Phases of bosonic strings and two dimensional gauge theories

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We propose that the extrinsic curvature and torsion of a bosonic string can be employed as Hamiltonian variables in a two dimensional Landau-Ginzburg gauge field theory. Their interpretation in terms of the Abelian Higgs multiplet leads to two different phases. In the phase with unbroken gauge symmetry the ground state describes open strings while in the phase with broken gauge symmetry the ground state involves closed strings. Finally, we relate aspects of the extrinsic geometry to the spectral properties of a Dirac operator, minimally coupled to the Abelian Higgs multiplet.

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I. INTRODUCTION

In the Wilsonian approach to renormalization group equations the Polyakov action [1] is a relevant term for describing the high energy limit of a bosonic string. But at lower energies there can be corrections such as the extrinsic curvature term [2]. Here we shall be interested in additional corrections that emerge from the extrinsic geometry of the string and that may affect its low energy behavior. For definiteness we consider a bosonic string in three spatial dimensions; even though the quantization of the Polyakov action dictates $D=26$, the role of a critical dimension becomes less obvious when higher order corrections are included. This is because such corrections in general fail to be conformally invariant on the worldsheet. We shall suggest that in three spatial dimensions the extrinsic geometry leads to effective two dimensional Landau-Ginzburg gauge field theories such as the Abelian Higgs model.

The Abelian Higgs model is particularly interesting as it comes in two different phases. We propose that the phase with unbroken gauge symmetry is natural for describing open strings. The phase where the U(1) gauge symmetry becomes spontaneously broken is then more natural for describing closed strings. In this phase the Polyakov action admits an interpretation as the vacuum expectation value of the Higgs field; it gives a mass to the two dimensional vector field. Finally, we consider coupling of fermions to the Higgs multiplet and find that the index of an appropriate two dimensional Dirac operator can be employed to inspect the geometric properties of the string.

We note that previously somewhat similar relations between the extrinsic geometry of a bosonic string and certain field theory models have been considered in [3]. Our construction is somewhat different, and our results are to be viewed as complementary to those in [3].

II. BOSONIC STRING AND ABELIAN HIGGS HAMILTONIAN

We start from the static limit of the $3+1$ dimensional classical bosonic string action

$$S=\mu^2\int d\sigma dt\sqrt{-g}g^{ab}\partial_a X^\mu\partial_b X^\mu\ (\mu=0,1,2,3). \tag{1}$$

We choose a proper time gauge where the worldsheet coordinate t becomes proportional to X^0; thus in the static limit the $t\propto X^0$ dependence can be ignored. The remaining spatial variables $X^i(\sigma)(i=1,2,3)$ then describe the string as a curve which is embedded in R^3. We choose σ so that it coincides with the R^3 arclength of the string, $\sigma\to s\in[0,L]$. Here L is the (variable) total length of the string in R^3, a remnant of the modular parameters on the string worldsheet. The static energy now reduces to

$$E=\mu^2\int_0^L ds\,\partial_s X^i\partial_s X^i=\mu^2\cdot L. \tag{2}$$

Note that the energy of the string depends only on the string tension μ^2 and the (modular) length L.

The three component unit vector

$$\mathbf{t}=\frac{d\mathbf{X}}{ds} \tag{3}$$

is tangent to the curve $X^i(s)$ in R^3. Together with the unit normal $\mathbf{n}$ and the unit binormal $\mathbf{b}=\mathbf{t}\times\mathbf{n}$, we then have an orthonormal frame in R^3, at each point along the string. In terms of the complex combination $\mathbf{e}_F^{\pm}=\frac{1}{2}(\mathbf{n}\pm i\mathbf{b})$ these vectors are subject to the Frenet equations

$$\frac{d\mathbf{t}}{ds}=\frac{1}{2}\kappa(\mathbf{e}_F^{+}+\mathbf{e}_F^{-}), \tag{4}$$

$$\frac{d\mathbf{e}_F^{\pm}}{ds}=-\kappa\mathbf{t}\mp i\tau\mathbf{e}_F^{\pm}. \tag{5}$$

Here κ is the extrinsic curvature and τ is the torsion of the string,

$$\kappa\equiv\kappa_{\pm}=\mathbf{e}_F^{\pm}\cdot\partial_s\mathbf{t}, \tag{6}$$

$$\tau=\frac{i}{2}\mathbf{e}_F^{-}\cdot\partial_s\mathbf{e}_F^{+}. \tag{7}$$

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Notice that we write $\kappa_\pm$ to distinguish the two different but equivalent representations of the curvature in terms of the complex Frenet frame $(\mathbf{t},\mathbf{e}_F^\pm)$. Obviously any physical property of the string should be independent of the choice of basis vectors on the normal planes. Instead of the Frenet combination $\mathbf{e}_F^\pm$ we may then introduce an arbitrary complex orthonormal frame $\mathbf{e}^\pm$ that relates to the Frenet frame $\mathbf{e}_F^\pm$ by a rotation with an angle $\theta(s)$ on the normal planes,

$$\mathbf{e}_F^\pm \rightarrow \mathbf{e}^\pm = e^{\pm i\theta}\mathbf{e}_F^\pm . \tag{8}$$

This rotation sends $\kappa_\pm$ to the complex conjugate pair

$$\kappa_\pm \rightarrow e^{\pm i\theta}\kappa_\pm \tag{9}$$

while for the torsion we get

$$\tau \rightarrow \tau - \partial_s\theta. \tag{10}$$

In Eqs. (9),(10) we identify the gauge transformation structure of a two dimensional, Hamiltonian Abelian Higgs multiplet (ϕ,A_i): The frame rotation (8) corresponds to a static U(1) gauge transformation, κ_+ together with its complex conjugate κ_- corresponds to the complex scalar field $\phi \sim \kappa_+$, and τ to the spatial component $A_1 \sim \tau$ of the U(1) gauge field in the Hamiltonian $A_0=0$ gauge. Since the physical properties of the string are independent of the local frame, they should remain invariant under the U(1) transformations (9),(10). In particular, any Landau-Ginzburg energy of the string which involves the multiplet $(\kappa_+,\tau) \sim (\phi,A_1)$ should be U(1) gauge invariant.

III. GEOMETRY AND PHASE STRUCTURE

The Abelian Higgs model admits two different phases. Consequently, we have two alternatives for the static U(1) invariant Landau-Ginzburg energy. We first consider the phase where the gauge symmetry remains unbroken. It is described by the following Landau expansion:

$$E_I = \mu^2\int_0^L ds(\partial_s X^i)^2 + \int_0^L ds\{\alpha|\phi|^2 + \beta|(\partial_s + iA_1)\phi|^2 + \cdots\}. \tag{11}$$

In the ground state the average value of local curvature along the string vanishes,

$$\langle|\phi|^2\rangle = 0. \tag{12}$$

Since $L\neq 0$ this means that in the ground state the string cannot form a closed curve. Consequently, this phase is appropriate for describing open strings. Note that the first two terms in Eq. (11) reproduce the rigid string action of [2], and the third term is a higher derivative correction. Consequently we expect that Eq. (11) describes strings in the same universality class with the action in [2].

The phase where the gauge symmetry becomes broken by the Higgs effect can be described by the following Landau-Ginzburg energy,

$$E_{II} = \int_0^L ds[|(\partial_s + iA_1)\phi|^2 + \lambda(|\phi|^2 - a^2)^2]. \tag{13}$$

Notice that, unlike in Eq. (11), now we do not introduce (1) explicitly. This term is in fact already contained in the field independent part of the potential in Eq. (13), which according to Eq. (2) can be rewritten as

$$\int_0^L ds\lambda(|\phi|^2 - a^2)^2 = \int_0^L ds\lambda[|\phi|^4 - 2a^2|\phi|^2\sqrt{\partial_s X^i\partial_s X^i} + a^4\partial_s X^i\partial_s X^i].$$

In particular the (static) string action (1) is proportional to the ground state expectation value of the complex Higgs scalar ϕ,

$$\langle|\phi|^2\rangle = a^2\sqrt{\partial_s X^i\partial_s X^i}. \tag{14}$$

With $a^2\neq 0$ we have spontaneous symmetry breaking, and Eq. (13) describes the string in a phase which is different from that described by Eq. (11). Notice that in the ground state (14) the energy (13) vanishes. In particular, this ground state energy is independent of the length (modular parameter) L. The U(1) gauge invariant variables (ρ,C) are defined by

$$\phi = \rho e^{i\chi} \quad \text{and} \quad \tau = C + \partial_s\chi. \tag{15}$$

In terms of these we have for the energy

$$E_{II} = \int_0^L ds\{(\partial_s\rho)^2 + \rho^2 C^2 + \lambda(\rho^2 - a^2)\}. \tag{16}$$

The variable ρ describes the curvature of the string, and C describes the (frame independent) torsion. There is an interplay between the parameters L and a and the ground state geometry of the string: When $L\cdot a = 2\pi n$ the ground state of the string has $\rho^2 = a^2$ and $C=0$ which corresponds to a circular planar unknot in R^3. For other values of $L\cdot a$ the ground state geometry becomes more involved. With $L\cdot a \geqslant 2\pi$ the ground state string can form a closed curve in R^3, and the Landau-Ginzburg energy (13) is appropriate for describing closed strings. But for $0 < L\cdot a < 2\pi$ the ground state string with vanishing energy is an open, bent string.

The phase factor χ in Eq. (15) is defined modulo $2\pi k$, with k an integer. In the Abelian Higgs model this integer labels the different instanton vacua. In the present case k counts the number of times the (normal frame of the) string rotates around its axis when we move once around the string. Since χ is absent in Eq. (16) these rotations of the normal frame have no effect on the energy. In order to relate the strings with different k to the instanton vacua, we add to the arclength s an additional coordinate t, which we view as the Euclidean time coordinate and expand Eq. (13) to the Abelian Higgs model

$$S = \int d^2x\left[\frac{1}{4}F_{\mu\nu}^2 + |(\partial_\mu + iA_\mu)\phi|^2 + \lambda(|\phi|^2 - a^2)^2\right]. \tag{17}$$

We interpret this as a (covariantized) Euclidean time dependent action for the string worldsheet. Its Hamiltonian clearly yields Eq. (13) for a static string in the $A_0=0$ gauge. This action is known to support two dimensional vortices as instantons. We assume that $L\cdot a=2\pi$ so that the ground state string is a circular planar unknot; it corresponds to a vacuum state of the Higgs action (17). We introduce an instanton that interpolates between two such ground state string vacua at $t=\pm T\to\pm\infty$ with different integers k. The instanton has a nontrivial first Chern character ($=$ integer m), and in the $A_0=0$ gauge

$$\mathcal{C}_1(F)=\frac{1}{2\pi}\int F=\frac{1}{2\pi}\oint_{+T}ds\,\tau(s)-\frac{1}{2\pi}\oint_{-T}ds\,\tau(s)=m. \tag{18}$$

This coincides with the difference in the integers k that count the number of times the frames of the ground state strings at $t=\pm T$ rotate around their axis when we move once around the strings.

IV. COUPLING TO FERMIONS

In general, the string can self-link, and in particular a closed string can form a knot. The Calugareanu theorem [4,5] states that the (integer valued) self-linking number $\mathcal{L}$ of a knotted string K equals the sum of its twist $\mathcal{T}$ and writhe $\mathcal{W}$,

$$\mathcal{L}(K)=\mathcal{T}(K)+\mathcal{W}(K).$$

This resembles an index theorem for the instanton in the Abelian Higgs model (17). For this we consider a Seifert surface $\mathcal{S}$ of K. This is a smooth orientable Riemann surface in R^3 with one boundary component that coincides with the string K. We reinterpret Eq. (17) as a (static) Hamiltonian on the Seifert surface, with $\kappa_\pm$ and τ appropriately extended into an Abelian Higgs multiplet (ϕ,A_i) on $\mathcal{S}$ so that τ coincides with the component of A_i which is tangential to K along the boundary of $\mathcal{S}$, and ϕ is equal to κ_+ on the boundary. We define a two dimensional Dirac operator $\not{D}=\gamma^a e^i_a(\partial_i+A_i+\omega_i)$ on $\mathcal{S}$, with ω_i the spin connection. The Atiyah-Patodi-Singer index theorem computes the index of this Dirac operator on $\mathcal{S}$ in terms of the first Chern character (18) over $\mathcal{S}$ and the η invariant of the restriction of $\not{D}$ on the boundary of $\mathcal{S}$,

$$\text{Index}\,\not{D}=\frac{1}{2\pi}\int_{\mathcal{S}}F+\frac{1}{2}\eta=\frac{1}{2\pi}\oint_K\tau+\frac{1}{2}\eta.$$

Presumably this coincides with the Calugareanu relation: The self-linking of the string equals the index of the Dirac operator, its twist equals the integral of $F=dA$ over the Seifert surface, and the writhe coincides with the η-invariant of the boundary Dirac operator. Notice that the twist can also be interpreted physically as the magnetic flux through the Seifert surface.

V. CONCLUSIONS

In conclusion, we have investigated the extrinsic geometry of bosonic strings in $3+1$ dimensions. In particular, we have proposed that the extrinsic curvature and torsion can be viewed as variables in a two dimensional gauge field theory. This leads to the Abelian Higgs model as a Landau-Ginzburg description of the string, with its two phases relating to open and closed strings in a rather natural fashion. Furthermore, we have studied the coupling of fermions to the Abelian Higgs multiplet. The ensuing Dirac operator can then be employed to inspect the extrinsic geometric properties of the string.

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