Three-loop radiative-recoil corrections to hyperfine splitting in muonium

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We calculate three-loop radiative-recoil corrections to hyperfine splitting in muonium generated by the diagrams with the first order electron and muon polarization loop insertions in graphs with two exchanged photons. These corrections are enhanced by the large logarithm of the electron-muon mass ratio. The leading logarithm squared contribution was obtained a long time ago. Here we calculate the single-logarithmic and nonlogarithmic contributions. We previously calculated the three-loop radiative-recoil corrections generated by two-loop polarization insertions in the exchanged photons. The current paper therefore concludes calculation of all three-loop radiative-recoil corrections to hyperfine splitting in muonium generated by diagrams with closed fermion loop insertions in the exchanged photons. The new results obtained here improve the theory of hyperfine splitting, and affect the value of the electron-muon mass ratio extracted from experimental data on the muonium hyperfine splitting.

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I. INTRODUCTION

Recently we initiated a program of calculating all three-loop radiative-recoil corrections to hyperfine splitting (HFS) in muonium [1]. Three-loop radiative-recoil corrections are enhanced by the presence of the cube of the large logarithm of the electron-muon mass ratio [2]. All leading logarithm cubed and logarithm squared contributions of this order were calculated a long time ago [2–4] (see also reviews in [1,5]). As the first step of our program we obtained in [1] previously unknown single-logarithmic and nonlogarithmic radiative-recoil corrections of order $\alpha^2(Z\alpha)(m/M)\tilde{E}_F$[1] generated by graphs with two-loop polarization insertions (irreducible and reducible) in the two-photon exchange diagrams. As the next logical stage in implementing our program, we present below the calculation of all single-logarithmic and nonlogarithmic three-loop radiative-recoil corrections generated by diagrams with one-loop electron and muon polarization insertions in the exchanged photons. There are four gauge invariant sets of such three-loop diagrams, and we calculate all their contributions.

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[1]We define the Fermi energy as

$$\tilde{E}_F=\frac{16}{3}Z^4\alpha^2\frac{m}{M}\left(\frac{m_r}{m}\right)^3 chR_\infty, \quad (1)$$

where m and M are the electron and muon masses, m_r is the reduced mass of an electron-muon system, α is the fine structure constant, c is the velocity of light, h is the Planck constant, R_∞ is the Rydberg constant, and Z is the nucleus charge in terms of the electron charge ($Z=1$ for muonium). The Fermi energy $\tilde{E}_F$ does not include the muon anomalous magnetic moment a_μ which does not factorize in the case of recoil corrections, and should be considered on the same grounds as other corrections to hyperfine splitting.

II. RADIATIVE-RECOIL CORRECTION OF ORDER $\alpha(Z\alpha)(m/M)\tilde{E}_F$

All four sets of diagrams considered below can be obtained from the two-photon exchange diagrams with the radiative photons in the electron or muon lines by insertions of the one-loop electron or muon polarization operators. As was discussed in [1], it is sufficient to calculate contributions of these diagrams in the scattering approximation. In the calculations below we use the approach developed earlier for analytic calculation of the two-loop radiative-recoil corrections of orders $\alpha(Z\alpha)(m/M)\tilde{E}_F$ and $(Z^2\alpha)(Z\alpha)(m/M)\tilde{E}_F$ in [6,7] (these corrections were also calculated numerically in [8]). To make this paper self-contained we first briefly remind the reader of the main steps in the calculation of the corrections induced by the radiative photon insertions in the electron line.

The integral representation for the radiative corrections of order $\alpha(Z\alpha)(m/M)\tilde{E}_F$ generated by the graphs with radiative insertions in the electron line in Fig. 1 has the form [9,10]

$$\delta E^{e\text{-}line}=\alpha(Z\alpha)\tilde{E}_F\frac{1}{8\pi^2\mu}\int_0^1 dx\int_0^x dy\int \frac{d^4k}{i\pi^2(k^2+i0)^2}\left(\frac{1}{k^2+\mu^{-1}k_0+i0}+\frac{1}{k^2-\mu^{-1}k_0+i0}\right)\Bigg\{(3k_0^2-2\mathbf{k}^2)$$
$$\times\left[\frac{c_1\mathbf{k}^2+c_2(k^2)^2}{(-k^2+2bk_0+a^2)^3}+\frac{c_3k^2+c_4 2k_0}{(-k^2+2bk_0+a^2)^2}\right]-3k_0\left[\frac{c_5k^2+c_6k^2 2k_0}{(-k^2+2bk_0+a^2)^2}+\frac{c_7k^2}{-k^2+2bk_0+a^2}\right]\Bigg\}\equiv\sum_1^7 \delta E_i^{e\text{-}line}, \tag{2}$$

where the dimensionless integration momentum k is measured in units of the electron mass, the small parameter μ is defined as half the ratio of the electron and muon masses [$\mu=m/(2M)$], the auxiliary functions of the Feynman parameters $a(x,y)$ and $b(x,y)$ are defined by the relationships

$$a^2=\frac{x^2}{y(1-y)},\qquad b=\frac{1-x}{1-y}, \tag{3}$$

and explicit expressions for the coefficient functions c_i are collected in Table I. We used the Yennie gauge for the radiative photons in the derivation of the explicit expressions for these coefficient functions c_i.

After the Wick rotation and transition to four-dimensional spherical coordinates, the expression in Eq. (2) acquires the form

$$\delta E^{e-line}=\frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}\tilde{E}_F\frac{1}{8\pi\mu^2}\int_0^\pi d\theta\sin^2\theta\int_0^\infty dk^2\mathcal{D}(k,\theta)$$
$$\times\frac{k^2}{k^2+\mu^{-2}\cos^2\theta}\,\frac{1}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}, \tag{4}$$

where

$$\mathcal{D}(k,\theta)=\int_0^1 dx\int_0^x dy\Bigg\{(2+\cos^2\theta)\left[(c_1\sin^2\theta+c_2k^2)\left(\frac{\partial}{\partial a^2}\right)^2\right.$$
$$\left.+2c_3\frac{\partial}{\partial a^2}\right](k^2+a^2)-8bc_4(2+\cos^2\theta)\cos^2\theta\frac{\partial}{\partial a^2}$$
$$+12b\cos^2\theta\left(c_5\frac{\partial}{\partial a^2}-c_7\right)$$
$$-12c_6\cos^2\theta\frac{\partial}{\partial a^2}(k^2+a^2)\Bigg\}. \tag{5}$$

We introduced derivatives with respect to a^2 in order to reduce the powers in the denominators before integration over angles.

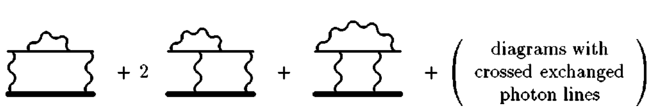

FIG. 1. Electron-line radiative-recoil corrections.

Next we separate the μ-dependent and μ-independent terms in the integrand with the help of the identity

$$\frac{1}{(k^2+\mu^{-2}\cos^2\theta)[(k^2+a^2)^2+4b^2k^2\cos^2\theta]}$$
$$=\frac{\mu^2}{(k^2+a^2)^2-4\mu^2b^2k^4}\left[\frac{1}{\mu^2k^2+\cos^2\theta}\right.$$
$$\left.-\frac{4b^2k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\right]$$
$$\approx\frac{\mu^2}{(k^2+a^2)^2}\left[\frac{1}{\mu^2k^2+\cos^2\theta}-\frac{4b^2k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\right], \tag{6}$$

where, at the last step, we omitted the term proportional to μ^2 in the denominator before the square bracket since, as can

TABLE I. Coefficients in the electron-line factor.

c_1	$\frac{16}{y(1-y)^3}[(1-x)(x-3y)-2y\ln x]$
c_2	$\frac{4}{y(1-y)^3}[-(1-x)(x-y-2y^2/x)+2(x-4y+4y^2/x)\ln x]$
c_3	$\frac{1}{y(1-y)^2}[1-6x-2x^2-(y/x)(26-6y/x-37x-2x^2+12xy+16\ln x)]$
c_4	$\frac{1}{y(1-y)^2}(2x-4x^2-5y+7xy)$
c_5	$\frac{1}{y(1-y)^2}(6x-3x^2-8y+2xy)$
c_6	$-b^2\frac{x-y}{x^2}$
c_7	$2\frac{1-x}{x}$

be shown, this term generates recoil corrections which are at least quadratic in the recoil factor μ. Then the integral for the radiative corrections in Eq. (4) becomes a sum of μ-dependent and μ-independent integrals

$$\delta E^{e-line}\approx \frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}\tilde{E}_F\frac{1}{8\pi}\int_0^{\pi}d\theta\sin^2\theta\int_0^{\infty}dk^2\mathcal{D}(k,\theta)\times\frac{k^2}{(k^2+a^2)^2}\left[\frac{1}{\mu^2k^2+\cos^2\theta}-\frac{4b^2k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\right]=\delta E^{e-line}_{\mu i}+\delta E^{e-line}_{ci},\tag{7}$$

which we will further call μ integrals and c integrals, respectively.

It is convenient to write the angular integrals in Eq. (7) in terms of the standard functions Φ^i_n defined as

$$\Phi_n(k)\equiv\frac{1}{\pi\mu^2}\int_0^{\pi}d\theta\sin^2\theta\cos^{2n}\theta\times\frac{(k^2+a^2)^2-4\mu^2b^2k^4}{(k^2+\mu^{-2}\cos^2\theta)[(k^2+a^2)^2+4b^2k^2\cos^2\theta]}=\frac{1}{\pi}\int_0^{\pi}d\theta\sin^2\theta\cos^{2n}\theta\left[\frac{1}{\mu^2k^2+\cos^2\theta}-\frac{4b^2k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\right]=\Phi^S_n(k)+\Phi^{\mu}_n(k)+\Phi^C_n(k).\tag{8}$$

Explicit expressions for these functions and their properties are collected in Appendix A (see also [10]).

The c integrals with the functions $\Phi^C_n(k)$ do not contain any free parameters and generate only contributions linear in m/M to HFS. These integrals are pure numbers, which admit analytic calculation. The μ integrals with the functions $\Phi^S_n(k)$, $\Phi^{\mu}_n(k)$ parametrically depend on μ and generate both nonrecoil and recoil contributions to HFS. Contributions of a fixed order in the small mass ratio (which are often enhanced by the large logarithms of the mass ratio) can be extracted from the μ integrals with the help of an auxiliary parameter σ chosen such that it satisfies the inequality $1\ll\sigma\ll\mu^{-1}$. The parameter σ is used to separate the momentum integration into two regions, a region of small momenta $0\leqslant k\leqslant\sigma$, and a region of large momenta $\sigma\leqslant k<\infty$. In the region of small momenta one uses the condition $\mu k\ll 1$ to simplify the integrand, and in the region of large momenta the same goal is achieved with the help of the condition $k\gg 1$. Note that for $k\simeq\sigma$ both conditions on the integration momenta are valid simultaneously, so in the sum of the low-momenta and high-momenta integrals all σ-dependent terms cancel and one obtains a σ-independent result for the total momentum integral (for a more detailed discussion of this method see, e.g., [11]). All contributions, up to and including the corrections quadratic in the small mass ratio m/M, were analytically calculated earlier in this framework [6,10] (see also [11,5]).

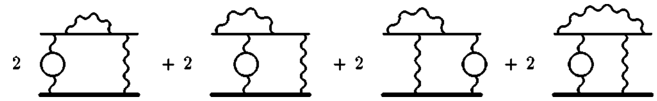

FIG. 2. Electron line and electron vacuum polarization.

III. DIAGRAMS WITH RADIATIVE PHOTONS IN THE ELECTRON LINE AND ELECTRON VACUUM POLARIZATION

The general expression for the contribution to HFS arising from the diagrams in Fig. 2[2] is obtained from the integral in Eq. (4) by insertion in the integrand of the doubled one-loop electron polarization $(\alpha/\pi)k^2I_e(k)$

$$2\left(\frac{\alpha}{\pi}\right)k^2I_e(k)=2\left(\frac{\alpha}{\pi}\right)k^2\int_0^1dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)},\tag{9}$$

where the additional multiplicity factor 2 corresponds to the fact that we can insert the vacuum polarization in each of the exchanged photons.

Then all contributions to HFS generated by the diagrams in Fig. 2 are given by the integral

$$\delta E^{ee}=\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F\frac{1}{4\pi\mu^2}\int_0^{\pi}d\theta\sin^2\theta\int_0^{\infty}dk^2\mathcal{D}(k,\theta)\times\frac{k^4}{k^2+\mu^{-2}\cos^2\theta}\frac{1}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\times\int_0^1dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}.\tag{10}$$

To obtain the radiative-recoil corrections of order $\alpha^2(Z\alpha)(m/M)\tilde{E}_F$ contained in this integral we follow the route described in the previous section. First we write the integral in Eq. (10) as a sum of seven μ- and seven c-integrals. These integrals are collected in Table II and Table III, where the dimensionless contributions $\delta\epsilon$ to the energy shifts are defined as $\delta E_i=\delta\epsilon_i\alpha^2(Z\alpha)(m/M)\tilde{E}_F/\pi^3$.

The c integrals in Table III automatically contain only contributions linear in the recoil factor m/M, and can be immediately calculated numerically. The respective results are again presented in Table III, and the total contribution of all c integrals is

[2]The graphs with the crossed exchanged photons are not shown explicitly in this figure and similar figures below.

TABLE II. μ integrals for the diagrams in Fig. 2.

$\delta\epsilon^{ee}_{\mu1}$	$\frac{1}{2}\int_0^1 dx\int_0^x dyc_1\int_0^\infty dk^2\frac{k^4}{(k^2+a^2)^3}[2\Phi_0^s(k)+2\Phi_0^\mu(k)-\Phi_1^\mu(k)-\Phi_2^\mu(k)]\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$
$\delta\epsilon^{ee}_{\mu2}$	$\frac{1}{2}\int_0^1 dx\int_0^x dyc_2\int_0^\infty dk^2\frac{k^6}{(k^2+a^2)^3}[2\Phi_0^s(k)+2\Phi_0^\mu(k)+\Phi_1^\mu(k)]\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$
$\delta\epsilon^{ee}_{\mu3}$	$-\frac{1}{2}\int_0^1 dx\int_0^x dyc_3\int_0^\infty dk^2\frac{k^4}{(k^2+a^2)^2}[2\Phi_0^s(k)+2\Phi_0^\mu(k)+\Phi_1^\mu(k)]\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$
$\delta\epsilon^{ee}_{\mu4}$	$4\int_0^1 dx\int_0^x dybc_4\int_0^\infty dk^2\frac{k^4}{(k^2+a^2)^3}[2\Phi_1^\mu(k)+\Phi_2^\mu(k)]\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$
$\delta\epsilon^{ee}_{\mu5}$	$-6\int_0^1 dx\int_0^x dybc_5\int_0^\infty dk^2\frac{k^4}{(k^2+a^2)^3}\Phi_1^\mu(k)\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$
$\delta\epsilon^{ee}_{\mu6}$	$3\int_0^1 dx\int_0^x dyc_6\int_0^\infty dk^2\frac{k^4}{(k^2+a^2)^2}\Phi_1^\mu(k)\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$
$\delta\epsilon^{ee}_{\mu7}$	$-3\int_0^1 dx\int_0^x dybc_7\int_0^\infty dk^2\frac{k^4}{(k^2+a^2)^2}\Phi_1^\mu(k)\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$

$$\delta E^{ee}_c=\delta\epsilon^{ee}_c\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\widetilde{E}_F=6.96182(3)\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\widetilde{E}_F. \tag{11}$$

The situation with the μ integrals is more complicated. Besides the recoil contributions linear in the small mass ratio m/M, they contain both nonrecoil contributions and the recoil contributions of higher order in the mass ratio. We would like to remove the already known nonrecoil contributions, to extract in analytic form all coefficients before the logarithmically enhanced terms linear in the recoil factor, and to calculate numerically the nonlogarthmic term which is linear in the recoil factor. To this end, we need to throw away the recoil contributions of higher orders which are also contained in the μ integrals and which are too small from the phenomenological point of view. If preserved, these higher order recoil contributions result only in the loss of accuracy in the numerical integrations. It is easy to see that the integrals with the function $\Phi_0^s(k)$ generate only already known nonrecoil contributions and can be safely omitted for our present goals. We extract the contributions linear (and logarithmically enhanced) in the recoil factor from the μ integrals by separating the integration region with the help of an auxiliary parameter σ as described in the previous section. All logarithms of the mass ratio originate from the high momentum parts of the μ integrals, where we can use the high energy asymptotic expansion of the polarization operator. Thus we obtain in the analytic form all coefficients before the logarithms of the mass ratio. The radiative-recoil corrections of order $\alpha(Z\alpha)(m/M)\widetilde{E}_F$ in Eq. (10), generated by the insertions of the radiative photons in the electron line, are linear in the large logarithm of the mass ratio. Hence the corrections of order $\alpha^2(Z\alpha)(m/M)\widetilde{E}_F$ are quadratic in this large logarithm. The coefficient before the logarithm squared, which we obtain in this way, coincides with the one obtained earlier [3], and the analytically obtained coefficient before the single-logarithmic term, as well as the nonlogarithmic contribution, are new. The results of the calculation of the μ integrals are collected in Table IV.

As an illustration of our methods let us obtain the first three entries in Table IV. We start with the integral in the first line in Table II. It contains a nonrecoil contribution

$$\delta\epsilon^{ee}_{\mu1}(\text{nonrecoil})=\int_0^1 dx\int_0^x dyc_1\int_0^\infty dk^2\frac{k^4}{(k^2+a^2)^3}\Phi_0^s(k)\times\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$$
$$=\frac{1}{\mu}\int_0^1 dx\int_0^x dyc_1\int_0^\infty dk^2\frac{k^3}{(k^2+a^2)^3}\times\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}. \tag{12}$$

This integral, as well as all other nonrecoil contributions of order $\alpha^2(Z\alpha)E_F$, was calculated analytically in [12], and

TABLE III. c integrals for the diagrams in Fig. 2.

$\delta\epsilon^{ee}_{c1}$	$-\frac{1}{4}\int_0^1 dx\int_0^x dyc_1\int_0^\infty dk^2\left(\frac{\partial}{\partial a^2}\right)^2\frac{k^4}{k^2+a^2}\frac{1}{\pi}\int_0^\pi d\theta\sin^2\theta$ $\times(2-\cos^2\theta-\cos^4\theta)\frac{4b^2k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	-0.916881 (3)
$\delta\epsilon^{ee}_{c2}$	$-\frac{1}{4}\int_0^1 dx\int_0^x dyc_2\int_0^\infty dk^2\left(\frac{\partial}{\partial a^2}\right)^2\frac{k^6}{k^2+a^2}\frac{1}{\pi}\int_0^\pi d\theta\sin^2\theta$ $\times(2+\cos^2\theta)\frac{4b^2k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	1.76474 (1)
$\delta\epsilon^{ee}_{c3}$	$-\frac{1}{2}\int_0^1 dx\int_0^x dyc_3\int_0^\infty dk^2\frac{\partial}{\partial a^2}\frac{k^4}{k^2+a^2}\frac{1}{\pi}\int_0^\pi d\theta\sin^2\theta$ $\times(2+\cos^2\theta)\frac{4b^2k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	5.93600 (5)
$\delta\epsilon^{ee}_{c4}$	$2\int_0^1 dx\int_0^x dybc_4\int_0^\infty dk^2\frac{\partial}{\partial a^2}\frac{k^4}{(k^2+a^2)^2}\frac{1}{\pi}\int_0^\pi d\theta\sin^2\theta\cos^2\theta$ $\times(2+\cos^2\theta)\frac{4b^2k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	0.0312952 (2)
$\delta\epsilon^{ee}_{c5}$	$-3\int_0^1 dx\int_0^x dybc_5\int_0^\infty dk^2\frac{\partial}{\partial a^2}\frac{k^4}{(k^2+a^2)^2}\frac{1}{\pi}\int_0^\pi d\theta\sin^2\theta$ $\times\cos^2\theta\frac{4b^2k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	0.0003483 (2)
$\delta\epsilon^{ee}_{c6}$	$3\int_0^1 dx\int_0^x dyc_6\int_0^\infty dk^2\frac{\partial}{\partial a^2}\frac{k^4}{k^2+a^2}\frac{1}{\pi}\int_0^\pi d\theta\sin^2\theta$ $\times\cos^2\theta\frac{4b^2k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	0.0543901 (2)
$\delta\epsilon^{ee}_{c7}$	$3\int_0^1 dx\int_0^x dybc_7\int_0^\infty dk^2\frac{k^4}{(k^2+a^2)^2}\frac{1}{\pi}\int_0^\pi d\theta\sin^2\theta$ $\times\cos^2\theta\frac{4b^2k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	0.0919342 (3)

numerically in [13], and we will not consider it here. We write the recoil part of the integral in the first line in Table II as a sum of low-momentum and high-momentum integrals[3]

$$\delta\epsilon^{ee}_{\mu1}=\frac{1}{2}\int_0^1 dx\int_0^x dyc_1\int_0^\infty dk^2\frac{k^4}{(k^2+a^2)^3}[2\Phi_0^\mu(k)-\Phi_1^\mu(k)-\Phi_2^\mu(k)]\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}=\delta\epsilon^{ee>}_{\mu1}+\delta\epsilon^{ee<}_{\mu1}. \tag{13}$$

First we estimate the high momentum part $\delta\epsilon^{ee>}_{\mu1}$ considering

[3] Here, as well as in similar cases below, we allow ourselves a slightly confusing notation, using the same symbol $\delta\epsilon^{ee}_{\mu1}$ both for the total contribution of the μ integral and for its recoil part. We hope this will not lead to any misunderstanding since in this paper we are interested only in the recoil corrections.

TABLE IV. Simplified μ integrals for the diagrams in Fig. 2.

$\delta\epsilon^{ee}_{\mu 1}$	$-\frac{21}{16}\int_0^1 dx\int_0^x dyc_1\int_0^\infty dk^2\frac{k^4}{(k^2+a^2)^3}\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	$-2.105912\ (2)$
$\delta\epsilon^{ee}_{\mu 2a+\mu 3a}$	$\left(\frac{\pi^2}{3}-\frac{5}{3}\right)\ln^2\frac{M}{m}+\left(\frac{4\pi^2}{9}-\frac{20}{9}\right)\ln\frac{M}{m}+\frac{\pi^4}{18}-\frac{5\pi^2}{18}+\frac{14(\pi^2-5)}{27}+\frac{3}{4}\int_0^1 dx\int_0^\infty dk^2\Big\{(-1+6x+2x^2)\Big[\ln k^2-\ln(k^2+a_1^2)-\frac{a_1^2}{k^2+a_1^2}\Big]-4x(-1+x+2\ln x)\Big[-\ln k^2+\ln(k^2+a_1^2)+\frac{2a_1^2}{k^2+a_1^2}-\frac{1}{2}\frac{a_1^4}{(k^2+a_1^2)^2}\Big]\Big\}\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	$\left(\frac{\pi^2}{3}-\frac{5}{3}\right)\ln^2\frac{M}{m}+\left(\frac{4\pi^2}{9}-\frac{20}{9}\right)\ln\frac{M}{m}+\frac{\pi^4}{18}-\frac{5\pi^2}{18}+0.4698395\ (7)$
$\delta\epsilon^{ee}_{\mu 2b}$	$\left(-\frac{5\pi^2}{3}+\frac{271}{18}\right)\ln^2\frac{M}{m}+\left(-\frac{20\pi^2}{9}+\frac{542}{27}\right)\ln\frac{M}{m}-\frac{1}{54}(30\pi^2-271)(8\ln 2-3)-\frac{3}{4}\int_0^1 dx\int_0^x dyc_{2b}\times\int_0^\infty dk^2\left[\frac{k^6}{(k^2+a^2)^3}-\frac{k^2}{k^2+4}\right]\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	$\left(-\frac{5\pi^2}{3}+\frac{271}{18}\right)\ln^2\frac{M}{m}+\left(-\frac{20\pi^2}{9}+\frac{542}{27}\right)\ln\frac{M}{m}-\frac{5\pi^4}{18}+\frac{271\pi^2}{108}-0.797177\ (4)$
$\delta\epsilon^{ee}_{\mu 3b}$	$\left(\frac{4\pi^2}{3}-\frac{89}{9}\right)\ln^2\frac{M}{m}+\left(\frac{16\pi^2}{9}-\frac{356}{27}\right)\ln\frac{M}{m}+\frac{2\pi^4}{9}-\frac{89\pi^2}{54}+\frac{1}{27}(12\pi^2-89)(8\ln 2-3)+\frac{3}{4}\int_0^1 dx\int_0^x dyc_{3b}\times\int_0^\infty dk^2\left[\frac{k^4}{(k^2+a^2)^2}-\frac{k^2}{k^2+4}\right]\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	$\left(\frac{4\pi^2}{3}-\frac{89}{9}\right)\ln^2\frac{M}{m}+\left(\frac{16\pi^2}{9}-\frac{356}{27}\right)\ln\frac{M}{m}+\frac{2\pi^4}{9}-\frac{89\pi^2}{54}+5.398568\ (7)$
$\delta\epsilon^{ee}_{\mu 4}$	$\frac{9}{2}\int_0^1 dx\int_0^x dybc_4\int_0^\infty dk^2\frac{k^4}{(k^2+a^2)^3}\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	$-0.3738824\ (5)$
$\delta\epsilon^{ee}_{\mu 5}$	$-3\int_0^1 dx\int_0^x dybc_5\int_0^\infty dk^2\frac{k^4}{(k^2+a^2)^3}\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	$0.127357\ (3)$
$\delta\epsilon^{ee}_{\mu 6}$	$\left(\frac{\pi^2}{3}-\frac{7}{2}\right)\ln^2\frac{M}{m}+\left(-\frac{8\pi^2}{9}+\frac{28}{3}\right)\ln\frac{M}{m}+\frac{\pi^4}{18}-\frac{5\pi^2}{36}-\frac{14}{3}+\frac{1}{18}(2\pi^2-21)(8\ln 2-3)+\frac{3}{2}\int_0^1 dx\int_0^x dyc_6\times\int_0^\infty dk^2\left[\frac{k^4}{(k^2+a^2)^2}-\frac{k^2}{k^2+4}\right]\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	$\left(\frac{\pi^2}{3}-\frac{7}{2}\right)\ln^2\frac{M}{m}+\left(-\frac{8\pi^2}{9}+\frac{28}{3}\right)\ln\frac{M}{m}+\frac{\pi^4}{18}-\frac{5\pi^2}{36}-\frac{14}{3}-0.1944535\ (6)$
$\delta\epsilon^{ee}_{\mu 7}$	$\left(-\frac{\pi^2}{3}+\frac{5}{2}\right)\ln^2\frac{M}{m}+\left(\frac{8\pi^2}{9}-\frac{20}{3}\right)\ln\frac{M}{m}-\frac{\pi^4}{18}-\frac{\pi^2}{36}+\frac{10}{3}-\frac{1}{18}(2\pi^2-15)(8\ln 2-3)-\frac{3}{2}\int_0^1 dx\int_0^x dybc_7\times\int_0^\infty dk^2\left[\frac{k^4}{(k^2+a^2)^2}-\frac{k^2}{k^2+4}\right]\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	$\left(-\frac{\pi^2}{3}+\frac{5}{2}\right)\ln^2\frac{M}{m}+\left(\frac{8\pi^2}{9}-\frac{20}{3}\right)\ln\frac{M}{m}-\frac{\pi^4}{18}-\frac{\pi^2}{36}+\frac{10}{3}-0.8472862\ (4)$

the leading term in the integrand in the high momentum limit (more technical details on similar estimates can be found in [10])

$$\delta\epsilon^{ee>}_{\mu 1}=\frac{1}{2}\int_0^1 dx\int_0^x dy\, c_1\int_{\sigma^2}^\infty dk^2\frac{k^4}{(k^2+a^2)^3}[2\Phi^\mu_0(k)-\Phi^\mu_1(k)-\Phi^\mu_2(k)]\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$$
$$\sim\int_\sigma^\infty\frac{dk^2}{k^4}\ln^2 k^2[2\Phi^\mu_0(k)-\Phi^\mu_1(k)-\Phi^\mu_2(k)]$$
$$\sim\frac{1}{\sigma^2}\ln^2\sigma. \tag{14}$$

We see that the high momentum contribution is suppressed like $1/\sigma^2$. We have already extracted the recoil factor from $\delta\epsilon^{ee}_{\mu 1}$ explicitly, so now we are looking only for such contributions which are not additionally suppressed. Hence, in the leading order in the recoil parameter we can omit the high-momentum contribution to $\delta\epsilon^{ee}_{\mu 1}$, and in our approximation the total contribution to the energy shift is given by the low-momentum integral

$$\delta\epsilon^{ee<}_{\mu 1}=\frac{1}{2}\int_0^1 dx\int_0^x dy\, c_1\int_0^{\sigma^2} dk^2\,\frac{k^4}{(k^2+a^2)^3}[2\Phi^\mu_0(k)-\Phi^\mu_1(k)-\Phi^\mu_2(k)]\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}. \tag{15}$$

To extract the leading nonvanishing contribution to this integral at $\mu\to 0$ we substitute in the integrand the leading terms in the small μ expansion of the functions $\Phi^\mu_i(k)$ (see Appendix A) and obtain

$$\delta\epsilon^{ee<}_{\mu 1}\approx-\frac{21}{16}\int_0^1 dx\int_0^x dy\, c_1\int_0^{\sigma^2} dk^2\frac{k^4}{(k^2+a^2)^3}\times\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}. \tag{16}$$

The momentum integral is ultravioletly convergent, so the magnitude of the integral with an infinite upper limit differs from the integral in Eq. (16) only by inverse powers of σ, which we omit anyway. Hence, the total $\mu 1$ contribution to the energy shift is given by the integral

$$\delta\epsilon^{ee}_{\mu 1}=-\frac{21}{16}\int_0^1 dx\int_0^x dy\, c_1\int_0^\infty dk^2\,\frac{k^4}{(k^2+a^2)^3}\times\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}, \tag{17}$$

which is just the first entry in Table IV.

Let us turn now to the next integrals in Table II. Each of the integrals $\delta\epsilon^{ee}_{\mu 2}$ and $\delta\epsilon^{ee}_{\mu 3}$ generates spurious logarithm cubed contributions which cancel in the sum of these terms. It is therefore convenient to rearrange these terms in such way that spurious logarithm cubed terms do not arise at all. To this end we write the coefficient functions c_2 in the form

$$c_2=\frac{4(1-2y)}{y(1-y)}x(-1+x+2\ln x)+\frac{4}{(1-y)^3}[-(1-x)\times(-1-2y/x)+2(-4+4y/x)\ln x+x(-1+x+2\ln x)\times(4-5y+2y^2)]$$
$$\equiv c_{2a}+c_{2b}. \tag{18}$$

We have chosen c_{2a} essentially as the singular part of the function c_2 at $y\simeq 0$ and multiplied this singular part by the factor $(1-2y)/(1-y)$ in order to simplify integration over y. Since $(1-2y)dy/[y(1-y)]=-da^2/a^2$ this makes integration over y trivial. The spurious logarithm cubed term originates only from the integral of the function c_{2a}. On the other hand, the integrals over x and y with the function c_{2b} remain finite even if we omit a^2 in the denominator $(k^2+a^2)^3$, which leads to significant simplifications in the high-momentum part of the integral $\delta\epsilon^{ee}_{\mu 2}$.

Following the same logic, we represent the coefficient functions c_3 also as a sum of two functions c_{3a} and c_{3b}

$$c_3=\frac{1-2y}{y(1-y)}(1-6x-2x^2)+\frac{1}{(1-y)^2}\{2(1-y)\times(1-6x-2x^2)+(1-y)(8+2x)-(1/x)[26(1-x)+2x(1-x)-6(1-x^2)y/x-6x(1-y)+16\ln x]\}$$
$$\equiv c_{3a}+c_{3b}. \tag{19}$$

Now the sum of the recoil contributions to HFS generated by the $\mu 2$ and $\mu 3$ integrals in Table II may be written in the form

$$\delta\epsilon^{ee}_{\mu 2}+\delta\epsilon^{ee}_{\mu 3}=\delta\epsilon^{ee}_{\mu 2a+\mu 3a}+\delta\epsilon^{ee}_{\mu 2b}+\delta\epsilon^{ee}_{\mu 3b}. \tag{20}$$

Let us consider calculation of the $\delta\epsilon^{ee}_{\mu 2a+\mu 3a}$ contribution

$$\delta\epsilon^{ee}_{\mu 2a+\mu 3a}=\frac{1}{2}\int_0^1 dx\int_0^x dy\int_0^\infty dk^2\left[\frac{c_{2a}k^6}{(k^2+a^2)^3}-\frac{c_{3a}k^4}{(k^2+a^2)^2}\right]\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}[2\Phi_0^\mu(k)+\Phi_1^\mu(k)]=\epsilon^{ee>}_{\mu 2a+\mu 3a}+\epsilon^{ee<}_{\mu 2a+\mu 3a}. \tag{21}$$

As mentioned above, calculation of the y integral becomes trivial after the change of variables $y\to a^2$. In the high-momentum integral we next expand over the inverse powers of k^2, and use the asymptotic expansions of the arising functions V_{lnm} (these functions were defined and their properties were discussed in [10])

$$\epsilon^{ee>}_{\mu 2a+\mu 3a}=\frac{1}{2}\int_{\sigma^2}^\infty\frac{dk^2}{k^2}\left[\left(-\frac{16}{3}\ln k-\frac{2\pi^2}{3}+10\right)-\left(-\frac{16}{3}\ln k+\frac{20}{3}\right)\right]\left(\frac{2}{3}\ln k-\frac{5}{9}\right)[2\Phi_0^\mu(k)+\Phi_1^\mu(k)]$$
$$\approx\left(-\frac{\pi^2}{6}+\frac{5}{6}\right)\left\{\frac{4}{3}[2V_{110}+V_{111}]-\frac{10}{9}[2V_{100}+V_{101}]\right\}$$
$$\approx\left(\frac{\pi^2}{3}-\frac{5}{3}\right)\ln^2\frac{M}{m}+\left(\frac{4\pi^2}{9}-\frac{20}{9}\right)\ln\frac{M}{m}+\frac{\pi^4}{18}-\frac{5\pi^2}{18}+\left(-\frac{\pi^2}{3}+\frac{5}{3}\right)\ln^2\sigma+\left(\frac{5\pi^2}{9}-\frac{25}{9}\right)\ln\sigma. \tag{22}$$

In the low-momentum part of the integral we again preserve only the leading terms in the small μ expansion of the functions $\Phi_i^\mu(k)$

$$\epsilon^{ee<}_{\mu 2a+\mu 3a}=\frac{3}{4}\int_0^1 dx\int_0^x dy\int_0^{\sigma^2} dk^2\left[-\frac{c_{2a}k^6}{(k^2+a^2)^3}+\frac{c_{3a}k^4}{(k^2+a^2)^2}\right]\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)} \tag{23}$$

and integrate explicitly over y

$$\epsilon^{ee<}_{\mu 2a+\mu 3a}=\left(\frac{\pi^2}{3}-\frac{5}{3}\right)\ln^2\sigma+\left(-\frac{5\pi^2}{9}+\frac{25}{9}\right)\ln\sigma+\frac{14(\pi^2-5)}{27}+\frac{3}{4}\int_0^1 dx\int_0^\infty dk^2\left\{(-1+6x+2x^2)\left[\ln k^2-\ln(k^2+a_1^2)-\frac{a_1^2}{k^2+a_1^2}\right]-4x(-1+x+2\ln x)\left[-\ln k^2+\ln(k^2+a_1^2)+\frac{2a_1^2}{k^2+a_1^2}-\frac{1}{2}\frac{a_1^4}{(k^2+a_1^2)^2}\right]\right\}\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}, \tag{24}$$

where $a_1^2=x/(1-x)$. Note that we again extended the momentum integration to infinity in the finite integrals for the low-momentum contribution. Finally, summing the high- and low-momentum contributions we obtain the σ-independent result

$$\epsilon^{ee}_{\mu 2a+\mu 3a}=\left(\frac{\pi^2}{3}-\frac{5}{3}\right)\ln^2\frac{M}{m}+\left(\frac{4\pi^2}{9}-\frac{20}{9}\right)\ln\frac{M}{m}+\frac{\pi^4}{18}-\frac{5\pi^2}{18}+\frac{14(\pi^2-5)}{27}+\frac{3}{4}\int_0^1 dx\int_0^\infty dk^2\left\{(-1+6x+2x^2)\left[\ln k^2-\ln(k^2+a_1^2)-\frac{a_1^2}{k^2+a_1^2}\right]-4x(-1+x+2\ln x)\left[-\ln k^2+\ln(k^2+a_1^2)+\frac{2a_1^2}{k^2+a_1^2}-\frac{1}{2}\frac{a_1^4}{(k^2+a_1^2)^2}\right]\right\}\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}, \tag{25}$$

which is convenient for numerical calculations.

For our final example let us derive an expression for $\delta\epsilon^{ee}_{\mu 2b}$ in the third line in Table IV. We start with the general expression for this contribution from Table II

$$\delta\epsilon^{ee}_{\mu 2b}=\frac{1}{2}\int_0^1 dx\int_0^x dy\int_0^\infty dk^2\,\frac{c_{2b}k^6}{(k^2+a^2)^3}\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}[2\Phi_0^\mu(k)+\Phi_1^\mu(k)]. \tag{26}$$

As we already mentioned above in this case we are free to omit a^2 in comparison with k^2 in the denominator in the high-momentum part of this integral

$$\epsilon^{ee>}_{\mu 2b}=\frac{1}{2}\int_0^1 dx\int_0^x dy\int_{\sigma^2}^{\infty}dk^2 c_{2b}\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}[2\Phi_0^{\mu}(k)+\Phi_1^{\mu}(k)]$$
$$\approx\frac{1}{2}\int_{\sigma}^{\infty}\frac{dk^2}{k^2}\left(\frac{10\pi^2}{3}-\frac{271}{9}\right)\left(\frac{2}{3}\ln k-\frac{5}{9}\right)[2\Phi_0^{\mu}(k)+\Phi_1^{\mu}(k)]$$
$$\approx\left(\frac{5\pi^2}{6}-\frac{271}{36}\right)\left\{\frac{4}{3}[2V_{110}+V_{111}]-\frac{10}{9}[2V_{100}+V_{101}]\right\}$$
$$\approx\left(-\frac{5\pi^2}{3}+\frac{271}{18}\right)\ln^2\frac{M}{m}+\left(-\frac{20\pi^2}{9}+\frac{542}{27}\right)\ln\frac{M}{m}-\frac{5\pi^4}{18}+\frac{271\pi^2}{108}+\left(\frac{5\pi^2}{3}-\frac{271}{18}\right)\ln^2\sigma$$
$$+\left(-\frac{25\pi^2}{9}+\frac{1355}{54}\right)\ln\sigma. \quad (27)$$

Dealing with the low-momentum part of the integral in Eq. (26) we would like to extract analytically leading logarithms of σ. This would allow us to get rid of any trace of the parameter σ in the final expression for this integral, making it much more suitable for further numerical calculations. The double integral over x and y, with the integrand including the function c_{2b}, cannot be calculated analytically as easily as the integral with the function $\epsilon^{ee}_{\mu 2a+\mu 3a}$ above. In order to overcome this difficulty we use the identity

$$\frac{k^4}{(k^2+a^2)^3}=\frac{1}{k^2+4}+\left[\frac{k^4}{(k^2+a^2)^3}-\frac{1}{k^2+4}\right] \quad (28)$$

in the low-momentum part of the integral in Eq. (26). Analytic calculation of the integral with the first term on the right-hand side (RHS) is simple. On the other hand, all logarithms of σ are supplied by this integral since the second term on the RHS in Eq. (28) decreases at large k faster than the first term. Then we obtain the low-momentum contribution to the integral in Eq. (26) in the form

$$\delta\epsilon^{ee<}_{\mu 2b}=\left(-\frac{5\pi^2}{3}+\frac{271}{18}\right)\ln^2\sigma+\left(\frac{25\pi^2}{9}-\frac{1355}{54}\right)\ln\sigma$$
$$-\frac{1}{54}(30\pi^2-271)(8\ln 2-3)$$
$$-\frac{3}{4}\int_0^1 dx\int_0^x dy c_{2b}\int_0^{\infty}dk^2\left[\frac{k^6}{(k^2+a^2)^3}\right.$$
$$\left.-\frac{k^2}{k^2+4}\right]\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}. \quad (29)$$

Finally, the total expression for the $\mu 2b$ integral has the form

$$\delta\epsilon^{ee}_{\mu 2b}=\left(-\frac{5\pi^2}{3}+\frac{271}{18}\right)\ln^2\frac{M}{m}+\left(-\frac{20\pi^2}{9}+\frac{542}{27}\right)\ln\frac{M}{m}$$
$$-\frac{5\pi^4}{18}+\frac{271\pi^2}{108}-\frac{1}{54}(30\pi^2-271)(8\ln 2-3)$$
$$-\frac{3}{4}\int_0^1 dx\int_0^x dy c_{2b}\int_0^{\infty}dk^2\left[\frac{k^6}{(k^2+a^2)^3}\right.$$
$$\left.-\frac{k^2}{k^2+4}\right]\int_0^1 dv\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}. \quad (30)$$

Calculation of the remaining μ integrals goes along similar lines, and does not require any additional new tricks. Calculating the nonlogarithmic contributions numerically and collecting all μ-integrals from Table IV, we obtain the total contribution of all μ integrals

$$\delta\epsilon^{ee}_{\mu}=\frac{5}{2}\ln^2\frac{M}{m}+\frac{22}{3}\ln\frac{M}{m}+4.45606(1). \quad (31)$$

The sum of μ and c integrals in Eq. (31) and Eq. (11) gives all radiative-recoil corrections of order $\alpha^2(Z\alpha)(m/M)\widetilde{E}_F$ generated by the diagrams in Fig. 2,

$$\delta\epsilon^{ee}=\delta\epsilon^{ee}_{\mu}+\delta\epsilon^{ee}_{c}=\frac{5}{2}\ln^2\frac{M}{m}+\frac{22}{3}\ln\frac{M}{m}+11.41788(3). \quad (32)$$

IV. DIAGRAMS WITH RADIATIVE PHOTONS IN THE ELECTRON LINE AND MUON VACUUM POLARIZATION

Let us consider now the diagrams in Fig. 3. The only difference between these diagrams and the diagrams in Fig. 2

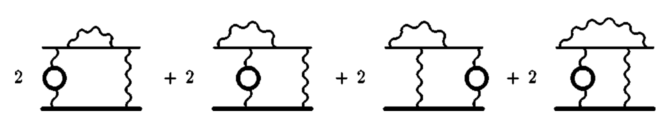

FIG. 3. Electron line and muon vacuum polarization.

from the previous section is that they contain the muon vacuum polarization insertion,[4]

$$2\left(\frac{\alpha}{\pi}\right)k^2I_\mu(k)=2\left(\frac{\alpha}{\pi}\right)k^2\int_0^1 dv\frac{v^2(1-v^2/3)}{\mu^{-2}+k^2(1-v^2)}, \quad (33)$$

instead of the electron vacuum polarization, and the respective contribution to HFS has the form [compare Eq. (10)]

$$\delta E^{e\mu}=\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F\frac{1}{4\pi\mu^2}\int_0^\pi d\theta\sin^2\theta\int_0^\infty dk^2\mathcal{D}(k,\theta)$$
$$\times\frac{k^4}{k^2+\mu^{-2}\cos^2\theta}\frac{1}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}$$
$$\times\int_0^1 dv\frac{v^2(1-v^2/3)}{\mu^{-2}+k^2(1-v^2)}. \quad (34)$$

We transform the integrand in the sum of μ and c integrals exactly as in the previous section, the only difference being that each integrand now contains as a factor the muon vacuum polarization insertion from Eq. (33) instead of the electron vacuum polarization insertion. This immediately leads to significant simplification of further calculations in comparison to the case of the electron polarization. The muon polarization insertion in Eq. (34) is suppressed as $k^2\mu^2$ at integration momenta much smaller than the muon mass. But the characteristic integration momenta in the c integrals are of order one in our dimensionless units, and hence now all c integrals do not generate corrections linear in the recoil factor m/M. Low-momenta parts of the μ integrals, where integration goes over momenta $k\leqslant\sigma\ll 1/\mu$, are also suppressed as $\sigma^2\mu^2$. Only the high-momenta parts of the μ integrals, where integration effectively goes over the momenta comparable to $1/\mu$, generate contributions linear in $m/M=2\mu$. Such contributions are present only in the μ integrals connected with the coefficient functions c_2, c_3, c_6, and c_7. Note that the integrals with the coefficient functions c_2 and c_3 contain the function $\Phi_0^s(k)=1/(\mu k)$ in the integrand. In the case of the electron vacuum polarization (and the absence of any polarization at all) the integrals with this function generated nonrecoil corrections only. This essentially happened because the characteristic integration momenta were of order one, and in this region the contribution of the function $1/(\mu k)$ is obviously enhanced as $1/\mu$. In the present case, characteristic integration momenta are about $1/(\mu k)$, and the integrals with the function $\Phi_0^s(k)$ generate recoil corrections on a par with the integrals with the other functions $\Phi(k)$.

Integration over the Feynman parameters x and y in the high-momenta integrals simplifies and may be performed analytically, and we obtain all one-dimensional momenta integrals which generate radiative-recoil corrections in the case of muon polarization insertions

$$\delta\epsilon_{\mu 2}^{e\mu>}\approx 2\int_{\sigma^2}^\infty dk^2\left[\left(-\frac{4}{3}\ln k-\frac{\pi^2}{6}+\frac{5}{2}\right)+\left(\frac{5\pi^2}{6}-\frac{271}{36}\right)\right]$$
$$\times[2\Phi_0^s(k)+2\Phi_0^\mu(k)+\Phi_1^\mu(k)]$$
$$\times\int_0^1 dv\frac{v^2(1-v^2/3)}{\mu^{-2}+k^2(1-v^2)}, \quad (35)$$

$$\delta\epsilon_{\mu 3}^{e\mu>}\approx 2\int_{\sigma^2}^\infty dk^2\left[\left(\frac{4}{3}\ln k-\frac{5}{3}\right)+\left(-\frac{2\pi^2}{3}+\frac{89}{18}\right)\right][2\Phi_0^s(k)$$
$$+2\Phi_0^\mu(k)+\Phi_1^\mu(k)]\int_0^1 dv\frac{v^2(1-v^2/3)}{\mu^{-2}+k^2(1-v^2)}, \quad (36)$$

$$\delta\epsilon_{\mu 6}^{e\mu>}\approx\left(\pi^2-\frac{21}{2}\right)\int_\sigma^\infty dk^2\Phi_1^\mu(k)\int_0^1 dv\frac{v^2(1-v^2/3)}{\mu^{-2}+k^2(1-v^2)}, \quad (37)$$

$$\delta\epsilon_{\mu 7}^{e\mu>}\approx\left(-\pi^2+\frac{15}{2}\right)\int_\sigma^\infty dk^2\Phi_1^\mu(k)\int_0^1 dv\frac{v^2(1-v^2/3)}{\mu^{-2}+k^2(1-v^2)}. \quad (38)$$

The sum of all these contributions has the form

$$\delta\epsilon_\mu^{e\mu>}\approx-\int_{\sigma^2}^\infty dk^2\left\{\frac{7}{2}[2\Phi_0^s(k)+2\Phi_0^\mu(k)+\Phi_1^\mu(k)]\right.$$
$$\left.+3\Phi_1^\mu(k)\right\}\int_0^1 dv\frac{v^2(1-v^2/3)}{\mu^{-2}+k^2(1-v^2)}, \quad (39)$$

or explicitly

$$\delta\epsilon_\mu^{e\mu>}\approx-\int_{\sigma^2}^\infty dk^2\left\{\frac{7}{2}\left[\frac{2}{\mu k}(\sqrt{1+\mu^2k^2}-\mu k)\right.\right.$$
$$\left.+\left(-\mu k\sqrt{1+\mu^2k^2}+\mu^2k^2+\frac{1}{2}\right)\right]$$
$$\left.+3\left(-\mu k\sqrt{1+\mu^2k^2}+\mu^2k^2+\frac{1}{2}\right)\right\}$$
$$\times\int_0^1 dv\frac{v^2(1-v^2/3)}{\mu^{-2}+k^2(1-v^2)}. \quad (40)$$

We can extend the momentum integration region in this integral to zero, since contributions from small momenta are additionally suppressed by powers of $\sigma\mu$. It is also natural to rescale the integration momentum $k\to\mu k$. Then the expression for the radiative-recoil corrections has the form

[4]We ascribe an extra factor Z to each photon emission by the heavy line, but being somewhat inconsequential, do not write any Z factor in the muon vacuum polarization.

$$\delta\epsilon_\mu^{e\mu}=-\int_0^\infty dk\left\{7\left[2(\sqrt{1+k^2}-k)+k\left(-k\sqrt{1+k^2}+k^2+\frac{1}{2}\right)\right]+6k\left(-k\sqrt{1+k^2}+k^2+\frac{1}{2}\right)\right\}\times\int_0^1 dv\,\frac{v^2(1-v^2/3)}{1+k^2(1-v^2)}. \quad (41)$$

The integral in the first line of this equation is proportional to the integral for the muon polarization contribution of order $\alpha(Z\alpha)(m/M)\tilde{E}_F$ in [10], and the contribution of the integral in the second line may be calculated in the same way. Finally we obtain

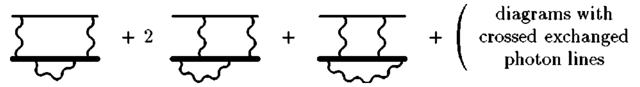

FIG. 4. Muon-line radiative-recoil corrections.

$$\delta\epsilon_\mu^{e\mu}=-\frac{5\pi^2}{12}+\frac{1}{18}=-4.0567796\ldots. \quad (42)$$

V. DIAGRAMS WITH RADIATIVE PHOTONS IN THE MUON LINE AND MUON VACUUM POLARIZATION

The expression in Eq. (2), for the radiative corrections generated by the diagrams with the radiative photon insertions in the electron line Fig. 1, was obtained without expansion in the mass ratio. Hence, after the substitutions $m\leftrightarrow M$ and $\alpha\to Z^2\alpha$ it goes into the expression for the corrections generated by the diagrams in Fig. 4,[5]

$$\delta E^{\mu\text{-}line}=\frac{(Z^2\alpha)(Z\alpha)}{\pi^2}\frac{m}{M}E_F\frac{1}{4}\int_0^1 dx\int_0^x dy\int\frac{d^4k}{i\pi^2(k^2+i0)^2}\left(\frac{1}{k^2+4\mu k_0+i0}+\frac{1}{k^2-4\mu k_0+i0}\right)\left\{(3k_0^2-2\mathbf{k}^2)\times\left[\frac{c_1\mathbf{k}^2+c_2(k^2)^2}{(-k^2+2bk_0+a^2)^3}+\frac{c_3k^2+c_42k_0}{(-k^2+2bk_0+a^2)^2}\right]-3k_0\left[\frac{c_5k^2+c_6k^22k_0}{(-k^2+2bk_0+a^2)^2}+\frac{c_7k^2}{-k^2+2bk_0+a^2}\right]\right\}. \quad (43)$$

The dimensionless integration momentum here is measured in muon mass units, and apparent incomplete symmetry with the expression in Eq. (2) is due to asymmetry of the parameter $\mu=m/(2M)$.

Unlike the case of the radiative photon insertions in the electron line in Sec. II, the integral in Eq. (43) does not generate nonrecoil contributions [7,14]. This happens because the radiative photon insertions in the muon line suppress the contribution from the integration momenta of order of the electron mass which were responsible for the nonrecoil correction by an extra factor μ^2. Hence, for calculation of the leading order recoil corrections we can safely allow $\mu\to 0$ in Eq. (43),

$$\delta E^{\mu\text{-}line}\approx\frac{(Z^2\alpha)(Z\alpha)}{\pi^2}\frac{m}{M}E_F\frac{1}{2}\int_0^1 dx\int_0^x dy\int\frac{d^4k}{i\pi^2(k^2+i0)^3}\left\{(3k_0^2-2\mathbf{k}^2)\left[\frac{c_1\mathbf{k}^2+c_2(k^2)^2}{(-k^2+2bk_0+a^2)^3}+\frac{c_3k^2+c_42k_0}{(-k^2+2bk_0+a^2)^2}\right]-3k_0\left[\frac{c_5k^2+c_6k^22k_0}{(-k^2+2bk_0+a^2)^2}+\frac{c_7k^2}{-k^2+2bk_0+a^2}\right]\right\}. \quad (44)$$

After the Wick rotation and transition to the four-dimensional spherical coordinates we have

$$\delta E^{\mu\text{-}line}=\frac{(Z^2\alpha)(Z\alpha)}{\pi^2}\frac{m}{M}E_F\frac{1}{2\pi}\int_0^\pi d\theta\sin^2\theta\times\int_0^\infty dk^2\mathcal{D}(k,\theta)\frac{1}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}, \quad (45)$$

where the differential operator $\mathcal{D}(k,\theta)$ was defined in Eq. (5). There are no μ-dependent terms in Eq. (45), and this expression for the energy shift is similar to the c integrals in Eq. (7). It can be formally obtained from the expression in Eq. (7) by the substitution

$$-\frac{b^2k^4}{(k^2+a^2)^2}\to Z^2.$$

All corrections to HFS described by the integral in Eq. (45) were analytically calculated in [7]. Contributions to the energy shift generated by the diagrams in Fig. 5 are obtained from the expression in Eq. (45) by insertion in the integrand of the doubled muon vacuum polarization

$$2\left(\frac{\alpha}{\pi}\right)k^2I_\mu(k)=2\left(\frac{\alpha}{\pi}\right)k^2\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}. \quad (46)$$

[5]We use the convention that the heavy line charge is Ze.

TABLE V. c integrals for the diagrams in Fig. 5.

$\delta\epsilon_1^{\mu\mu}$	$\int_0^1 dx \int_0^x dy c_1 \int_0^\infty dk^2 \left(\frac{\partial}{\partial a^2}\right)^2 \frac{1}{\pi}\int_0^\pi d\theta \sin^2\theta \times (2-\cos^2\theta-\cos^4\theta)\frac{k^2(k^2+a^2)}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv \frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	0.297870 (1)
$\delta\epsilon_2^{\mu\mu}$	$\int_0^1 dx \int_0^x dy c_2 \int_0^\infty dk^2 \left(\frac{\partial}{\partial a^2}\right)^2 \frac{1}{\pi}\int_0^\pi d\theta \sin^2\theta \times (2+\cos^2\theta)\frac{k^4(k^2+a^2)}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv \frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	-0.759522 (5)
$\delta\epsilon_3^{\mu\mu}$	$2\int_0^1 dx \int_0^x dy c_3 \int_0^\infty dk^2 \frac{\partial}{\partial a^2} \frac{1}{\pi}\int_0^\pi d\theta \sin^2\theta \times (2+\cos^2\theta)\frac{k^2(k^2+a^2)}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv \frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	-1.11839 (2)
$\delta\epsilon_4^{\mu\mu}$	$-8\int_0^1 dx \int_0^x dy b c_4 \int_0^\infty dk^2 \frac{\partial}{\partial a^2} \frac{1}{\pi}\int_0^\pi d\theta \sin^2\theta \cos^2\theta \times (2+\cos^2\theta)\frac{k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv \frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	-0.0484183 (2)
$\delta\epsilon_5^{\mu\mu}$	$12\int_0^1 dx \int_0^x dy b c_5 \int_0^\infty dk^2 \frac{\partial}{\partial a^2} \frac{1}{\pi}\int_0^\pi d\theta \sin^2\theta \times \cos^2\theta\frac{k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv \frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	-0.0027928 (4)
$\delta\epsilon_6^{\mu\mu}$	$-12\int_0^1 dx \int_0^x dy c_6 \int_0^\infty dk^2 \frac{\partial}{\partial a^2} \frac{1}{\pi}\int_0^\pi d\theta \sin^2\theta \times \cos^2\theta\frac{k^2(k^2+a^2)}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv \frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	-0.0251636 (1)
$\delta\epsilon_7^{\mu\mu}$	$-12\int_0^1 dx \int_0^x dy b c_7 \int_0^\infty dk^2 \frac{1}{\pi}\int_0^\pi d\theta \sin^2\theta \times \cos^2\theta\frac{k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv \frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}$	-0.1453343 (4)

This expression is apparently different from the respective expression in Eq. (33) because now our dimensionless momenta are measured in terms of the muon mass. Finally, the integral for the contribution to HFS generated by the diagrams in Fig. 5 has the form

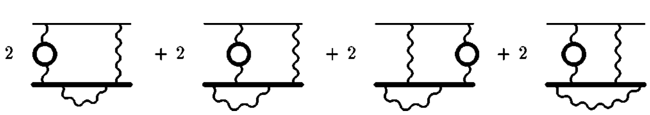

FIG. 5. Muon line and muon vacuum polarization.

$$\delta E^{\mu\mu} = \frac{\alpha(Z^2\alpha)(Z\alpha)}{\pi^3}\frac{m}{M}E_F \frac{1}{\pi}\int_0^\pi d\theta \sin^2\theta \int_0^\infty dk^2 \mathcal{D}(k,\theta) \times \frac{k^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\int_0^1 dv \frac{v^2(1-v^2/3)}{4+k^2(1-v^2)} = \left(\sum_1^7 \delta\epsilon_i^{\mu\mu}\right)\frac{\alpha(Z^2\alpha)(Z\alpha)}{\pi^3}\frac{m}{M}E_F . \tag{47}$$

All seven contributions are collected in Table V, and the total contribution to HFS generated by the diagrams in Fig. 5 is

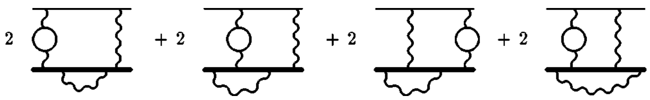

FIG. 6. Muon line and electron vacuum polarization.

$$\delta\epsilon^{\mu\mu} = -1.80176(2). \tag{48}$$

VI. DIAGRAMS WITH RADIATIVE PHOTONS IN THE MUON LINE AND ELECTRON VACUUM POLARIZATION

Let us turn now to the diagrams in Fig. 6. The only difference between these diagrams and the diagrams in Fig. 5 is that now we have electron polarization insertions instead of the muon polarization insertions. The respective analytic expression is obtained from the one in Eq. (47) by substitution of the electron polarization instead of the muon polarization

$$2k^2\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)} \to 2k^2\int_0^1 dv\frac{v^2(1-v^2/3)}{4(m/M)^2+k^2(1-v^2)}. \tag{49}$$

The typical integration momenta in Eq. (45) are of order of the muon mass and, since we are calculating to linear accuracy in m/M, it is sufficient to substitute instead of the electron polarization operator the leading terms in its expansion over m/M:

$$2k^2\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4(m/M)^2+k^2(1-v^2)} \approx \frac{2}{3}\ln k^2+\frac{4}{3}\ln\frac{M}{m}-\frac{10}{9}. \tag{50}$$

Then the total contribution to HFS generated by the diagrams in Fig. 6 is given by the integral

$$\delta E^{\mu e} = \frac{\alpha(Z^2\alpha)(Z\alpha)}{\pi^3}\,\frac{m}{M}E_F\,\frac{1}{2\pi}\int_0^\pi d\theta\sin^2\theta \times\int_0^\infty dk^2\mathcal{D}(k,\theta)\frac{1}{(k^2+a^2)^2+4b^2k^2\cos^2\theta} \times\left(\frac{2}{3}\ln k^2+\frac{4}{3}\ln\frac{M}{m}-\frac{10}{9}\right) = (\delta\epsilon_c^{\mu e}+\delta\epsilon_s^{\mu e})\frac{\alpha(Z^2\alpha)(Z\alpha)}{\pi^3}\,\frac{m}{M}E_F, \tag{51}$$

where the contribution $\delta\epsilon_c^{\mu e} = \Sigma_i\epsilon_{ci}^{\mu e}$ corresponds to the first term in the last brackets, and the contribution $\delta\epsilon_s^{\mu e}$ corresponds to the second and third terms in the same brackets. We calculate $\delta\epsilon_c^{\mu e}$ numerically and collect all seven contributions to this integral in Table VI. The sum of all these contributions is

$$\delta\epsilon_c^{\mu e} = 12.94447(4). \tag{52}$$

The integral for $\delta\epsilon_s^{\mu e}$ is proportional to the integral for the radiative-recoil corrections generated by the radiative photon insertions in the muon line, which is known analytically [7,14]. Hence, we can immediately put down an analytic result for $\delta\epsilon_s^{\mu e}$:

$$\delta\epsilon_s^{\mu e} = \left(\frac{4}{3}\ln\frac{M}{m}-\frac{10}{9}\right)\left(\frac{9}{2}\zeta(3)-3\pi^2\ln 2+\frac{39}{8}\right). \tag{53}$$

Then the total contribution to HFS generated by the diagrams in Fig. 6 is equal to

$$\delta\epsilon^{\mu e} = \left(6\zeta(3)-4\pi^2\ln 2+\frac{13}{2}\right)\ln\frac{M}{m}+24.32115(4). \tag{54}$$

VII. DISCUSSION OF RESULTS

Collecting the results in Eqs. (32), (42), (48), and (54) we obtain the three-loop single-logarithmic and -nonlogarithmic corrections generated by the one-loop electron and muon polarization insertions in the exchanged photons

$$\delta E = \delta\epsilon^e\,\frac{\alpha^2(Z\alpha)}{\pi^3}\,\frac{m}{M}\tilde{E}_F+\delta\epsilon^\mu\,\frac{\alpha(Z^2\alpha)(Z\alpha)}{\pi^3}\,\frac{m}{M}\tilde{E}_F, \tag{55}$$

where

$$\delta\epsilon^e = \frac{5}{2}\ln^2\frac{M}{m}+\frac{22}{3}\ln\frac{M}{m}+7.36110(3), \tag{56}$$

$$\delta\epsilon^\mu = \left(6\zeta(3)-4\pi^2\ln 2+\frac{13}{2}\right)\ln\frac{M}{m}+22.51939(5). \tag{57}$$

If we recall that in the real muonium $Z=1$ and that the logarithm-squared term was already calculated earlier [3] then the new three-loop single-logarithmic and -nonlogarithmic corrections obtained above may be written as

$$\delta E_f = \left[\left(6\zeta(3)-4\pi^2\ln 2+\frac{83}{6}\right)\ln\frac{M}{m} +29.88049(6)\right]\frac{\alpha^3}{\pi^3}\,\frac{m}{M}\tilde{E}_F. \tag{58}$$

Combining this result with the result of our earlier paper [1] we obtain all three-loop single-logarithmic and -nonlogarithmic corrections generated by the electron and muon polarization insertions in the exchanged photons

$$\delta E_{tot} = \left[\left(-4\pi^2\ln 2+\frac{67}{12}\right)\ln\frac{M}{m}+9.59318(6)\right]\frac{\alpha^3}{\pi^3}\,\frac{m}{M}\tilde{E}_F, \tag{59}$$

or, numerically

TABLE VI. c integrals for the diagrams in Fig. 6.

$\delta\epsilon_{c1}^{\mu e}$	$\frac{1}{3}\int_0^1 dx\int_0^x dyc_1\int_0^\infty dk^2\left(\frac{\partial}{\partial a^2}\right)^2\frac{1}{\pi}\int_0^\pi d\theta\sin^2\theta \times(2-\cos^2\theta-\cos^4\theta)\frac{k^2+a^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\ln k^2$	-15.49349 (3)
$\delta\epsilon_{c2}^{\mu e}$	$\frac{1}{3}\int_0^1 dx\int_0^x dyc_2\int_0^\infty dk^2\left(\frac{\partial}{\partial a^2}\right)^2\frac{1}{\pi}\int_0^\pi d\theta\sin^2\theta \times(2+\cos^2\theta)\frac{k^2(k^2+a^2)}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\ln k^2$	-1.235177 (4)
$\delta\epsilon_{c3}^{\mu e}$	$\frac{2}{3}\int_0^1 dx\int_0^x dyc_3\int_0^\infty dk^2\frac{\partial}{\partial a^2}\frac{1}{\pi}\int_0^\pi d\theta\sin^2\theta \times(2+\cos^2\theta)\frac{k^2+a^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\ln k^2$	26.74813 (3)
$\delta\epsilon_{c4}^{\mu e}$	$-\frac{8}{3}\int_0^1 dx\int_0^x dybc_4\int_0^\infty dk^2\frac{\partial}{\partial a^2}\frac{1}{\pi}\int_0^\pi d\theta\sin^2\theta\cos^2\theta \times(2+\cos^2\theta)\frac{1}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\ln k^2$	1.304129 (5)
$\delta\epsilon_{c5}^{\mu e}$	$4\int_0^1 dx\int_0^x dybc_5\int_0^\infty dk^2\frac{\partial}{\partial a^2}\frac{1}{\pi}\int_0^\pi d\theta\sin^2\theta \times\cos^2\theta\frac{1}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\ln k^2$	0.411544 (4)
$\delta\epsilon_{c6}^{\mu e}$	$-4\int_0^1 dx\int_0^x dyc_6\int_0^\infty dk^2\frac{\partial}{\partial a^2}\frac{1}{\pi}\int_0^\pi d\theta\sin^2\theta \times\cos^2\theta\frac{k^2+a^2}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\ln k^2$	-0.0503852 (6)
$\delta\epsilon_{c7}^{\mu e}$	$-4\int_0^1 dx\int_0^x dybc_7\int_0^\infty dk^2\frac{1}{\pi}\int_0^\pi d\theta\sin^2\theta \times\cos^2\theta\frac{1}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}\ln k^2$	1.259721 (4)

$$\delta E=-0.0288\ \text{kHz}. \tag{60}$$

This correction has the same scale as some other corrections to hyperfine splitting in muonium calculated recently [10,15,16]. All these results improve the accuracy of the theory of hyperfine splitting and affect the value of the electron-muon mass ratio derived from the experimental data [17] on hyperfine splitting (see, e.g., reviews in [5,18]). We postpone discussion of the phenomenological implications of the result above until the completion of the calculations of the remaining three-loop radiative recoil corrections.

ACKNOWLEDGMENTS

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APPENDIX: STANDARD AUXILIARY FUNCTIONS

Calculations described in this paper are greatly facilitated by the use of the auxiliary functions $\Phi_n(k)(n=0,1,2,3)$ defined by the relationship

$$\Phi_n(k)\equiv\frac{1}{\pi\mu^2}\int_0^\pi d\theta\sin^2\theta\cos^{2n}\theta \times\frac{(k^2+a^2)^2-4\mu^2b^2k^4}{(k^2+\mu^{-2}\cos^2\theta)[(k^2+a^2)^2+4b^2k^2\cos^2\theta]}. \quad \text{(A1)}$$

The integral over angles may be explicitly calculated, and the result of the integration is conveniently written as a sum

$$\Phi_n(k)\equiv\Phi_n^S(k)+\Phi_n^\mu(k)+\Phi_n^C(k), \quad \text{(A2)}$$

where

$$\Phi_n^S(k)=\frac{\delta_{n0}}{\mu k}, \quad \text{(A3)}$$

$$\Phi_0^\mu(k)=W(\xi_\mu)-\frac{1}{\sqrt{\xi_\mu}}, \quad \text{(A4)}$$

$$\Phi_1^\mu(k)=-\xi_\mu W(\xi_\mu)+\frac{1}{2}, \quad \text{(A5)}$$

$$\Phi_2^\mu(k)=\xi_\mu\left(\xi_\mu W(\xi_\mu)-\frac{1}{2}\right)+\frac{1}{8}, \quad \text{(A6)}$$

$$\Phi_3^\mu(k)=-\xi_\mu\left[\xi_\mu\left(\xi_\mu W(\xi_\mu)-\frac{1}{2}\right)+\frac{1}{8}\right]+\frac{1}{16}, \quad \text{(A7)}$$

$$\Phi_0^C(k)=-W(\xi_C), \quad \text{(A8)}$$

$$\Phi_1^C(k)=\xi_C W(\xi_C)-\frac{1}{2}, \quad \text{(A9)}$$

$$\Phi_2^C(k)=-\xi_C\left(\xi_C W(\xi_C)-\frac{1}{2}\right)-\frac{1}{8}, \quad \text{(A10)}$$

$$\Phi_3^C(k)=\xi_C\left[\xi_C\left(\xi_C W(\xi_C)-\frac{1}{2}\right)+\frac{1}{8}\right]-\frac{1}{16}. \quad \text{(A11)}$$

The standard function $W(\xi)$ has the form

$$W(\xi)=\sqrt{1+\frac{1}{\xi}}-1 \quad \text{(A12)}$$

and

$$\xi_\mu=\mu^2k^2, \quad \xi_C=\frac{(k^2+a^2)^2}{4b^2k^2}. \quad \text{(A13)}$$

One may easily obtain asymptotic expressions for the function $W(\xi)$

$$\lim_{\xi\to 0}W(\xi)\to\frac{1}{\sqrt{\xi}},$$

$$\lim_{\xi\to\infty}W(\xi)\to\frac{1}{2\xi}. \quad \text{(A14)}$$

High- and low-momentum asymptotic expressions for the functions $\Phi_i(k)$ may also be easily calculated. Let us cite low-momentum expansions, which were used in the main text for calculation of the contributions to the hyperfine splitting of relative order μ^2

$$\Phi_0^\mu(k)\approx-1+\frac{\mu k}{2}, \quad \text{(A15)}$$

$$\Phi_1^\mu(k)\approx\frac{1}{2}-\mu k+(\mu k)^2, \quad \text{(A16)}$$

$$\Phi_2^\mu(k)\approx\frac{1}{8}-\frac{(\mu k)^2}{2}. \quad \text{(A17)}$$

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