

# Generalized parton distributions and double distributions for $q\bar{q}$ pions

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We consider two simple covariant models for pions (one with scalar and the other with spin- $\frac{1}{2}$  constituents). Pion generalized parton distributions are derived by integration over the light-cone energy. The model distributions are consistent with all known properties of generalized parton distributions, including positivity. We also construct corresponding double distributions by appealing to the Lorentz invariance of the form factor. These ostensibly constructed double distributions lead to incorrect generalized parton distributions that need not respect the positivity constraints. This inconsistency arises from the ambiguity inherent in defining double distributions in the standard one-component formalism (even in the absence of the Polyakov-Weiss term). We demonstrate that the correct model double distributions can be calculated from nondiagonal matrix elements of twist-two operators.

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## I. INTRODUCTION

In recent years generalized parton distributions (GPDs) [1] have generated a considerable amount of attention. These distributions stem from hadronic matrix elements that are both nondiagonal with respect to hadron states and involve quark and gluon operators separated by a lightlike distance. Thus physics of both inclusive (parton distributions, e.g.) and exclusive (form factors, e.g.) reactions is contained in the GPDs. At the leading-twist level, these new structure functions describe the soft physics of a variety of hard exclusive processes (see the reviews [2]).

Since lightlike correlation functions are involved in the description of deeply virtual Compton scattering (DVCS), e.g., there exists a simple decomposition of these matrix elements in terms of the light-cone Fock space wave functions of the initial and final states [3,4]. This representation of GPDs is ideal for physical intuition; however, comparatively little has been done to show that the light-cone wave function representation is consistent with the reduction properties required of the generalized parton distributions. Below we undertake a simpler task of presenting covariant models for the pion which respect the properties of GPDs. Albeit simple, these models illustrate the utility and physicality of the light-cone Fock representation as well as provide a guide to understanding how the reduction relations arise in this formalism which will be useful when nonperturbative solutions for the Fock components in QCD become available. The scalar constituent model which we consider is merely the triangle graph with pointlike vertices (this has been considered in [5], see also [6]). The spin- $\frac{1}{2}$  model is based on an earlier exactly soluble,  $(1+1)$ -dimensional light-front model [7]. An extension of this model to  $3+1$  dimensions, which involves regularizing the divergent light-front current, has been done in [8].

Another approach is to use the formalism of double distributions (DDs) [9,10]. This formalism elegantly explains the polynomiality conditions required of GPDs and thus gives one the ability to construct models consistent with known properties—although insight into model construction has often been limited to factorization *Ansätze*. Recently two-body light-front wave function models of the pion have

been used to obtain GPDs [11] based on DDs. Without modifying the quark distribution, the resulting GPDs violate the positivity constraints [12]. This inconsistency is attributed to missing contributions from non-wave function vertex diagrams which are absent when one uses noncovariant vertices. In general these diagrams are a substitute for higher Fock components, see [13]. The covariant models used here, however, allow us to test the uniqueness of this ostensible construction.

Indeed we find that appealing to Lorentz invariance is enough to determine only one component of the double distribution in the two-component formalism (even for  $C$ -odd distributions, where the Polyakov-Weiss  $D$  term [14] is absent). Moreover, the component determined from the reduction relations is ambiguous. We show that in the scalar constituent model, missing the second component leads to incorrect GPDs. The same is true for the spin- $\frac{1}{2}$  constituent model, where additionally the positivity constraint is violated. The correct DDs unique to each model can be calculated from nondiagonal matrix elements of twist-two operators which we demonstrate. On the other hand, exploiting the ambiguity inherent in defining one component DDs (which is akin to gauge freedom [15]) one can generate infinitely many different GPD models which share the same form factor and quark distribution as well as satisfy polynomiality (and likely positivity).

The organization of the paper is as follows. First in Sec. II, we explicitly derive the GPD for the scalar triangle diagram with pointlike vertices. Next we show the DD for this model extracted from the form factor in the Drell-Yan frame leads to incorrect GPDs. Having encountered this problem, we review definitions of the double distributions in Sec. III. Here we also calculate the missing component of the DD for the scalar triangle diagram. In Sec. IV, we present the model for the spin- $\frac{1}{2}$  case. Next in Sec. IV B, we regularize the current and then extract this model's GPD. Although not manifest, this model satisfies polynomiality, which is demonstrated in Sec. IV C. Using [11] as a guide, we construct a DD for this model in Sec. IV D. Similar to Sec. II, this one-component DD too gives rise to a different GPD than the light-front projection. Additionally positivity is not satisfied by this one-component DD (Sec. IV E). We calculate the

complete two-component DD from matrix elements of twist-two operators in Sec. IV F. Lastly we conclude with a brief summary (Sec. V).

## II. PION WITH SCALAR CONSTITUENTS

For the pion model with scalar constituents, we choose the pointlike Bethe-Salpeter vertex  $\Gamma(k, P) = 1$ , where the coupling constant is assumed to be absorbed into the overall normalization. Furthermore, we choose derivative coupling of the photon to charged scalar particles.

The pion electromagnetic form factor for this model can be calculated from the Feynman triangle diagram. In order to derive the GPD, however, we need to choose the kinematics specified in Fig. 1 with  $k$  as the momentum of the struck quark. Using the stated pion vertex and taking the plus-component of the current,<sup>1</sup> we have

$$F(t) = \frac{-i|N|^2}{(1-\zeta/2)P^+} \int d^4k (2k^+ + \Delta^+) [k^2 - m^2 + i\epsilon]^{-1} \times [(k + \Delta)^2 - m^2 + i\epsilon]^{-1} [(P - k)^2 - m^2 + i\epsilon]^{-1}, \quad (1)$$

where the momentum transfer is  $t = \Delta^2$  and the skewness is defined relative to the initial state  $\Delta^+ = -\zeta P^+ < 0$ . Physically  $\zeta$  plays the role of Bjorken variable for DVCS. Additionally we work in the frame where  $\mathbf{P}^\perp = 0$ .

To turn Eq. (1) into an expression for the GPD  $H(x, \zeta, t)$ , we insert  $\delta(k^+ / P^+ - x)$  to fix the momentum of the struck quark and keep  $\zeta \neq 0$ . This forces

$$F(t) = \int H(x, \zeta, t) dx. \quad (2)$$

Lastly we integrate over  $k^-$  to project onto the light cone. Doing the contour integration to extract  $H(x, \zeta, t)$  in Eq. (1), we are confronted with the poles

$$\begin{aligned} k_a^- &= k_{\text{on}}^- - \frac{i\epsilon}{x}, \\ k_b^- &= P^- + (k - P)_{\text{on}}^- - \frac{i\epsilon}{x-1}, \\ k_c^- &= -\Delta^- + (k + \Delta)_{\text{on}}^- - \frac{i\epsilon}{x'}, \end{aligned} \quad (3)$$

where the on-shell energies are  $p_{\text{on}}^- = (\mathbf{p}^\perp + m^2) / 2p^+$  and the abbreviation  $x' = (x - \zeta) / (1 - \zeta)$  is used. Thus the nonvanishing contribution to the integral is

$$2\pi i \theta[x(1-x)] [\theta(x-\zeta) \text{Res}(k_b^-) - \theta(\zeta-x) \text{Res}(k_a^-)], \quad (4)$$

<sup>1</sup>For any vector  $a^\mu$ , we define the light-cone variables  $a^\pm \equiv 1/\sqrt{2}(a^0 \pm a^3)$ .

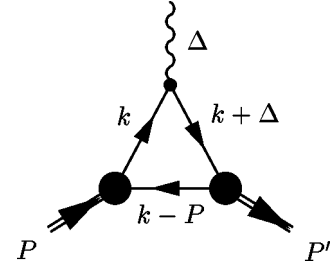


FIG. 1. Covariant triangle diagram for the pion electromagnetic form factor.

which leads to

$$(1 - \zeta/2)H(x, \zeta, t) = \theta(x - \zeta)H_1(x, \zeta, t) + \theta(\zeta - x)H_2(x, \zeta, t). \quad (5)$$

Using  $\mathbf{k}'^\perp = \mathbf{k}^\perp + (1 - x')\Delta^\perp$  for the relative transverse momentum of the final state, the functional forms are

$$H_1(x, \zeta, t) = (2x - \zeta)|N|^2 \int d\mathbf{k}^\perp D_W(x, \mathbf{k}^\perp | M_\pi^2) \times D_W(x', \mathbf{k}'^\perp | M_\pi^2) / x(1-x)x' \quad (6)$$

$$H_2(x, \zeta, t) = (2x - \zeta)|N|^2 \int d\mathbf{k}^\perp D_W(x, \mathbf{k}^\perp | M_\pi^2) \times D_W(x'', \mathbf{k}''^\perp | t) / \zeta x''(1-x'')(1-x), \quad (7)$$

where  $x'' \equiv x/\zeta$  and  $\mathbf{k}''^\perp \equiv \mathbf{k}^\perp + x''\Delta^\perp$  are the relative momenta of the photon. Additionally, we use the replacement

$$D_W(x, \mathbf{k}^\perp | M^2)^{-1} = M^2 - \frac{\mathbf{k}^\perp + m^2}{x(1-x)}, \quad (8)$$

which is the propagator of the Weinberg equation [16].

Comments about the GPD  $H(x, \zeta, t)$  in Eq. (5) are in order. Firstly, the model is covariant and thus the sum rule and polynomiality conditions are met (see Sec. IV C below for clarification). We have checked this explicitly and suitable discussion can be found in [5,6]. Secondly,  $H_1(x, \zeta, t)$  appearing in Eq. (6) satisfies the relevant positivity constraint for a compound scalar of scalar constituents (which appears in the Appendix of [12]) which is clear from inspection.

Consideration of this model was first done from the perspective of DDs, see e.g. the toy model of [9]. This DD model was revisited recently with derivative coupling at the photon vertex in the Appendix of [11] and the same DD also appears in [5]. To derive the DD for this simple model, we appeal to Lorentz invariance as suggested in [11], recalling along the way the relevant properties of DDs.

First consider the form factor. In the  $\zeta = 0$  frame, Eqs. (6) and (7) reduce to the Drell-Yan formula [17] via the definition in Eq. (2):

$$F(t) = |N|^2 \int \frac{dx d\mathbf{k}^\perp}{x(1-x)} D_W(x, \mathbf{k}^\perp | M_\pi^2) \times D_W(x, \mathbf{k}^\perp + (1-x)\mathbf{\Delta}^\perp | M_\pi^2), \quad (9)$$

where  $t = -\mathbf{\Delta}_\perp^2$ . The form factor is Lorentz invariant and has a decomposition in terms of the Lorentz invariant DD  $F(x, y; t)$ : namely,

$$F(t) = \int_0^1 \int_0^1 F(x, y; t) dy dx. \quad (10)$$

The DD satisfies the following relations [10]: support property

$$F(x, y; t) \propto \theta(1-x-y), \quad (11)$$

reduction to the quark distribution at zero momentum transfer

$$q(x) = \int_0^{1-x} F(x, y; 0) dy \quad (12)$$

and is *München* symmetric [18]

$$F(x, y; t) = F(x, 1-x-y; t). \quad (13)$$

Using Eq. (9), we can write  $F(t)$  in the form (10) with

$$F(x, y; t) = \frac{x|N|^2 \theta(1-x-y)}{m^2 - x(1-x)M_\pi^2 - y(1-x-y)t}, \quad (14)$$

which satisfies Eqs. (11)–(13). The ingenuity of DDs comes about when we construct the GPD via

$$H(x, \zeta, t) = \int_0^1 \int_0^1 F(z, y; t) \delta(x-z-\zeta y) dy dz. \quad (15)$$

In this form the sum rule and polynomiality conditions follow trivially.

In Fig. 2, we plot the GPD Eq. (5) as well as the GPD derived from DD via Eq. (15). Surprisingly the two are different despite the fact both models are covariant and possess

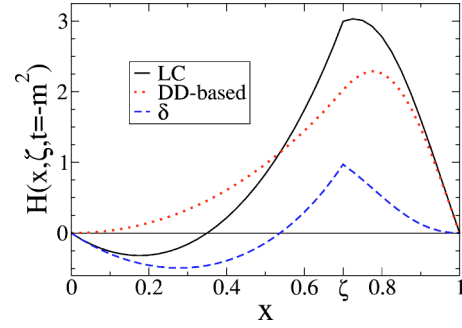


FIG. 2. Comparison of covariant GPDs for the scalar triangle diagram. The GPDs Eq. (5) (denoted LC) and Eq. (15) (DD-based) are plotted as a function of  $x$  for fixed  $\zeta=0.7$  and  $t=-m^2$  for the mass  $M_\pi=m<2m$ . We also plot the difference between the two curves ( $\delta$ ). The area under the curves is identically  $F(-m^2)$  for LC and DD-based GPDs, and hence zero for their difference  $\delta$ .

the same form factor and quark distribution. Additionally we plot their difference ( $\delta$ ) as a function of  $x$ .

### III. DEFINING DOUBLE DISTRIBUTIONS

In this section we define DDs via their moments by following the two-component formalism of [14,15]. Focusing on the ambiguities inherent in defining one-component DDs from two-component objects, we will understand why the DD in Eq. (14) leads to an incorrect GPD. Moreover, we shall calculate the correct DD for the scalar triangle diagram from matrix elements of twist-two operators. We remark in passing that the two components of the DD (below  $F$  and  $G$ , or  $F$  and  $D$  term in the standard formalism) can be viewed as projections of a single function of two variables [19].

#### A. Definitions

Above we have worked in Radyushkin's asymmetrical frame and asymmetrical variables which are ideal for perturbation theory and are a natural generalization of ordinary parton distributions. The nondiagonal matrix elements of twist-two operators are, however, more conveniently expressed in variables symmetric with respect to initial and final states.<sup>2</sup> To this end we define the average momentum  $\bar{P}^\mu = (P + P')^\mu / 2$ . Let  $\vec{D}^\mu = \frac{1}{2}(\vec{\partial}^\mu - \vec{\partial}'^\mu)$ . Then for a pion of spin- $\frac{1}{2}$  constituents we have

$$\begin{aligned} \langle \bar{P} + \Delta/2 | \bar{\psi}(0) \gamma^{\mu_1} i \vec{D}^{\mu_1} \dots i \vec{D}^{\mu_n} \psi(0) | \bar{P} - \Delta/2 \rangle = 2 \bar{P}^{[\mu} \sum_{k=0}^n \frac{n!}{k!(n-k)!} A_{nk}(t) \bar{P}^{\mu_1} \dots \bar{P}^{\mu_{n-k}} \left(-\frac{\Delta}{2}\right)^{\mu_{n-k+1}} \dots \left(-\frac{\Delta}{2}\right)^{\mu_n]} \\ - \Delta^{[\mu} \sum_{k=0}^n \frac{n!}{k!(n-k)!} B_{nk}(t) \bar{P}^{\mu_1} \dots \bar{P}^{\mu_{n-k}} \left(-\frac{\Delta}{2}\right)^{\mu_{n-k+1}} \dots \left(-\frac{\Delta}{2}\right)^{\mu_n]}, \end{aligned} \quad (16)$$

<sup>2</sup>Good discussion of the conversion from symmetrical and asymmetrical variables and distributions can be found in [20]. Additionally advantages and disadvantages of both are presented.

where the action of  $\dots$  on Lorentz indices produces only the symmetric traceless part and  $T$  invariance restricts  $k$  in the first sum to be even and odd in the second. For a pion of scalar constituents, replace  $\gamma^\mu$  with  $2i\vec{D}^\mu$ . As it stands there is considerable freedom in the above decomposition, e.g. one could rewrite the above with  $kB_{n,k-1}(t)/(n-k+1)$  as a contribution to  $A_{nk}(t)$ . Carrying this out for all  $k$ , puts the bulk in the first term and renders the second term proportional only to the symmetric traceless part of  $(n+1)\Delta$ 's—moments of the Polyakov-Weiss  $D$  term [14]. This is the usually encountered form of the DD with  $D$  term. Calculationally, however, we find Eq. (16) is the most useful.

The  $F$  and  $G$  DDs can be defined as generators of the coefficient functions

$$A_{nk}(t) = \int_{-1}^1 d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \beta^{n-k} \alpha^k F(\beta, \alpha; t) \quad (17)$$

$$B_{nk}(t) = \int_{-1}^1 d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \beta^{n-k} \alpha^k G(\beta, \alpha; t). \quad (18)$$

As a consequence of the restrictions on  $k$  in the sums, the function  $F(\beta, \alpha; t)$  is even in  $\alpha$  while  $G(\beta, \alpha; t)$  is odd. Also for  $n$ -even, there is no contribution from the  $D$  term to the function  $G(\beta, \alpha; t)$ .

These functions then appear in the decomposition of matrix elements of lightlike separated operators

$$\begin{aligned} \langle \bar{P} + \Delta/2 | \bar{\psi}(-z^-/2) \not{t} \psi(z^-/2) | \bar{P} - \Delta/2 \rangle &= 2\bar{P} \cdot z \int_{-1}^1 d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha e^{-i\beta\bar{P} \cdot z + i\alpha\Delta \cdot z/2} F(\beta, \alpha; t) \\ &\quad - \Delta \cdot z \int_{-1}^1 d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha e^{-i\beta\bar{P} \cdot z + i\alpha\Delta \cdot z/2} G(\beta, \alpha; t), \end{aligned} \quad (19)$$

where  $z^2=0$ .

Denoting  $\xi = -\Delta^+ / 2\bar{P}^+$ , the GPD in symmetric variables reads

$$H(\tilde{x}, \xi, t) = \frac{1}{4\pi} \int dz^- e^{i\tilde{x}\bar{P}^+ z^-} \langle \bar{P} + \Delta/2 | \bar{\psi}(-z^-/2) \gamma^+ \psi(z^-/2) | \bar{P} - \Delta/2 \rangle. \quad (20)$$

Inserting Eq. (19) into this definition yields

$$H(\tilde{x}, \xi, t) = \int_{-1}^1 d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \delta(\tilde{x} - \beta - \xi\alpha) [F(\beta, \alpha; t) + \xi G(\beta, \alpha; t)]. \quad (21)$$

By integrating over  $\tilde{x}$ , we uncover *two* sum rules: the sum rule for the form factor

$$\int d\beta \int d\alpha F(\beta, \alpha; t) = F(t) \quad (22)$$

and what we call the  $G$ -sum rule

$$\int d\beta \int d\alpha G(\beta, \alpha; t) = 0, \quad (23)$$

which is trivial since  $G$  is an odd function of  $\alpha$ . Equation (23) has nontrivial consequences however, e.g. it shows the method employed by [11] leads only to an  $F$ -type DD in Eq. (19). This function integrates to the form factor via Eq. (22) and in the forward limit  $\{\xi, t \rightarrow 0\}$  reduces to the quark distribution (when integrated over  $\alpha$ ). Thus  $F(\beta, \alpha; t)$  should be properly termed the forward-visible DD, which encompasses *more* than just neglecting the  $D$  term. From Eq. (16),

we see that  $G(\beta, \alpha; t)$  does affect higher moments of the GPD. This is the source of the discrepancy shown in Fig. 2 as we now demonstrate.

### B. Scalar model, revisited

To derive both  $F$  and  $G$  DDs for the scalar triangle diagram of Sec. II, we must now consider the action of the operator  $2i\vec{D}^{\mu_1} i\vec{D}^{\mu_2} \dots i\vec{D}^{\mu_n}$ . This produces a factor

$$\frac{1}{2^n} (2k + \Delta)^{\mu_1} (2k + \Delta)^{\mu_2} \dots (2k + \Delta)^{\mu_n} \quad (24)$$

in the integrand of Eq. (9), which we now take in the symmetric frame. After the integration over  $k$  is performed, we are left only with

$$\begin{aligned} \int_0^1 d\beta \int_{-1+\beta}^{1-\beta} d\alpha D(\beta, \alpha; t) (2\beta\bar{P} - \alpha\Delta)^{\mu_1} \\ \times (\beta\bar{P} - \alpha\Delta/2)^{\mu_2} \dots (\beta\bar{P} - \alpha\Delta/2)^{\mu_n}, \end{aligned} \quad (25)$$

where we have used the replacement

$$D(\beta, \alpha; t) = \frac{1}{m^2 - \beta(1 - \beta)M_\pi^2 - t[(1 - \beta)^2 - \alpha^2]/4}. \quad (26)$$

Using the binomial expansion, we can identify  $F(\beta, \alpha; t)$  and  $G(\beta, \alpha; t)$  via Eqs. (17) and (18), namely<sup>3</sup>

$$F(\beta, \alpha; t) = \beta\theta(1 - \beta - \alpha)D(\beta, \alpha; t)|N|^2 \quad (27)$$

$$G(\beta, \alpha; t) = \alpha\theta(1 - \beta - \alpha)D(\beta, \alpha; t)|N|^2. \quad (28)$$

To compare with the results of Sec. II, we must revert to asymmetric variables which is accomplished by  $\{\beta \rightarrow x, y \rightarrow (\alpha - \beta + 1)/2\}$ . The denominator common to both terms becomes

$$D(x, y; t) = \frac{1}{m^2 - x(1 - x)M_\pi^2 - y(1 - x - y)t}. \quad (29)$$

Thus we have

$$F(x, y; t) = x\theta(1 - x - y)D(x, y; t)|N|^2 \quad (30)$$

$$G(x, y; t) = (2y + x - 1)\theta(1 - x - y)D(x, y; t)|N|^2. \quad (31)$$

Notice the function  $G(x, y; t)$  is *München* antisymmetric which is required because  $G(\beta, \alpha; t)$  is odd with respect to  $\alpha$ .

To construct the GPD  $H(x, \zeta, t)$  we must also convert Eq. (21) to asymmetric variables.

$$H(x, \zeta, t) = \int_0^1 dz \int_0^{1-z} dy \delta(x - z - \zeta y) \times \left[ F(x, y; t) + \frac{\zeta}{2 - \zeta} G(x, y; t) \right]. \quad (32)$$

We can now plot the GPD in Eq. (32) using the two DDs in Eqs. (30) and (31). The result agrees with Eq. (5) depicted in Fig. 2. Moreover, the contribution from  $[\zeta/(2 - \zeta)]G$  is identically the difference  $\delta$  plotted in the figure. In the DD formalism, the reduction relations alone do not determine the GPD. We illustrate this further with a spin- $\frac{1}{2}$  pion model.

#### IV. PION WITH SPIN- $\frac{1}{2}$ CONSTITUENTS

##### A. Bethe-Salpeter amplitude and pion wave function

For the spin- $\frac{1}{2}$  model, we choose the trivially  $q\bar{q}$  symmetric Bethe-Salpeter vertex

<sup>3</sup>Notice  $F$  and  $G$  are projections of a single function  $D(\beta, \alpha; t) = f(\beta, \alpha)$  in the notation of [19]. Determination of  $f(\beta, \alpha)$  for the spin- $\frac{1}{2}$  model of Sec. IV, however, cannot be done by inspection.

$$\Gamma(k, P) = -ig\gamma^5. \quad (33)$$

Here we have assumed only  $\gamma^5$  coupling at the quark-pion vertex with coupling constant  $g$ , whereas four Dirac structures exist [21]. This simple coupling is suggested by an effective interaction Lagrangian of the form (see, e.g. [22])

$$\mathcal{L}_1 = -ig\pi \cdot \bar{q}\gamma^5\pi q, \quad (34)$$

where the coupling constant  $g = m/f_\pi$ , with  $m$  the constituent mass and  $f_\pi$  the pion decay constant. Notice the (ladder approximation) kernel is independent of light-cone time. Thus this model (as well as the scalar triangle model in Sec. II) is a special case of the instantaneous formalism described by [23] in the impulse approximation. The relation of the vertex to the Bethe-Salpeter wave function is given by

$$\Psi(k, P) = \frac{i}{\not{k} - m + i\epsilon} (-ig)\gamma^5 \frac{i}{\not{k} - \not{P} - m + i\epsilon}. \quad (35)$$

The valence wave function can be found by projecting the Bethe-Salpeter wave function onto the light-cone  $x^+ = 0$ , see e.g. [24]. Using the normalization convention of [25], we have

$$\psi(x, \mathbf{k}_{\text{rel}}^\perp; \lambda, \lambda') = \frac{1}{2P^+} \int \frac{dk^-}{2\pi} \frac{\bar{u}_\lambda(xP^+, \mathbf{k}^\perp)}{\sqrt{x}} \gamma^+ \times \Psi(k, P) \gamma^+ \frac{v_{\lambda'}[(1-x)P^+, \mathbf{P}^\perp - \mathbf{k}^\perp]}{\sqrt{1-x}}, \quad (36)$$

where  $x$  is the fraction of the pion's plus momentum carried by the quark ( $x = k^+/P^+$ ), and the relative transverse momentum is  $\mathbf{k}_{\text{rel}}^\perp = \mathbf{k}^\perp - x\mathbf{P}^\perp$ . The valence wave function is found from Eq. (36) to be

$$\psi(x, \mathbf{k}^\perp; \lambda, \lambda') = \frac{g\sqrt{N_c}/2C}{x(1-x)} D_W(x, \mathbf{k}^\perp | M_\pi^2) \times [k_{-\lambda} \delta_{\lambda, \lambda'} - \lambda m \delta_{\lambda, -\lambda'}], \quad (37)$$

where we have employed the notation  $k_\lambda = k^+ + i\lambda k^2$ . As a result of the contour integration, we have a factor of  $\theta[x(1-x)]$  implicitly in Eq. (37). Additionally the wave function is symmetric under interchange of  $x$  and  $1-x$ . As is known, the simplistic form of this wave function leads to divergent quark distributions and form factors which will be handled below. Since this model is nonrenormalizable, the choice of regularization scheme influences the dynamics.

##### B. Form factor and generalized parton distribution

The pion electromagnetic form factor for this model can be calculated from the Feynman triangle diagram. In order to extract the GPD, however, we need to choose the kinematics specified in Fig. 1 with  $k$  as the momentum of the struck quark. As it stands, using the wave function Eq. (37), the triangle diagram diverges. Following the approach of [8], we

covariantly smear the pointlike photon vertex in Fig. 1 in a way reminiscent of Pauli-Villars regularization

$$\gamma^\mu \rightarrow \Gamma_\Lambda^\mu = \frac{\Lambda^2}{k^2 - \Lambda^2 + i\epsilon} \gamma^\mu \frac{\Lambda^2}{(k + \Delta)^2 - \Lambda^2 + i\epsilon}. \quad (38)$$

Equation (38) is a simple way to model non- $q\bar{q}$  components of the wave function. Alternatively one could smear the  $q\bar{q} - \pi$  vertex in a covariant manner [26,27]. This smearing should additionally respect the  $q\bar{q}$  symmetry of the vertex. We do not pursue this option here since positivity constraints (see Sec. IV E) are likely to be violated. On the other hand, one could use Pauli-Villars subtractions to regulate the theory, however, positivity would also be put into question.

Because our concern is with model comparisons not phenomenology, we shall choose  $\Lambda = m$  for simplicity. Although not obvious from inspection, results for  $\Lambda \neq m$  exhibit the same features investigated below. Most noteworthy, positivity remains satisfied when  $\Lambda \neq m$ .

Considering matrix elements of the current operator  $J^\mu$  between pion states, the model (38) conserves current. This can be demonstrated most easily by calculating  $\Delta \cdot J$  in the Breit frame. Additionally since the model is fully covariant, we can extract the electromagnetic form factor from any component of the current. In particular, potential end-point singularities present in matrix elements of  $J^-$  have been removed by the photon vertex smearing Eq. (38) [8]. Using the plus component of the current, we have the expression

$$F(t) = \frac{ig^2 N_c |C|^2 m^4}{1 - \frac{\zeta}{2}} \int \frac{d^4 k}{(2\pi)^4 P^+} \frac{\text{Tr}[(\mathbf{k} + m)\gamma^5(\mathbf{k} - \mathbf{P} + m)\gamma^5(\mathbf{k} + \mathbf{\Delta} + m)\gamma^+]}{[k^2 - m^2 + i\epsilon]^2 [(k + \Delta)^2 - m^2 + i\epsilon]^2 [(P - k)^2 - m^2 + i\epsilon]}, \quad (39)$$

where the momentum transfer is  $t = \Delta^2$  and the skewness  $\zeta$  is defined relative to the initial state:  $\Delta^+ = -\zeta P^+ < 0$ .

To calculate the GPD, we follow the procedure described in Sec. II. The result can be written as

$$(1 - \zeta/2)H(x, \zeta, t) = \theta(x - \zeta)H_{\text{eff}}(x, \zeta, t) + \theta(\zeta - x)[H_{\text{inst}}(x, \zeta, t) + H_{\text{nval}}(x, \zeta, t)], \quad (40)$$

where  $H_{\text{eff}}$  is the piece determined by the effective two-body wave function,  $H_{\text{inst}}$  is the contribution from instantaneous propagation of the spectator quark, and the remaining contributions we term nonvalence (although strictly speaking the instantaneous piece is also of the nonvalence variety). It is a peculiarity of this model that explicit instantaneous terms are not present for  $x > \zeta$ .

The functional forms are

$$H_{\text{eff}}(x, \zeta, t) = \int \frac{d\mathbf{k}^\perp}{(2\pi)^3} \sum_{\lambda, \lambda'} \psi_{\text{eff}}^*(x', \mathbf{k}'^\perp; \lambda, \lambda') \psi_{\text{eff}}(x, \mathbf{k}^\perp; \lambda, \lambda'), \quad (41)$$

$$H_{\text{inst}}(x, \zeta, t) = -\mathcal{A} \int d\mathbf{k}^\perp \frac{4\zeta D_W^3(x'', \mathbf{k}^\perp | t)}{(1-x)x''(1-x'')} \quad (42)$$

$$H_{\text{nval}}(x, \zeta, t) = -\mathcal{A} \int d\mathbf{k}^\perp \frac{2(\mathbf{k}^\perp \cdot \mathbf{k}'^\perp + m^2) D_W(x, \mathbf{k}^\perp | M_\pi^2) D_W^2(x'', \mathbf{k}''^\perp | t)}{x(1-x)x''(1-x'')x'(1-x')(1-\zeta)} [2\zeta D_W(x'', \mathbf{k}''^\perp | t) + D_W(x, \mathbf{k}^\perp | M_\pi^2)], \quad (43)$$

where we have defined the effective wave function

$$\psi_{\text{eff}}(x, \mathbf{k}^\perp; \lambda, \lambda') = \frac{g\sqrt{N_c/2} C m^2}{x^2(1-x)} [k_{-\lambda} \delta_{\lambda, \lambda'} - \lambda m \delta_{\lambda, -\lambda'}] D_W^2(x, \mathbf{k}^\perp | M_\pi^2) \quad (44)$$

and made the abbreviation  $\mathcal{A} = g^2 |C|^2 N_c m^4 / 2(2\pi)^3$ . It is sensible to think of Eq. (44) as an effective wave function since  $x \rightarrow 1-x$  symmetry has been lost. Moreover the wave function vanishes at  $x=1$  and is nonvanishing at  $x=0$ . This is a desirable addition to the dynamics stemming from the regularization. Notice the ladder kernel (34) is momentum

independent and hence does not vanish at  $x=0,1$ . This is the dynamical reason why the un-regularized wave function Eq. (37) does not vanish at the end points. Continuity of the GPD at  $x=\zeta$  follows directly from Eqs. (41)–(43).

In Fig. 3, we plot the GPD for the parameters:  $M = 0.14$  GeV and  $m = 0.33$  GeV. On the left, the graph

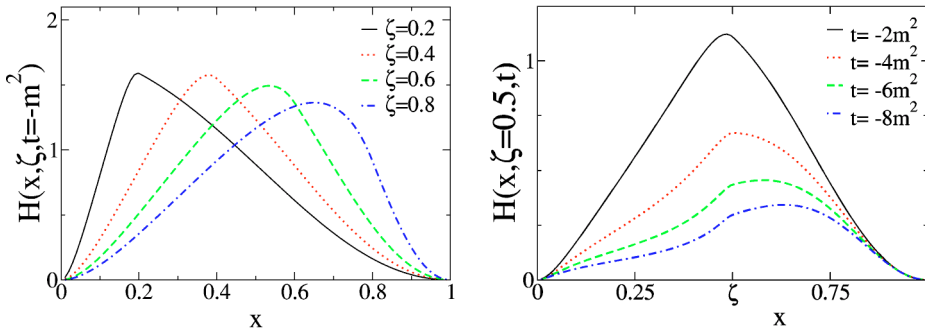


FIG. 3. On the left, the GPD Eq. (40) is plotted at fixed  $t = -m^2$  for a few values of  $\zeta$ . On the right, the same GPD is plotted for fixed  $\zeta = 0.5$  for a few values of  $t$ . The model parameters are arbitrarily chosen as:  $M = 0.14$  GeV and  $m = 0.33$  GeV.

shows the GPD for a few values of  $\zeta$  as a function of  $x$  for fixed  $t$ , while on the right we have fixed  $\zeta$  and  $t$  varying.

### C. Sum rule and polynomiality

Since not manifest, one should check the covariance of the model Eq. (40). With the covariant starting point Eq. (39), we anticipate polynomiality will be satisfied which provides a useful check on our expressions Eqs. (41)–(43). First we define the moments of the GPD with respect to asymmetric variables

$$P_n(\zeta, t) = \int x^n H(x, \zeta, t) dx. \quad (45)$$

Polynomiality requires the moments  $P_n$  to be of the form

$$P_n(\zeta, t) = \sum_{j=0}^n a_{nj}(t) \zeta^j. \quad (46)$$

The zeroth moment is merely the sum rule for the form factor, hence  $a_{00}(t) = F(t)$ . For simplicity, we check a few of the lowest moments for the polynomiality condition Eq. (46) at  $t=0$ . In Fig. 4, we plot the moments  $P_n(\zeta, 0)$  for  $n = 0, 1, 2, 3$  which appear as smooth functions. Additionally we plot simple  $(n+1)$ -point polynomial fits to the moments, which line up nicely with the integrals (45).

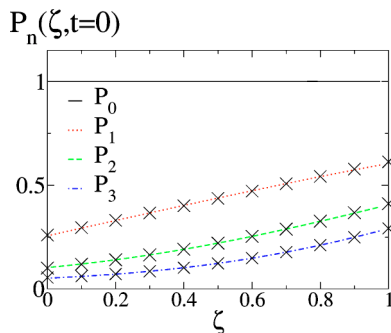


FIG. 4. Polynomiality conditions checked for the GPD Eq. (40). The moments  $P_n(\zeta, t=0)$  from Eq. (45) are plotted as a function of  $\zeta$  for  $n = 0, 1, 2, 3$ . Additionally  $\times$ 's denote the simple  $(n+1)$ -point polynomial fit to the moment  $P_n$ .

### D. Double distribution

To construct the DD, we shall first proceed incorrectly by appealing to the Lorentz invariance of the form factor as in Sec. II. This will at least lead to one component of the DD satisfying the reduction relations and can be compared to Eq. (40).

Using Eq. (41) in the  $\Delta^+ = 0$  (Drell-Yan) frame, we can write  $F(t)$  in the form (10) with

$$F(x, y; t) = [3m^2 - M_\pi^2 x(1-x) + y(1-x-y)t] \times \frac{2\pi\mathcal{A}\theta(1-x-y)y(1-x-y)}{(1-x)[m^2 - M_\pi^2 x(1-x) - y(1-x-y)t]^3}. \quad (47)$$

Aside from factors arising from spin, this DD is basically the same as that considered in [11] which one can realize by utilizing  $\lambda^2 = -M_\pi^2/4 + m^2$ . Not surprisingly, then, this DD satisfies Eqs. (11)–(13). For reference we give the quark distribution function

$$q(x) = \frac{2\pi\mathcal{A}}{6} \frac{(1-x)^2 [3m^2 - M_\pi^2 x(1-x)]}{[m^2 - M_\pi^2 x(1-x)]^3}, \quad (48)$$

which could be calculated directly from  $\psi_{\text{eff}}$  in Eq. (44).

In Fig. 5, we plot the GPD Eq. (40) as well as the GPD derived from DD Eq. (47) via Eq. (15). As in Sec. II, the two are different despite the fact both models are covariant and possess the same form factor and quark distribution. Additionally we plot their difference as a function of  $x$ .

### E. Positivity constraints

Here we demonstrate another difference between the GPD in Eq. (40) and the one stemming from the one-component DD Eq. (47). To do so, we look at the positivity constraints. Originally these constraints appeared in [10,28] and were derived from the positivity of the density matrix by restricting the final-state parton to have positive plus-momentum [and ignoring the contribution from  $E(x, \zeta, t)$  for the spin- $\frac{1}{2}$  case]. Although the matrix elements involved for GPDs are off diagonal, they are still restricted by positivity and their diagonal elements. Correcting the constraints for the presence of the  $E$  distribution was first done in [4]. By considering the positivity of the norm on Hilbert space, stricter con-

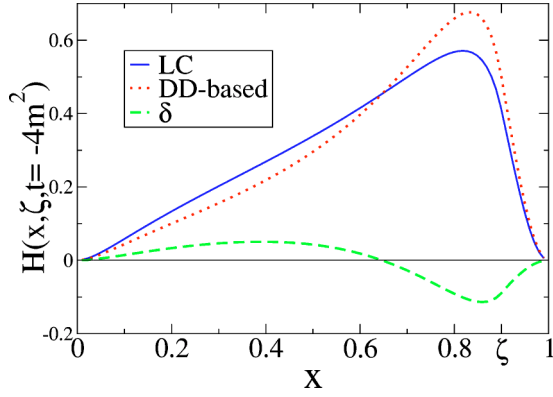


FIG. 5. Comparison of covariant GPDs for the spinor triangle diagram. The GPDs Eq. (40) (denoted LC) and Eq. (47) (DD-based) are plotted as a function of  $x$  for fixed  $\zeta=0.9$  and  $t=-4m^2$  for the mass  $M_\pi=0.15m$ . We also plot the difference between the two curves ( $\delta$ ). The area under the curves is identically  $F(-4m^2)$  for LC and DD-based GPDs, and hence zero for their difference  $\delta$ .

straints for the spin- $\frac{1}{2}$  distributions  $H$  and  $E$  have recently appeared as well as constraints for the full set of twist-two GPDs [29].

For the scalar pion case there is of course no contribution from the nonexistent  $E$  distribution and hence the original bounds are actually correct (modulo factors due to the difference of a spin- $\frac{1}{2}$  proton versus a spin-0 pion). Given the matrix element definition of the GPD consistent with Eq. (2), namely

$$H(x, \zeta, t) = \frac{1}{1 - \zeta/2} \int \frac{dz^-}{4\pi} e^{ixP^+ z^-} \langle \pi(P') | \bar{\psi}(0) \gamma^+ \psi(z^-) | \pi(P) \rangle, \quad (49)$$

the spin-0 positivity constraint (for  $x > \zeta$ ) reads

#### F. Derivation of the correct DDs

To derive both  $F$  and  $G$  DDs for the spin- $\frac{1}{2}$  pion model, we must consider the action of the operator  $\gamma^{\mu} i \vec{D}^{\mu_1} \dots i \vec{D}^{\mu_n}$  between nondiagonal pion states. Inserted into Eq. (39) which is now taken in the symmetrical frame, this operator produces

$$\frac{i\mathcal{A}}{\pi} \int d^4k \frac{\text{Tr}\{(\mathbf{k}+m) \gamma^5 (\mathbf{k}-\bar{\mathbf{P}}+\mathbf{\Delta}/2+m) \gamma^5 (\mathbf{k}+\mathbf{\Delta}+m) \gamma^{\mu}\} (k+\mathbf{\Delta}/2)^{\mu_1} \dots (k+\mathbf{\Delta}/2)^{\mu_n}}{[k^2-m^2+i\epsilon]^2 [(k+\mathbf{\Delta})^2-m^2+i\epsilon]^2 [(k-\bar{\mathbf{P}}+\mathbf{\Delta}/2)^2-m^2+i\epsilon]}. \quad (51)$$

The presence of the trace

$$\text{Tr}\{\dots\}^{\mu} = 4 \left[ \bar{P}^{\mu} (m^2 - k^2 - k \cdot \mathbf{\Delta}) + \frac{\Delta^{\mu}}{2} (m^2 - k^2 + 2k \cdot \bar{\mathbf{P}}) + k^{\mu} \left( m^2 - k^2 - \frac{t}{2} + 2k \cdot \bar{\mathbf{P}} - k \cdot \mathbf{\Delta} \right) \right] \quad (52)$$

complicates evaluating Eq. (51) by requiring contributions from diagrams reduced by one propagator. Since we have smeared the photon via Eq. (38), the reduced diagrams are finite. Let us denote the propagators simply by  $\mathfrak{A} = (k - \bar{\mathbf{P}} + \mathbf{\Delta}/2)^2 - m^2 + i\epsilon$ ,  $\mathfrak{B} = (k + \mathbf{\Delta})^2 - m^2 + i\epsilon$  and  $\mathfrak{C} = k^2 - m^2 + i\epsilon$ . To correctly evaluate Eq. (51), we must write the trace as

$$\text{Tr}\{\dots\}^{\mu} = 4 \left[ \bar{P}^{\mu} \left( \frac{t}{2} - \frac{1}{2} (\mathfrak{B} + \mathfrak{C}) \right) + \frac{\Delta^{\mu}}{2} \left( M^2 - \frac{t}{4} - \mathfrak{A} + \frac{1}{2} (\mathfrak{B} - \mathfrak{C}) \right) + k^{\mu} \left( M^2 - \frac{t}{2} - \mathfrak{A} \right) \right] \quad (53)$$

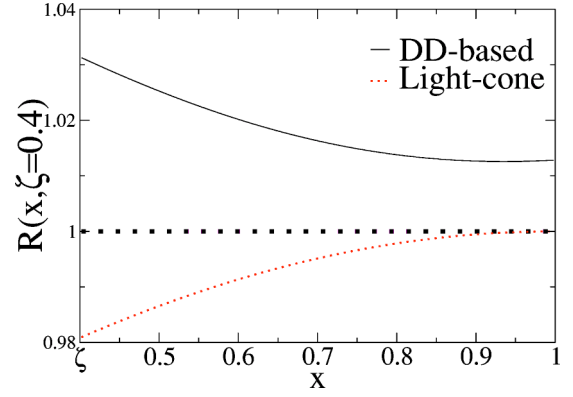


FIG. 6. Comparison of GPDs: GPD calculated from the DD Eq. (47) via Eq. (15) (DD-based) compared with the light-cone projection of the form factor Eq. (40) (light-cone) for fixed  $\zeta=0.4$  at  $t=0$ . Here we plot the ratio  $R(x, \zeta)$  appearing in Eq. (50) as a function of  $x > \zeta$ . Positivity constrains this ratio to be less than 1.

$$R(x, \zeta) \equiv (1 - \zeta/2) \frac{|H(x, \zeta, 0)|}{\sqrt{q(x)q(x')}} \leq 1, \quad (50)$$

where  $q(x)$  is the model distribution function in Eq. (48). Of course the above result holds for finite  $-t$ , however since the function  $\mathcal{F}$  decreases with  $-t$ , Eq. (50) is the tightest constraint. Notice for  $M_\pi^2 \neq 0$ , the limit  $t=0$  is in an unphysical region. If we treat this limit as formal, however, and analytically continue our expressions, we can use Eq. (50). Such continuation is consistent with the light-cone Fock space representation of GPDs [3,4].

Given the constraint Eq. (50), we can test whether GPDs calculated from the light-cone projection (41) and DD (47) satisfy positivity. In Fig. 6, we plot  $R(x, \zeta)$  for each GPD as a function of  $x$  for the fixed value of  $\zeta=0.4$ . There is noticeably different behavior in the figure: positivity is violated by the DD-based model. As above (Sec. III), we must carefully derive contributions from the other component  $G(x, y; t)$ .



and evaluate each term separately canceling propagators in the denominator of Eq. (51). These integrals can easily be evaluated using Feynman parameters. For example, let us consider the nonreduced contribution from Eq. (53). The denominator of Eq. (51) appears as  $\mathfrak{A}\mathfrak{B}^2\mathfrak{C}^2$  and so we introduce two Feynman parameters  $\{x, y\}$  to render the denominator specifically in the form  $[x\mathfrak{A} + y\mathfrak{B} + (1-x-y)\mathfrak{C}]^{-5}$ . One then translates  $k^\mu$  to render the integral (hyper-) spherically symmetric via the definition  $k^\mu = l^\mu + \beta\bar{\mathbf{P}}^\mu - (\alpha+1)\Delta^\mu/2$ . Here  $\beta=x$  and  $\alpha=x+2y-1$ . Using a Wick rotation to evaluate the resulting integral over  $l$ , we can cast the contribution to Eq. (51) from nonreduced terms in the form

$$\begin{aligned} \pi\mathcal{A} \int_0^1 d\beta \int_{-1+\beta}^{1-\beta} d\alpha \frac{1}{4} [(1-\beta)^2 - \alpha^2] D(\beta, \alpha; t)^3 \{2\bar{\mathbf{P}}[\beta M_\pi^2 + (1-\beta)t/2] - \Delta\alpha(M_\pi^2 - t/2)\}^{l_\mu} \\ \times \sum_{k=0}^n \frac{n!}{k!(n-k)!} \beta^{n-k} \alpha^k \bar{\mathbf{P}}^{\mu_1} \dots \bar{\mathbf{P}}^{\mu_{n-k}} \left(-\frac{\Delta}{2}\right)^{\mu_{n-k+1}} \dots \left(-\frac{\Delta}{2}\right)^{\mu_n} \end{aligned} \quad (54)$$

with  $D(\beta, \alpha; t)$  given by Eq. (26). Given the form of Eq. (54), we can identify  $\{\beta, \alpha\}$  as DD variables and hence read off contributions to  $F$  and  $G$  DDs.

$$\delta F(\beta, \alpha; t) = \pi\mathcal{A}[(1-\beta)^2 - \alpha^2] D(\beta, \alpha; t)^3 [\beta M_\pi^2 + (1-\beta)t/2] \quad (55)$$

$$\delta G(\beta, \alpha; t) = \pi\mathcal{A}\alpha[(1-\beta)^2 - \alpha^2] D(\beta, \alpha; t)^3 (M_\pi^2 - t/2). \quad (56)$$

Notice that these contributions respect the properties of DDs, namely  $\delta F$  is even in  $\alpha$  while  $\delta G$  is odd. This need not be the case, however, for each intermediate step of the calculation, e.g. contributions from  $\mathfrak{B}$ -reduced terms and  $\mathfrak{C}$ -reduced terms are individually neither even nor odd in  $\alpha$  while their sum is even and their difference is odd.

Ignoring for the moment contributions from  $\mathfrak{A}$ -reduced terms, we arrive at the DDs

$$F(\beta, \alpha; t) = \pi\mathcal{A}D(\beta, \alpha; t)^3 \{(1-\beta)m^2 - \beta\alpha^2 M_\pi^2 + (1-\beta)[(1-\beta)^2 - \alpha^2]t/4\} \quad (57)$$

$$G(\beta, \alpha; t) = -\pi\mathcal{A}\alpha D(\beta, \alpha; t)^3 \{m^2 - M_\pi^2(1-\beta - \alpha^2) + [(1-\beta)^2 - \alpha^2]t/4\}. \quad (58)$$

The contribution from  $\mathfrak{A}$ -reduced terms has the form of a  $D$ -term. Using Feynman parameters for the denominator  $\mathfrak{B}^2\mathfrak{C}^2$  and suitable changes of variables, we arrive at the contribution to Eq. (51)

$$-\Delta^{l_\mu} \int_{-1}^1 d\alpha \frac{\pi\mathcal{A}\alpha^{n+1}(1-\alpha^2)}{[m^2 - (1-\alpha^2)t/4]^2} \left(-\frac{\Delta}{2}\right)^{\mu_1} \dots \left(-\frac{\Delta}{2}\right)^{\mu_n} \quad (59)$$

from which we can identify the  $D$ -term

$$D(\alpha; t) = \pi\mathcal{A} \frac{\alpha(1-\alpha^2)}{[m^2 - (1-\alpha^2)t/4]^2}. \quad (60)$$

Although strictly speaking, the  $D$  term is a contribution to the  $G$  DD, we shall treat it separately for ease.

Switching now to asymmetric variables, we have

$$F(x, y; t) = \pi\mathcal{A}D(x, y; t)^3 [(1-x)m^2 - x(x+2y-1)^2 M_\pi^2 + (1-x)y(1-x-y)t] \quad (61)$$

$$G(x, y; t) = -\pi\mathcal{A}(x+2y-1)D(x, y; t)^3 \{m^2 + y(1-x-y)t - M_\pi^2[1-x-(x+2y-1)^2]\} \quad (62)$$

$$D(y; t) = \pi\mathcal{A}y(1-y)(2y-1)[m^2 - y(1-y)t]^{-2} \quad (63)$$

where the function  $D(x,y;t)$  is given by Eq. (29). Accordingly  $F(x,y;t)$  is *München* symmetric and  $G(x,y;t)$  is anti-symmetric, while  $D(y;t)$  is antisymmetric about  $y=1/2$ . Notice  $F(x,y;t)$  in Eq. (61) is not that of Eq. (47). Given the ambiguity inherent in defining  $F$  versus  $G$  DDs [cf. Eq. (23)], there is no reason to believe the  $F$ s would be the same. In principle, we could construct a gauge transformation [15] to render the  $F$ s the same. This would enable identification of the missing  $G$  function unique to Sec. IV D. We shall not pursue this tangential point.<sup>4</sup> It suffices to note: the procedure in [11] leads only to one component of the DD and precisely which component [in the decomposition of Eq. (16)] is unknown.

The function  $F(x,y;t)$  satisfies the reduction relations: it reduces to the quark distribution via Eq. (12) and integrates to the form factor (13)—the latter can only be checked numerically. Lastly then it remains to see whether the DD-based GPD lines up with true GPD calculated in Sec. IV B. To construct the GPD we use the form of Eq. (32) modified to handle the  $D$  term in Eq. (63) separately

$$\begin{aligned}
 H(x,\zeta,t) &= \int_0^1 dz \int_0^{1-z} dy \delta(x-z-\zeta y) \\
 &\times \left[ F(x,y;t) + \frac{\zeta}{2-\zeta} G(x,y;t) \right] \\
 &+ \frac{\theta[x''(1-x'')]}{2(2-\zeta)} D(x'';t). \quad (64)
 \end{aligned}$$

In Fig. 7, we plot the GPD Eq. (40) and the DD-based Eq. (64). They are identical. We also plot the individual  $F$  and  $G+D$  contributions to the GPD. Even in the absence of the  $D$  term, the contribution from  $G$  cannot be neglected in ascertaining the DD. Again calculation of the GPD from the full two-component DD agrees with the light-cone GPD definition. The argument of Lorentz invariance used to calculate an  $F$  DD that satisfies the reduction relations does not determine the GPD.

For a given model, there exists a unique two-component DD in the decomposition of Eq. (16). This DD produces a GPD which is then consistent with the light-cone projection and satisfies all the necessary reduction relations. On the other hand, if one is interested in constructing *different* GPD models satisfying the reduction relations, polynomiality, etc., one could exploit the ambiguity of one-component DDs. Consider the following. The contribution to the GPD from  $F$  in Fig. 7 is markedly different from Eq. (40) and the analogous contribution from Eq. (47). These three GPDs, how-

<sup>4</sup>Notice the contribution to the GPD from the  $D$  term resembles that of  $H_{\text{inst}}$  in Eq. (42) but is not identical. Both terms originate from a reduction of the spectator's propagator. In the case of the  $D$  term, the spectator's propagator is completely removed by the  $\mathfrak{A}$  reduction. For  $H_{\text{inst}}$ , there is residual  $x$  dependence stemming from the light-cone instantaneous propagator  $\gamma^+/2P^+(1-x)$ .

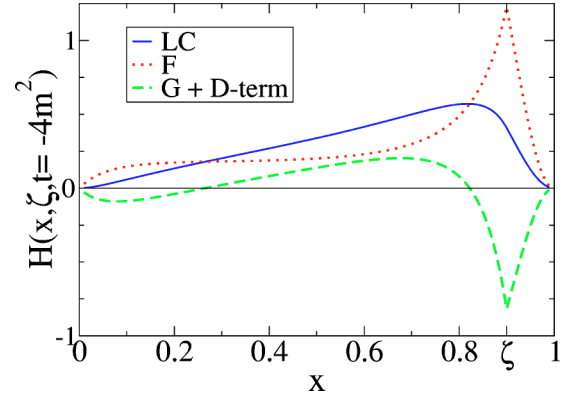


FIG. 7. Contributions to the covariant GPD from DDs. The light-cone GPD Eq. (40) and the DD-based Eq. (64) are identical, denoted (LC) and plotted as a function of  $x$  for fixed  $\zeta=0.9$  and  $t=-4m^2$  for the mass  $M_\pi=0.15m$ . We also plot the individual contributions from  $F$  in Eq. (61) and from  $[\zeta/(2-\zeta)]G + \{1/[2(2-\zeta)]\}D$  in Eqs. (62) and (63), denoted (G+D term).

ever, share the same form factor and quark distribution, since their respective  $G$  functions do not contribute to the reduction relations. Thus one can always use DD-gauge freedom [15] to transform a given  $F$  into a new function and throw away contributions from  $G$ . The result is an infinite set of different GPDs with identical form factors and quark distributions. As pointed out in Sec. IV E, one must be careful to maintain positivity although it is likely that there still is an infinite set of GPDs which would.

## V. SUMMARY

Above we consider two covariant models for pions: one with scalar constituents and the other with spin- $\frac{1}{2}$ . The spin- $\frac{1}{2}$  model requires regularization and we choose the method in [8] in order to maintain positivity. For each case we derive the GPD from its matrix element definition which forces us to consider the triangle diagram for the form factor with the plus momentum of the struck quark kept fixed in a general  $\Delta^+ \neq 0$  frame.

We also construct the DDs for each model. The approach of [11] leads only to one component of the DD (the “forward-visible” piece) which is itself ambiguous, thus resulting GPDs are incorrect and need not satisfy positivity. This fact remains true even for  $C$ -odd distributions. This means the reduction relations alone are not enough from which to calculate a model’s DD (and hence GPD). To obtain both components of the DD unambiguously, we calculate the matrix elements of twist-two operators. The resulting DD-based GPDs then agree with those calculated on the light cone. The “gauge freedom” inherent in defining  $F$  vs  $G$  DDs could be exploited, however, for phenomenological studies where one is interested in a mathematical fit to data, rather than a calculation of a given model’s DD.

## ACKNOWLEDGMENTS

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- [1] D. Müller, D. Robaschik, B. Geyer, F.M. Dittes, and J. Hořejši, Fortschr. Phys. **42**, 101 (1994); X.-D. Ji, Phys. Rev. Lett. **78**, 610 (1997); Phys. Rev. D **55**, 7114 (1997); A.V. Radyushkin, Phys. Lett. B **380**, 417 (1996); **385**, 333 (1996).
- [2] X.-D. Ji, J. Phys. G **24**, 1181 (1998); A.V. Radyushkin, hep-ph/0101225; K. Goeke, M.V. Polyakov, and M. Vanderhaeghen, Prog. Part. Nucl. Phys. **47**, 401 (2001).
- [3] S.J. Brodsky, M. Diehl, and D.S. Hwang, Nucl. Phys. **B596**, 99 (2001).
- [4] M. Diehl, T. Feldmann, R. Jakob, and P. Kroll, Nucl. Phys. **B596**, 33 (2001).
- [5] P.V. Pobylitsa, Phys. Rev. D **67**, 094012 (2003).
- [6] L. Theussl, S. Noguera, and V. Vento, nucl-th/0211036.
- [7] M. Sawicki and L. Mankiewicz, Phys. Rev. D **37**, 421 (1988); L. Mankiewicz and M. Sawicki, *ibid.* **40**, 3415 (1989).
- [8] B.L. Bakker, H.-M. Choi, and C.-R. Ji, Phys. Rev. D **63**, 074014 (2001).
- [9] A.V. Radyushkin, Phys. Rev. D **56**, 5524 (1997).
- [10] A.V. Radyushkin, Phys. Rev. D **59**, 014030 (1999).
- [11] A. Mukherjee, I.V. Musatov, H.C. Pauli, and A.V. Radyushkin, Phys. Rev. D **67**, 073014 (2003).
- [12] B.C. Tiburzi and G.A. Miller, Phys. Rev. D **67**, 013010 (2003).
- [13] B.C. Tiburzi and G.A. Miller, hep-ph/0205109; Phys. Rev. D **67**, 054014 (2003); **67**, 054015 (2003).
- [14] M.V. Polyakov and C. Weiss, Phys. Rev. D **60**, 114017 (1999).
- [15] O.V. Teryaev, Phys. Lett. B **510**, 125 (2001).
- [16] S. Weinberg, Phys. Rev. **150**, 1313 (1966).
- [17] S.D. Drell and T.-M. Yan, Phys. Rev. Lett. **24**, 181 (1970); G.B. West, *ibid.* **24**, 1206 (1970).
- [18] L. Mankiewicz, G. Piller, and T. Weigl, Eur. Phys. J. C **5**, 119 (1998).
- [19] A.V. Belitsky, D. Müller, A. Kirchner, and A. Schäfer, Phys. Rev. D **64**, 116002 (2001).
- [20] K.J. Golec-Biernat and A.D. Martin, Phys. Rev. D **59**, 014029 (1999).
- [21] C.H. Llewellyn-Smith, Ann. Phys. (N.Y.) **53**, 521 (1969).
- [22] T. Frederico and G.A. Miller, Phys. Rev. D **45**, 4207 (1992); **50**, 210 (1994).
- [23] B.C. Tiburzi and G.A. Miller, Phys. Rev. D **65**, 074009 (2002).
- [24] H.H. Liu and D.E. Soper, Phys. Rev. D **48**, 1841 (1993).
- [25] G.P. Lepage and S.J. Brodsky, Phys. Rev. D **22**, 2157 (1980).
- [26] J.P.B.C. de Melo, H.W. Naus, and T. Frederico, Phys. Rev. C **59**, 2278 (1999); J.P.B.C. de Melo, T. Frederico, E. Pace, and G. Salmè, Nucl. Phys. **A707**, 399 (2002).
- [27] W. Jaus, Phys. Rev. D **60**, 054026 (1999).
- [28] B. Pire, J. Soffer, and O. Teryaev, Eur. Phys. J. C **8**, 103 (1999).
- [29] P.V. Pobylitsa, Phys. Rev. D **65**, 077504 (2002); **65**, 114015 (2002).