Inflationary models inducing non-Gaussian metric fluctuations

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We construct explicit models of multifield inflation in which the primordial metric fluctuations do not necessarily obey Gaussian statistics. These models are realizations of mechanisms in which non-Gaussianity is first generated by a light scalar field and then transferred into curvature fluctuations. The probability distribution functions of the metric perturbation at the end of inflation are computed. This provides a guideline for designing strategies to search for non-Gaussian signals in future CMB and large scale structure surveys.

Inflation generically predicts Gaussian initial metric fluctuations with an almost scale invariant power spectrum [1]. It is interesting to determine how general this property is. With the advent of large scale structure and cosmic microwave background (CMB) surveys it will be possible to test in detail whether or not the initial conditions were Gaussian and in the latter case how they can be characterized. At the moment, no non-Gaussian signal has been detected in CMB data [2] but the number of modes that can be probed is still small. In large-scale structure surveys the number of independent modes that are observed is large but the difficulty arises from the nonlinear gravitational dynamics [3] which can shadow the primordial non-Gaussianity [4]. Having at our disposal models of inflation in which non-Gaussian adiabatic metric fluctuations are generated can then serve as a guideline for designing strategies for detecting primordial non-Gaussianities.

Indeed, the detection of non-Gaussianity is a difficult task by principle. The reason is that neither the bi- nor tri-spectra are fully measurable; only some configuration averages of those are. This is the reason why it is not conceivable to design a universal estimator of non-Gaussianity. They have to be constructed in light of the type of non-Gaussianities that are aimed at. One merit for investigating explicit models is therefore to characterize the type of non-Gaussianity that can be present in the data.

In single field inflation, the slow-roll conditions, if valid throughout the period during which the seeds of the large scale structures are generated, prevent the generation of observable primordial non-Gaussianities. The reason is that the potential needs to be both flat enough for the fluctuations to develop and steep enough for the non-linearity to be significant. Note however that starting from a nonvacuum initial state [5] or allowing for sharp features in the potential [6] could lead to primordial non-Gaussianities. In those cases these features will be localized in a narrow band of wavelengths and will affect the shape of the density fluctuation power spectrum.

The situation is richer when more than one light scalar field is present during inflation. In multifield inflation models, a mixture of adiabatic and isocurvature fluctuations [7], that can be correlated or not [8], are generically pro-duced. It opens the door to the generation of primordial non-Gaussianity [9] because the nonlinear couplings can be much stronger in the isocurvature direction than in the (inflaton) adiabatic direction [9,10]. When the fields are coupled, exchange between isocurvature and adiabatic modes can be important [11] which makes it possible to generate non-Gaussianity on a large band of wavelengths while preserving an adiabatic slow-roll type primordial power spectrum [12], which is not the case of the original model of Ref. [9].

In a previous study [12], we have presented the building blocks for such models. It is based on the generation of non-Gaussian isocurvature fluctuations which are subsequently transferred to the adiabatic mode due to a bend in the classical inflaton trajectory. The necessary ingredients for such models are (i) the existence of a light scalar field that has nonlinear coupling, typically quartic, and (ii) a coupling term in the potential leading to isocurvature-adiabatic mode mixing. This latter mechanism can be given a simple geometrical interpretation. On super Hubble scales, curvature fluctuations are induced by fluctuations of the total expansion in different parts of the universe. The relative duration of their inflationary phase can be affected if inflaton trajectory is bent. Fluctuations in the isocurvature modes generate a bundle of parallel trajectories along which inflation lasts longer or shorter depending on whether they lie in the outer or inner part of the bent (see Fig. 1 of Ref. [12]).

In Ref. [12], we characterized the expected statistical properties of the metric fluctuations that were shown to be the superposition of a Gaussian and a non-Gaussian contribution of the same variance [16]. The relative weight of the two contributions is related to the bending angle in field space. We explicitly computed the probability distribution function (PDF) of the non-Gaussian contribution which appears to be described by a single new parameter, ν_3 [see Eq. (6) below]. These results were obtained without the use of any explicit model. This Rapid Communication is dedicated to the building of this kind of inflationary model involving either two or three fields. The shape of their PDF will be computed and compared to our previous analytical expression [12].

In general the curvature fluctuations, $\mathcal{R}$, can be computed on superhorizon scales as [13]

$$\mathcal{R} = H\,\delta t. \tag{1}$$

Isocurvature fluctuations can then induce curvature fluctuations only if in some way or another they can change the relative total number of e-folds.

This can actually be obtained in a very simple way in hybrid type model of inflation with three fields. In that case, one field, ϕ, is the inflaton; the second field is a light scalar, χ, with quartic coupling $\lambda \lesssim 1$ and the third field, σ, is coupled to the two others so that the end of inflation is triggered when σ undergoes a phase transition. To be more explicit, let us examine the following model

$$V(\phi,\chi,\sigma) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\chi^4 + \frac{\mu}{2}(\sigma^2 - \sigma_0^2)^2$$

$$+ \frac{g}{2}\sigma^2(\phi\cos\alpha + \chi\sin\alpha)^2 \tag{2}$$

where σ_0 is the final vev of σ and α is the mixing angle between ϕ and χ in their coupling to σ.

Inflation ends when the effective mass of σ vanishes, that is when

$$g(\phi\cos\alpha + \chi\sin\alpha)^2 - 2\mu\sigma_0^2 = 0. \tag{3}$$

The value of ϕ at the end of inflation is

$$\phi_{\mathrm{end}} \equiv \frac{\pm\sqrt{2\mu/g}\,\sigma_0 - \chi\sin\alpha}{\cos\alpha}. \tag{4}$$

For $\phi > \phi_{\mathrm{end}}$, $\sigma = 0$ and the two fields evolve independently: ϕ drives the inflation while χ develops non-Gaussianity. The amount of non-Gaussianity of χ then depends only on λ and on the total number of e-folds between horizon crossing and the end of inflation.

When α is nonzero, fluctuations of χ induce curvature fluctuations because they change the time at which the phase transition occurs. Thus the χ-induced curvature fluctuations read (assuming H is basically constant during the inflationary period),

$$\mathcal{R} \simeq H\,\delta t_{\mathrm{end}} \simeq -\frac{3H^2}{V_{,\phi}}\frac{\sin\alpha}{\cos\alpha}\chi. \tag{5}$$

We present in Fig. 1 the result of a numerical investigation in which the super-Hubble evolution of the χ modes are explicitly computed. The Klein-Gordon equations for the two fields and the Friedmann equations are integrated for a set of initial conditions leading to a bundle of classical trajectories. The parameters of this example were taken to be $\lambda = 10^{-2}$ and an evolution for a number of e-folds equal to $N_e = 50$. The shape of the PDF is compared to our analytical expression [12],

$$P(\chi)d\chi = \sqrt{\frac{1}{2\pi}\left|\frac{1-\chi^2\nu_3}{(1+\chi^2\nu_3/3)^3}\right|}$$

$$\times\exp\left[-\frac{3\chi^2}{(6+2\chi^2\nu_3)\sigma_\chi^2}\right]\frac{d\chi}{\sigma_\chi}. \tag{6}$$
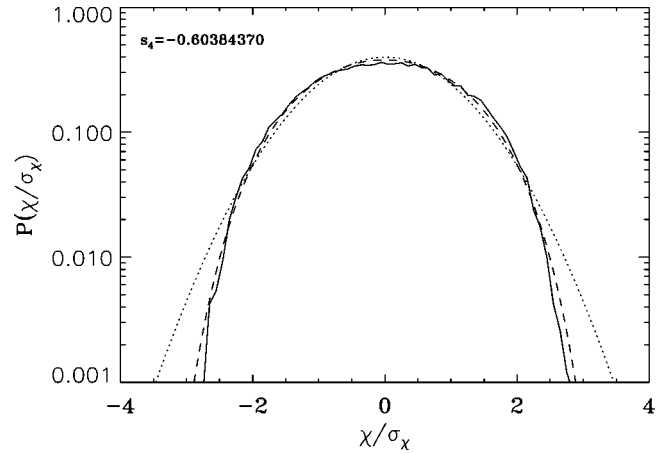

FIG. 1. The shape of the probability distribution function of the isocurvature modes in case of a quartic coupling, 2 (solid line) compared to the analytic PDF of Eq. (6) (dashed line) and to Gaussian distribution (dotted line). In this example, $\lambda = 10^{-2}$ and $N_e = 50$.

which depends only on the variance of χ and on the parameter ν_3 quantifying the amount of non-Gaussianity. The variance is directly proportional to the value of the Hubble constant at horizon crossing, H_c. The parameter ν_3 is proportional to λ and is explicitly given by

$$\nu_3 = -\lambda N_e/(3H_c^2) \approx -0.60. \tag{7}$$

In Fig. 1, the numerical results are compared to the expression (6) and to a Gaussian distribution of the same variance. It confirms that the analytic form (6) is a very good approximation of the probability distribution function. The value of ν_3 in this plot is [17] $\nu_3 = -0.15/\sigma_\chi^2$, which implies that the value of the kurtosis of distribution is

$$s_4 \equiv \langle\chi^4\rangle/\langle\chi^2\rangle^2 - 3 = 4\nu_3\sigma_\chi^2. \tag{8}$$

The curvature fluctuation PDF is related to the PDF of the field χ. In the limit where the inflation ends abruptly the relation is straightforward, Eq. (5). It is more complex in general and depends on the details of the phase of inflation.

The parameters of the potential (2) are tuned, as usual, so that the amplitude of the fluctuations are compatible with the observations. No further fine-tunings are required for the initial conditions: whatever the initial value for the field χ, it rolls down rapidly towards $\chi = 0$ to follow the valley bottom, as described in the previous paragraphs.

The model (2) reads as a simple polynomial potential of order four but involves three fields. It is actually possible to build models involving only two fields. In this case, it is clear that the χ trajectory cannot be straight and has to bend. A possible explicit model can be obtained when the zero-mode trajectory is given by

$$\chi_0(t) = \chi_\infty\tanh\left(\frac{\alpha\phi_0(t)}{\chi_\infty}\right) \tag{9}$$

where α and χ_∞ are free parameters. A model to get such a trajectory can be constructed by considering a term of the
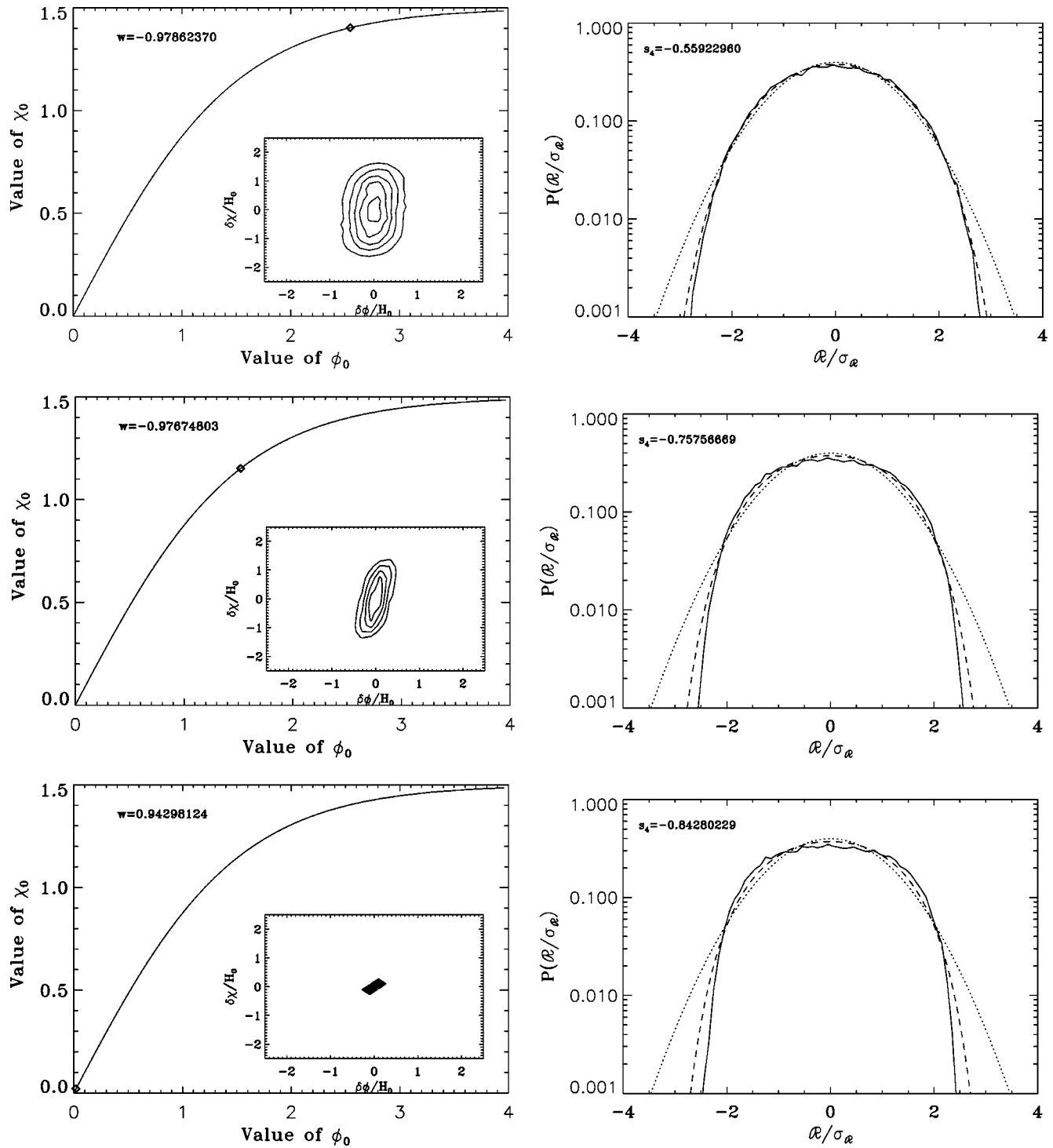
FIG. 2. PDF for the two-field model with potential (11) at different time step with the parameters (12),(13). The left panel describes the field trajectory and the shape of the wave packet and the right panels compare the numerically obtained PDF (solid) to a Gaussian (dotted) and to the analytical PDF (6) (dash). w is the effective equation of state parameter and s_4 is defined in Eq. (8).

form $(\chi - \chi_0)^4$ in the potential, as first advocated in Ref. [12], so that the potential might take the form

$$V(\phi, \chi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \left(\chi \cosh\left[\frac{\alpha \phi}{\chi_\infty}\right] - \chi_\infty \sinh\left[\frac{\alpha \phi}{\chi_\infty}\right] \right)^4.$$

$$(10)$$

Although it gives the adequate trajectory (9) in field space, it is non-polynomial and does not induce significant metric non-Gaussianities. The reason is that the χ fluctuations are squashed as soon as the trajectory is bent, the bundle of trajectories being drastically focused much before any significant bending. This is a generic feature of all models in

which the bending of the trajectory is driven by the non-linear coupling term.

A way to construct a model in which the χ fluctuations are not squashed is to consider the potential

$$V(\phi,\chi) = \frac{1}{2}m^2\left(\chi^2 + \frac{\chi_\infty^2}{\alpha^2}\sinh^2\left[\frac{\alpha\phi}{\chi_\infty}\right]\right)$$
$$+ \frac{\lambda}{4!}\left(\chi\cosh\left[\frac{\alpha\phi}{\chi_\infty}\right] - \chi_\infty\sinh\left[\frac{\alpha\phi}{\chi_\infty}\right]\right)^4. \quad (11)$$

Now, the zero mode trajectory (ϕ_0,χ_0) is not determined by the nonlinear coupling. At lowest order in the coupling parameter λ and in the slow-roll regime, the Klein-Gordon equations for ϕ and χ can be integrated and lead to the trajectory (9).

In Fig. 2, we depict the behavior of the super-horizon ϕ and χ fluctuations (left panels) and the curvature fluctuations that are induced by the initial χ fluctuations (right panels). The parameters of the model for the simulation of Fig. 2 were chosen to be

$$m = 10^{-7}M_{\text{Pl.}}, \quad \alpha = 1 \quad (12)$$

$$\chi_\infty = 1.5M_{\text{Pl.}}, \quad \lambda = 5\times10^{-5}, \quad (13)$$

so that it gives realistic fluctuation amplitudes and e-fold number. The initial conditions are set up at $\phi_{\text{init}} = 4M_{\text{Pl.}}$. At this time the coupling constant is then about 0.15 and the number of e-folds during the inflationary period is about 72. We integrate the Klein-Gordon equations and the Friedmann equation for the set of classical trajectories starting from different initial conditions around $(\phi_{\text{init}},0)$ Gaussian distributed with $\langle(\phi-\phi_{\text{init}})^2\rangle \sim \langle\chi^2\rangle \sim H_{\text{init}}^2$.

As can be observed in the left panels the joint evolution of the ϕ and χ fluctuations is quite complex. This is due to the fact that the fluctuation distribution is shaped by the nonlinear couplings during the trajectory bend. The resulting curvature fluctuations can however be simply described.

They are given by the sum of those induced by the ϕ fluctuations, $\mathcal{R}_\phi$, (Gaussian distributed) and those induced by the χ fluctuations, $\mathcal{R}_\chi$. Both distributions have approximately the same variance. For the parameters (12), (13), our numerical computations show that actually the rms of the former is about $18H_c$, the rms of the latter $15H_c$. The right panels show the PDF shape of the non-Gaussian component of the curvature fluctuations, $\mathcal{R}_\chi$. They follow the distribution function given in Eq. (6) as long as $|s_4| \lesssim 0.7$ at least for less than 3σ events. The complete PDF of the curvature fluctuations can then be obtained as the convolution of the PDFs of the two components, $\mathcal{R}_\phi$ and $\mathcal{R}_\chi$.

In this Rapid Communication, we have introduced a series of multifield inflationary models leading to the generation of non-Gaussian primordial curvature fluctuations. We have been able to build explicit models with either two or three scalar fields. In the case of two-field models, it is difficult to design a simple potential: it has to be constructed in such a way that the nonlinear term does not damp the curvature fluctuations. In the case of three fields, we have shown that models as simple as polynomial potentials of order four work. In both cases we have illustrated our models by numerical computations and have shown that the analytic result (6) derived in Ref. [12] is a very good approximation to the primordial curvature fluctuation PDF when non-Gaussianities in the χ distribution remain modest. Let us emphasize that in the two-field models the potential has to be tuned but that this is not the case for the three-field model. Moreover, the initial conditions are not required to be fine-tuned, as explained before, to produce a reasonable level of non-Gaussianity.

Our calculations are valid when λ is smaller than unity. Recent CMB constraints [15,16] suggest that at the relative level of non-Gaussianity of order 10^{-3} is within range of detection. This would correspond to $\lambda \sim 10^{-3}$ but a proper investigation remains to be done. This description of the primordial non-Gaussianity is actually the starting point of further studies on its various observational aspects, both for CMB and large scale structure surveys [14].

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[16] We implicitly assumed throughout this work that no isocurvature modes will survive the reheating phase ignoring the fact that the two scalar fields may decay in different species. This would imply further signatures in the CMB (see Refs. [8]).

[17] Note that the variance of the field value evolves due to the nonzero effective mass of χ.