Large N dynamics in QED in a magnetic field

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The expression for the dynamical mass of fermions in QED in a magnetic field is obtained for a large number of the fermion flavor N in the framework of $1/N$ expansion. The existence of a threshold value N_{thr}, dividing the theories with essentially different dynamics, is established. For the number of flavors $N \ll N_{thr}$, the dynamical mass is very sensitive to the value of the coupling constant α_b, related to the magnetic scale $\mu = \sqrt{|eB|}$. For N of the order of N_{thr} or larger, a dynamics similar to that in the Nambu–Jona-Lasinio model with a cutoff of the order of $\sqrt{|eB|}$ and the dimensional coupling constant $G \sim 1/(N|eB|)$ takes place. In this case, the value of the dynamical mass is essentially α_b independent (the dynamics with an infrared stable fixed point). The value of N_{thr} separates a weak coupling dynamics (with $\tilde{\alpha}_b \equiv N\alpha_b \ll 1$) from a strong coupling one (with $\tilde{\alpha}_b \gtrsim 1$) and is of the order of $1/\alpha_b$.

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I. INTRODUCTION

The phenomenon of the magnetic catalysis of dynamical symmetry breaking was established as a universal phenomenon in a wide class of $(2+1)$- and $(3+1)$-dimensional relativistic models in Refs. [1,2] (for earlier consideration of dynamical symmetry breaking in a magnetic field see Refs. [3,4]). The general result states that a constant magnetic field B leads to the generation of a fermion dynamical mass (a gap in a one-particle energy spectrum) even at the weakest attractive interaction between fermions. The essence of this effect is the dimensional reduction $D \to D-2$ in the dynamics of fermion pairing in a magnetic field. At weak coupling, this dynamics is dominated by the lowest Landau level (LLL) which is essentially $D-2$ dimensional [1,2]. The applications of this effect have been considered both in condensed matter physics [5,6] and cosmology (for reviews see Ref. [7]).

The phenomenon of the magnetic catalysis was studied in gauge theories, in particular, in QED [8–14] and in QCD [15–18]. In Ref. [9], the present authors derived an asymptotic expression for the fermion dynamical mass in the chiral limit in QED, reliable for a weak coupling α_b and for the number of charged fermions N being not too large (here α_b is the running coupling related to the magnetic scale $\mu^2 \sim |eB|$). Specifically, when the parameter $\tilde{\alpha}_b \equiv N\alpha_b$ is small, i.e., $\tilde{\alpha}_b \ll 1$, the fermion dynamical mass is [9]

$$m_{dyn} = C_1 \sqrt{|eB|} F(\tilde{\alpha}_b) \exp\left[-\frac{\pi N}{\tilde{\alpha}_b \ln(C_2/\tilde{\alpha}_b)} \right], \qquad (1)$$

where $F(\tilde{\alpha}_b) \simeq (\tilde{\alpha}_b)^{1/3}$, and the constants C_1 and C_2 are of order one.

In this paper, we will extend the analysis of Ref. [9] to the case with a large coupling $\tilde{\alpha}_b$. As will be discussed below, such a strong coupling regime can be put under control for large values of N in the framework of $1/N$ expansion. It will be shown that the expression for the dynamical mass in this dynamical regime is essentially different from that in Eq. (1) and it reads

$$m_{dyn} \simeq \sqrt{|eB|} \exp(-N). \qquad (2)$$

It is noticeable that this expression of m_{dyn} is α_b independent. As will be shown below, the origin of such a dramatic change of the form of the dynamical mass is intimately connected with the dynamics of the screening of the photon interactions in a magnetic field in the region of momenta relevant for the chiral symmetry breaking dynamics, $m_{dyn}^2 \ll |k^2| \ll |eB|$. In this region, photons acquire a mass M_γ of order $\sqrt{N\alpha_b|eB|}$. More rigorously, M_γ is the mass of a fermion-antifermion composite state coupled to the photon field. The appearance of such mass resembles the pseudo Higgs effect in $(1+1)$-dimensional *massive* QED (massive Schwinger model) [19] (see below). The crossover from the dynamics corresponding to expression (1) to that corresponding to expression (2) occurs for such a threshold value of N_{thr} when the mass $M_\gamma \sim \sqrt{N_{thr}\alpha_b|eB|}$ becomes of order $\sqrt{|eB|}$, i.e., for $\tilde{\alpha}_b^{thr} \equiv N_{thr}\alpha_b \sim 1$.

Let us consider this point in more detail. There are generically three different scales, $\sqrt{|eB|}$, M_γ, and m_{dyn}, in this problem. These scales correspond to the following four, dynamically different, energy regions. The first one is the region with the energy scale above the magnetic scale $\sqrt{|eB|}$. In that region, the dynamics is essentially the same as in

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QED without a magnetic field. In particular, the running coupling increases logarithmically with increasing the energy scale there. The second region is that with the energy scale below the magnetic scale $\sqrt{|eB|}$ but larger than the photon mass M_γ. In that region the photon can be considered as approximately massless. The next, third, region is the region with the energy scale less than the photon mass M_γ but larger than the fermion mass m_{dyn}. In this region, the photon is heavy, and the interaction is similar to that in the Nambu–Jona-Lasinio (NJL) model (with the current-current interaction) in a magnetic field. The important point is that just those third and second regions are relevant for spontaneous chiral symmetry breaking in this problem. At last, the fourth region is the region with the energy scale E less than the fermion mass m_{dyn}. In that region, fermions decouple and their interaction is suppressed by powers of the ratio E/m_{dyn}.

Now, when N grows up to $N_{thr}\sim 1/\alpha_b$, the photon mass M_γ becomes of the order of the scale $\sqrt{|eB|}$ and, therefore, the third region, between M_γ and $\sqrt{|eB|}$, shrinks and disappears. Thus for N of order N_{thr} or larger, the dynamics of spontaneous chiral symmetry breaking is solely provided in the second region, through the interaction with a heavy photon. As a result, the dynamics becomes similar to that in the NJL model in a magnetic field with cutoff $\sqrt{|eB|}$ and the dimensional coupling constant $G\simeq \alpha_b/M_\gamma^2\sim 1/N|eB|$. This implies that this dynamics is α_b independent and, therefore, corresponds to an infrared stable fixed point. It also explains the origin of the threshold value $N_{thr}\sim 1/\alpha_b$. In the rest of the paper, we will derive expression (2) and justify this qualitative dynamical picture.

II. MAGNETIC CATALYSIS IN QED

We begin by considering the Schwinger-Dyson (SD or gap) equation for the fermion propagator. It has the following form:

$$G^{-1}(x,y)=S^{-1}(x,y)+4\pi\alpha_b\gamma^\mu \times\int G(x,z)\Gamma^\nu(z,y,z')\mathcal{D}_{\nu\mu}(z',x)d^4zd^4z', \tag{3}$$

where $S(x,y)$ and $G(x,y)$ are the bare and full fermion propagators in an external magnetic field, $\mathcal{D}_{\nu\mu}(x,y)$ is the full photon propagator and $\Gamma^\nu(x,y,z)$ is the full amputated vertex function.

Let us first consider the weak coupling dynamics ($\alpha_b \ll 1$) with the number of fermion flavors N of order one. In this case, one might think that the rainbow (ladder) approximation is reliable in this problem. However, this is not the case. Because of the $(1+1)$-dimensional form of the fermion propagator in the LLL approximation, there are relevant higher order contributions [8,9]. In particular, there is a large contribution of fermions to the polarization operator. Fortunately, one can solve this problem [9]. Let us discuss this in more detail.

First of all, one can show that the dynamics of the fermion-antifermion pairing is mainly induced in the region of momenta k much less than $\sqrt{|eB|}$ and much larger than the dynamical mass m_{dyn}, i.e., in the the second and third scale regions discussed in the Introduction. In particular, this implies that the magnetic scale $|eB|$ yields a dynamical ultraviolet cutoff in this problem.

The important ingredient of this dynamics is a large contribution of fermions to the polarization operator. It is large because of an (essentially) $(1+1)$-dimensional form of the fermion propagator in a strong magnetic field. Its explicit form in the one-loop approximation is [9]:

$$\mathcal{P}^{\mu\nu}\simeq \frac{\alpha_b N}{3\pi}(k_\parallel^\mu k_\parallel^\nu-k_\parallel^2 g_\parallel^{\mu\nu})\frac{|eB|}{m_{dyn}^2}, \tag{4}$$

for $|k_\parallel^2|\ll m_{dyn}^2$, and

$$\mathcal{P}^{\mu\nu}\simeq -\frac{2\alpha_b N}{\pi}(k_\parallel^\mu k_\parallel^\nu-k_\parallel^2 g_\parallel^{\mu\nu})\frac{|eB|}{k_\parallel^2}, \tag{5}$$

for $m_{dyn}^2\ll|k_\parallel^2|\ll|eB|$, where $g_\parallel^{\mu\nu}\equiv\mathrm{diag}(1,0,0,-1)$ is the projector onto the longitudinal subspace, and $k_\parallel^\mu\equiv g_\parallel^{\mu\nu}k_\nu$ (note that the magnetic field is in the x^3 direction). Similarly, we introduce the orthogonal projector $g_\perp^{\mu\nu}\equiv g^{\mu\nu}-g_\parallel^{\mu\nu}=\mathrm{diag}(0,-1,-1,0)$ and $k_\perp^\mu\equiv g_\perp^{\mu\nu}k_\nu$ that we shall use below. Notice that fermions in a strong magnetic field do not couple to the transverse subspace spanned by $g_\perp^{\mu\nu}$ and $k_\perp^\mu$. This is because in a strong magnetic field only the fermions from the LLL matter and they couple only to the longitudinal components of the photon field. The latter property follows from the fact that spins of the LLL fermions are polarized along the magnetic field [8].

The expressions (4) and (5) coincide with those for the polarization operator in the massive $(1+1)$-dimensional QED, QED_{1+1} (Schwinger model) [19] if the parameter $2\alpha_b|eB|$ here is replaced by the dimensional coupling e_1^2 of QED_{1+1}. As in the Schwinger model, Eq. (5) implies that there is a massive resonance in the $k_\parallel^\mu k_\parallel^\nu-k_\parallel^2 g_\parallel^{\mu\nu}$ component of the photon propagator. Its mass is

$$M_\gamma^2=\frac{2N\alpha_b}{\pi}|eB|. \tag{6}$$

This is reminiscent of the pseudo Higgs effect in the $(1+1)$-dimensional massive QED. It is not the genuine Higgs effect because there is no complete screening of the electric charge in the infrared region with $|k_\parallel^2|\ll m_{dyn}^2$. This can be seen clearly from Eq. (4). Nevertheless, the pseudo Higgs effect is manifested in creating a massive resonance and this resonance provides the dominant forces leading to chiral symmetry breaking.

Now, the main points of the analysis of the weak coupling dynamics in QED in a magnetic field are [9]: (i) the so called improved rainbow approximation is reliable in this problem provided a special nonlocal gauge is used, and (ii) the relevant region of momenta in this problem is $m_{dyn}^2\ll|k^2|\ll|eB|$. We recall that in the improved rainbow approxima-

tion the vertex $\Gamma^{\nu}(x,y,z)$ is taken to be bare and the photon propagator is taken in the one-loop approximation. For a weak coupling dynamics, this approximation is reliable since in that special gauge the loop contributions in the vertex are suppressed by powers of α_b. [It is appropriate to call this approximation the ''strong-magnetic-field-loop improved rainbow approximation.'' It is an analog of the hard-dense-loop improved rainbow approximation in QED or QCD with a nonzero baryon density [20].] This leads us to the expression (1) for the dynamical gap.

Let us now turn to the case with a large number of fermion flavors N. The crucial point is that the improved rainbow approximation is still reliable in this case. The essential difference, however, is that now one has to consider not the conventional loop expansion (with a small α_b) but the $1/N$ expansion (with a small $1/N$). It is well known [21] that in this expansion the coupling constant $\tilde{\alpha}_b \equiv N\alpha_b$ has to be kept fixed as $N\to\infty$. A great advantage of the $1/N$ expansion is that now one can treat the dynamics with an arbitrary value of $\tilde{\alpha}_b$: it could be small ($\tilde{\alpha}_b \ll 1$), intermediate ($\tilde{\alpha}_b \sim 1$), or large ($\tilde{\alpha}_b \gg 1$). Indeed, independently of the value of $\tilde{\alpha}_b$, the loop corrections in the vertex are suppressed by powers of $1/N$ and, therefore, the improved rainbow approximation is indeed reliable for large N.

Let us now proceed to the analysis of the SD equation for the dynamical mass of fermions in QED in a magnetic field for a large number of flavors N. In the improved rainbow approximation, the SD equation reads in Euclidean space [see Eq. (54) in Ref. [9]]

$$B(p^2) = \frac{\alpha_b}{2\pi^2}\int \frac{d^2q B(q^2)}{q^2+m_{dyn}^2}\int_0^{\infty}\frac{dx\,\exp(-x/2|eB|)}{x+(\mathbf{q}-\mathbf{p})^2+M_{\gamma}^2}, \quad (7)$$

where $B(q^2)$ is the fermion mass function and the two-dimensional vector $\mathbf{q}$ is $\mathbf{q}=(q_4,q_3)$ with $q_4=-iq_0$.

As we mentioned in the Introduction, in the limit of small coupling constant $\tilde{\alpha}_b$, the above SD equation was solved in Ref. [9], using numerical as well as approximate analytical methods. The result for the dynamical mass of fermions is quoted in Eq. (1). Here we would like to comment on the nature of the interaction, provided by photons, in this weak coupling regime. One could easily check that the dominant interaction is provided by the photons with the (''longitudinal'') momenta in the following range: $m_{dyn}^2 \lesssim (\mathbf{q}-\mathbf{p})^2 \lesssim |eB|$. Then, by noticing that the photon mass also lies in the same range of momenta, i.e., $m_{dyn}^2 \ll M_{\gamma}^2 \ll |eB|$, one finds that the degree of importance of the photon mass is changing when the values of momenta are sweeping the relevant range of momenta. While in the near-infrared region with $m_{dyn}^2 \lesssim (\mathbf{q}-\mathbf{p})^2 \lesssim M_{\gamma}^2$ (the third region, in the nomenclature of Introduction) the interaction is local with a good precision, it becomes essentially nonlocal in the intermediate range of momenta where $M_{\gamma}^2 \lesssim (\mathbf{q}-\mathbf{p})^2 \lesssim |eB|$ (the second region in that nomenclature).

In the opposite limit, $\tilde{\alpha}_b \gtrsim 1$, the structure of the SD equation (7) considerably simplifies. The simplification comes due to the new hierarchy of scales, $|eB| \lesssim M_{\gamma}^2$ [see Eq. (6)]. From physical point of view, this hierarchy means that the photon mass is so large that the interaction leading to fermion pairing is essentially local. Therefore, by neglecting $(\mathbf{q}-\mathbf{p})^2$ term in the denominator of the second integral on the right hand side of Eq. (7), we derive an approximate algebraic form of the gap equation that works rather well at large values of $\tilde{\alpha}_b$,

$$-\frac{\tilde{\alpha}_b}{2\pi N}\exp\left(\frac{\tilde{\alpha}_b}{\pi}\right)\mathrm{Ei}\left(-\frac{\tilde{\alpha}_b}{\pi}\right)\ln\frac{|eB|}{m^2} = 1, \quad (8)$$

where $\mathrm{Ei}(z)$ is the exponential integral function. By making use of the asymptotic expansion of the exponential integral function at large $\tilde{\alpha}_b$, this equation is further simplified, and the following result for the dynamical mass of fermions is obtained:

$$m_{dyn} \simeq \sqrt{|eB|}\exp(-N), \quad \text{for} \quad \tilde{\alpha}_b \gg 1. \quad (9)$$

Notice that this regime with large $\tilde{\alpha}_b$ is qualitatively the same as in the NJL model with the cutoff of order $|eB|$ and the dimensional coupling constant $G \simeq \alpha_b/M_{\gamma}^2 = \pi/(2N|eB|)$.

Therefore our analysis shows that there are two qualitatively different regimes of dynamics of spontaneous symmetry breaking in QED in a magnetic field at large number of flavors. The first of them, which develops for $\tilde{\alpha}_b \ll 1$, is essentially the same as the weakly coupled regime with a small number of fermion flavors N. The other limiting case appears when $\tilde{\alpha}_b \gtrsim 1$, and it is characterized by pairing dynamics governed by an almost local interaction. In terms of the number of fermion flavors, these two regimes occur for $N \ll 1/\alpha_b$ and for $N \gtrsim 1/\alpha_b$, respectively.

III. CONCLUSION

QED in an external magnetic field yields an example of a rich dynamics. It is important that this dynamics can be taken under control both for a weak coupling constant $\tilde{\alpha}_b$ with N of order one and for an arbitrary value of $\tilde{\alpha}_b$ when $N \gg 1$. In accordance with the general analysis of Refs. [1,2], the phenomenon of the magnetic catalysis in QED is universal, although its dynamics varies dramatically with increasing N.

In this paper we did not discuss the dynamical regime with a strong coupling constant $\tilde{\alpha}_b$ and N of order one (genuine strong coupling regime). Although in this case the dynamics does not admit a controllable approximation, one should expect that spontaneous chiral symmetry breaking in this regime takes place even without an external magnetic field [22]. An external magnetic field should presumably enhance the value of the dynamical mass for fermions, as it happens for example in the supercritical phase of the NJL model (see Ref. [2] and the second paper in Ref. [8]).

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