A}+b^{A}{}_{BC}\epsilon_{\mu\nu\rho}\tilde{\omega}_a{}^{\nu B}\tilde{\chi}^{\rho C} \tag{2.48}$$

and the field equations

$$E_{a\mu}{}^{A}=\partial_{[a}\tilde{\omega}_{b]}{}^{\nu A}\epsilon_{\mu\nu}{}^{b}, \tag{2.49}$$

where $\tilde{\omega}_a{}^{\nu A}=\epsilon^{\nu\mu\rho}\omega_{a\mu\rho}{}^{A}$ satisfies the following quadratic spin-connection equation:

$$\partial_{[a}h_{b]\mu}{}^{A}=\epsilon_{\mu\nu\rho}(\sigma_{[a}{}^{\nu}\tilde{\omega}_{b]}{}^{\rho A}+b^{A}{}_{BC}\tilde{\omega}_{[a}{}^{\nu B}\tilde{\omega}_{b]}{}^{\rho C}. \tag{2.50}$$

It is possible to solve Eq. (2.50) to obtain

$$\tilde{\omega}_{a\nu}{}^{A}=\Omega^{1/2}{}_{a\nu}{}^{A}-\tfrac{1}{2}\sigma_{a\nu}\mathbb{1}^{A} \tag{2.51}$$

in terms of the square root of

$$\Omega_{a\nu}{}^{A}=-2\epsilon_a{}^{bc}\partial_b h_{c\nu}{}^{A}+\tfrac{1}{2}\sigma_{a\nu}\mathbb{1}^{A} \tag{2.52}$$

as defined by

$$\epsilon_c{}^{ab}\epsilon_\nu{}^{\mu\rho}\Omega^{1/2}{}_{a\mu}{}^{B}\Omega^{1/2}{}_{b\rho}{}^{C}=-\Omega_{c\nu}{}^{A}, \tag{2.53}$$

where $\mathbb{1}^A$ is a unit element (appended if necessary) in algebra $(b^{A}{}_{BC},\mathbb{R}^N)$. (Note, with indices suppressed, this square root satisfies the algebraic relation $\Omega^{1/2}\times\Omega^{1/2}=-\Omega$ with $\times$ being a symmetric product on $\mathbb{R}^3\otimes\mathbb{R}^3\otimes\mathbb{R}^N$ given by the tensor product of three-dimensional cross-products ϵ combined with the algebra product b.) This theory can be formulated in terms of ordinary spin-two fields $\gamma'{}_{ab}{}^{A}=h'{}_{(ab)}{}^{A}$ analogously to the five-dimensional theory by a suitable field redefinition using the finite gauge transformations

$$h'{}_{a\mu}{}^{A}=h_{a\mu}{}^{A}+\partial_a\xi_\mu{}^{A}+\epsilon_{a\mu\nu}\tilde{\chi}^{\nu A} + b^{A}{}_{BC}\epsilon_{\mu\nu\rho}(\tfrac{1}{2}\partial_a\tilde{\chi}^{\nu B}+\tilde{\omega}_a{}^{\nu B})\tilde{\chi}^{\rho C} \tag{2.54}$$

generated from the gauge symmetries on solutions of the field equations.

III. CONCLUDING REMARKS

It is worth investigating to what extent the complete five-dimensional nonlinear spin-two theory derived here is sensible as a classical field theory in view of its unusual features.

First, the field equations (2.46) possess a well-posed initial value formulation. The linear part of Eq. (2.46) is given by the free Fierz-Pauli spin-two equations (2.7), which are a second order hyperbolic system whose characteristic directions coincide with the Minkowski light cones, when suitable gauge conditions are imposed. While the nonlinear part of Eq. (2.46) also involves second order derivatives and *a priori* might be expected to alter the characteristic directions, in fact, the nilpotency of the internal algebra $(c^{A}{}_{BC},\mathbb{R}^N)=\mathcal{A}$ implies that the full field equations are a decoupled triangular system of semilinear Fierz-Pauli equations. In particular, consider $\mathcal{A}_{(1)}=[\mathcal{A},\mathcal{A}]$ where the brackets denote the algebra product $c^{A}{}_{BC}$ on the internal normed vector space $(\mathbb{R}^N,\delta_{AB})$. Because of the nilpotency and indefinite sign of the norm of $\mathcal{A}$, $\mathcal{A}_{(1)}$ is a null, Abelian subalgebra of $\mathcal{A}$, with a null complement $\mathcal{A}'_{(1)}$ such that $\mathcal{A}_{(1)}\oplus\mathcal{A}'_{(1)}=\mathcal{A}$ (once we divide any trivial Abelian factors of $\mathcal{A}$). Since $\mathcal{A}_{(1)}$ and $\mathcal{A}'_{(1)}$ obey $[\mathcal{A}'_{(1)},\mathcal{A}'_{(1)}]\subseteq\mathcal{A}_{(1)}$ and $[\mathcal{A}_{(1)},\mathcal{A}_{(1)}]=[\mathcal{A}_{(1)},\mathcal{A}'_{(1)}]=0$, the subset of spin-two fields associated with $\mathcal{A}'_{(1)}$ [i.e. $\mathcal{P}_{\mathcal{A}'_{(1)}}(h_{a\mu}{}^{A})$ where $\mathcal{P}_{\mathcal{A}'_{(1)}}=\mathcal{P}^{\mathrm{T}}_{\mathcal{A}_{(1)}}$ is the transpose of the projector $\mathcal{P}_{\mathcal{A}_{(1)}}$ onto $\mathcal{A}_{(1)}$] satisfies free Fierz-Pauli equations (2.7), and the remaining subset of spin-two fields associated with $\mathcal{A}_{(1)}$ [i.e. $\mathcal{P}_{\mathcal{A}_{(1)}}(h_{a\mu}{}^{A})$] satisfies inhomogeneous Fierz-Pauli equations with quadratic source terms that involve only the free spin-two fields. For example, consider the case of the algebra given by $c_{ABC}=u_{[A}v_B w_{C]}$ on

$$(\mathbb{R}^6,\delta_{AB}=2u_{(A}u'{}_{B)}+2v_{(A}v'{}_{B)}+2w_{(A}w'{}_{B)})$$

with a null vector basis $u_A, v_B, w_C, u'{}_A, v'{}_B, w'{}_C$ whose only non-zero inner products in this basis are $u'{}_A u^A=v'{}_A v^A=w'{}_A w^A=1$. The inhomogeneous Fierz-Pauli equations hold for the fields $\mathcal{P}^A_B h_{a\mu}{}^{B}$ given by the null projector $\mathcal{P}^A_B=u^A u'{}_B+v^A v'{}_B+w^A w'{}_B$, while the free spin-two Fierz-Pauli equations involve the fields $\mathcal{P}^{\mathrm{T}A}_{B}h_{a\mu}{}^{B}$ given in terms of the transpose null projector $\mathcal{P}^{\mathrm{T}A}_{B}=\delta_{BC}\delta^{AD}\mathcal{P}^C_D=u'^A u_B+v'^A v_B+w'^A w_B$.

As a consequence of this decoupling feature, the well-posedness property of the field equations (2.46) is insensitive to the lack of positivity of energy [16] arising from the conserved stress-energy tensor of the Lagrangian (2.40) through the indefinite sign of the norm on algebra $\mathcal{A}$.

Finally, although the theory exists only in five dimensions, it is relevant for four dimensions if a Kaluza-Klein reduction is considered. We begin with a product decomposition of five-dimensional Minkowski spacetime $(\mathbb{R}^5,\eta_{ab})=(\mathbb{R}^4,\overline{\eta}_{ab})\times(\mathbb{R},y_a y_b)$ where $\overline{\eta}_{ab}$ is the four-dimensional Minkowski metric and y_a is a spacelike unit vector orthogonal to $\overline{\eta}_{ab}$. Note we have the $4+1$ decompositions

$$\eta_{ab}=\overline{\eta}_{ab}+y_a y_b,\quad \epsilon_{abcde}=5\overline{\epsilon}_{[abcd}y_{e]}, \tag{3.1}$$

where $\overline{\epsilon}_{abcd}$ is the four-dimensional volume form. (Throughout, a bar will denote a tensor or field variable on $\mathbb{R}^4$.) Now we decompose the spin-two field variables $\gamma'{}_{ab}{}^{A}$ in the field equations (2.46) into the $4+1$ form

$$\gamma'{}_{ab}{}^{A}=\overline{\gamma}_{ab}{}^{A}+\overline{A}_{(a}{}^{A}y_{b)}+\overline{\phi}^{A}y_a y_b \tag{3.2}$$

with the fields $\overline{\gamma}_{ab}{}^{A}$, $\overline{A}_a{}^{A}$, $\overline{\phi}^{A}$ taken to have no dependence on the y_a coordinate:

$$y^c\partial_c\overline{\gamma}_{ab}{}^{A}=y^c\partial_c\overline{A}_a{}^{A}=y^c\partial_c\overline{\phi}^{A}=0. \tag{3.3}$$

The components of the five-dimensional field equations (2.46) under the decompositions (3.1)–(3.3) yield four-dimensional field equations consisting of

$$\overline{E}^A = E'_{ab}{}^A y^a y^b = 0$$

for the spin-zero fields $\overline{\phi}^A$, and

$$\overline{E}_a{}^A = E'_{ab}{}^A y^b - \overline{E}^A y_a = 0$$

for the spin-one fields $\overline{A}_a{}^A$, in addition to

$$\overline{E}_{ab}{}^A = E'_{ab}{}^A - 2\overline{E}_{(a}{}^A y_{b)} - \overline{E}^A y_a y_b = 0$$

for the spin-two fields $\overline{\gamma}_{ab}{}^A$. There is a corresponding decomposition of the five-dimensional gauge symmetries (2.45) in terms of the parameters $\overline{\xi}^A = \xi_a{}^A y^a$ and $\overline{\xi}_a{}^A = \xi_a{}^A - \overline{\xi}^A y_a$, which also are taken to have no dependence on the y_a coordinate,

$$y^c \partial_c \overline{\xi}^A = y^c \partial_c \overline{\xi}_a{}^A = 0. \tag{3.4}$$

The resulting four-dimensional gauge theory is a nonlinear deformation of the combined linear theory of scalar fields

$$\overset{(1)}{\overline{E}}{}^A = -\tfrac{1}{2}\overline{\eta}^{cd}\overline{\partial}_c\overline{\partial}_d\overline{\phi}^A, \quad \overset{(0)}{\delta_{\overline{\xi}}}\overline{\phi}^A = 0, \tag{3.5}$$

Maxwell gauge fields

$$\overset{(1)}{\overline{E}}{}_a{}^A = \overline{\eta}^{cd}\overline{\partial}_c\overline{\partial}_{[d}\overline{A}_{a]}{}^A, \quad \overset{(0)}{\delta_{\overline{\xi}}}\overline{A}_a{}^A = \overline{\partial}_a\overline{\xi}^A, \tag{3.6}$$

and linearized graviton fields

$$\overset{(1)}{\overline{E}}{}_{ab}{}^A = -2\overline{\eta}^{cd}\overline{\partial}_{[c|}\overline{\partial}_{[d}\overline{\gamma}_{b]|a]}{}^A, \quad \overset{(0)}{\delta_{\overline{\xi}}}\overline{\gamma}_{ab}{}^A = \overline{\partial}_{(a}\overline{\xi}_{b)}{}^A. \tag{3.7}$$

A Lagrangian formulation is readily obtained by decomposing the five-dimensional linearized frames $h_{ab}{}^A = \sigma_b{}^\mu h_{a\mu}{}^A$ and linearized spin-connections $\overset{(1)}{\omega}{}_{abc}{}^A = \sigma_b{}^\mu \sigma_c{}^\nu \overset{(1)}{\omega}{}_{a\mu\nu}{}^A$ into the $4+1$ form

$$h_{ab}{}^A = \overline{h}_{ab}{}^A + \overline{A}_a{}^A y_b + \overline{\phi}^A y_a y_b, \tag{3.8}$$

$$\overset{(1)}{\omega}{}_{abc} = \overline{\omega}_{abc}{}^A - \overline{F}_{bc}{}^A y_a + 2\overline{F}_{a[b}{}^A y_{c]} - 2\overline{H}_{[b}{}^A y_{c]} y_a, \tag{3.9}$$

where

$$\overline{\omega}_{abc}{}^A = 3\overline{\partial}_{[a}\overline{h}_{bc]}{}^A - 2\overline{\partial}_{[b}\overline{h}_{c]a}{}^A \tag{3.10}$$

represents the four-dimensional linearized spin-connection, and

$$\overline{F}_{ab}{}^A = \overline{\partial}_{[a}\overline{A}_{b]}{}^A, \quad \overline{H}_a{}^A = \overline{\partial}_a\overline{\phi}^A \tag{3.11}$$

represent the four-dimensional spin-one field strength and spin-zero field strength. The linearized spin-connection and field strengths here have the role of auxiliary fields (analogous to a "1.5" formalism in supergravity [12]). Then, through decompositions (3.1), (3.8), and (3.9), the five-dimensional Lagrangian (2.40) reduces to form $\overline{L} = \overset{(2)}{\overline{L}} + \overset{(3)}{\overline{L}}$, where

$$\overset{(2)}{\overline{L}} = -\tfrac{1}{2}(\overline{h}_a{}^{aA}(\overline{\partial}_b\overline{\omega}_c{}^{bcB} - \overline{\partial}_b\overline{H}^{bB}) + \overline{h}_b{}^{aA}(2\overline{\partial}_{[c}\overline{\omega}_{a]}{}^{bcB} + \overline{\partial}_a\overline{H}^{bB}) - \overline{\omega}_{[a}{}^{caA}\overline{\omega}_{b]c}{}^{bB} - \overline{A}_b{}^A\overline{\partial}_c\overline{F}^{bcB} + \tfrac{1}{2}\overline{F}_{ab}{}^A\overline{F}^{abB})\delta_{AB}, \tag{3.12}$$

and

$$\overset{(3)}{\overline{L}} = -\tfrac{1}{2}((\overline{\omega}_a{}^{pqA}\overline{\omega}_{bpq}{}^B - 2\overline{F}_a{}^{pA}\overline{F}_{bp}{}^B)\overline{\omega}_{bcd}{}^C - (4\overline{\omega}_{ab}{}^{pA}\overline{\omega}_{cd}{}^{qB} + 2\overline{\omega}_a{}^{pqA}\overline{\omega}_{bcd}{}^B)\overline{F}_{pq}{}^C - 4\overline{H}^{pA}\overline{F}_{ap}{}^B\overline{\omega}_{bcd}{}^C - 4\overline{H}_{(p}{}^A\overline{F}_{a)b}{}^B\overline{\omega}_{cd}{}^{pC})\overline{\epsilon}^{abcd}c_{ABC}. \tag{3.13}$$

This four-dimensional Lagrangian $\overline{L} = \overset{(2)}{\overline{L}} + \overset{(3)}{\overline{L}}$ is invariant to within a total divergence under the gauge symmetries

$$\delta_{\overline{\xi}}\overline{h}_{ab}{}^A = \overline{\partial}_a\overline{\xi}_b{}^A, \quad \delta_{\overline{\chi}}\overline{h}_{ab}{}^A = \overline{\chi}_{ab}{}^A + 4c^A{}_{BC}\overline{\epsilon}_{bcpq}\overline{F}_a{}^{cB}\overline{\chi}^{pqC}, \tag{3.14}$$

$$\delta_{\overline{\xi}}\overline{A}_a{}^A = \overline{\partial}_a\overline{\xi}^A, \delta_{\overline{\chi}}\overline{A}_a{}^A = c^A{}_{BC}(\overline{\epsilon}_{bcpq}\overline{\omega}_a{}^{bcB} + 2\overline{\epsilon}_{abpq}\overline{H}^{bB})\overline{\chi}^{pqC}, \tag{3.15}$$

$$\delta_{\overline{\xi}}\overline{\phi}^A = 0, \quad \delta_{\overline{\chi}}\overline{\phi}^A = -c^A{}_{BC}\overline{\epsilon}_{cdpq}\overline{F}^{cdB}\overline{\chi}^{pqC}, \tag{3.16}$$

whose parameters $\overline{\xi}^A$, $\overline{\xi}_a{}^A$, $\overline{\chi}_{ab}{}^A = \overline{\chi}_{[ab]}{}^A$ are arbitrary functions of the four-dimensional spacetime coordinates, as obtained by decomposing the five-dimensional gauge symmetries (2.40) and simplifying various terms via the nilpotency (2.32) of $c^A{}_{BC}$.

Thus we have obtained a four-dimensional parity-violating nonlinear gauge theory of a massless coupled set of spin-two fields, spin-one fields, and spin-zero fields. This theory can be generalized to include an algebra-valued gravitational coupling, where the internal algebras $(c^A_{BC}, \mathbb{R}^N)$ and $(a^A_{BC}, \mathbb{R}^N)$ underlying the respective parity-violation coupling and gravity coupling in the theory satisfy the necessary conditions stated in Theorem 2. A simple example for these algebras consists of taking the vector space $(\mathbb{R}^6, \delta_{AB} = \text{diag}(+1,+1,+1,-1,-1,-1))$ with c_{ABC} being the skew product of three mutually orthogonal null vectors u_A, v_B, w_C, and a_{ABC} being the sum of any symmetric products of these same vectors.

[1] S. C. Anco, Ann. Phys. (N.Y.) **270**, 52 (1998).
[2] N. Boulanger, T. Damour, L. Gualtieri, and M. Henneaux, Nucl. Phys. **B597**, 127 (2001).
[3] S. C. Anco, Contemp. Math. **132**, 27 (1992).

[4] M. Henneaux, Contemp. Math. **219**, 93 (1998).
[5] G. Barnich, F. Brandt, and M. Henneaux, Phys. Rep. **338**, 439 (2000).
[6] C. Cutler and R. M. Wald, Class. Quantum Grav. **4**, 1267 (1987).
[7] R. M. Wald, Class. Quantum Grav. **4**, 1279 (1987).
[8] S. C. Anco, Class. Quantum Grav. **19**, 6445 (2002).
[9] N. Boulanger and L. Gualtieri, Class. Quantum Grav. **18**, 1485 (2001).
[10] E. Witten, Nucl. Phys. **B311**, 46 (1988); S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).
[11] Latin and greek indices represent spacetime coordinates and frame components, respectively; upper case indices represent internal labels on the fields.
[12] P. Van Nieuwenhuizen, Phys. Rep. **68**, 189 (1980).
[13] S. Deser, Gen. Relativ. Gravit. **1**, 9 (1970).
[14] A. Sagle and R. Walde, *Introduction to Lie Groups and Lie Algebras* (Academic, New York, 1973).
[15] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[16] The effect of an indefinite-sign energy will be important for perturbative quantization of the classical field theory, since it implies that negative-energy ghosts will occur in setting up the one-particle Hilbert space for the free theory. Such ghosts can be expected to lead to difficulties similar to the instability of the ground state as found for an algebra-valued scalar field theory studied in an earlier work: S. C. Anco and R. M. Wald, Phys. Rev. D **39**, 2297 (1989).