Color SU(3) symmetry, confinement, stability, and clustering in the $q^2\overline{q}^2$ system

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We examine the assumptions underlying the (model-dependent) predictions of $q^2\overline{q}^2$ or *tetraquark* spectra. The models implemented so far have used only two-body interactions proportional to the color charges; that assumption is the source of many serious shortcomings. We extend the analysis to three- and four-body interactions based on color SU(3) algebra, while including all relevant information one has about three-quark forces from lattice QCD. Thus we find that (quasi)stable tetraquarks are not necessarily a consequence of color SU(3) dynamics, let alone of QCD. We make this statement and the conditions under which it holds more precise in the text. In the process we are led to a set of sufficient conditions for a mathematical description of the hadronic world as we know it, i.e., of baryons and $q\overline{q}$ mesons, without going into the question of tetraquark existence. These conditions are as follows. (1) Stability: All the (colorless and colored) states' energies must be bounded from below. (2) Confinement: A color singlet $q\overline{q}$ potential energy must (infinitely) rise with the separation distance. (3) Color ordering: Colored states must be heavier than color-neutral ones. (4) Clustering: Any multiquark color-singlet state Hamiltonian must turn into a sum of three-quark (baryons) and quark-antiquark (mesons) cluster Hamiltonians, in the limit of asymptotically large separations. We discuss the consistency of these four requirements with color SU(3) symmetry and with each other.

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I. INTRODUCTION

In few-quark systems with the number of constituent quarks (n_q) + the number of constituent antiquarks ($\overline{n}_q$) larger than 3 ($N=n_q+\overline{n}_q>3$) there are additional color singlets that have not been experimentally observed (as yet). The simplest such system is made of two quarks and two antiquarks: the tetraquark ($q^2\overline{q}^2$). It can be in either of the two linearly independent, mutually orthogonal color singlet states: one that is a mere product of two ordinary meson $q\overline{q}$ color singlets, which we term the ‘‘two-meson state,’’ and another one that is a singlet combination of two $q\overline{q}$ color octets, which we call the ‘‘true tetraquark state.’’

The color SU(3) Yang-Mills (‘‘gauge’’) field dynamics, also known as quantum chromodynamics (QCD), has been proposed as the solution to all of the quark dynamical problems. The QCD equations of motion are nonlinear and strongly coupled, so no exact solution has been found to date. In the following we shall use only QCD's exact (‘‘unbroken’’) color SU(3) symmetry, which is beyond doubt, but not its equations of motion, to constrain and/or predict the properties of the mathematically allowed dynamical solutions. Thus all of our conclusions must also hold in QCD, though we shall not attempt to derive them explictly.

A number of (model-dependent) calculations, Refs. [1–4], have predicted numerous tetraquark resonances and even a few (quasi)stable tetraquarks, and these states have been experimentally sought for well over 20 years, but to no avail [5]. So it appears to be high time that one critically considers the assumptions under which tetraquark states exist. This paper is a continuation of the research started in Ref. [6] and extended to six-quark systems by Pepin and Stancu [7].

In this paper we examine the dynamical assumptions underlying various model calculations of tetraquark spectra in the literature and compare them with purely algebraic color SU(3) predictions or arguments, while including what little information one has about three-quark forces from lattice QCD [8–10]. Thus we find that (quasi)stable tetraquarks are not necessarily a consequence of color SU(3) dynamics, let alone of QCD. We shall make this statement and the conditions under which it holds, more precise in the text.

In the process we found certain generic difficulties in the (realistic) models based on color SU(3), and in QCD in particular, in meeting the four basic conditions of *rational color quark dynamics*. (2) Confinement: A color singlet $q\overline{q}$ potential energy that (linearly) rises at least up to the two-meson production threshold energy and probably also higher (to explain the Regge recurrences).[1] (3) Color ordering: Colored states must be heavier than color-neutral ones. (4) (Asymptotic) clustering: Any multiquark (more than three-quark) color singlet Hamiltonian must reduce to a sum of q^3 (baryon) and $q\overline{q}$ (meson) color singlet Hamiltonians in the limit of large interquark separations. This property sometimes goes by the name of ‘‘color saturation’’ and should not be confused with the stronger demand that there be no ‘‘strong van der Waals forces.’’ We add to these yet another demand that is so natural as to often be forgotten, and yet one that is not always met by color SU(3) models: (1) Stability: All the (colorless and colored) states' energies must be bounded from below.

We show that no color SU(3) symmetric interaction can satisfy these four requirements in the $q^2\overline{q}^2$ system. First we

[1]There is a school of thought that puts all the blame for the failure of quark potential models on the very concept of potential. This school would have us believe that all would be well if only one used string dynamics (in spite of the great difficulties of four-dimensional string theories). There is no evidence supporting that opinion, however.

show, under the assumption of color ordered stable two- and three-quark interactions, that at least some four-quark interaction is necessary to ensure clustering of the $q^2\overline{q}^2$ system into two $q\overline{q}$ mesons. The resulting Hamiltonian leads to asymptotic *anticonfinement* of the two color singlet $q\overline{q}$ states, however. And vice versa: if we adjust the free coupling parameters so as to avoid the anticonfinement problem, there can be no clustering. We compare this situation with the one in perturbative and lattice QCD and the $F_i \cdot F_j$ model, and find that the source of the problems lies either with the asymptotically rising confining potentials or with the color ordering.

We also find a correlation between colored state (dis)ordering and tetraquark binding: The lower the $q\overline{q}$ octet's energy is compared to that of the singlet, the more likely is the tetraquark to bind, at least in the (vector interaction) $F_i \cdot F_j$ model, whose stability is assured by Rosina's conjecture. We also show how the strength of the three-body force can be adjusted so as to bind or unbind the tetraquark, without changing the baryon energies.

This paper falls into five sections. After the Introduction, in Sec. II we give a reminder of the basic facts regarding the $q^2\overline{q}^2$ system's color SU(3) symmetry and clustering. Then in Sec. III we examine the predictions of the (standard) $F_i \cdot F_j$ model of quark interactions for tetraquarks. In Sec. IV, we look at the general two-, three-, and four-quark forces allowed by the color SU(3) symmetry. Then we show that at least some four-quark interactions are necessary to ensure clustering of the $q^2\overline{q}^2$ system into two $q\overline{q}$ mesons, though that also leads to a breakdown of confinement, and vice versa. There we compare our results with those of perturbative QCD. Finally, in Sec. V we draw our conclusions.

II. BASIC FACTS ABOUT THE $q^2\overline{q}^2$ SYSTEM

A. Color singlet states and their mixing

In the $q^2\overline{q}^2$ system, there are two linearly independent and mutually orthogonal color singlets. One can designate them, for example, according to their symmetry properties under interchange of the two quark or antiquark indices: one state ($|\mathbf{6}_{12}\overline{\mathbf{6}}_{34}\rangle$) is symmetric, another ($|\overline{\mathbf{3}}_{12}\mathbf{3}_{34}\rangle$) antisymmetric; see, e.g., Ref. [11]. The asymptotic "two-meson" color singlet state is a linear combination of the two:

$$|\mathbf{1}_{13}\mathbf{1}_{24}\rangle = \frac{1}{\sqrt{3}}|\overline{\mathbf{3}}_{12}\mathbf{3}_{34}\rangle + \sqrt{\frac{2}{3}}|\mathbf{6}_{12}\overline{\mathbf{6}}_{34}\rangle. \tag{1}$$

The indices 1,2 and 3,4 denote all other quantum numbers, such as flavor and spin, of the two quarks and antiquarks, respectively. Thus it ought to be clear that there is nothing special about the state $|\mathbf{1}_{13}\mathbf{1}_{24}\rangle$, one can equally well use the state

$$|\mathbf{1}_{14}\mathbf{1}_{23}\rangle = -\frac{1}{\sqrt{3}}|\overline{\mathbf{3}}_{12}\mathbf{3}_{34}\rangle + \sqrt{\frac{2}{3}}|\mathbf{6}_{12}\overline{\mathbf{6}}_{34}\rangle \tag{2}$$

in all subsequent developments. Clearly there is another linearly independent color singlet state, the

$$|\mathbf{8}_{13}\mathbf{8}_{24}\rangle = -\sqrt{\frac{2}{3}}|\overline{\mathbf{3}}_{12}\mathbf{3}_{34}\rangle + \frac{1}{\sqrt{3}}|\mathbf{6}_{12}\overline{\mathbf{6}}_{34}\rangle, \tag{3}$$

that is orthogonal to the first one, and similarly in the (14) basis

$$|\mathbf{8}_{14}\mathbf{8}_{23}\rangle = \sqrt{\frac{2}{3}}|\overline{\mathbf{3}}_{12}\mathbf{3}_{34}\rangle + \frac{1}{\sqrt{3}}|\mathbf{6}_{12}\overline{\mathbf{6}}_{34}\rangle. \tag{4}$$

The Pauli principle applies only to identical particles, i.e., it antisymmetrizes either only quark or only antiquark pairs ("diquarks" and "antidiquarks"), but not to the $q\overline{q}$ pairs. For this reason, the unphysical basis spanned by $|\overline{\mathbf{3}}_{12}\mathbf{3}_{34}\rangle$ and $|\mathbf{6}_{12}\overline{\mathbf{6}}_{34}\rangle$ is better suited to the application of the Pauli principle than the ("physical") asymptotic basis. The linear independence and orthogonality of the two color singlet representations, however, provide an additional permutation symmetry constraint, even on those pairs (such as $q\overline{q}$) to which the Pauli principle does not apply.

No signature other than the flavor structure is available to differentiate between "genuine tetraquark," or hidden-color states, Eq. (3) and "accidental" resonances in ordinary "two-meson" states, Eq. (1). The mixing of these two classes of states, if it exists, affects the observables' expectation values: If the Hamiltonian does not connect (mix) these two flavor subspaces of Hilbert space, then the "genuine tetraquark" class of states remains unobservable in experiments based on "ordinary meson" states, such as elastic meson-meson scattering. Such a "decoupling" of the hidden color state corresponds precisely to the (phenomenological) Freund-Waltz-Rosner (duality) "rule" [12]. Therefore we must carefully reconsider mixing of the two color singlets.

1. Color, spin, and flavor mixing

First, we remember that there is one mundane source of mixing: symmetry breaking. The color and angular momentum are good symmetries, of course, but flavor is not. When a symmetry is broken, say SU(N_f), the physical states are generally mixtures of two or more submultiplets of the broken symmetry multiplets that are eigenstates of the good residual symmetry, e.g., SU(N_f-1). Thus, there will surely be some mixing among the flavor multiplets belonging to either of the two color singlets, but not among the flavor states belonging to different color singlets, as the quark mass Hamiltonian that breaks the flavor symmetry in QCD does not depend on color.[2] This means that the physical (mixed) states corresponding to two color singlets still belong to different permutation symmetry classes even after flavor symmetry breaking induced mixing.

[2]Of course, flavor symmetry might also be broken by secondary effects such as the strong hyperfine interaction, which does depend on color.

The number of states and their quantum numbers do not change under the mixing, but their energies (masses) generally do. Moreover, the mixed states are orthogonal and may be subject to special selection rules. Quantum mechanical mixing of the two color singlet states is allowed if (a) the color-dependent interaction Hamiltonian (the potential V) connects the two states

$$\langle \mathbf{8}_{13}\mathbf{8}_{24}|V|\mathbf{1}_{13}\mathbf{1}_{24}\rangle_C \neq 0,$$

and one of the following conditions holds: either (b) all other quantum numbers (flavor, spin) of the two states are identical, or (c) the mixing potential $V=V_C V_F V_S$ also depends on flavor and/or spin, such that the corresponding flavor matrix element does not vanish either. The Pauli principle for quarks or antiquarks and the orthogonality of the two overall color singlet wave functions demand that the flavor and/or spin wave functions of multiplets belonging to two color singlets be orthogonal to each other, as well. This means that the mixing Hamiltonian has to depend on both the color and the flavor (or spin) in such a way that it connects different permutation symmetry wave functions,[3] and yet preserves the color (and preferably also flavor) symmetry, if it is to yield mixing. That is a powerful constraint that prevents the two-color-singlet representation from mixing in the exact flavor symmetry limit and thus precludes the observability of genuine tetraquarks in elastic meson scattering phase shifts for the most commonly used two-quark interactions, as these are either only color [1–3] or only flavor dependent [13].

2. The color exchange or $F_i \cdot F_j$ interaction

Now, the so-called $F_i \cdot F_j$ color dependent two-quark interaction

$$V_{ij} = F_i \cdot F_j \mathcal{V}_{ij}, \tag{5}$$

leads to mixing of the two color singlets without breaking of the color SU(3) symmetry. In applications of this model there were basically two schools: (a) the MIT bag model [3], which dealt (mostly schematically) with consequences of the Breit interaction, Eq. (8), between relativistic quarks confined in a spherical bag; and (b) the nonrelativistic constituent quark model [1,2,4], which assumes a confining two-body potential $\mathcal{V}_{12}^{\mathrm{conf}}$, usually the harmonic oscillator, or the linear one

$$-\mathcal{V}_{12}^{\mathrm{conf}} = \begin{cases} \frac{1}{2} m\omega^2 (\mathbf{r}_1 - \mathbf{r}_2)^2, \\ \lambda |\mathbf{r}_1 - \mathbf{r}_2| \end{cases} \tag{6}$$

The ‘‘realistic’’ potential consists of the linear + Coulomb + constant + Breit (see below) terms [14]

$$-\mathcal{V}_{12}^{\mathrm{real}} = -\frac{\alpha_C}{r_{12}} + \lambda r_{12} + \Lambda + \mathcal{V}_{12}^{\mathrm{Breit}}, \tag{7}$$

where α_C is the strong fine structure constant. Note the overall minus sign that cancels the negative sign of the color factor in the color singlet $q\bar{q}$, and the $\bar{\mathbf{3}}$ diquark state.[4]

The color singlet mixing is due to the fact that the color factor $F_i \cdot F_j$ is a part of the $i \leftrightarrow j$ two-quark color *exchange* operator P_{ij}^C.[5] That does not mean that color SU(3) symmetry is broken, however.

The color exchange nature of the confining interaction, however, is not enough: one must have simultaneous color and flavor or spin dependence. Most of the two-quark potentials are assumed to be flavor and spin independent. One exception is the ‘‘strong hyperfine’’ (Breit) interaction (which does not confine, however)

$$\mathcal{V}_{12}^{\mathrm{Breit}} = -\frac{\kappa}{m_1 m_2} (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) \delta(\mathbf{r}_1 - \mathbf{r}_2), \tag{8}$$

which has explicit color and spin (here $\boldsymbol{\sigma}_i$ are the Pauli matrices and κ is a constant proportional to α_C) exchange dependence and implicit (by way of quark masses) flavor dependence. The Breit interaction is a standard part of the (higher order in v/c) nonrelativistic reduction of the Lorentz vector two-body potential, i.e., of the one-gluon exchange potential. Jaffe [3] has emphasized the Breit interaction as an important force in hadron spectroscopy [SU(6) symmetry breaking] and as the source of the attraction that lowers the mass of a scalar tetraquark flavor nonet (under the tacit assumption of a color independent confining potential, which, however, makes the subsequent predictions unrealistic). The detailed calculations of Ref. [4] do not agree with the schematic results of Jaffe. Moreover, the calculations of the tetraquark energy in Ref. [4] were done in an unphysical color basis without considering color mixing, and no attempt had been made to calculate the meson-meson scattering matrix. Thus, it is not clear if the tetraquark states calculated there are observable or not.

Proper asymptotic behavior of the $q^2\bar{q}^2$ system imposes an additional ‘‘clustering’’ condition on its Hamiltonian that has also gone largely unnoticed.

B. Clustering in the $q^2\bar{q}^2$ system

Technically, in this case clustering[6] means that the ‘‘two-meson’’ color singlet state potential must be equal to the sum of two-body potentials in the two separate mesons in the limit of asymptotically large cluster center-of-mass (c.m.) separations $\Delta R = |\Delta \mathbf{R}| = \frac{1}{2}|\mathbf{r}_1 + \mathbf{r}_3 - \mathbf{r}_2 - \mathbf{r}_4|$:

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[3]One may say that it is a permutation symmetry changing or an ‘‘exchange’’ Hamiltonian.

[4]The constant term Λ has an interesting role: it effectively changes the total mass of the hadron (or the gravitational mass of the constituent quark, but not its inertial mass) in different color states. For example, a negative Λ lowers the color singlet $q\bar{q}$ mass and increases the color octet one; and similarly for q^3 states.

[5]One can also construct three-body color exchange operators from SU(3) invariant products of three-quark color charge matrices [15].

[6]This property also goes by the name of ‘‘color saturation,’’ for historical reasons, named after similarities with the saturation of the nucleon interactions.

$$\lim_{\Delta R\to\infty}(\langle V\rangle_{11}\equiv\langle \mathbf{1}_{13}\mathbf{1}_{24}|V|\mathbf{1}_{13}\mathbf{1}_{24}\rangle)$$
$$=\langle \mathbf{1}_{13}|V_{13}|\mathbf{1}_{13}\rangle+\langle \mathbf{1}_{24}|V_{24}|\mathbf{1}_{24}\rangle \tag{9}$$

where

$$V=V^{2b}+V^{3b}+V^{4b} \tag{10}$$

$$V^{2b}=\sum_{i<j}^{4}V_{ij}, \tag{11}$$

$$V^{3b}=\sum_{i<j<k}^{4}V_{ijk}, \tag{12}$$

$$V^{4b}=V_{1234}. \tag{13}$$

In order to verify clustering Eq. (9) in QCD, one must know the exact forms of the two-, three-, and four-body potentials. That is impossible at this stage, either empirically or theoretically. In Ref. [6] we made some *Ansätze* for the two- and three-quark potentials, and constrained them by the requirements of confinement, stability and proper color ordering in the $q\bar{q}$ and q^3 systems. That case will be discussed in Sec. IV, but first we look at the $F_i\cdot F_j$ model, which obeys clustering.

III. THE $F_i\cdot F_j$ MODEL IN THE $q^2\bar{q}^2$ SYSTEM

Several simple models of the quark color SU(3) dynamics have been used so far. They are all variations of the $F_i\cdot F_j$ model, defined by Eq. (5). Many light unstable resonant states and several (almost) stable heavy tetraquarks have been predicted in the MIT and the constituent quark versions of this model. Very few experimental candidates for the alleged tetraquarks have appeared to date, and they are all light $[f_0,a_0(980)]$ and nonexotic. The experimental hunt for exotic heavy tetraquarks began in the mid 1990s [16] at FERMILAB and will continue at CERN, but with no apparent success thus far. So it appears that this model is in conflict with the paucity of observed states, i.e., we may have to look for an additional selection rule or a new dynamical principle that forbids the tetraquark state.

A. Advantages and disadvantages of the $F_i\cdot F_j$ model

The two main advantages of the $F_i\cdot F_j$ model are the following.

(1) It predicts stable, confined color singlet $q\bar{q}$ and q^3 states. In other words, it satisfies condition (2) in Sec. I.

(2) It leads to clustering of all hadronic states into color singlet q^3 and $q\bar{q}$ states. In other words, it satisfies condition (4) in Sec. I, or Eq. (9).

The $F_i\cdot F_j$ model is also a sentimental favorite of many a quark modeller due to the validity of the "$V_{qq}=\frac{1}{2}V_{q\bar{q}}$ rule." This "rule" cannot be directly checked, of course,[7] but can be related to the energy separations between the ground and excited states for mesons and baryons, under the assumption of (only) two-body interactions. The experimental data can be reproduced, however, with (a continuous infinity) of three-body potentials, so this property need not be considered as an advantage.

As already pointed out in Refs. [6,2,1] the $F_i\cdot F_j$ model suffers from a number of weaknesses.

(1) It violates condition (1) in Sec. I, i.e., it predicts unstable colored q^2 and $q\bar{q}$ states (this is the "color dissolution/anticonfinement" problem of Ref. [2]).

(2) It violates condition (3) in Sec. I, i.e., the colored states are not properly ordered; the octets are sometimes lighter than the singlet(s).

(3) It does not take advantage of the full mathematical range of the color symmetry: $SU_C(3)$ is a rank-2 Lie group, which means that it has two invariants, or Casimir operators. The $F_i\cdot F_j$ model uses only one invariant, however.

(4) It predicts hosts of new (as yet) unobserved states, the tetraquarks $q^2\bar{q}^2$ being just one example.

The standard "solution" to problem (2) in the literature, the assumption that only color singlet states exist, is entirely *ad hoc* and thus unsatisfactory. That can be remedied by invoking a special initial condition (a color-neutral Universe) and Rosina's conjecture [17] which ensures the stability of all the (color singlet) states that can be produced or otherwise reached from this initial state, which is valid for a certain class of two-body potentials (to be specified below). Problem (3) has been solved in Ref. [6] by adding a color-independent two-quark force and a three-quark force, i.e., by extending the model. That procedure violates the clustering condition, however, as will be shown in Sec. IV. It is important to note here that adding a three-quark force cannot solve all the problems: color octet $q\bar{q}$ states and color sextet diquarks are "anticonfined" in the $F_i\cdot F_j$ model and will remain so irrespective of any three-body force. The Rosina scenario has its own problems, too: it turns out to be in conflict with (1) the observed *LS* meson splitting, i.e., with the mass splitting between the scalar, axial-vector, and tensor mesons, and with (2) the existence of three-quark interactions.

B. Three-body potential

Lattice calculations [8–10] indicate the existence of a three-quark potential in the color singlet state. The three-quark potential can be factored into a color part $\mathcal{C}_{123}$ and a spin-spatial part $\mathcal{V}_{123}$:

$$V_{123}=\mathcal{C}_{123}\mathcal{V}_{123}. \tag{14}$$

As the lattice calculations have been done only in the (total) color singlet state, one cannot determine its color structure (except for the fact that it does not vanish in the said state). We shall use only one (of several possible; see Sec. IV below and Ref. [6]) color factor for the three-body potential that ensures clustering and thus can naturally be viewed as a part of the $F_i\cdot F_j$ model. It is

$$\mathcal{C}_{123}=c\,d^{abc}F_1^aF_2^bF_3^c, \tag{15}$$

[7] Potentials cannot be directly measured here.

TABLE I. Diagonal matrix elements of the three-body color operators for variously colored q^3 states. Of course, there are two distinct **8** states, but they are equivalent in this regard.

q^3 state	$\|\mathbf{1}\rangle$	$\|\mathbf{8}\rangle$	$\|\mathbf{10}\rangle$
$\langle \Sigma_{i<j}^3 F_i \cdot F_j \rangle$	-2	$-\frac{1}{2}$	1
$\langle d^{abc} F_1^a F_2^b F_3^c \rangle$	$\frac{10}{9}$	$-\frac{5}{36}$	$\frac{1}{9}$

where $F^a = \frac{1}{2}\lambda^a$ is the quark color charge matrix, the lower index indicates the number of the quark, λ^a are the Gell-Mann matrices, and f^{abc}, d^{abc} are the SU(3) structure constants, and an as yet undetermined strength c. The color factor Eq. (15) is an SU(3) invariant, i.e., it can be expressed in terms of Casimir operators as follows:

$$d^{abc} F_1^a F_2^b F_3^c = \frac{1}{6}\left[C^{(2)}_{1+2+3} - \frac{5}{2} C^{(1)}_{1+2+3} + \frac{20}{3} \right], \tag{16}$$

where $1+2+3$ stands for the (elastic) matrix element in the three-quark color state and the two Casimir operators of SU(3) are

$$C^{(1)} = F^a F^a \equiv F \cdot F \equiv F^2, \tag{17}$$

$$C^{(2)} = d^{abc} F^a F^b F^c. \tag{18}$$

Note that the color factor Eq. (16), depends on the cubic Casimir operator $C^{(2)}$. This leads to the results shown in Table I. For simplicity's sake, and in accord with some lattice results [8,10], we make the linearly additive *Ansatz*, i.e., we assume that the spatial part of the three-quark potential is the sum of the confining parts of the two-body potentials[8] (the Coulomb, the constant, and the Breit parts do not appear in the perturbative QCD three-quark potential; see Sec. IV E):

$$\mathcal{V}_{ijk} = \sum_{i<j}^{k'} \mathcal{V}_{ij}^{\text{conf}} \equiv \mathcal{V}_{ij}^{\text{conf}} + \mathcal{V}_{jk}^{\text{conf}} + \mathcal{V}_{ik}^{\text{conf}}. \tag{19}$$

This (''linear additivity'') assumption is necessary only when the spatial part of the three-body potential is confining and symmetric under permutations of the quark indices;[9] moreover, it is an insufficient condition for clustering without appropriate spatial and color dependencies of the four-quark potential (see Sec. IV C).

Before we can write down the Hamiltonians for the tetraquark system and then solve for their spectra we must look at the C-conjugation properties of these interactions.

[8]This assumption corresponds, perhaps a bit loosely, to the Δ *Ansatz* in the string picture of confinement. At present there is no consensus on the issue of the Δ versus the *Y Ansatz* on the lattice. The problem is made more difficult by similar functional dependencies of the two *Ansätze* in spatially symmetric configurations.

[9]There is no phenomenological reason to have a *confining* three-body potential at this time. Permutation symmetry of the three-body potential is even more difficult to ascertain on the basis of hadron data.

C. *C* conjugation and the Lorentz scalar vs vector potentials

As both the Lorentz scalar and the Lorentz vector two-body interactions reduce to the same form in the (lowest order) nonrelativistic limit, the distinction between them may seem an academic point. That is indeed so for interactions solely between quarks, or solely between antiquarks, but when it comes to quark-antiquark interactions, the vector and scalar interactions differ by an overall sign, i.e., if one is attractive, the other is repulsive. That is a consequence of the opposite C-conjugation properties of Lorentz scalars and Lorentz vectors. This leads to opposite signs in $q\bar{q}$ potentials: For scalar vertices

$$\bar{\mathcal{C}}_{12} = \begin{cases} -F_1 \cdot \bar{F}_2, \\ \bar{F}_1 \cdot \bar{F}_2, \end{cases} \tag{20}$$

whereas, for the vector ones,

$$\bar{\mathcal{C}}_{12} = \begin{cases} F_1 \cdot \bar{F}_2, \\ \bar{F}_1 \cdot \bar{F}_2, \end{cases} \tag{21}$$

where the antiquark color factor is defined by

$$\bar{F^a} = -\frac{1}{2}\boldsymbol{\lambda}^{aT} = -\frac{1}{2}\boldsymbol{\lambda}^{a*}. \tag{22}$$

Therefore, of course, the difference cannot be seen in systems made up entirely of (constituent) quarks, such as baryons. Nor can it be seen in the $q\bar{q}$ system alone, because the sign of this interaction can be fixed to agree with experiment. It is first in the tetraquark system that the distinction between scalar and vector interactions leads to dramatic differences.

We have discussed the importance of the Lorentz-scalar-like origin of the three-body interaction in Ref. [6]. There we also showed the explicit C-conjugation properties of the three-body force

$$\bar{\mathcal{C}}_{123} = \begin{cases} -d^{abc} F_1^a F_2^b \bar{F}_3^c, \\ d^{abc} F_1^a \bar{F}_2^b \bar{F}_3^c, \end{cases} \tag{23}$$

Thus we see that C conjugation is also important for the two-body force, and that yet another tacit assumption was made in previous tetraquark studies: that of Lorentz-vector-like behavior of the two-body force under C conjugation. We shall show that Rosina's conjecture does not hold for Lorentz scalar two-body interactions, whereas it does for Lorentz vector ones.

D. Stability and color ordering

A commonly neglected aspect of the colored quark model is the stability of the colored states. It was noticed in Ref. [2] that the color nonsinglet states have lower energy than the color singlet ones, or even that the nonsinglets are unstable. At first this problem was simply ignored with words to the effect that one assumes that only color singlets may exist. Rosina [17] made the first step toward a rational explanation, in that he showed that, for certain classes of power-law (lin-

TABLE II. Diagonal matrix elements of the three-body color operators for variously colored $q^2\bar{q}$ states.

$q^2\bar{q}$ state	$\|\mathbf{3}_a\rangle$	$\|\mathbf{3}_s\rangle$	$\|\overline{\mathbf{6}}\rangle$	$\|\mathbf{15}\rangle$
$\langle\Sigma_{i<j}^3 F_i\cdot F_j\rangle$	$-\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
$\langle d^{abc}F_1^aF_2^bF_3^c\rangle$	$\frac{5}{9}$	$-\frac{5}{18}$	$-\frac{5}{18}$	$\frac{1}{18}$

ear and square) interactions, all color singlets ought to be stable. For example, even though one color octet $q\bar{q}$ pair is unstable, two such pairs in the total color singlet state are stable due to their mutual interactions. If the initial state is a singlet then, by exact color SU(3) conservation all subsequent states must also be color singlets and, according to Rosina's conjecture, stable. Thus, in Rosina's scenario the stability problem has been turned into an initial condition one: the basic question then becomes why was the Universe created in a color-neutral state?

We shall show, however, that there are several tacit assumptions underlying this conjecture that spoil its general validity: (i) only two-body interactions are assumed; (ii) these two-body interactions are assumed to be of the Lorentz vector type; (iii) no constituent gluons are allowed.

The color ordering and stability problems have not been solved in the Rosina scenario, but only pushed under the rug. The nonzero color quark states may still be lighter than the singlets, or even unstable, although apparently inaccessible from this Universe. There is one possible caveat to this unobservability of colored quark states, however: if one allows for the existence of constituent gluons, then the quarks can be in a color octet state and the ensuing instability may prove fatal. Moreover, we shall show that the ''saturating'' three-body force may violate Rosina's conjecture.

1. The $q^2\bar{q}^2$ Hamiltonian

Using Table II we find the following color singlet diagonal and off-diagonal potentials in the $q^2\bar{q}^2$ system with Lorentz vector two-body interactions and Eq. (15) three-body potential color factor:

$$V_{\bar{\mathbf{3}}\mathbf{3}}=-\frac{1}{3}[\mathcal{V}_{13}+\mathcal{V}_{14}+\mathcal{V}_{23}+\mathcal{V}_{24}]-\frac{2}{3}[\mathcal{V}_{12}+\mathcal{V}_{34}]-\frac{5}{9}c[\mathcal{V}_{123}+\mathcal{V}_{134}+\mathcal{V}_{234}+\mathcal{V}_{124}], \tag{24}$$

$$V_{\mathbf{6}\bar{\mathbf{6}}}=-\frac{5}{6}[\mathcal{V}_{13}+\mathcal{V}_{14}+\mathcal{V}_{23}+\mathcal{V}_{24}]+\frac{1}{3}[\mathcal{V}_{12}+\mathcal{V}_{34}]+\frac{5}{18}c[\mathcal{V}_{123}+\mathcal{V}_{134}+\mathcal{V}_{234}+\mathcal{V}_{124}], \tag{25}$$

$$V_{\mathbf{36}}=-\frac{1}{\sqrt{2}}[\mathcal{V}_{13}+\mathcal{V}_{24}-\mathcal{V}_{23}-\mathcal{V}_{14}]. \tag{26}$$

For Lorentz scalar interactions flip the sign of the $q\bar{q}$ terms, i.e., of the $[\mathcal{V}_{13}\pm\mathcal{V}_{14}\pm\mathcal{V}_{23}+\mathcal{V}_{24}]$ terms. Now use Eq. (1) to find

$$\begin{aligned}\langle V\rangle_{11}&=\langle\mathbf{1}_{13}\mathbf{1}_{24}|V|\mathbf{1}_{13}\mathbf{1}_{24}\rangle\\&=\frac{1}{3}(V_{\bar{\mathbf{3}}\mathbf{3}}+2V_{\mathbf{6}\bar{\mathbf{6}}}+2\sqrt{2}V_{\mathbf{36}})\\&=-\frac{4}{3}(\mathcal{V}_{13}+\mathcal{V}_{24})\\&=\langle\mathbf{1}_{13}|V_{13}|\mathbf{1}_{13}\rangle+\langle\mathbf{1}_{24}|V_{24}|\mathbf{1}_{24}\rangle,\end{aligned} \tag{27}$$

which proves clustering in this model. The factor $-\frac{4}{3}$ is just the value of the color factors $F_i\cdot\overline{F}_j$, for $(i=1,j=3)$ and $(i=2,j=4)$ pairs, in their respective color singlet states. Together with the overall minus sign in the confining potential Eq. (6); this yields positive confining two-body potentials for the two $q\bar{q}$ pairs in accord with Rosina's conjecture.

Similarly, for the ''hidden-color'' state

$$\begin{aligned}\langle V\rangle_{88}&=\langle\mathbf{8}_{13}\mathbf{8}_{24}|V|\mathbf{8}_{13}\mathbf{8}_{24}\rangle\\&=\frac{1}{3}(2V_{\bar{\mathbf{3}}\mathbf{3}}+V_{\mathbf{6}\bar{\mathbf{6}}}-2\sqrt{2}V_{\mathbf{36}})\\&=\frac{1}{6}(\mathcal{V}_{13}+\mathcal{V}_{24})-\frac{7}{6}(\mathcal{V}_{14}+\mathcal{V}_{23})-\frac{1}{3}(\mathcal{V}_{12}+\mathcal{V}_{34})\\&\quad-\frac{5}{18}c[\mathcal{V}_{123}+\mathcal{V}_{134}+\mathcal{V}_{234}+\mathcal{V}_{124}],\end{aligned} \tag{28}$$

which is directly affected by the three-body force. The coupling constant c is free, except for the stability requirements discussed below, but could be fixed on the lattice.[10]

Assuming stability of the two-body Hamiltonian (for a critical discussion see Sec. III D 2) and the additive *Ansatz* Eq. (19) for the three-body part, we may read off the necessary, though perhaps insufficient, conditions for stability of the two- plus three-body Hamiltonian as

$$c>-\frac{3}{5}, \tag{29}$$

$$c>-\frac{21}{10}, \tag{30}$$

$$c>\frac{3}{10}. \tag{31}$$

Note that these three inequalities are not in conflict, as was the case in the q^3 system [6]; they are all satisfied when inequality (31) is satisfied.

[10] Indeed, there are a couple of recent three-body potential lattice calculations [9,10] but they were done only in the color singlet state, however. This means that the color dependence of the force is undetermined as of now. This does not prevent one from assuming the color dependence of Eq. (15) as a first guess.

2. Stability of the two-body Hamiltonian

Rosina [17] conjectured and showed that, under certain restrictions already discussed above, for the first and second power-law interactions, all color singlet (confining) two-body potentials ought to be positive semidefinite. We shall check Rosina's conjecture explicitly for quadratic (harmonic oscillator) potentials in both color singlets ($|\mathbf{6}_{12}\overline{\mathbf{6}}_{34}\rangle$ and $|\overline{\mathbf{3}}_{12}\mathbf{3}_{34}\rangle$). It is best to go to the center-of-mass and relative (Jacobi) coordinates $\boldsymbol{\sigma},\boldsymbol{\sigma}',\boldsymbol{\lambda}$ (Ref. [11]) defined by

$$\mathbf{r}_{13}=\boldsymbol{\lambda}+\frac{1}{\sqrt{2}}(\boldsymbol{\sigma}-\boldsymbol{\sigma}'),\tag{32}$$

$$\mathbf{r}_{14}=\boldsymbol{\lambda}+\frac{1}{\sqrt{2}}(\boldsymbol{\sigma}+\boldsymbol{\sigma}'),\tag{33}$$

$$\mathbf{r}_{23}=\boldsymbol{\lambda}-\frac{1}{\sqrt{2}}(\boldsymbol{\sigma}+\boldsymbol{\sigma}'),\tag{34}$$

$$\mathbf{r}_{24}=\boldsymbol{\lambda}-\frac{1}{\sqrt{2}}(\boldsymbol{\sigma}-\boldsymbol{\sigma}'),\tag{35}$$

$$\mathbf{r}_{12}=\sqrt{2}\,\boldsymbol{\sigma},\tag{36}$$

$$\mathbf{r}_{34}=\sqrt{2}\,\boldsymbol{\sigma}'.\tag{37}$$

Thus we find the following (vector) potentials (remember that Rosina's conjecture holds only for two-body potentials):

$$V^{2b}_{\mathbf{33}}=\frac{1}{3}m\omega^2[3(\boldsymbol{\sigma}^2+\boldsymbol{\sigma}'^2)+2\boldsymbol{\lambda}^2]\geqslant 0,\tag{38}$$

$$V^{2b}_{\mathbf{66}}=\frac{1}{6}m\omega^2[3(\boldsymbol{\sigma}^2+\boldsymbol{\sigma}'^2)+10\boldsymbol{\lambda}^2]\geqslant 0,\tag{39}$$

$$V^{2b}_{\mathbf{36}}=-\sqrt{2}m\omega^2(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'),\tag{40}$$

from which we can see that both color singlet potentials are positive semidefinite. (Rosina's conjecture does not say anything about off-diagonal potentials.)

Let us now turn to the Lorentz scalar potentials, which are phenomenologically preferable to the vector ones due to the absence of LS coupling terms: Flipping the signs as described above we find

$$V^{2b}_{\mathbf{33}}=\frac{1}{3}m\omega^2[(\boldsymbol{\sigma}^2+\boldsymbol{\sigma}'^2)-2\boldsymbol{\lambda}^2],\tag{41}$$

$$V^{2b}_{\mathbf{66}}=-\frac{1}{6}m\omega^2[7(\boldsymbol{\sigma}^2+\boldsymbol{\sigma}'^2)+10\boldsymbol{\lambda}^2]\leqslant 0,\tag{42}$$

$$V^{2b}_{\mathbf{36}}=\sqrt{2}m\omega^2(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'),\tag{43}$$

which clearly shows that scalar interactions do not obey Rosina's conjecture.

E. Discussion

Thus we conclude that scalar confining potentials are not allowed in Rosina's scenario in this model. Note that this is more than an academic point: the well-known problem of large LS coupling with vector potentials demands a Lorentz scalar confining two-body interaction [8].

Both the Lorentz vector and scalar $F_i\cdot F_j$ interactions induce a nonzero "permutation symmetry breaking parameter χ" introduced by Richard [18], as a measure of the likelihood of tetraquark binding. Only the Lorentz scalar, color-independent interaction leads to $\chi=0$, i.e., to no tetraquark attraction. As this specific kind of interaction is ruled out by the previous analysis, we are led to the conclusion that (some) attraction in the tetraquark channel is necessarily a consequence of the $F_i\cdot F_j$ two-body interaction, both vector and scalar. This fact can also be understood in the following way: as there is no color ordering in the $F_i\cdot F_j$ model, e.g., the color octet $q\overline{q}$ is lighter than the corresponding singlet, the second (tetraquark) color singlet that consists of two such light octets may have a mass that is comparable to or even smaller than the ordinary "asymptotic state" color singlet. This is an intuitive explanation of the connection between color ordering and (light) tetraquark binding.

It also ought to be clear from Eqs. (24),(25) that the three-body color singlet potentials can be of either sign, thus also potentially undermining Rosina's scenario for simultaneous confinement, stability, and clustering in the (Lorentz vector) $F_i\cdot F_j$ model extended to include color staturating three-quark forces. Thus we conclude that the class of two- and three-body potentials that automatically, i.e., by way of their color SU(3) structure, satisfy the clustering condition Eq. (9) does not necessarily also obey the stability, color ordering, and confinement postulates. Hence we shall look at the most general SU(3) symmetric case.

IV. THE GENERAL SU(3) SYMMETRIC INTERACTION IN THE $q^2\overline{q}^2$ SYSTEM

Hence we shall seek the most general quark dynamics that is consistent with the basic requirements (1–4) that lead to the solution of the confinement problem(s). We will have to limit ourselves to dynamics with a definite number of (constituent) quarks (in this case four), i.e., we do not allow for pair creation or annihilation, nor for constituent gluons. We consider the displacement of colored states to (arbitrarily) high energies or masses as a solution to the color confinement problem.

In a recent attempt to ensure correct color ordering and confinement of quarks with SU(3) symmetric color dynamics, we were forced to modify the usual $F_i\cdot F_j$ two-quark interaction and introduce a new three-quark one [6]. This new interaction ensures that the color singlets are the lowest energy states in both the $q\overline{q}$ and the q^3 systems in addition to confinement of these systems. In the $q^2\overline{q}^2$ system this three-quark force splits the energies of the two color singlet states, as it does in the q^6 system [7]. That is, however, not enough to make this dynamics viable: the dynamics has to allow for the observed clustering of quarks and antiquarks into mesons (and baryons) at asymptotic center-of-mass (c.m.) separations.

We shall start here the study of clustering in the simplest nontrivial system: $q^2\overline{q}^2$ ought to cluster into two $q\overline{q}$ mesons. Clustering is automatic with the $F_i \cdot F_j$ two-quark interaction, but the new color-independent two-body interaction is additive, i.e., it does not saturate, thus spoiling the clustering. The new three-quark interaction introduced in Ref. [6] does saturate; indeed it vanishes entirely in the two-meson color singlet state [7]. Thus, we must look for other ways to cancel the additive two-quark force in this channel. Several possibilities arise: (1) a nonsaturating three-quark force, which, however, spoils the good confinement properties of the q^3 system, or (2) a nonsaturating four-quark force. We shall focus here on the latter.

A. Clustering with general two- and three-body interactions

In Ref. [6] we made general *Ansätze* for the two- and three-quark potentials and constrained them by the requirements of stability, proper ordering, and confinement in the $q\overline{q}$ and q^3 systems.[11] Thus we found

$$V_{ij} = \sum_\alpha \mathcal{C}^\alpha_{ij}\mathcal{V}_{ij} = \left[c_1 + \frac{4}{3} + F_i \cdot F_j\right]\mathcal{V}_{ij}, \tag{44}$$

$$V_{ijk} = \sum_\alpha \mathcal{C}^\alpha_{ijk}\mathcal{V}_{ijk} = c d^{abc} F^a_i F^b_j F^c_k \mathcal{V}_{ijk}, \tag{45}$$

where c_1 and c are constants. Note that $\mathcal{V}_{ij}$ in Eq. (44) have the opposite sign to the ones in Eqs. (5),(6),(7). With the assumption Eq. (19) we find that the $F_i \cdot F_j$ model two-body interaction leads to the same form of the effective potential in the q^3 system as the three-body force with the analogous color factor. (This makes an unambiguous identification of the Δ three-quark force on the lattice particularly difficult.) Similar statements hold for the color-independent two- and three-body potentials. For this reason there is no need to introduce such two- and three-body potentials separately, but only one of a kind, i.e., only a two-body or only a three-body potential.

We have shown in Ref. [6] that a color-independent two-body potential is necessary for the absolute stabilization of both $q\overline{q}$ and q^3 spectra. For the above discussed reasons we shall not introduce a separate color-independent and $\sum_{i<j}^{k'} F_i \cdot F_j$ three-body potentials. If we further assume the harmonic oscillator potential *Ansatz* for $\mathcal{V}_{ij}$, Eq. (6), *but with opposite overall sign*, the coupling constants become constrained to $c_1 > 0$, usually taken as $c_1 = 1$ or $\frac{4}{3}$, and $\frac{2}{5} > c > -\frac{3}{2}$ for $c_1 = 1$. Straightforward evaluation of the two- and three-quark potential matrix elements yields

$$\langle V\rangle^{2b+3b}_{11} = \langle \mathbf{1}_{13}\mathbf{1}_{24}|V|\mathbf{1}_{13}\mathbf{1}_{24}\rangle$$
$$= \left(c_1 + \frac{4}{3}\right)\sum_{i<j}^{4} \mathcal{V}_{ij} - \frac{4}{3}(\mathcal{V}_{13} + \mathcal{V}_{24}). \tag{46}$$

This potential manifestly does not satisfy the clustering condition, Eq. (9), except when $c_1 = -\frac{4}{3}$, which case is explicitly excluded by the requirement of confinement in the $q\overline{q}$ sector. Thus we must conclude that either some modification of the three-quark potential, or a (new) four-quark potential is necessary. The former would spoil the confinement of the q^3 system (see Ref. [6]) so the latter is left as our only choice.

B. The four-quark potential

First we make a general SU(3) symmetric *Ansatz* for the four-quark potential. Then we will show that several kinds of four-quark force can lead to clustering of $q^2\overline{q}^2$, but always at the price of unconfining the asymptotic meson states.

The four-quark potential can be factored into a color part $\mathcal{C}_{1234}$ and a spin-spatial part $\mathcal{V}_{1234}$:

$$V_{1234} = \sum_\alpha \mathcal{C}^\alpha_{1234}\mathcal{V}_{1234}. \tag{47}$$

As we are primarily interested in the scalar channel ground state, i.e., in the static case, we may neglect the spin and momentum dependencies of the potential. We shall take only color factors $\mathcal{C}^\alpha_{1234}$ that are symmetric under the interchange of any pair of indices $i \leftrightarrow j$. Then the corresponding spin-spatial potentials $\mathcal{V}_{1234}$ must also be symmetric under the same interchange.[12] Then the following four-body SU(3) symmetric color factors can be written down:

$$\mathcal{C}_{1234} = \begin{cases} a_4 \sum_{i<j}^{4} F_i \cdot F_j, \\ b_4 \sum_{i<j<k}^{4} d^{abc} F^a_i F^b_j F^c_k, \\ c_4 \sum_{i<j<k<l}^{4} (F_i \cdot F_j)(F_k \cdot F_l), \\ d_4 \sum_{i<j<k<l}^{4} d^{abf} F^a_i F^b_j d^{cdf} F^c_k F^d_l, \end{cases} \tag{48}$$

[11] We confine ourselves to statics, so we may neglect momentum- and spin-dependent potentials. This does not represent a loss of generality as confinement is believed to be spin and momentum independent. The strong hyperfine (Breit) interaction cannot change our conclusions because (a) it is of short range, so it automatically clusters and does not confine; and (b) it is of the $F_i \cdot F_j$ type which (also) automatically clusters.

[12] This choice is sufficient, but not strictly necessary: only the complete potential V_{1234} has to be symmetric under such particle permutations. Thus, other types of "mixed symmetry" color factors and spin-spatial potentials are mathematically allowed. However, for spin- and momentum-independent potentials such mixed symmetry spin-spatial potentials vanish identically.

where $F^a=\frac{1}{2}\lambda^a$ is the quark color charge, the lower index indicates the number of the quark, λ^a are the Gell-Mann matrices, d^{abc} are the symmetric SU(3) structure constants defined by the anticommutators of the Gell-Mann matrices, and summation over repeated SU(3) indices is understood.

Only three of the four color factors in Eq. (48) are linearly independent, however, as the following identity holds:

$$\sum_{i<j<k<l}^{4} d^{abf}F_i^aF_j^bd^{cdf}F_k^cF_l^d=\frac{1}{3}\sum_{i<j<k<l}^{4}(F_i\cdot F_j)(F_k\cdot F_l). \tag{49}$$

For this reason we may set $d_4\equiv 0$ without loss of generality. The remaining three color operators can be expressed in terms of the two Casimir operators as follows:

$$\sum_{i<j}^{4} F_i\cdot F_j=\frac{1}{2}C^{(1)}_{1+2+3+4}-\frac{8}{3}, \tag{50}$$

$$\sum_{i<j<k}^{4} d^{abc}F_j^aF_j^bF_k^c=\frac{1}{6}\left[C^{(2)}_{1+2+3+4}-\frac{5}{2}C^{(1)}_{1+2+3+4}+\frac{80}{9}\right], \tag{51}$$

$$\sum_{i<j<k<l}^{4}(F_i\cdot F_j)(F_k\cdot F_l)=\frac{1}{8}(C^{(1)}_{1+2+3+4})^2-\frac{19}{24}C^{(1)}_{1+2+3+4}+\frac{10}{9}-\frac{1}{4}C^{(2)}_{1+2+3+4}, \tag{52}$$

where $1+2+3+4$ stands for the (total) color of the four-quark state and the two Casimir operators as defined by Eqs. (17) and (18). As discussed in Sec. IV A, one may set $b_4=0$ with impunity, because it essentially duplicates the three-body force contribution.

C. Clustering with the four-quark potential

Taking into account the C conjugation, we must use Eqs. (23),(23) in the definition of the color factor

$$\sum_{i<j<k}^{4}\overline{\mathcal{C}}_{ijk}=d^{abc}(F_1^a+F_2^a)\overline{\mathrm{F}}_3^b\overline{\mathrm{F}}_4^c-d^{abc}(\overline{\mathrm{F}}_3^a+\overline{\mathrm{F}}_4^a)F_1^bF_2^c. \tag{53}$$

Once again, we can express the three independent SU(3) invariant color factors in Eq. (48) in terms of the two Casimir operators. The first factor remains unchanged:

$$\sum_{i<j}^{4} F_i\cdot F_j=\frac{1}{2}C^{(1)}_{1+2+3+4}-\frac{8}{3}, \tag{54}$$

TABLE III. Diagonal matrix elements of the four-body color operators for the two distinct color singlet $q^2\overline{q}^2$ states.

$q^2\overline{q}^2$ state	$\Sigma_{i<j}^4 F_i\cdot F_j$	$\Sigma_{i<j<k}^4\overline{\mathcal{C}}_{ijk}$	$\Sigma_{i<j<k<l}^4(F_i\cdot F_j)(\overline{\mathrm{F}}_k\cdot\overline{\mathrm{F}}_l)$
$\lvert 1_{13}1_{24}\rangle$	$-\frac{8}{3}$	0	$\frac{20}{9}$
$\lvert 8_{13}8_{24}\rangle$	$-\frac{8}{3}$	$-\frac{10}{9}$	$\frac{35}{18}$

whereas the second one can be evaluated using Eqs. (23) and (25) in Ref. [6], and the third one[13] is

$$\begin{aligned}\sum_{i<j<k<l}^{4}\overline{\mathcal{C}}_{ijkl}&\equiv\sum_{i<j<k<l}^{4}(F_i\cdot F_j)(\overline{\mathrm{F}}_k\cdot\overline{\mathrm{F}}_l)\\&=\frac{1}{8}\left(C^{(1)}_{1+2+3+4}-\frac{16}{3}\right)^2+\frac{5}{24}(C^{(1)}_{1+2+3+4}\\&\quad-C^{(1)}_{1+2}-C^{(1)}_{3+4})+\frac{1}{2}\sum_{i<j<k}^{4}\overline{\mathcal{C}}_{ijk}\\&\quad-\frac{1}{6}\sum_{i<j}^{4}F_i\cdot F_j-\frac{2}{3},\end{aligned} \tag{55}$$

where $1+2+3+4$ stands for the (total) color of the four-quark state. This leads to the results shown in Table III, using which we find

$$\begin{aligned}\langle V\rangle_{11}&=\langle\mathbf{1}_{13}\mathbf{1}_{24}|V|\mathbf{1}_{13}\mathbf{1}_{24}\rangle\\&=\left(c_1+\frac{4}{3}\right)\sum_{i<j}^{4}\mathcal{V}_{ij}-\frac{4}{3}(\mathcal{V}_{13}+\mathcal{V}_{24})\\&\quad+\left(-\frac{8}{3}a_4+\frac{20}{9}c_4\right)\mathcal{V}_{1234}.\end{aligned} \tag{56}$$

Making the *Ansatz* $\mathcal{V}_{1234}=\Sigma_{i<j}^4\mathcal{V}_{ij}$, we find the saturation condition

$$c_1+\frac{4}{3}-\frac{8}{3}a_4+\frac{20}{9}c_4=0, \tag{57}$$

which is the principal result of this paper. Note, however, that in that case we are left with

$$\begin{aligned}\langle V\rangle_{11}&=-\frac{4}{3}(\mathcal{V}_{13}+\mathcal{V}_{24})\\&=-\frac{4}{3c_1}[\langle\mathbf{1}_{13}|V_{13}|\mathbf{1}_{13}\rangle+\langle\mathbf{1}_{24}|V_{24}|\mathbf{1}_{24}\rangle],\end{aligned} \tag{58}$$

the right hand side of which has the physically wrong negative sign, i.e., the two independent $q\overline{q}$ states are anticonfined.

[13]There is no difference between the Lorentz scalar and vector couplings here due to the even number of antiquarks.

D. Consequences

Note the consequences of Eq. (57).

(1) Some four-quark interaction is necessary to achieve clustering: one cannot satisfy Eq. (57) with $a_4 = c_4 = 0$, because $c_1 > 0$. Note that one may have exact cluster separation of the Hamiltonian, not only asymptotically, but at all distances. That, however, would also imply absence of interaction between the two $q\bar{q}$ clusters (mesons), except by way of quark exchange. One may, however, modify the $\mathcal{V}_{1234} = \Sigma_{i<j}^4 \mathcal{V}_{ij}$ *Ansatz* at short distances to introduce some meson-meson interaction without spoiling clustering.

(2) Of all the $q^2\overline{q}^2$ states the "two-meson" color singlet $|\mathbf{1}_{13}\mathbf{1}_{24}\rangle$ has the lowest energy. Unfortunately this state is also deconfined [due to the minus sign in Eq. (58)]: each of the two independent $q\bar{q}$ pairs is unbound in an "upside-down" confining (concave) two-body potential. This problem cannot be avoided: if we change the overall sign of the color-dependent two-body interaction, the color octet $q\bar{q}$ state becomes deconfined. Thus we have found a paradox: if both color singlet and octet $q\bar{q}$ pairs are to be confined by two-body forces, then two color singlet $q\bar{q}$ pairs are deconfined due to the influence of the four-quark force. If we eliminate the four-quark force, then the $q^2\overline{q}^2$ system, though confined, *cannot* cluster into two mesons. These constraints are only a consequence of the assumed SU(3) symmetry.

(3) Clearly, the clustering condition Eq. (56) is met by a continuous infinity of a_4, c_4 coefficients/four-body potentials. In order to narrow down this (theoretical) uncertainty one may play the same kind of game as with the three-quark potential: constrain the free parameters by demanding proper ordering of colored states. That procedure, however, *cannot* solve the problem in point 2, as that depends only on the two-quark interaction.

(4) Even if one had clustering in the $q^2\overline{q}^2$ system, that would not necessarily ensure the $q^4\bar{q} \to (q^3) + (q\bar{q})$ clustering, nor that of $q^6 \to (q^3) + (q^3)$. Thus we may have to consider the latter two cases separately and introduce a five- and a six-quark interaction to ensure clustering.

Our results are general, as they depend only on the assumption of exact color SU(3) symmetry and that quarks transform as the fundamental irreducible representation (**3**) of SU(3). Thus, our results must hold in all SU(3) symmetric theories, *inter alia* also in QCD, no matter what the spatial parts of the potentials may be. (The assumption of an additive spatial four-quark potential is necessary to achieve clustering. A similar *Ansatz* for the three-body potential is sufficient, though perhaps not necessary.) The conflict between clustering and confinement/stability found here was not expected, at least to the present author's knowledge. For this reason we wish to know how things stand in perturbative QCD (PQCD), in particular, if there is a similar conflict between clustering and stability/confinement.

E. Comparison with perturbative QCD

Of course, the tree-level PQCD two-body potential is just the Coulombic one. But, at the one-loop level new color-independent terms appear: after renormalization they lead to a (Lorentz scalar) r^{-3} potential. That is all we need to know for clustering purposes, as this term vanishes much more quickly than the Coulomb one in the $r \to \infty$ limit. Thus, clustering is maintained in one-loop PQCD.

As already announced, the Born approximation gluon-exchange graphs lead to the *momentum-dependent* three-body potential

$$\mathcal{V}_{ijk} = \alpha_S^2 \sum_{i<j}^{k'} \frac{\mathbf{v}_j \cdot \mathbf{v}_k}{m_i r_{ij} r_{ik}}, \tag{59}$$

where α_S is the QCD fine coupling constant,[14] with the color factor Eq. (15), which automatically leads to clustering, and $\mathbf{v}_i = \mathbf{p}_i / m_i$ is the ith quark's velocity; as well as to the three-body force

$$V_{ijk} = \frac{2}{3} \alpha_S^2 \sum_{i<j}^{k'} F_i \cdot F_j \frac{\mathbf{v}_j \cdot \mathbf{v}_k}{m_i r_{ij} r_{ik}}, \tag{60}$$

which also leads to clustering in the $q^2\overline{q}^2$ system, but not due to the SU(3) algebraic properties of its color factor. Rather, this term would cluster due to the (double) $1/r$ asymptotic behavior (vanishing) of the spatial part of the potential, even if it were not momentum dependent. Such strong r behavior of the potential is potentially dangerous, as it may lead to an instability of the Schrödinger equation, or "fall to the center" classical-mechanically. Fortunately, this potential vanishes altogether in static situations due to its velocity dependence.

Because of the r^{-3} behavior of the color-independent two-body potential and the momentum dependence of the three-body force, the PQCD two- and three-quark potentials lead to clustering (in the one-loop approximation). Here we can easily see that a straightforward extension to "confining" (infinitely rising) potentials is not possible, as the asymptotic behavior of the spatial part of the potential plays a crucial role in clustering.

Classically, the string model may solve our problems because the effective range of its interaction ("potential") extends only up to the string-breaking point, i.e., a (short) finite distance. Thus, the $\Delta R = |\mathbf{\Delta R}| = \frac{1}{2}|\mathbf{r}_1 + \mathbf{r}_3 - \mathbf{r}_2 - \mathbf{r}_4| \to \infty$ limit in the clustering condition Eq. (9) becomes trivial in the string model. A similar spatial "cutoff" principle, however, may be adopted in potential models, as well.

The common thread to both the potential and the string kinds of models is the SU(3) color symmetry: the quark potentials depend on functions of SU(3) generators, whereas the string dynamics depend on the "Chan-Paton" factors [19]. Thus, the energetics of multiquark states in both kinds of models depend crucially on the color SU(3) symmetry factors. Of course, the stability and color ordering problems may have been exacerbated in this way, for it is not physically clear what negative string tension would mean in the

[14] Note the violation of the linearly additive *Ansatz* Eq. (19) and the absence of the static three-body "Coulomb" potential.

TABLE IV. Table of validity of the four basic requirements (''axioms'') in models with various color-dependent forces. The asterisk on the $\times$ mark and the OK in parentheses in the first column, second row, indicate (conditional) stability of color singlets with Lorentz *vector* interactions in the $F_i \cdot F_j$ model due to Rosina's theorem (see text).

color dependence	Stability (**1**)	Color ordering (**2**)	Confinement (**3**)	Clustering (**4**)
constant	OK	$\times$	OK	$\times$
$F_i \cdot F_j$	$\times$* (OK)	$\times$	OK	OK
$F_i \cdot F_j$ + const=two-body	OK	OK	OK	$\times$
$F_i \cdot F_j$ + const two-+three-body	OK	OK	OK	$\times$
$F_i \cdot F_j$ + const two-+three-+four-body	$\times$	OK	OK	OK
PQCD	OK	OK	$\times$	OK

case when a Chan-Paton factor is negative. Another accompanying problem is that one must have a fully Lorentz covariant string theory to account for string breaking, i.e., for meson production. Note further that this must be a consistent *quantum* string theory, because the classical string breaks at the string breaking length with certitude, i.e., with unit probability. That implies that no radially excited state with radius larger than the string breaking length (or energy larger than the two-meson threshold) can exist in classical string theory. This conflict with experiment can be removed only by a consistent (unitary) relativistic quantum string theory, or quantum mechanics with confining potentials, for example, neither of which exists at the moment.

V. CONCLUSIONS

We have considered the stability, confinement, clustering, and SU(3) color state ordering in the simplest and extended color exchange ($F_i \cdot F_j$) model, in PQCD and in the general color SU(3) symmetric case. Most of the results in Sec. III and all the results in Sec. IV are new, so far as we know. We shall not repeat here the (partial) summaries given in Secs. III E and IV D, but briefly conclude that we have invariably found that at least one of these four simplest requirements is not satisfied by any confining color SU(3) symmetric Hamiltonian with a fixed number of quark (see Table IV). The deeper source of the problem appears to be the assumption of interminably rising confining potentials, as one can simultaneously satisfy the remaining three conditions with Coulomb-like few-body potentials (see PQCD).

We may view this paper as an attempt at establishing a rational color quark *dynamics*, by which we mean constructing a (classical) mechanical model based either on potentials (which case includes the cavity/MIT bag models), or on elastic string (later interpreted as ''flux tube'') dynamics with exact color SU(3) symmetry. We say classical mechanics here, though, of course, we wish to do quantum mechanics, because of the well known difficulties in quantizing string models; potential models should present few or no problems in this regard. Unfortunately, we saw that even the *statics* present some serious difficulties. Extension to relativistic dynamics appears to be necessary.

Clearly, new ideas and better input from lattice QCD are needed here. In particular, a conclusive study of ''Casimir scaling'' in the three-body sector on the lattice would clarify the color structure of the three-quark interaction in QCD.

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