Spectrum of $\mathcal{N}=1$ massive super Yang-Mills theory with fundamental matter in 1+1 dimensions

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We consider $\mathcal{N}=1$ supersymmetric Yang-Mills theory with fundamental matter in the large-N_c approximation in 1+1 dimensions. We add a Chern-Simons term to give the adjoint partons a mass and solve for the meson bound states. Here ''mesons'' are color-singlet states with two partons in the fundamental representation. The spectrum is exactly supersymmetric, and there is complete degeneracy between the fermion and boson bound states. We find that the mass spectrum is composed of two distinct bands. We analyze the properties of the bound states in each band and find a number of interesting properties of these states. In both bands, some of the states are nearly pure quark-gluon bound states while others are nearly pure squark-gluon bound states. The structure functions of many of the bound states found are very strongly peaked near $x=0$. The convergence of the numerical approximation appears to be very good in all cases.

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I. INTRODUCTION

At this point there is no direct evidence for supersymmetry. Nevertheless supersymmetry is such a beautiful symmetry and provides such elegant solutions to a host of theoretical problems [1] that many believe that it must be present in nature. It is a pressing experimental issue to see if nature takes advantage of this elegant option. Of course, we already know that supersymmetry is rather badly broken, since we do not see any superpartners for the particles of the standard model. It is assumed that all of the superpartners are in fact very heavy and that we will see them as we go to higher energies in accelerators. It is therefore extremely interesting to investigate what the properties of supersymmetric bound states might look like. There are indications that some unusual things can happen in a theory with supersymmetry [2,3].

We present here the solution of an $\mathcal{N}=1$ super Yang-Mills (SYM) theory with fundamental matter in 1+1 dimensions. We also add a Chern-Simons (CS) term to give the adjoint matter a mass. There are several motivations to look at (1+1)-dimensional models. 't Hooft [4] showed long ago that two-dimensional models can be powerful laboratories for the study of the bound-state problem. These models remain popular to this day because they are easy to solve and share many properties with their four-dimensional cousins, most notably stable bound states. Supersymmetric two-dimensional models are particularly attractive, since they are also super-renormalizable. Given that the dynamics of the gauge field is responsible for the strong interaction and for the formation of bound states, it comes as no surprise that a great deal of effort has gone into the investigation of bound states of pure glue in supersymmetric models [5,6]. Extensive study of the meson spectrum of non-supersymmetric theories has been done (see [7] for a review). Recently an initial study addressed some of these states in the context of supersymmetric models [8]. In addition to these reasons for studying (1+1)-dimensional models, we are also interested in investigating them as the first step toward solving (2+1)-dimensional models. A (2+1)-dimensional model can be much more realistic, since it includes some transverse dynamics.

Currently there are several numerical approaches to solving field theories. For QCD-like theories, lattice gauge theory is probably the most popular approach, since the lattice approximation does not break the most important symmetry, gauge symmetry. Similarly for supersymmetric theories, supersymmetric discrete light-cone quantization (SDLCQ) [5,7,9,10] is probably the most powerful approach since the discretization does not break the most important symmetry, supersymmetry [11]. In this paper we consider supersymmetric theories and follow this latter approach. To simplify the calculation we will consider only the large-N_c limit [4], which has proven to be a powerful approximation for bound-state calculations. While baryons can be constructed in this limit [12], they have an infinite number of partons, and thus practical calculations for such states are complicated.

Throughout this paper we use the word ''meson'' to indicate the group structure of the state. Namely, we define a meson as a bound state whose wave function can be written as a linear combination of parton chains, each chain starting and ending with a creation operator in the fundamental representation. In supersymmetric theories, the states defined this way can have either bosonic or fermionic statistics.

Previously we saw that the lightest bound states in $\mathcal{N}=1$ supersymmetric theories are very interesting [2,3]. In SYM theories in 1+1 and 2+1 dimensions, the lightest bound states in the spectrum are massless Bogomol'nyi-Prasad-Sommerfield (BPS) bound states [5]. These states are exactly massless at all couplings. When we add a CS term to the (1+1)-dimensional SYM theory, which gives a mass to the constituents, we find approximate BPS states. The masses of these states are approximately independent of the cou-

pling, and at strong coupling these states are the lightest bound states in the theory [2]. In (2+1)-dimensional SYM-CS theory we also find that at strong coupling there are anomalously light bound states [3]. In both 2+1 and 1+1 dimensions, these interesting states appear because of the exact BPS symmetry in the underlying SYM theory. We have also looked at the lightest bound states of SYM-CS theory with fundamental matter [13]. Again we see that the lightest bound states are significantly lighter than one would have naively expected. We found that the lightest state is nearly massless and well below threshold.

We will see that this model has a number of interesting bound states in addition to the lowest mass state that we studied previously [13]. The bound states separate into two bands, a low-mass band and a high-mass band. Interestingly the low mass permits two solutions for bound states. The preferred solution has an unusual oscillatory convergence. Some of the states in the low-mass band are very light and well below threshold. The upper band has the standard linear convergence. In both bands, some of the states are nearly pure quark-gluon states, and some are nearly pure squark and gluons. Some have structure functions that are very sharply peaked at small longitudinal momentum fractions, and some have several of these properties.

Throughout this paper we completely ignore the zero-mode problem [14,15]; however, it is clear that considerable progress on this issue could be made following our earlier work on the zero modes of the two-dimensional supersymmetric model with only adjoint fields [16].

The paper has the following organization. In Sec. II A we consider three-dimensional supersymmetric QCD (SQCD) with a CS term and dimensionally reduce it to 1+1 dimensions. We perform the light-cone quantization of the resulting theory by applying canonical commutation relations at fixed $x^+ \equiv (x^0 + x^1)/\sqrt{2}$ and choosing the light-cone gauge ($A^+ = 0$) for the vector field. After solving the constraint equations, we obtain a model containing four dynamical fields. We construct the supercharge for this dimensionally reduced theory. In Sec. II B we discuss the structure of the lighter meson bound states in the large-N_c approximation. In Sec. II C we discuss the addition of a CS term to the supercharge [17–19]. We explain that, in the context of this model, this is equivalent to adding a mass to the partons in the adjoint representation. In Sec. III we discuss the bound-state solutions that we find, including the mass spectrum, structure functions and the numerical convergence of the various bound states. In Sec. IV we discuss our results and the future directions that are indicated by this research.

II. SUPERSYMMETRIC SYSTEMS WITH FUNDAMENTAL MATTER

A. Construction of the supercharge

We consider the supersymmetric models in two dimensions which can be obtained as the result of dimensional reduction of SQCD_{2+1}. Our starting point is the three-dimensional action

$$S = \int d^3x \mathrm{tr}\Big(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\Lambda}\bar{\Gamma}^\mu D_\mu \Lambda + D_\mu \xi^\dagger D^\mu \xi + i\bar{\Psi} D_\mu \Gamma^\mu \Psi - g[\bar{\Psi}\Lambda\xi + \xi^\dagger \bar{\Lambda}\Psi]\Big). \tag{2.1}$$

This action describes a system of a gauge field A_μ, representing gluons, and its superpartner Λ, representing gluinos, both taking values in the adjoint representation of $SU(N_c)$, and two complex fields, a scalar ξ representing squarks and a Dirac fermion Ψ representing quarks, transforming according to the fundamental representation of the same group. In matrix notation the covariant derivatives are given by

$$D_\mu \Lambda = \partial_\mu \Lambda + ig[A_\mu, \Lambda], \quad D_\mu \xi = \partial_\mu \xi + igA_\mu \xi,$$
$$D_\mu \Psi = \partial_\mu \Psi + igA_\mu \Psi. \tag{2.2}$$

The action (2.1) is invariant under the following supersymmetry transformations, which are parametrized by a two-component Majorana fermion ε:

$$\delta A_\mu = \frac{i}{2}\bar{\varepsilon}\bar{\Gamma}_\mu \Lambda, \quad \delta\Lambda = \frac{1}{4} F_{\mu\nu}\Gamma^{\mu\nu}\varepsilon,$$
$$\delta\xi = \frac{i}{2}\bar{\varepsilon}\bar{\Psi}, \quad \delta\Psi = -\frac{1}{2}\Gamma^\mu \varepsilon D_\mu \xi. \tag{2.3}$$

Using standard techniques one can construct the Noether current corresponding to these transformations as

$$\bar{\varepsilon} q^\mu = \frac{i}{4}\bar{\varepsilon}\bar{\Gamma}^{\alpha\beta}\Gamma^\mu \mathrm{tr}(\Lambda F_{\alpha\beta}) + \frac{i}{2} D^\mu \xi^\dagger \bar{\varepsilon}\bar{\Psi} + \frac{i}{2}\xi^\dagger \bar{\varepsilon}\bar{\Gamma}^{\mu\nu} D_\nu \Psi - \frac{i}{2}\bar{\Psi}\varepsilon D^\mu \xi + \frac{i}{2} D_\nu \bar{\Psi}\Gamma^{\mu\nu}\varepsilon\xi. \tag{2.4}$$

We will consider the reduction of this system to two dimensions, which means that the field configurations are assumed to be independent of the transverse coordinate x^2. In the resulting two-dimensional system we will implement light-cone quantization, where the initial conditions as well as canonical commutation relations are imposed on a light-like surface $x^+ = \text{const}$. In particular, we construct the supercharge by integrating the current (2.4) over the light-like surface to obtain

$$\bar{\varepsilon} Q = \int dx^- dx^2 \Big(\frac{i}{4}\bar{\varepsilon}\bar{\Gamma}^{\alpha\beta}\Gamma^+ \mathrm{tr}(\Lambda F_{\alpha\beta}) + \frac{i}{2} D_- \xi^\dagger \bar{\varepsilon}\bar{\Psi} + \frac{i}{2}\xi^\dagger \bar{\varepsilon}\bar{\Gamma}^{+\nu} D_\nu \Psi - \frac{i}{2}\bar{\Psi}\varepsilon D^+ \xi + \frac{i}{2} D_\nu \bar{\Psi}\Gamma^{+\nu}\varepsilon\xi \Big). \tag{2.5}$$

Since all fields are assumed to be independent of x^2, the integration over this coordinate gives just a constant factor, which we absorb by a field redefinition.

If we use the following specific representation for the Dirac matrices in three dimensions:

$$\Gamma^0 = \sigma_2, \quad \Gamma^1 = i\sigma_1, \quad \Gamma^2 = i\sigma_3, \tag{2.6}$$

the Majorana fermion Λ can be chosen to be real. It is also convenient to write the fermion fields and the supercharge in component form as

$$\Lambda=(\lambda,\tilde{\lambda})^T, \quad \Psi=(\psi,\tilde{\psi})^T, \quad Q=(Q^+,Q^-)^T. \quad (2.7)$$

In terms of this decomposition the superalgebra has an explicit (1,1) form

$$\{Q^+,Q^+\}=2\sqrt{2}P^+, \quad \{Q^-,Q^-\}=2\sqrt{2}P^-,$$
$$\{Q^+,Q^-\}=0. \quad (2.8)$$

The SDLCQ method exploits this superalgebra by constructing P^- from a discrete approximation to Q^- [9], rather than directly discretizing P^-, as is done in ordinary DLCQ [7].

To begin to eliminate nondynamical fields, we impose the light-cone gauge ($A^+=0$). Then the supercharges are given by

$$Q^+=2\int dx^-\left(\lambda\partial_- A^2+\frac{i}{2}\partial_-\xi^\dagger\psi-\frac{i}{2}\psi^\dagger\partial_-\xi-\frac{i}{2}\xi^\dagger\partial_-\psi+\frac{i}{2}\partial_-\psi^\dagger\xi\right), \quad (2.9)$$

$$Q^-=-2\int dx^-\left(-\lambda\partial_- A^-+i\xi^\dagger D_2\psi-iD_2\psi^\dagger\xi+\frac{i}{\sqrt{2}}\partial_-(\tilde{\psi}^\dagger\xi-\xi^\dagger\tilde{\psi})\right). \quad (2.10)$$

Note that apart from a total derivative these expressions involve only left-moving components of the fermions (λ and ψ). In fact, in the light-cone formulation only these components are dynamical. To see this we consider the equations of motion that follow from the action (2.1), in light-cone gauge. Three of them serve as constraints rather than as dynamical statements; they are

$$\partial_-\tilde{\lambda}=-\frac{ig}{\sqrt{2}}([A^2,\lambda]+i\xi\psi^\dagger-i\psi\xi^\dagger), \quad (2.11)$$

$$\partial_-\tilde{\psi}=-\frac{ig}{\sqrt{2}}A^2\psi+\frac{g}{\sqrt{2}}\lambda\xi, \quad (2.12)$$

and

$$\partial_-^2 A^-=gJ, \quad (2.13)$$

with

$$J\equiv i[A^2,\partial_- A^2]+\frac{1}{\sqrt{2}}\{\lambda,\lambda\}-ih\partial_-\xi\xi^\dagger+i\xi\partial_-\xi^\dagger+\sqrt{2}\psi\psi^\dagger. \quad (2.14)$$

Apart from the zero-mode problem [14], one can invert the last constraint to write the auxiliary field A^- in terms of physical degrees of freedom.

In order to solve the bound-state problem $2P^+P^-|M\rangle=M^2|M\rangle$, we apply the methods of SDLCQ. Namely we compactify the two-dimensional theory on a light-like circle ($-L<x^-<L$), and impose periodic boundary conditions on all physical fields. This leads to the following mode expansions:

$$A^2_{ij}(0,x^-)=\frac{1}{\sqrt{4\pi}}\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}[a_{ij}(k)e^{-ik\pi x^-/L}+a^\dagger_{ji}(k)e^{ik\pi x^-/L}], \quad (2.15)$$

$$\lambda_{ij}(0,x^-)=\frac{1}{2^{1/4}\sqrt{2L}}\sum_{k=1}^{\infty}[b_{ij}(k)e^{-ik\pi x^-/L}+b^\dagger_{ji}(k)e^{ik\pi x^-/L}], \quad (2.16)$$

$$\xi_i(0,x^-)=\frac{1}{\sqrt{4\pi}}\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}[c_i(k)e^{-ik\pi x^-/L}+\tilde{c}^\dagger_i(k)e^{ik\pi x^-/L}], \quad (2.17)$$

$$\psi_i(0,x^-)=\frac{1}{2^{1/4}\sqrt{2L}}\sum_{k=1}^{\infty}[d_i(k)e^{-ik\pi x^-/L}+\tilde{d}^\dagger_i(k)e^{ik\pi x^-/L}]. \quad (2.18)$$

We drop the zero modes of the fields; including them could lead to new and interesting effects (see [16], for example), but this is beyond the scope of this work.

In the light-cone formalism one treats x^+ as the time direction, thus the commutation relations between fields and their momenta are imposed on the surface $x^+=0$. For the system under consideration this means that

$$[A^2_{ij}(0,x^-),\partial_- A^2_{kl}(0,y^-)]=i\left(\delta_{il}\delta_{kj}-\frac{1}{N}\delta_{ij}\delta_{kl}\right)\delta(x^--y^-), \quad (2.19)$$

$$\{\lambda_{ij}(0,x^-),\lambda_{kl}(0,y^-)\}=\sqrt{2}\left(\delta_{il}\delta_{kj}-\frac{1}{N}\delta_{ij}\delta_{kl}\right)\times\delta(x^--y^-), \quad (2.20)$$

$$[\xi_i(0,x^-),\partial_-\xi_j(0,y^-)]=i\delta_{ij}\delta(x^--y^-), \quad (2.21)$$

$$\{\psi_i(0,x^-),\psi_j(0,y^-)\}=\sqrt{2}\delta_{ij}\delta(x^--y^-). \quad (2.22)$$

These relations can be rewritten in terms of creation and annihilation operators as

$$[a_{ij},a^\dagger_{kl}]=\left(\delta_{il}\delta_{kj}-\frac{1}{N}\delta_{ij}\delta_{kl}\right),$$
$$\{b_{ij},b^\dagger_{kl}\}=\left(\delta_{il}\delta_{kj}-\frac{1}{N}\delta_{ij}\delta_{kl}\right), \quad (2.23)$$

$$[c_i,c^\dagger_j]=\delta_{ij}, \quad [\tilde{c}_i,\tilde{c}^\dagger_j]=\delta_{ij},$$
$$\{d_i,d^\dagger_j\}=\delta_{ij} \quad \{\tilde{d}_i,\tilde{d}^\dagger_j\}=\delta_{ij}. \quad (2.24)$$

In this paper we will discuss numerical results obtained in the large-N_c limit, i.e. we neglect $1/N_c$ terms in the above expressions. Although $1/N_c$ corrections may have interesting consequences, they are beyond the scope of this work.

Substituting the result into the expression for the supercharge and omitting the boundary term, we get

$$Q^- = Q_s^- + Q_1^- + Q_2^- + Q_3^-, \tag{2.25}$$

where Q_s^- is the supercharge of pure adjoint matter [9]. The three supercharges that govern the behavior of the fundamental matter in these states are

$$Q_1^- = -\frac{g}{\sqrt{2}}\int dx^-(i\sqrt{2}\xi\partial_-\xi^\dagger - i\sqrt{2}\partial_-\xi\xi^\dagger)\frac{1}{\partial_-}\lambda, \tag{2.26}$$

$$Q_2^- = -\frac{g}{\sqrt{2}}\int dx^-(2\psi\psi^\dagger)\frac{1}{\partial_-}\lambda, \tag{2.27}$$

$$Q_3^- = -2g\int dx^-(\xi^\dagger A^2\psi + \psi^\dagger A^2\xi). \tag{2.28}$$

After substituting the expansions (2.15)–(2.18), one gets the mode decomposition of the supercharge,

$$Q_\alpha^- = \frac{i2^{-1/4}g\sqrt{L}}{\pi}\sum_{k_1,k_2,k_3=1}^{\infty} q_\alpha^-(k_1,k_2,k_3), \tag{2.29}$$

with

$$\begin{aligned} q_1^- = \frac{(k_2+k_3)}{2k_1\sqrt{k_2k_3}}[&\tilde{c}_i^\dagger(k_2)\tilde{c}_j(k_3)\tilde{b}_{ji}(k_1) - \tilde{c}_i^\dagger(k_2)b_{ij}^\dagger(k_1)\tilde{c}_j(k_3) \\ &+ b_{ji}^\dagger(k_1)c_i^\dagger(k_2)c_j(k_3) \\ &- c_i^\dagger(k_2)b_{ij}(k_1)c_j(k_3)]\delta_{k_3,k_1+k_2}, \end{aligned} \tag{2.30}$$

$$\begin{aligned} q_2^- = \frac{1}{k_1}[&\tilde{d}_i^\dagger(k_2)b_{ij}^\dagger(k_1)\tilde{d}_j(k_3) + \tilde{d}_j^\dagger(k_3)\tilde{d}_i(k_2)b_{ij}(k_1) \\ &+ d_i^\dagger(k_2)b_{ij}^\dagger(k_1)d_j(k_3) \\ &+ d_j^\dagger(k_3)d_i(k_2)b_{ij}(k_1)]\delta_{k_3,k_1+k_2}, \end{aligned} \tag{2.31}$$

$$\begin{aligned} q_3^- = \frac{-i}{2\sqrt{k_2k_3}}\{[&d_j^\dagger(k_1)\tilde{c}_i(k_3)a_{ij}(k_2) + \tilde{c}_i^\dagger(k_3)a_{ij}^\dagger(k_2)d_j(k_1) \\ &+ \tilde{d}_i^\dagger(k_1)a_{ij}(k_2)c_j(k_3) \\ &+ a_{ij}^\dagger(k_2)c_j^\dagger(k_3)\tilde{d}_i(k_1)]\delta_{k_1,k_3+k_2} \\ &+ [\tilde{c}_j^\dagger(k_3)d_i(k_1)a_{ij}(k_2) + d_i^\dagger(k_1)a_{ij}^\dagger(k_2)\tilde{c}_j(k_3) \\ &+ c_i^\dagger(k_3)a_{ij}(k_2)\tilde{d}_j(k_1) \\ &+ a_{ij}^\dagger(k_2)\tilde{d}_j^\dagger(k_1)c_i(k_3)]\delta_{k_3,k_1+k_2}\}. \end{aligned} \tag{2.32}$$

We have dropped from these expression the terms that connect the states that are closed loops of adjoint partons to the string-like meson states. These interactions are of order $1/\sqrt{N_c}$ and can be neglected in the large-N_c approximation.

B. Meson structure

We will consider here only meson-like states. In the large-N_c approximation these are color-singlet states with exactly two partons in the fundamental representation. The boson bound states will have either two bosons or two fermions in the fundamental representation. In general a boson bound state will have a combination of these types of contributions. Because this theory and the numerical formalism are exactly supersymmetric, for each boson bound state there will be a degenerate bound state that is a fermion. The fermion bound state will have one fermion in the fundamental representation and one boson in the fundamental representation. In the string interpretation of these theories, such states would correspond to open strings with freely moving end points. In the language of QCD, the model corresponds to a system of interacting gluons and gluinos which bind dynamical (s)quarks and anti-(s)quarks. In the large-N_c limit we will have to consider only a single (s)quark–anti-(s)quark pair. Thus the Fock space is constructed from states of the following type:

$$\overline{f}_{i_1}^\dagger(k_1)a_{i_1i_2}^\dagger(k_2)\ldots b_{i_ni_{n+1}}^\dagger(k_{n-1})\ldots f_{i_p}^\dagger(k_n)|0\rangle. \tag{2.33}$$

Here $\overline{f}_i^\dagger$ and $f_i^\dagger$ each create one of the fundamental partons, and $|0\rangle$ is the vacuum annihilated by a_{ij}, b_{ij}, c_i, $\tilde{c}_i$, d_i, and $\tilde{d}_i$.

The other color-singlet bound states in this theory are states that are composed of traces of only adjoint mesons. These can be considered to be loops. At finite N_c this theory has interactions that break these loops and insert a pair of fundamental partons, making an open-string state. This type of interaction can, of course, also form loops from open strings and break one open string into two open strings. In principle, a calculation of the spectrum of such a finite-N_c theory is within the reach of SDLCQ. The only significant change is to include states in the basis with more than one color trace.

C. Supersymmetric Chern-Simons theory

The CS term we use in this calculation is obtained by starting with a CS term in 2+1 dimensions and reducing it to 1+1 dimensions. This term has the effect of adding a mass for the adjoint partons. In this calculation we are including fundamental matter because we are interested in QCD-like meson bound states. Without a mass for the adjoint matter, this theory is known to produce very long light chains of adjoint partons. In a finite-N_c calculation we would not have these very long chains because they would break, but in the large-N_c approximation they do not. While SDLCQ can be used to do finite-N_c calculations, it is much easier to add a mass to restrict the number of adjoint partons in our bound

states. We choose the CS mechanism to give the adjoint partons a mass because we can do this without breaking the supersymmetry.

The Lagrangian of this theory is

$$\mathcal{L}=\mathcal{L}_{\mathrm{SQCD}}+\frac{\kappa}{2}\mathcal{L}_{\mathrm{CS}}, \tag{2.34}$$

where $\mathcal{L}_{\mathrm{SQCD}}$ is the SQCD Lagrangian we discussed earlier, κ is the CS coupling, and

$$\mathcal{L}_{\mathrm{CS}}=\epsilon^{\mu\nu\lambda}\left(A_\mu\partial_\nu A_\lambda+\frac{2i}{3}gA_\mu A_\nu A_\lambda\right)+2\bar{\Psi}\Psi. \tag{2.35}$$

A trace of the color matrices is understood. The constraint equation (2.12) gains a third term of the form $-\kappa\lambda/\sqrt{2}$ on the right-hand side, and the definition of the current in Eq. (2.14) now has an additional term, $\kappa\partial_- A^2$. The discrete version of the CS part of the supercharge in $1+1$ dimensions is

$$Q^-_{CS}=\left(\frac{2^{-1/4}\sqrt{L}}{\sqrt{\pi}}\right)\sum_n \frac{\kappa}{\sqrt{n}}[a^\dagger_{ij}(n)b_{ij}(n)+b^\dagger_{ij}(n)a_{ij}(n)]. \tag{2.36}$$

It is informative to compare Q^-_{CS} with the one term in the supercharge for $\mathcal{N}=1$ SYM in $2+1$ dimensions [20] which has an explicit dependence on the transverse momentum $k_\perp$. This $k_\perp$-dependent term has the form

$$Q^-_\perp=i\left(\frac{2^{-1/4}\sqrt{L}}{\sqrt{\pi}}\right)\sum_{n,n_\perp}\frac{k_\perp}{\sqrt{n}}[a^\dagger_{ij}(n,n_\perp)b_{ij}(n,n_\perp) - b^\dagger_{ij}(n,n_\perp)a_{ij}(n,n_\perp)], \tag{2.37}$$

where $k_\perp=2\pi n_\perp/L_\perp$ is the discrete transverse momentum. Notice that $k_\perp$ and κ enter the supercharge in very similar ways. Addition of a CS term has the effect of changing $k_\perp$ to $k_\perp+i\kappa$. Because the light-cone energy is of the form $(k_\perp^2+m^2)/k^+$, $k_\perp$ behaves like a mass, and therefore κ also behaves in many ways like a mass for the adjoint particles.

The partons in the fundamental representation in this theory will remain massless. Of course, in a more physical theory the supersymmetry would be badly broken; the squark would acquire a large mass, and only the quarks would remain nearly massless.

III. NUMERICAL RESULTS

This SYM-CS theory with fundamental matter has two dimensionful parameters with the dimension of a mass squared, the YM coupling squared g^2N_c/π, and the CS coupling squared κ^2. The latter is also the mass squared of the partons in the adjoint representation. All of the masses in this paper will be given in units of $g\sqrt{N_c/\pi}$, which will usually be suppressed. Furthermore, we are only considering meson bound states. These are states of the form shown in Eq. (2.33) with two fundamental partons linked by partons in the adjoint representation. Since we are working in the large-N_c approximation, this class of states is disconnected from the other allowed class of pure adjoint matter bound states and multiparticle states. This theory also has a Z_2 symmetry [21] which is very useful in labeling the states and reducing the dimension of the Fock basis that one has to consider in any one diagonalization step. For this theory the Z_2 symmetry divides the basis into states with an even or odd number of gluons.

A. Spectra

We find that the spectra of meson bound states, for both odd and even Z_2 symmetry, divides into two bands of states as we increase κ, a light-mass band and a heavy-mass band, as can be seen in Fig. 1(a). As κ grows, a gap develops in the spectrum. This mass gap reflects in part the number of massive adjoint partons in the bound states. At very large κ, the low-mass band will comprise states with only fundamental partons. We will look at the bound states in both bands and consider various values of κ relative to $g\sqrt{N_c/\pi}$. In Fig. 1(b) we see that at $\kappa=1$ this gap grows with the coupling. The lowest mass state remains very light even for values of $g\sqrt{N_c/\pi}$ up to 2. As far as we can tell, this state remains very light even at very large couplings.

In Figs. 2(a) and 2(b) we show the spectrum in the Z_2-even sector, with $g\sqrt{N_c/\pi}=1$, as a function of resolution at $\kappa=0$ and 1, respectively. We see that at large resolution with $\kappa=0$ the two bands merge. It is unclear from this figure whether the states in the lower band mix with the states in the upper band in the region where their masses overlap. An inspection of the adjacent states seems to show that these states remain on smooth trajectories and do not repel, as one would expect if they were interacting. This seems to indicate that a separate analysis of these two sets of states is possible. For the most part, however, we will look at the situation where $\kappa=1$, as shown in Fig. 2(b). In this case the two bands are well separated even at the highest accessible resolution, and, therefore, there is not a problem considering the two bands separately.

Let us first consider the bound states in the lower band. To understand the physics underlying these bound states we will look at both the spectrum and the structure functions of individual states. In this theory we have four species of partons: adjoint fermions and bosons, and fundamental fermions and bosons. Therefore, we will look at four structure functions that give the probability of finding a particular variety of parton at a particular value of longitudinal momentum fraction in a particular bound state. We need all of this information about each bound state to properly analyze this theory.

Although the procedure that we follow for identifying bound states is the standard one in DLCQ, it is worthwhile to give a brief description of that procedure here because of the unusual nature of the lower band. To identify a bound state, we start at the lowest value of the resolution K, usually $K=3$, and select a particular eigenvalue, usually the lowest nonzero one. We then look at the properties of this state, which we always calculate along with the mass. In principle, we can look at the entire wave function, but for higher resolution there is more data than we can efficiently handle. It is

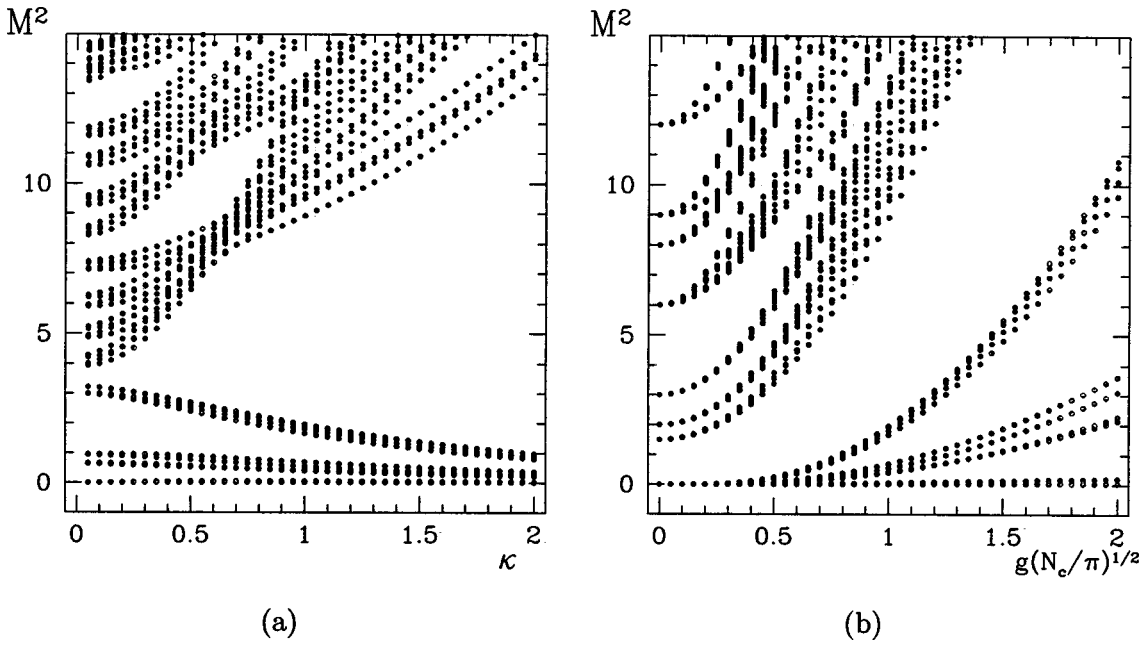

FIG. 1. The mass-squared spectrum, with Z_2 odd, in units of g^2N_c/π, at a resolution of $K=6$ as a function of (a) κ at $g\sqrt{N_c/\pi}=1$ and (b) $g\sqrt{N_c/\pi}$ at $\kappa=1$.

efficient to look at the average number of partons of the various types and sometimes the average momentum of some of these partons. We then move to the next higher resolution and try to identify an eigenstate that has the same or very similar properties to the state at lower resolution. We continue this through all the resolutions. This then gives us the mass squared of a particular bound state and the various properties of the state as functions of the resolution. If this history of a state as a function of the resolution makes sense, and can be extrapolated to infinite resolution, we say we have found a bound state. We then repeat the process to obtain another bound state. For the oscillatory states presented below, we actually started with the lightest state at the highest resolution and then followed the states down to low resolution. We then repeated the process with the next highest states at the highest resolution.

The lower band of this model is unique relative to all other models we have studied in SDLCQ or DLCQ. We find that there are two different ways of carrying out the above procedure to find the states in the lower band. We can follow

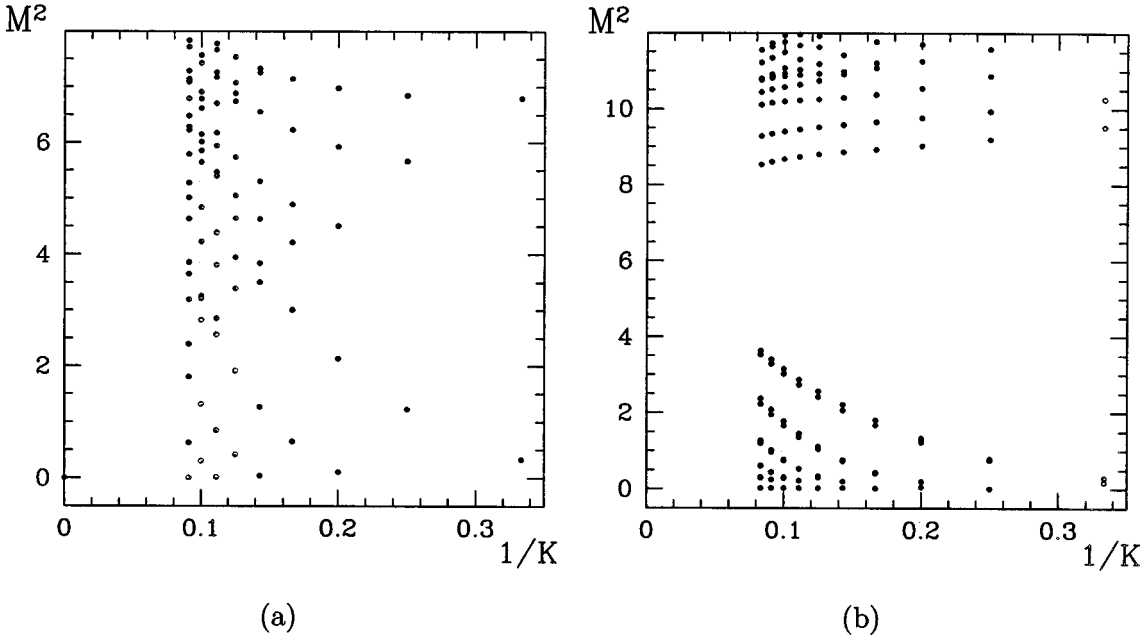

FIG. 2. The mass-squared spectrum, with Z_2 even, in units of g^2N_c/π, as a function of the resolution at (a) $g=0$ and (b) $g\sqrt{N_c/\pi}=1$.

TABLE I. Properties of the lowest mass boson bound state in the Z_2-even sector, including the average numbers of adjoint bosons *aB*, adjoint fermions *aF*, fundamental bosons *fB*, and fundamental fermions *fF*, for different values of the resolution *K*. The mass squared M^2 is given in units of g^2N_c/π. The CS coupling is $\kappa = g\sqrt{N_c/\pi}$.

K	M^2	$\langle n\rangle$	$\langle n_{aB}\rangle$	$\langle n_{fB}\rangle$	$\langle n_{aF}\rangle$	$\langle n_{fF}\rangle$
3	0.178	2.30	0.30	1.03	0.01	0.97
4	0.006	2.56	0.51	1.86	0.05	0.14
5	0.049	2.69	0.63	1.29	0.06	0.71
6	0.016	2.83	0.75	1.71	0.08	0.30
7	0.029	2.84	0.76	1.45	0.08	0.55
8	0.022	2.92	0.83	1.58	0.09	0.42
9	0.025	2.92	0.83	1.49	0.10	0.51
10	0.024	2.96	0.86	1.52	0.10	0.48
11	0.025	2.97	0.87	1.49	0.11	0.51
12	0.025	3.00	0.89	1.48	0.11	0.52
13	0.026	3.01	0.90	1.47	0.11	0.53

the eigenvalues from resolution to resolution in two ways. As a result we end up with two sets of bound states, all of whose properties are very smoothly behaved as functions of the resolution. Two such states and their properties are shown in Tables I and II. The masses in Table I are very small. We have discussed the lowest mass state in this spectrum in detail elsewhere [13]. The convergence as a function of the resolution is oscillatory, but clearly as we move to large resolution the convergence is very good. In Fig. 3(a) we present the spectrum as a function of $1/K$, and we see that the oscillations lead to good convergence that extrapolates nicely to $K=\infty$. These states have masses near zero. The lowest state has on average one adjoint parton; therefore, the threshold is at κ^2. Thus this is a deeply bound state. In Fig. 3(b) we show the spectrum for the next highest mass bound state, and again we see very good convergence to a very light state. In fact all of the data in the lower band describe a series of oscillatory states at higher masses.

Shown in Table II is another way of organizing the same set of data used to analyze the oscillatory states discussed above, and all these states also converge numerically, but to

TABLE II. Same as Table I, but for the lowest mass divergent state.

K	M^2	$\langle n\rangle$	$\langle n_{aB}\rangle$	$\langle n_{fB}\rangle$	$\langle n_{aF}\rangle$	$\langle n_{fF}\rangle$
3	0.29	2.39	0.33	1.01	0.06	0.99
4	0.75	2.58	0.47	0.91	0.11	1.09
5	1.23	2.73	0.59	0.87	0.14	1.14
6	1.67	2.82	0.66	0.85	0.17	1.15
7	2.06	2.88	0.70	0.84	0.18	1.16
8	2.41	2.92	0.72	0.84	0.20	1.16
9	2.73	2.94	0.73	0.84	0.21	1.16
10	3.02	2.96	0.74	0.84	0.22	1.16
11	3.28	2.97	0.74	0.84	0.23	1.16
12	3.53	2.98	0.74	0.84	0.23	1.16

infinite mass. As a function of the resolution we see almost no variation in the properties of this state. We will refer to these states as divergent states, and we will argue that they are divergent. This data and a fit to the data points are shown in Fig. 3(c), and there are no oscillations. In Fig. 3(d) we see another state of this type. It appears that the divergent states do extrapolate to an infinite mass as $K\to\infty$. The curves, however, can be fit by a variety of functions. In fact the functions $a+b/\sqrt{K}+1/K$ and $a+b\log K+1/K$ produce equally good fits. The fits shown are of the former form and have intercepts between 10 and 11. In Fig. 4 we put all four plots from Fig. 3 on the same graph, so that one can clearly see that at low resolution both curves involve the same data. In the DLCQ procedure a state must extrapolate nicely to $K=\infty$, if it is a true bound state of the theory. It is unclear from the data whether this is true for these states. Our prejudice is that this bound state diverges as $K\to\infty$.

In Fig. 5 we show similar results for the Z_2-odd sector. In Fig. 6 we again show the oscillatory and divergent fits on the same figure, but now we add a few more states to better show the interlacing of these fits to the same data. There are, of course, degenerate fermion bound states for all these states, whose properties we do not show.

The oscillatory behavior made this calculation particularly challenging numerically. We were forced to go to very high resolution, $K=13$, to be certain that the spectrum really converged. This was made even more difficult because we had four species of particles in the problem.

To get some insight into the behavior of the low-mass band, we recall the behavior of the two-particle continuum in DLCQ. The ''mass squared'' of two free partons of mass *m* at resolution *K* is

$$M^2=m^2K\left[\frac{1}{K-n}+\frac{1}{n}\right], \tag{3.1}$$

where one free parton has longitudinal momentum $K-n$ and the other has n [22]. In DLCQ this formula produces a band of states. It is interesting to compare the top and bottom of this band with the top and bottom of the low-mass band of bound states that we have been studying. The top of the band is obtained by fixing *n* at one. If we then take *K* large, we find $M^2=m^2(K+1+1/K+\ldots)$. Not surprisingly, the divergent curves in Figs. 3(c) and 3(d) and Figs. 5(c) and 5(d) can be fit with a function of the form $M^2=a+bK+c/K$. In essence, the form of the divergent state is comparable to the shape of the top of the continuum band. Similarly the bottom of the continuum band is obtained by taking $n=K/2$ for *K* even and $n=(K-1)/2$ for *K* odd. The bottom of the continuum band oscillates as a function of *K*. For *K* even it is $4m^2$, and for *K* odd it is $4m^2(1+1/K+\ldots)$. It again appears that the lowest mass states resemble the bottom of the continuum band. We would expect to see a connection between the weak-coupling bound states and the free spectrum, but this connection appears to extend to $g^2N_c/\pi=\kappa^2$. Based on this argument, the divergent states actually do diverge and are not true states of the theory.

The lowest mass Z_2-even state in the low-mass band has

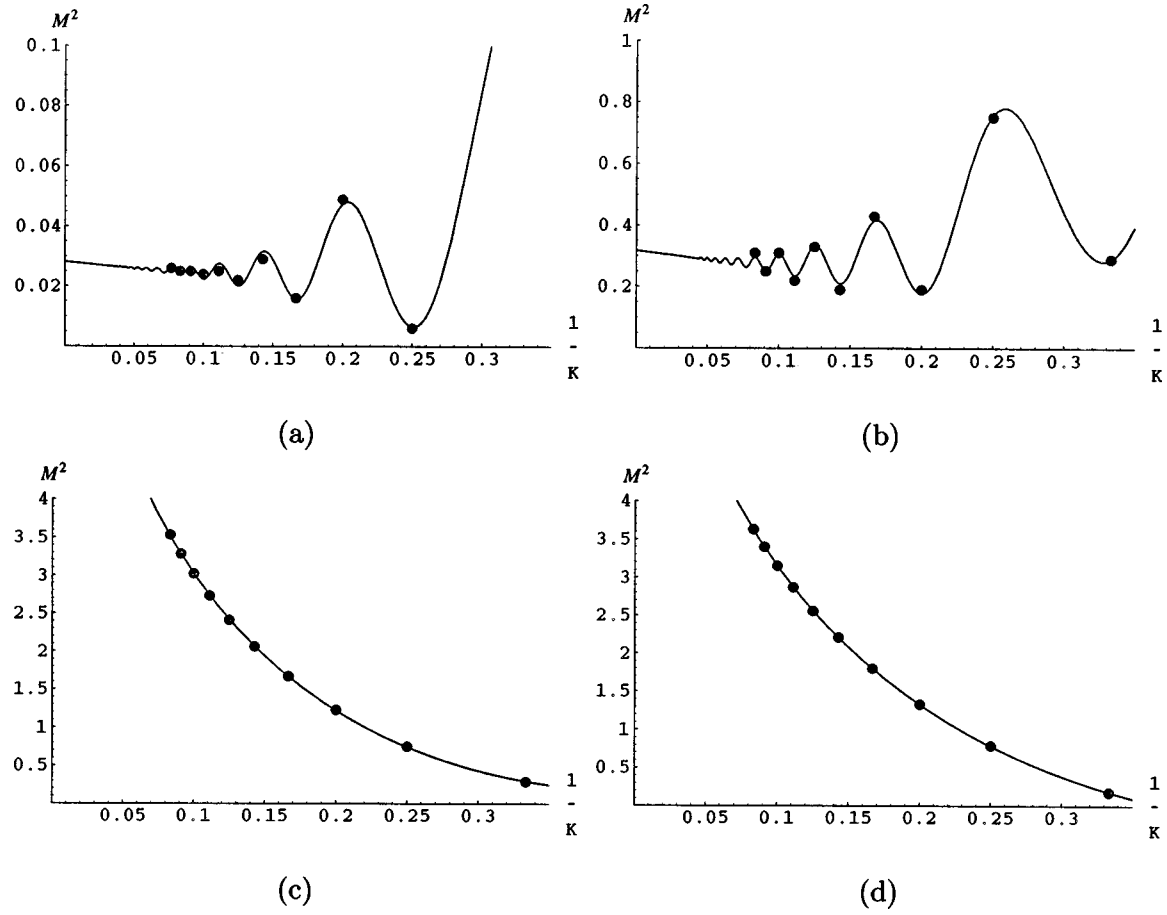

FIG. 3. The mass squared in units of g^2N_c/π, as a function of $1/K$ for $\kappa=g\sqrt{N_c/\pi}$ in the Z_2-even sector, of (a) the lowest mass oscillatory state, (b) the second lowest mass oscillatory state, (c) the first divergent state, and (d) the second divergent state. The solid curve is a fit to the computed points.

on average one adjoint parton, which has mass 1.0, and the fundamental partons are massless. One would therefore expect the lowest bound state in the spectrum to have a mass of order 1.0. In $\mathcal{N}=1$ SYM-CS theory [2,3], we previously found anomalously light states at strong coupling, but all of these states were at or close to threshold. Here the lowest mass state is anomalously light, in fact nearly massless, and therefore well below threshold.

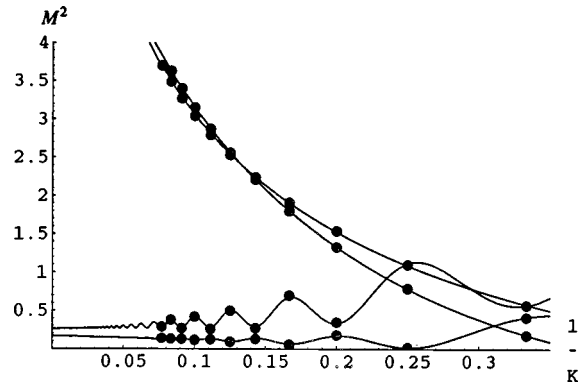

FIG. 4. The mass squared in units of g^2N_c/π, as a function of $1/K$ for $\kappa=g\sqrt{N_c/\pi}$ in the Z_2-even sector, of oscillatory and divergent mass fits.

At yet stronger coupling, $g^2N_c/\pi=10\kappa^2$, we find that oscillations are significantly stronger, as seen in Fig. 7, and at $K=12$ the curve is not totally converged. The mass squared at $K=12$ is 0.068, but without complete convergence we are not able to say anything precise about the variation of the bound-state mass as a function of the coupling from $g^2N_c/\pi=\kappa^2$ to $g^2N_c/\pi=10\kappa^2$.

B. Structure functions

The two largest structure functions for the lowest mass state are shown in Fig. 8. Even though this is an exactly supersymmetric theory, nearly all the adjoint partons are gluons. About 2/3 of the wave function of this state is composed of two fundamental bosons and an adjoint parton, and about 1/3 of the wave function is made of two fundamental fermions. Within the context of the standard model, this state is primarily a bound state of two squarks and a gluon. We see that both of these distributions are peaked at small x. This reflects strong binding of the fundamental partons, allowing them to be widely separated in momentum, combined with only a small contribution to the momentum from the adjoint boson.

The average number of partons in the next lowest state is about 4.5. For this state, 90% is composed of gluons and

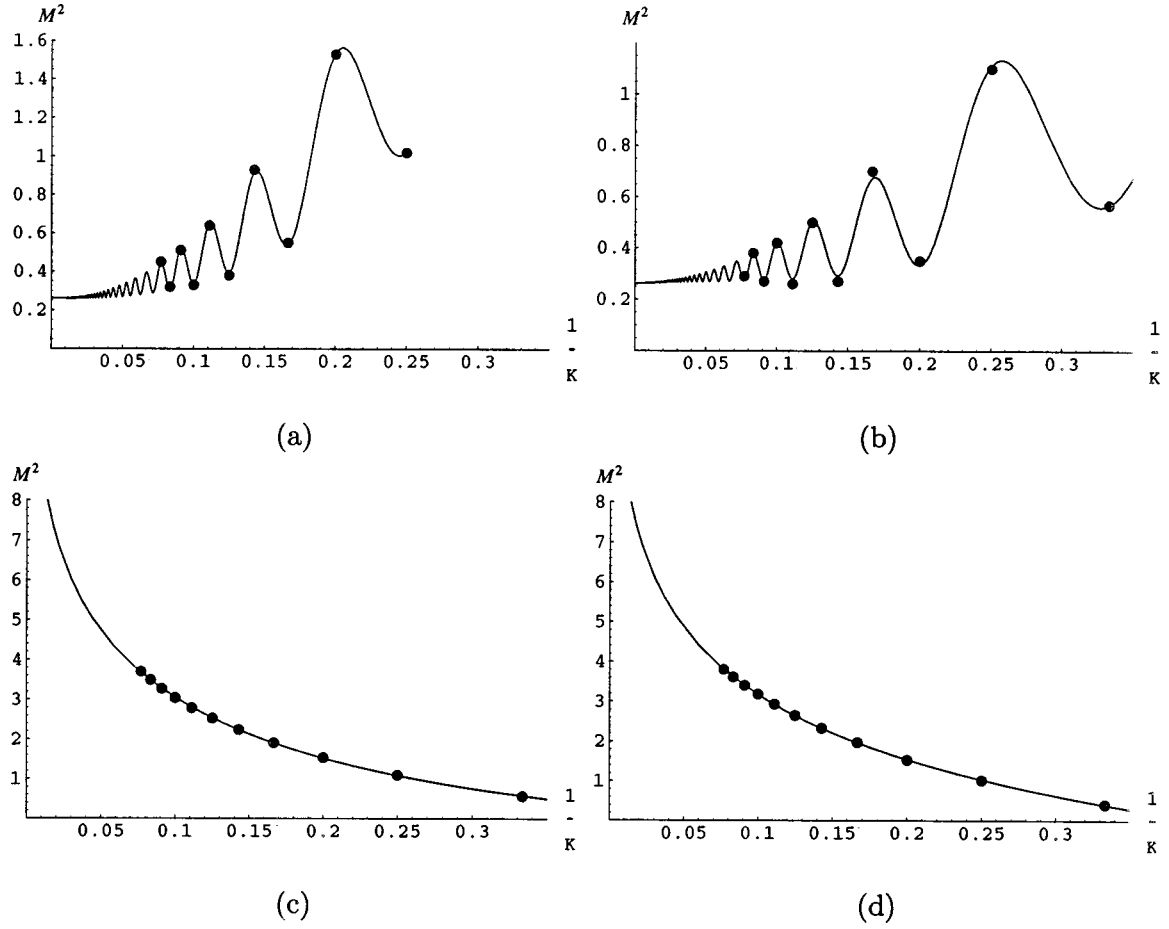

FIG. 5. Same as Fig. 3, but for the Z_2-odd sector.

squarks, and the structure functions are shown in Fig. 9. The squarks have a significantly wider distribution than in the lowest state. The gluons remain very sharply peaked at small x.

For the lowest state in the low-mass band with Z_2 odd, the average number of partons is slightly less that 4. Again the partons in the adjoint representation are almost entirely bosons. However, the partons in the fundamental representation are almost entirely quarks, so this is the lightest standard-model meson. This is a deeply bound state since the gluons have mass 1 and on average there are close to two gluons per bound state. The structure functions for the gluon and quarks are shown in Fig. 10. As we have seen in all the states that we have considered, the gluons are strongly peaked at x near zero. The quarks have a strong peak at small x but have a significant distribution at other values of x.

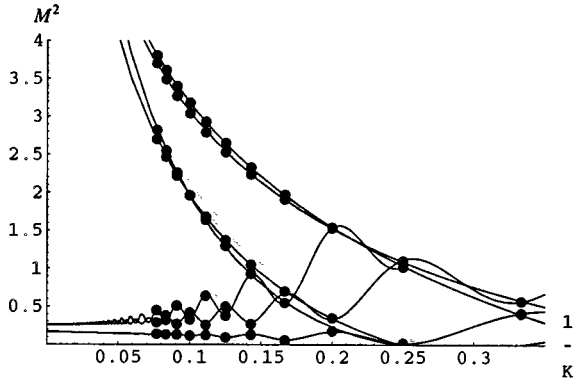

FIG. 6. Same as Fig. 4, but for the Z_2-odd sector.

C. Spectra and structure functions in the upper band

While the lower mass band appears to have a very unusual behavior as a function of the resolution, the bound

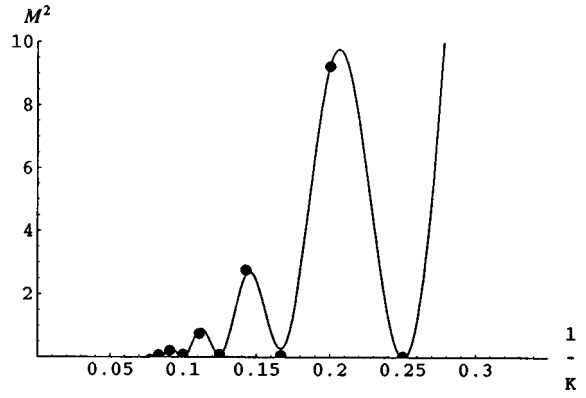

FIG. 7. The mass squared of the lowest mass oscillatory state in units of g^2N_c/π as a function of $1/K$ for $10\kappa^2=g^2N_c/\pi$ in the Z_2-even sector.

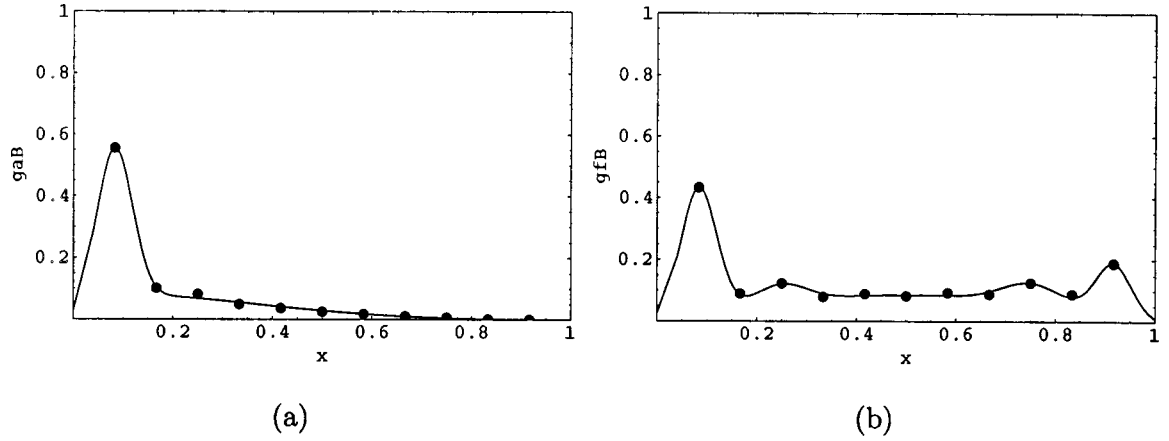

FIG. 8. Structure functions of the bound state $M^2=0.025$ at resolution $K=12$, with Z_2 even, for (a) adjoint bosons aB and (b) fundamental bosons fB, with the CS coupling fixed at $\kappa=g\sqrt{N_c/\pi}$. The solid curve is a fit to the computed points.

states in the upper band behave very similarly to the bound states found in most SDLCQ calculations. The convergence is excellent, starting from the lowest resolution, and the data are fit very well as a function of $1/K$ by a line with a small slope. In Fig. 11 we show that the lowest two bound states in the upper band converge to masses $M^2_\infty=8.31$ and $M^2_\infty=9.03$ at infinite resolution. We carry the calculation out to resolution $K=13$, but the structure functions are calculated at $K=12$. From the structure functions of the state at $M^2_\infty=8.31$, we find that this state is 85% quarks and gluons, so it is very much like a standard-model meson. The average number of partons is 3.98, so for the most part the bound state contains two gluons. It is very interesting that this bound state is very much like a QCD meson and is the solution of a supersymmetric field theory. The structure functions are shown in Fig. 12. While the gluons are peaked at small x, they have a much larger spread than we saw in the lower band. The fundamental fermions are very sharply peaked at small x. The structure functions of the state at $M^2_\infty=9.03$ are interesting because the state has nearly equal mixtures of all four constituents. The structure functions are shown in Fig. 13. The shapes are different, particularly for the distribution of adjoint fermions, which are spread over the region of larger x.

In Fig. 14 we see that in the Z_2-odd sector we again have an excellent linear fit. The two lowest states have masses of $M^2_\infty=8.93$ and $M^2_\infty=9.60$. The structure functions for this sector are similar to those for the Z_2-even states. The structure functions for the state with $M^2_\infty=8.93$ are shown in Fig. 15 and are very similar to the structure functions for the state with $M^2_\infty=9.03$ in the Z_2-even sector.

IV. DISCUSSION

In this paper we studied $\mathcal{N}=1$ SYM-CS theory with fundamental matter in $1+1$ dimensions. The CS term was included to give masses to the adjoint partons. The calculations were performed at large N_c in the framework of SDLCQ; namely, we compactified the light-like coordinate x^- on a finite circle and calculated the Hamiltonian as the square of a supercharge Q^-, which we then diagonalized numerically. We found that the spectrum of this theory has two bands, a lower mass band and an upper mass band. With a CS term present, we found that these bands separate for $g^2N_c/\pi=\kappa^2$, and we can easily study them separately.

For very massive adjoint partons, the lower mass band becomes a set of massless bound states composed of only fundamental partons. When the mass for the adjoint partons

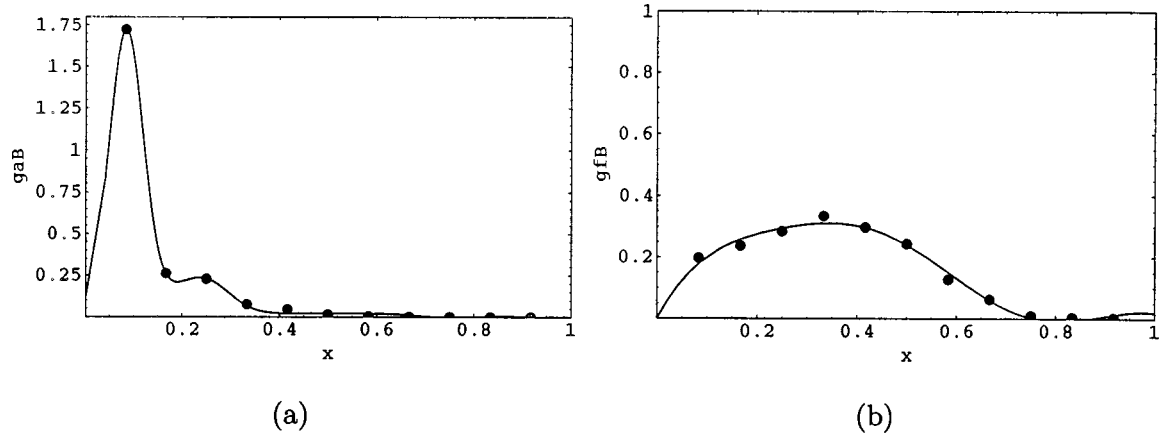

FIG. 9. Same as Fig. 8, but for $M^2=0.29$.

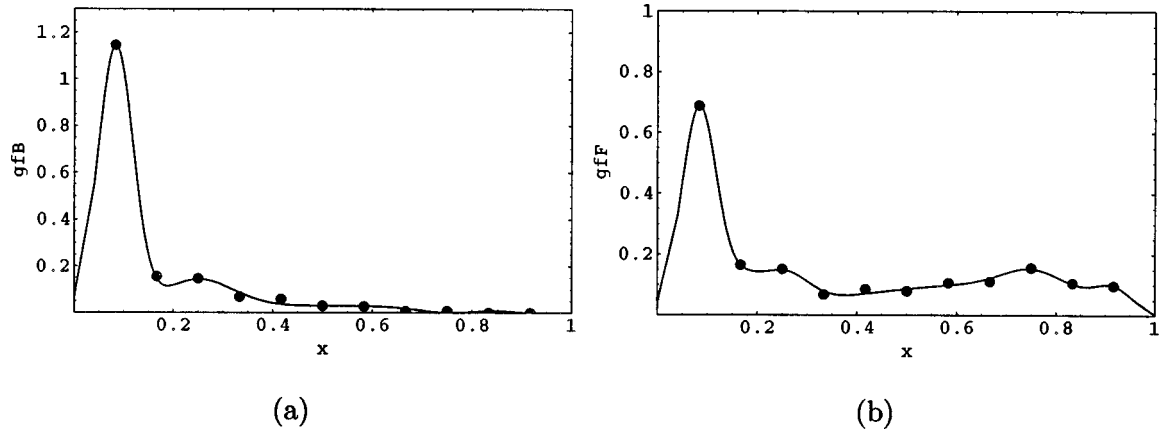

FIG. 10. Structure functions of the bound state $M^2=0.135$ at resolution $K=12$, with Z_2 odd, for (a) fundamental bosons *fB* and (b) fundamental fermions *fF*, with the CS coupling fixed at $\kappa=g\sqrt{N_c/\pi}$. The solid curve is a fit to the computed points.

is reduced, the massless states become the states of the lower mass band. The states in the low-mass band are unusual in that their convergence is oscillatory. We show that the upper and lower bounds of the low-mass band have the same shape as the DLCQ approximation to the two-particle free spectrum, which has an oscillatory behavior for its lower bound and a growing upper bound. We argue therefore that the oscillatory behavior is a numerical remnant of the free two-particle spectrum. In previous work [13] we found that at $g^2N_c/\pi=\kappa^2$ the lowest mass state of this theory was anomalously low, and in fact close to zero, while the threshold for this state is at $M^2=1$. On average this state has one massive gluon and two squarks. This is the fifth supersymmetric theory where we have found anomalously light states. In SYM theory in $1+1$ [6] and $2+1$ [23] dimensions, there are massless BPS states. In SYM-CS theory in $1+1$ dimensions we saw that at strong coupling the lightest states are approximately BPS states whose masses are independent of the YM coupling [2]. In SYM-CS theories in $2+1$ dimensions at strong coupling, there is again an anomalously light bound state [3]. The next lightest state in this band has about four partons. We carry the calculation to resolution $K=13$, so in principle these states could contain 13 partons. In addition, almost all of the partons in the adjoint representation are gluons rather than gluinos. Also, the content of the particles in the fundamental representation is rather pure, either squarks or quarks. Thus some of the boson bound states are rather pure combinations of quarks and glue, in spite of the fact that they are bound states of an exactly supersymmetric theory.

The bound states in the upper mass band are rather conventional SDLCQ bound states. They converge very quickly and have the conventional linear behavior in $1/K$. Similar to the lower mass band, most of the partons in the adjoint representation are gluons. In addition, the states are again rather pure quark or squark states. When we take the adjoint parton mass large, these are the states that have at least one adjoint parton.

We calculate the structure functions of the states in both bands for all four species of partons, and we find that most of the distributions are strongly peaked at x near zero. This is an indication that for $g^2N_c/\pi=\kappa^2$ these are strongly coupled

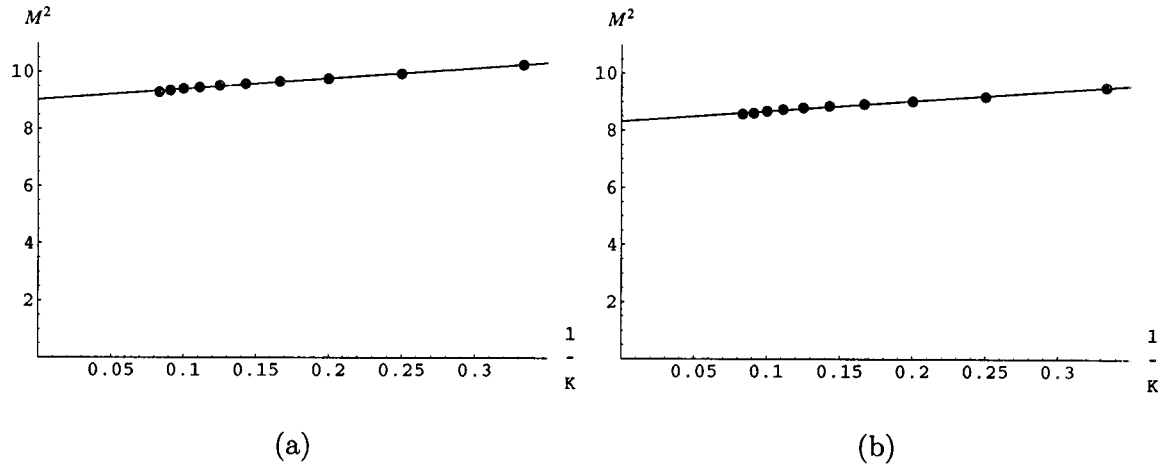

FIG. 11. The mass squared in units of g^2N_c/π, as a function of $1/K$ for $\kappa=g\sqrt{N_c/\pi}$ in the Z_2-even sector, of (a) the second lowest mass state of the upper band, with $M^2_\infty=9.03$, and (b) the lowest mass state of the upper band, with $M^2_\infty=8.31$. The solid curve is a fit to the computed points.

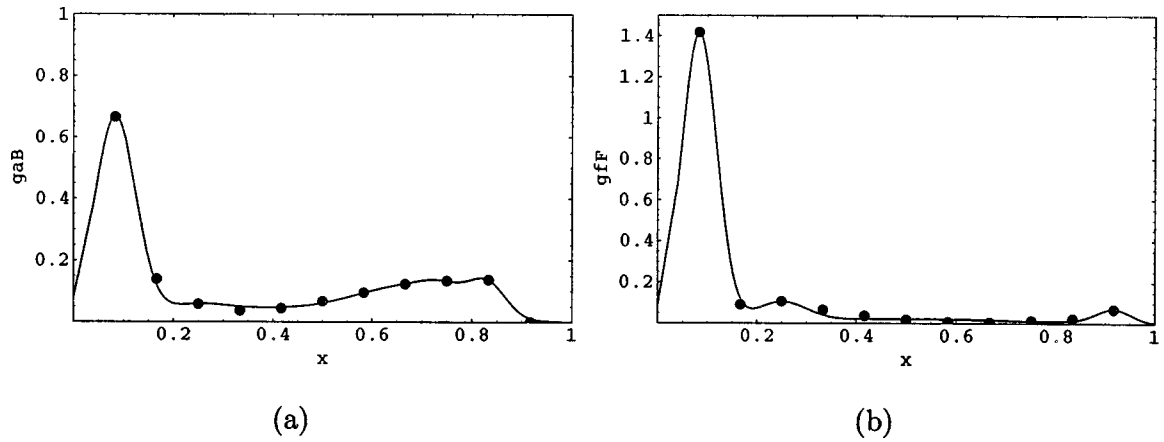

FIG. 12. Structure functions of the bound state $M_\infty^2=8.31$, with Z_2 even, for (a) adjoint bosons *aB* and (b) fundamental fermions *fF*, with the CS coupling fixed at $\kappa=g\sqrt{N_c/\pi}$. The solid curve is a fit to the computed points.

bound states. In a weakly coupled bound state one would, on average, expect the partons to share the momentum fraction equally, and therefore the structure function would be peaked at 1/3 for a state with three partons, for example. Since several of the states we consider are almost pure quark-gluon bound states, they might not be affected by supersymmetry breaking, which would give large mass to the squarks and gluinos. It would be interesting to investigate in more detail the properties of these states and compare them with QCD.

There remains a considerable amount of work to be done on SYM-CS theories with fundamental matter. The most straightforward extension of the present work is to consider calculations in 2+1 dimensions [20,23,24]. The $\mathcal{N}=1$ theory in 2+1 dimensions is easily within our reach. Beyond that the $\mathcal{N}=2$ theory [25] in 2+1 dimensions, which is the dimensional reduction of the $\mathcal{N}=1$ theory in 3+1 dimensions,

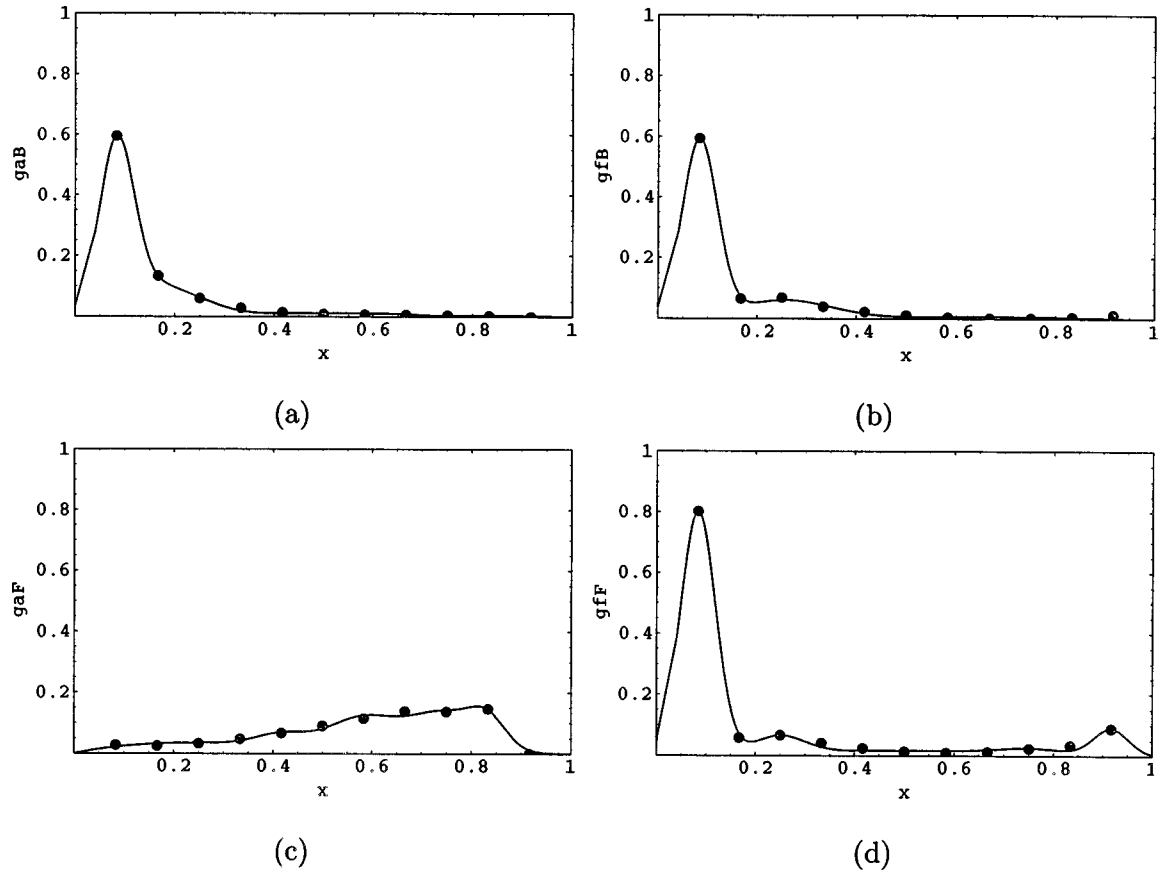

FIG. 13. Structure functions of the bound state $M_\infty^2=9.03$ for (a) adjoint bosons *aB*, (b) fundamental bosons *fB*, (c) adjoint fermions *aF*, and (d) fundamental fermions *fF*, with the CS coupling fixed at $\kappa=g\sqrt{N_c/\pi}$ and Z_2 even. The solid curve is a fit to the computed points.

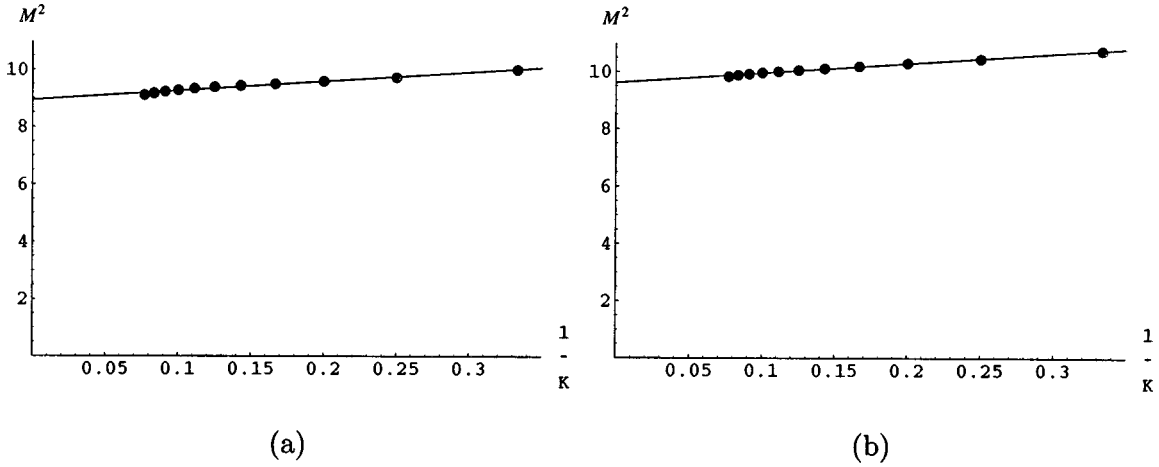

FIG. 14. The mass squared in units of g^2N_c/π, as a function of $1/K$ for $\kappa=g\sqrt{N_c/\pi}$ in the Z_2-odd sector, for (a) the lowest mass state of the upper band, with $M_\infty^2=8.93$, and (b) the second lowest mass state of the upper band, with $M_\infty^2=9.6$. The solid curve is a fit to the computed points.

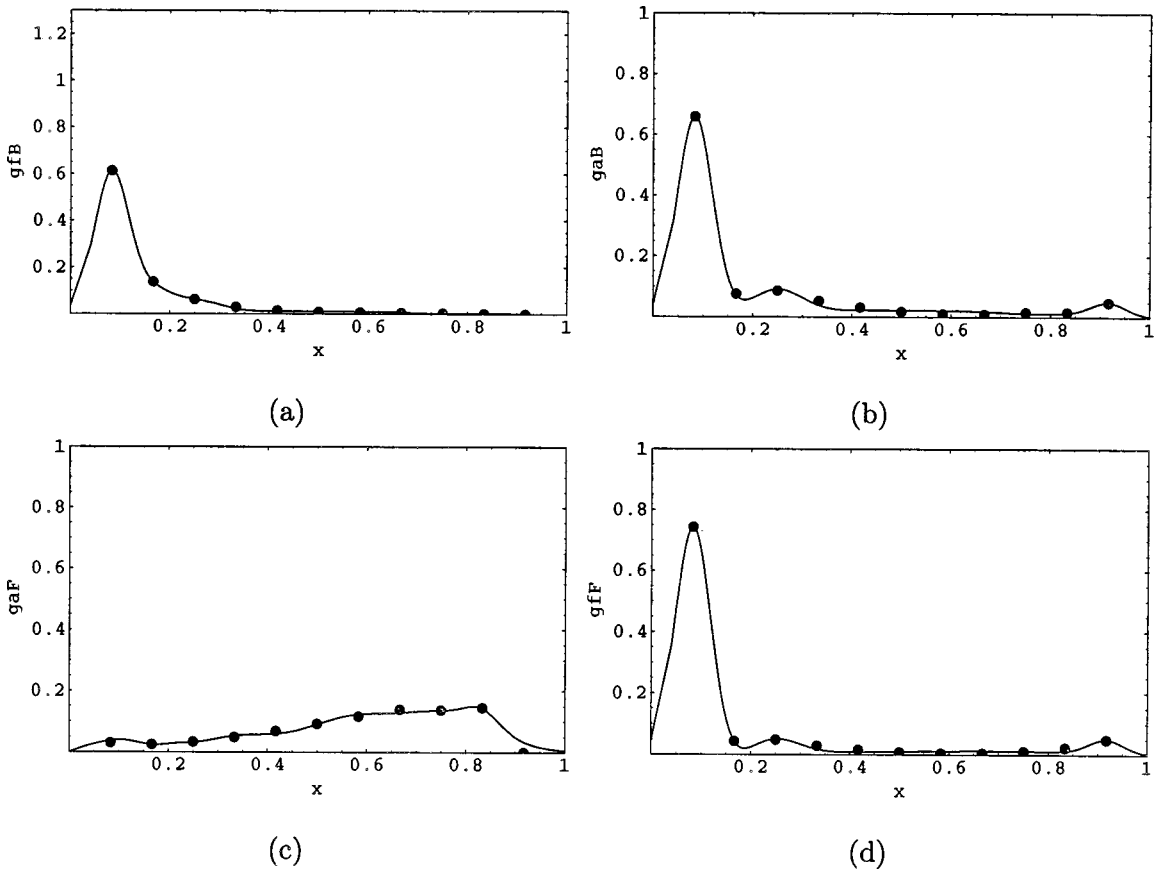

FIG. 15. Same as Fig. 13, but for $M_\infty^2=8.93$ and Z_2 odd.

will be very interesting. For finite N_c we could also look at the mixing of closed-loop states and finite string-like meson states. For $\kappa = 0$ the bound states of all these theories provide an interesting field-theoretic model for strings.

ACKNOWLEDGMENT

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