Local Casimir energy for solitons

Alfred Scharff Goldhaber,* Andrei Litvintsev,† and Peter van Nieuwenhuizen‡

C.N. Yang Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794
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Direct calculation of the one-loop contributions to the energy density of bosonic and supersymmetric ϕ^4 kinks exhibits the following. (1) *Local mode regularization*. Requiring the mode density in the kink and the trivial sectors to be equal at each point in space yields the anomalous part of the energy density. (2) *Phase space factorization*. A striking position-momentum factorization for reflectionless potentials gives the nonanomalous energy density a simple relation to that for the bound state. For the supersymmetric kink, our expression for the energy density (both the anomalous and nonanomalous parts) agrees with the published central charge density, whose anomalous part we also compute directly by point-splitting regularization. Finally we show that, for a scalar field with arbitrary scalar background potential in one space dimension, point-splitting regularization implies local mode regularization of the Casimir energy density.

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INTRODUCTION

Quantum corrections to solitons were of great interest in the 1970s and 1980s, and again in the past few years, due to the present activity in quantum field theories with dualities between extended objects and pointlike objects. Dashen, Hasslacher, and Neveu [1], in a 1974 article that has become a classic, computed the one-loop corrections to the mass of the bosonic kink in ϕ^4 field theory and to the bosonic soliton in sine-Gordon (SG) theory. For the latter, there exist exact analytical methods associated with the complete integrability of the system, authenticating the perturbative calculation. Our work here uses general principles but focuses on the kink, for which exact results are not available. Dashen *et al.* put the object (classical background field corresponding to a kink or SG soliton) in a box of length L to discretize the continuous spectrum, and used mode number regularization (equal numbers of modes in the topological and trivial sectors, including the zero mode in the topological sector in this counting) for the ultraviolet divergences. They imposed periodic boundary conditions (PBC) on the meson field which describes the fluctuations around the trivial or topological vacuum solutions, and added a logarithmically divergent mass counterterm whose finite part was fixed by requiring absence of tadpoles in the trivial background. They found, for the kink,

$$M^{(1)} = \sum \frac{1}{2}\hbar\omega_n - \sum \frac{1}{2}\hbar\omega_n^{(0)} + \Delta M$$
$$= -\hbar m\left(\frac{3}{2\pi} - \frac{\sqrt{3}}{12}\right)$$
$$< 0, \tag{1}$$

*Email address: goldhab@insti.physics.sunysb.edu
†Email address: litvint@insti.physics.sunysb.edu
‡Email address: vannieu@insti.physics.sunysb.edu

where m is the renormalized mass of the meson in the trivial background. This result remains unchallenged.

The supersymmetric (SUSY) case, as well as the general case including fermions, proved more difficult. The action and the kink solution read

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}U^2 - c\frac{1}{2}\frac{dU}{d\phi}\bar{\psi}\psi,$$
$$\tag{2}$$
$$\phi_{\text{kink}}(x) = \frac{\mu}{\sqrt{\lambda}}\tanh\frac{\mu x}{\sqrt{2}},$$

where $-\frac{1}{2}U^2 = -(\lambda/4)(\phi^2 - \mu^2/\lambda)^2$, the meson mass is $m = \mu\sqrt{2}$, and $c = 1$ for supersymmetry. Dashen *et al.* did not explicitly compute the fermionic corrections to the soliton mass, stating "The actual computation of [the contribution to] $M^{(1)}$ [due to fermions] can be carried out along the lines of the Appendix. As the result is rather complicated and not particularly illuminating we will not give it here" (p. 4137 of [1]).

Several authors later computed $M^{(1)}$ for the SUSY kink and found different answers. It became clear that the method of Dashen et al. yielded results that depended on the boundary conditions for the fluctuating fields. In fact, repeating exactly the same steps for the susy kink as taken by Dashen *et al.* for the bosonic kink [using PBC also for the fermions, $\psi_\pm(-L/2) = \psi_\pm(L/2)$], taking equal numbers of modes in all four sectors, including one term with $\omega \simeq 0$ (due to a periodic solution with $\omega^2 > 0$) in the bosonic kink sector and one term with $\omega = 0$ in the fermionic kink sector (explicitly, there are two real independent solutions with $\omega = 0$, one localized at the kink and one at the boundary, and the coefficient of each satisfies $c^2 = \frac{1}{2}$ [2]) gives $M^{(1)} = \hbar m((1/4) - (1/2\pi))$ [3]. We now know that this $M^{(1)}$ is the correct answer to an inappropriate question, because it includes boundary energy. Schonfeld [4] finessed the problem of a single kink with its sensitivity to boundary conditions by considering the kink-antikink system with PBC. Taking into account two terms with $\omega \sim 0$ in the bosonic kink-antikink sector (due to one periodic solution with ω^2

<0 and another antiperiodic solution with $\omega^2>0$) and one term with $\omega=0$ in the fermionic kink-antikink sector (due to two periodic solutions with $\omega=0$ [2]), he obtained what we now know to be the correct answer. The problem of boundary contributions was circumvented by other methods in [5] and [6]. The fermionic contribution to $M^{(1)}$ is given by

$$M_f^{(1)}=\hbar m\left(\frac{1}{\pi}-\frac{\sqrt{3}}{12}\right)>0, \qquad (3)$$

and the total one-loop correction is thus

$$M_{susy}^{(1)}=-\frac{\hbar m}{2\pi}. \qquad (4)$$

With attention restricted to the kink alone, it was shown in [2] that one could eliminate boundary contributions from the fermionic part of the energy by averaging over quartets of boundary conditions for the fermionic fluctuations—periodic, antiperiodic, twisted periodic and twisted antiperiodic, where twisting means interchange of the upper and lower components of the fermion wave function $\psi_{\pm}\to\psi_{\mp}$ [5]. This averaging is necessary to preserve certain discrete symmetries for fermions. The results in [2] give a complete, though intricate, way to calculate $M^{(1)}$ in terms of mode frequencies ω_n.

In light of the complexities which boundary conditions generate for the problem including fermions, the most important advance since [1] was the approach of Shifman, Vainshtein, and Voloshin [7], who used higher space-derivative regularization [with factors $(1-\partial_x^2/M^2)$ for the kinetic terms but not the interactions] to compute the central charge densities of the SUSY sine-Gordon soliton and kink. Their scheme is manifestly SUSY, canonical (no higher time derivatives), and independent of boundary conditions (because it yields a local density).[1] They argued that the energy density is equal to the central charge density (because the difference is a SUSY commutator) and they computed the latter—including an anomaly recognizable as an M^2/M^2 effect. They verified that the one-loop correction $Z^{(1)}$ to the integrated central charge of the kink comes only from the anomaly and is equal to Eq. (4). The presence of a topological anomaly was first conjectured in [5].

One may compute the energy density for the bosonic sine-Gordon soliton by mapping the system onto another one which exhibits supersymmetry, and computing the central charge density for that fictitious system [7]. Quite possibly similar techniques would work for the bosonic kink. Our approach here is instead to attack the Casimir energy density directly, freeing the calculation from dependence on supersymmetry. In doing so, we have found it helpful to introduce a simple rule for regularization of energy densities which we propose as a fundamental principle. At this point, the primary evidence for the validity of the principle is the agreement between the energy density we compute by its use with the

central charge density of [7]. Our proposed principle appears new in the literature, and yet has roots in early quantum physics. *Local mode regularization* (LMR), or mode density regularization, is the local counterpart of the familiar (global) mode regularization or mode number regularization. The principle is that, when fluctuations are expanded into normalized modes $\phi_n(x)$ [i.e., $\int dx\,\phi_n^*(x)\phi_n(x)=1$ for all n], the cutoff local mode density $\rho_N(x)=\Sigma_{n=1}^N\phi_n^*(x)\phi_n(x)$ should be background independent, i.e., the same in the trivial ($\rho^{(0)}$) and kink (ρ) sectors. For the bosonic kink this implies that one must truncate the sums at different upper bounds corresponding to different frequency cutoffs

$$\rho_\Lambda(x)=\rho_{\Lambda+\Delta\Lambda(x)}^{(0)}(x). \qquad (5)$$

Here the x-independent cutoff Λ is given by $\Lambda=2\pi N/L$ to an accuracy $\mathcal{O}(1/L)$. The above equation determines $\Delta\Lambda(x)$ in terms of Λ, and clearly $\Delta\Lambda(x)$ is x dependent. For the SUSY kink one can begin by fixing the mode number cutoff $N_b^{(0)}$ for the bosons and $N_f^{(0)}$ for the fermions in the trivial sector such that here the bosonic and fermionic mode densities are equal. From Eq. (5) one then obtains in the kink sector the requirement

$$\rho_{\Lambda,b}(x)=\rho_{\Lambda+\Delta\Lambda(x),f}(x), \qquad (6)$$

which again determines $\Delta\Lambda(x)$ in terms of Λ. We use this principle to compute the anomalous energy density of the bosonic kink, as well as of the supersymmetric kink, which as mentioned was obtained already in [7] through the equality of energy and central charge densities. We believe that LMR is sufficient for regularization of the one-loop Casimir energy density in one space dimension, and at least necessary in higher dimensions, where further conditions may be needed to specify the regularization completely. It is important to emphasize that LMR, like ordinary mode regularization, is not easily applied at arbitrary order in perturbation theory, because it is not directly expressed in terms of a modification of the action. Thus our claim is that for a specialized purpose, namely, calculation of localized Casimir energy, LMR is the ideal tool, providing maximal simplicity and efficiency. This claim is simultaneously modest, because it is restricted to the computation of one-loop energy densities, and substantial, because Casimir energy plays such a large role in quantum physics.

For the non-anomalous contributions to both the bosonic and SUSY kink densities, we find empirically another striking regularity, *phase space factorization*. The continuum contribution to the Casimir energy density in phase space exhibits a remarkable factorization, involving a few terms each with simple momentum-dependent factors multiplying functions related to the bound-state and zero-mode probability densities in coordinate space. We believe this factorization should hold for all reflectionless potentials, but might not extend farther. The factorization takes a particularly simple form for what we call below $\epsilon_{Cas}(x)$, a local density whose integral over a region containing the kink gives the total quantum correction to the mass of the kink. The local energy density has two contributions besides ϵ_{Cas}. First, the

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[1]As [7] points out, Yamagishi [8] was perhaps the first to study local densities in this context.

fundamental definition of Casimir energy density differs from the usual sum over zero-point energies by an extra piece which is a perfect differential of an expression vanishing far from the kink, so that including this piece does not alter the mass correction but does alter the local density. Secondly, there is an effect which perhaps is best viewed as vacuum polarization—the quantum fluctuations of the Bose and Fermi fields lead to a local shift in the classical background field defining the kink. Again, this changes the local density but not the total mass. We check explicitly that the energy density and the central charge density of the SUSY kink are equal.

After these calculations of the anomalous and nonanomalous contributions to the energy density of the bosonic (and SUSY) kink, we turn to the calculation of the central charge density of the SUSY kink. From its definition the central charge density at a point y is an x integral of a bilocal quantity depending on x and y times a delta function $\delta(x-y)$. Not setting $x=y$ too soon yields the anomaly in the central charge density near the kink, confirming [7]. Finally, we discuss the physical basis for LMR, observing in particular that point-splitting regularization for energy density implies lmr, at least for the bosonic case with arbitrary background potential.

BOSONIC KINK ENERGY DENSITY

For the energy density of the bosonic kink, one must evaluate sums (setting $\hbar=1$ from now on) $\Sigma \frac{1}{2}\omega_n \phi_n^*(x)\phi_n(x)$, where the modes $\phi_n(x)$ are each normalized to unity. As these sums clearly differ from the density sums $\Sigma \phi_n^*(x)\phi_n(x)$, one expects in general a nonvanishing one-loop correction to the energy density, and hence to the quantum mass. Let us begin with explicit expressions for the mode eigenfunctions, so that one may follow the argument in detail. The wave functions of the continuous spectrum [using $|\phi_n|^2(x)=1$ away from the kink to determine the normalization constant $\mathcal{N}$] obey

$$\phi(k,x)=\frac{e^{ikx}}{\mathcal{N}}\left[-3\tanh^2\frac{mx}{2}+1+4\left(\frac{k}{m}\right)^2+6i\frac{k}{m}\tanh\frac{mx}{2}\right],$$
(7)

with $\omega=\sqrt{k^2+m^2}$ and $\mathcal{N}^2=16(\omega^2/m^2)(\omega^2/m^2-\omega_B^2/m^2)$. The zero mode with $\omega_0=0$ is given by

$$\phi_0(x)=\sqrt{\frac{3m}{8}}\frac{1}{\cosh^2(mx/2)}.$$
(8)

The bound state with $\omega_B=(\sqrt{3}/2)m$ is given by

$$\phi_B(x)=\sqrt{\frac{3m}{4}}\frac{\sinh(mx/2)}{\cosh^2(mx/2)}.$$
(9)

The density of the continuous spectrum can be written as follows:

$$|\phi(k,x)|^2=\frac{1}{\mathcal{N}^2}\left[\frac{4\omega^2}{m^2}\left(\frac{4\omega^2}{m^2}-3\right)-12\frac{\omega^2}{m^2}\frac{1}{\cosh^2(mx/2)}\right.$$
$$\left.+9\frac{1}{\cosh^4(mx/2)}\right]$$
$$=1-\frac{m\phi_B^2(x)}{\omega^2-\omega_B^2}-\frac{2m\phi_0^2(x)}{\omega^2}.$$
(10)

As $\int_{-\infty}^{\infty}(dk/2\pi)[1/(\omega^2-\omega_B^2)]=1/m$, while $\int_{-\infty}^{\infty}(dk/2\pi)1/\omega^2=1/2m$, it is clear that the completeness relation

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}\{|\phi(k,x)|^2-1\}+\phi_0^2(x)+\phi_B^2(x)=0$$
(11)

is satisfied. Equation (10) may be written in a remarkable formula perhaps true for all reflectionless potentials, showing factorization of the difference in mode densities in phase space, where the position dependence of each term is given by the corresponding bound-state or zero-mode probability density,

Relation 1:

$$|\phi(k,x)|^2-1=-\sum_j \phi_j^2(x)\frac{2\sqrt{m^2-\omega_j^2}}{\omega^2-\omega_j^2},$$
(12)

satisfying the completeness relation, as one may check by performing the integration over k.

Note that all the above expressions for the density do not refer to any particular choice of boundary conditions, which of course do affect eigenenergies and the corresponding wave functions. The reason is that the choice of boundary conditions will contribute to the density away from the boundary at most terms of order $1/L$. In the large-L or continuum limit, in principle such terms might contribute to the total energy obtained by integration over the entire interval between the boundaries. We are unaware of any example of this phenomenon, as a previous claim in [2] has been shown to be incorrect [9].[2] Even if the phenomenon were to occur, for the integral just over a finite interval around the kink the effect would be negligible, so that the kink energy density and resulting energy can be computed reliably in terms of the continuum, modified-plane-wave solutions, unaffected by the choice of boundary conditions.

The requirement in Eq. (5) that the topological vacuum density and the trivial vacuum density be equal leads via Eq. (11) to

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[2]However, the latter work did identify a delocalized *momentum* for certain special boundary conditions.

$$\frac{\Delta\Lambda(x)}{\pi} = \int_\Lambda^\infty \frac{dk}{\pi} (1 - |\phi(k,x)|^2)$$

$$= \int_\Lambda^\infty dk \frac{2}{\pi} \sum_j \frac{\sqrt{m^2 - \omega_j^2}}{\omega^2 - \omega_j^2} \phi_j^2(x)$$

$$= \frac{m}{\pi\Lambda}(\phi_B^2(x) + 2\phi_0^2(x)) + \mathcal{O}\left(\frac{1}{\Lambda^2}\right)$$

$$= \frac{3m^2}{4\pi\Lambda} \frac{1}{\cosh^2(mx/2)} + \mathcal{O}\left(\frac{1}{\Lambda^2}\right). \tag{13}$$

With this result for $\Delta\Lambda(x)$ we can evaluate the energy density $\epsilon(x)$ in the kink sector. Adding also the counterterm[3]

$$\Delta M(x) = \sum_j 2\phi_j^2(x) \sqrt{m^2 - \omega_j^2} \int_0^\Lambda \frac{dk}{2\pi} \frac{1}{\omega}$$

$$= \frac{m^2}{4} \frac{1}{\cosh^2(mx/2)} \frac{3}{2\pi} \int_0^\Lambda \frac{dk}{\omega}, \tag{14}$$

and rewriting $1/\cosh^2(mx/2)$ as $(4/3m)\phi_B^2(x) + (8/3m)\phi_0^2(x)$ yields

$$\epsilon_{Cas}(x) = \epsilon(x) - \epsilon^{(0)}(x) = \frac{1}{2}\omega_B \phi_B^2(x)$$

$$+ 2 \int_0^\Lambda \frac{dk}{2\pi} |\phi(k,x)|^2 \frac{1}{2}\omega$$

$$- 2 \int_0^{\Lambda + \Delta\Lambda(x)} \frac{dk}{2\pi} \frac{1}{2}\omega + \Delta M(x)$$

$$= \frac{1}{2}\omega_B \phi_B^2(x) - \int_0^\Lambda \frac{dk}{2\pi} \left(\frac{m\phi_B^2(x)}{k^2 + m^2/4} + \frac{2m\phi_0^2(x)}{k^2 + m^2}\right)\omega$$

$$- \frac{\Delta\Lambda(x)}{2\pi}\Lambda + m(\phi_B^2(x) + 2\phi_0^2(x)) \int_0^\Lambda \frac{dk}{2\pi} \frac{1}{\omega}. \tag{15}$$

The two quadratic divergences proportional to $\int_0^\Lambda dk\,\omega$ have canceled because we subtracted the energy density of the trivial vacuum, while the counterterm cancels the remaining

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[3]The counterterm $\Delta M(x)$ usually is expressed in terms of the kink background field, but it also can be determined by noting that it should cancel the remaining logarithmic divergence in the difference of the sums over zero point energies $\int_{-\infty}^\infty (dk/2\pi)|\phi(k,x)|^2$ $(-\omega/2)$. Equating both expressions yields another formula perhaps valid for general reflectionless potentials,

Relation 2:

$$\frac{\sum_j 2\phi_j^2(x)\sqrt{m^2 - \omega_j^2}}{\sum_j 2\sqrt{m^2 - \omega_j^2}} = \frac{\phi_K^2(x) - \phi_K^2(\infty)}{\int_{-\infty}^\infty dx[\phi_K^2(x) - \phi_K^2(\infty)]}.$$

logarithmic divergence. Again, each term is proportional to a bound-state or zero-mode probability density.

The result is finite and reads

$$\epsilon_{Cas}(x) = \frac{1}{2}\omega_B \phi_B^2(x) - m\int_0^\Lambda \frac{dk}{2\pi}\left(\frac{\omega}{k^2 + m^2/4} - \frac{1}{\omega}\right)\phi_B^2(x)$$

$$- \frac{m}{2\pi}(\phi_B^2(x) + 2\phi_0^2(x)). \tag{16}$$

The last term is the contribution from the term due to $\Delta\Lambda(x)$, and is the analogue of the term in the central charge density for the SUSY case identified as the anomaly by [7]. Using the integral

$$\int_0^\infty \frac{dk}{2\pi}\left(\frac{1}{k^2 + m^2/4} - \frac{1}{k^2 + m^2}\right)\omega = \frac{1}{2\sqrt{3}}, \tag{17}$$

we obtain

$$\epsilon_{Cas}(x) = \left(\frac{1}{2}\omega_B - \frac{m}{2\sqrt{3}} - \frac{m}{2\pi}\right)\phi_B^2(x) - \frac{m}{\pi}\phi_0^2(x). \tag{18}$$

This formula can be rewritten as follows:

$$\epsilon_{Cas}(x) = \sum_j \frac{1}{2}\left(1 - \frac{2}{\pi}\arctan\frac{\omega_j}{\sqrt{m^2 - \omega_j^2}}\right)\omega_j\phi_j^2(x)$$

$$- \sum_j \frac{1}{\pi}\sqrt{m^2 - \omega_j^2}\,\phi_j^2(x), \tag{19}$$

where in the first sum the contribution with 1 comes from the bound states, and that with arctan comes from the continuum, while the second sum is the anomaly contribution. Such formulas for the total mass can be found in [10], though we are unaware of local versions in the literature. This kink example might be an illustration of a general factorization rule, valid for a wide class of reflectionless potentials. While we have not tested it for other cases, and do not know how to prove it other than by explicit computation, we believe that its simplicity and elegance make the rule worthy of further investigation.

Integration of $\epsilon_{Cas}(x)$ over x yields

$$M^{(1)} = \frac{1}{2}\omega_B\left(1 - \frac{2}{3}\right) - \frac{m}{2\pi} - \frac{m}{\pi} = \frac{\sqrt{3}m}{12} - \frac{3m}{2\pi}, \tag{20}$$

in agreement with Eq. (1). For convenience later, let us express Eq. (18) as a total derivative:

$$\epsilon_{Cas} = \frac{m}{2}\frac{d}{dx}\left\{\tanh(mx/2)\left[\frac{\sqrt{3}}{12}\tanh^2(mx/2) - \frac{3}{2\pi}\right]\right\}. \tag{21}$$

The first term gives the non-anomalous contribution, while the second yields that due to the anomaly.

Equations (19), (21) give compact expressions for the local energy density, which certainly provide the correct total

quantum energy of the bosonic kink. However, to obtain the correct local Casimir energy density, one must start with an expression for the energy density of each mode including a quadratic term in the gradient of the boson field $\frac{1}{2}(\partial_x \eta)^2$, whereas our formulas above, giving energies ω_n multiplying corresponding mode probability densities, implies instead the expression coming from the field equation,[4] $-\frac{1}{2}\eta\partial_x^2\eta$. Therefore we need to add to Eq. (19) the difference, a perfect differential of a function which vanishes far from the kink (and so does not change the computed mass correction for the kink),

$$\Delta\epsilon_{\mathrm{Cas}}(x) = \frac{1}{4}\partial_x^2\langle\eta^2(x)\rangle. \tag{22}$$

The propagator at equal times and positions $\langle\eta^2(x)\rangle$ (excluding the zero mode which solves the homogeneous equation) [7] can be obtained by integrating $|\phi|^2/2\omega$ in Eq. (10) with respect to $dk/2\pi$ and adding $\phi_B^2/2\omega_B$. The divergent part in $\langle\eta^2\rangle$ corresponds to setting $|\phi(k,x)|^2$ equal to its value at $|x|\to\infty$ in the k integral; as the divergence is x independent it cancels in Eq. (22). One thus obtains a finite (because the logarithmically divergent part has been subtracted) expectation value

$$\langle\eta^2(x)\rangle_{\mathrm{r}} \equiv \langle\eta^2(x)-\eta^2(x=\infty)\rangle$$

$$= \int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{1}{2\omega}[|\phi(k,x)|^2-1]+\frac{1}{2\omega_B}\phi_B^2(x)$$

$$= -\frac{3}{8\pi}\frac{1}{\cosh^4(mx/2)}+\frac{1}{4\sqrt{3}}\frac{\sinh^2(mx/2)}{\cosh^4(mx/2)}, \tag{23}$$

and therefore,

$$\Delta\epsilon_{\mathrm{Cas}} = \frac{d}{dx}\left\{\tanh(mx/2)\left[\frac{m}{4}\left(\frac{3}{4\pi}+\frac{1}{2\sqrt{3}}\right)\frac{1}{\cosh^4(mx/2)}\right.\right.$$

$$\left.\left.-\frac{m}{16\sqrt{3}}\frac{1}{\cosh^2(mx/2)}\right]\right\}. \tag{24}$$

As observed in [7], besides the Casimir energy density there is another consequence of the zero-point oscillations, namely, a position-dependent shift ϕ_1 in the classical background field. This in turn implies a further term in the local energy density, given by

$$\Delta\epsilon_{(\phi_1)}(x) = \partial_x\phi_1\partial_x\phi_{\mathrm{kink}}+\left(\frac{1}{2}U^2\right)'\phi_1 = \partial_x(\phi_1\partial_x\phi_{\mathrm{kink}}), \tag{25}$$

but of course no shift at this order in the total energy, because the classical energy is stationary with respect to arbitrary small variations of the classical field about its equilibrium form. Decomposing the Heisenberg field $\Phi(x,t)$ as $\phi_{\mathrm{kink}}(x)+\phi_1(x)+\eta(x,t)$, with the quantum fluctuation field obeying $\langle\eta\rangle=0$, and taking the expectation value of the Φ field equation $\langle-\partial_t^2\Phi+\partial_x^2\Phi-(\frac{1}{2}U^2)'\rangle=0$ gives

$$\partial_x^2\phi_1-\left(\frac{1}{2}U^2\right)''\phi_1 = \frac{1}{2!}\left(\frac{1}{2}U^2\right)'''\langle\eta^2\rangle-\frac{1}{2}\Delta m^2\phi_{\mathrm{kink}}. \tag{26}$$

This ϕ_1 is just what is needed to satisfy the no-tadpole condition in the kink background. As mentioned above, the singularity in $\langle\eta^2(x)\rangle$ is x independent and compensated by the Δm^2 term, yielding the quantity $\langle\eta^2(x)\rangle_{\mathrm{r}}$ of Eq. (23).[5] Solving Eq. (26) by the ansatz[6] $\phi_1 = Ax\phi_0(x)+B\partial_x\phi_0(x)$, and using the fact that ϕ_0 is proportional to $\partial_x\phi_{\mathrm{kink}}$, one finds

$$\phi_1 = \frac{\lambda}{m^2}\left[\left(\frac{1}{2\sqrt{3}}+\frac{3}{4\pi}\right)\frac{1}{\cosh^2(mx/2)}\right.$$

$$\left.-\frac{\sqrt{3}}{4}(m\partial_m+2\lambda\partial_\lambda)\right]\phi_{\mathrm{kink}}. \tag{27}$$

Through one-loop order (as in the SUSY case [7]), the effect of the second term is to replace the renormalized mass m and coupling λ in ϕ_{kink} with the bosonic pole mass $\bar{m}=m(1-\sqrt{3}(\lambda/4m^2))$ given in [3], Eq. (7) and the adjusted coupling $\bar{\lambda}=(\bar{m}^2/m^2)\lambda$. If we then rewrite the classical energy in terms of $\bar{m}$ and $\bar{\lambda}$, the classical mass is multiplied by a factor $1-\sqrt{3}(\lambda/4m^2)$. As ϕ_1 cannot shift the total mass, we know even without explicit calculation that the classical energy density in terms of the barred quantities must be renormalized by a compensating factor $1+\sqrt{3}(\lambda/4m^2)$. The first term in Eq. (27) is sensitive only to bosonic fluctuations and hence unchanged in the SUSY case [because, as we shall see, the fermionic source for ϕ_1 includes no terms with $1/\cosh^4(mx/2)$]; it contributes according to Eq. (25). The total one-loop bosonic energy density becomes

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[4]Our boson fluctuation field η is designated by χ in [7]. Note that, as utilized just below, for a mode of frequency ω_n our normalized mode function is given by $\phi_n=\sqrt{2\omega_n}\eta_n$.

[5]This procedure gives ϕ_1 as a finite, renormalized quantity, while [7] use an unrenormalized ϕ_1. Hence the "rescaling" part of their ϕ_1 gives the shift to the pole mass from the unrenormalized mass, but ours gives only the shift to the pole mass from the renormalized mass corresponding to the vanishing-tadpole condition in the trivial sector.

[6]The term with A is needed for the term in Eq. (23) proportional to $1/\cosh^2(mx/2)$ and the term with B is needed for terms with $1/\cosh^4(mx/2)$.

Relation 3:

$$\mathcal{E}(x) = U^2(\bar{\lambda}, \bar{m}, \phi_{\text{kink}}(\bar{\lambda}, \bar{m}, x)) \left(1 + \sqrt{3} \frac{\bar{\lambda}}{4\bar{m}^2} \right) + \epsilon_{\text{Cas}}(x)$$

$$+ \Delta \epsilon_{\text{Cas}}(x) + \frac{\bar{m}}{4} \left(\frac{3}{4\pi} + \frac{1}{2\sqrt{3}} \right) \partial_x \left(\frac{\tanh(\bar{m}x/2)}{\cosh^4(\bar{m}x/2)} \right).$$

(28)

The last term in Eq. (28) comes from the first term in Eq. (27). The second term in Eq. (27) renormalizes the classical contribution U^2, as seen in the first term in Eq. (28). The effect of this renormalization is that the classical energy density flattens out a bit. Besides the rescaling, all other contributions are of the form $\partial_x[\phi_{\text{kink}}(x)/\cosh^n(mx/2)]$, with $n = 0, 2, 4$.

SUSY KINK ENERGY DENSITY

For the SUSY kink we choose the cutoffs in the trivial sector in such a way that the bosonic and fermionic densities in that sector are equal. To make the bosonic and fermionic densities also equal in the topological sector, we use a cutoff Λ for the bosons and $\Lambda + \Delta \Lambda(x)$ for the fermions as in Eq. (6). The fermion is described by a Majorana two-component spinor $\psi = \binom{\psi_+}{\psi_-}$. As $\psi_+(k,x)$ is proportional to $\phi(k,x)$ while $\psi_-(k,x) = (i/\omega)(\partial_x + m \tanh(mx/2))\psi_+(k,x)$ for solutions proportional to $\exp(-i\omega t)$ according to the Dirac equation [3], one obtains for the wave functions of the continuous fermionic spectrum

$$\psi_+(k,x) = \frac{1}{\sqrt{2}} \phi(k,x)$$

(29)

$$\psi_-(k,x) = \frac{1}{\sqrt{2}} \frac{\omega}{\mathcal{N}m} \left(-4\frac{k}{m} - 2i \tanh\frac{mx}{2} \right) e^{ikx}.$$

(30)

In the difference of the densities the constant term of course cancels, giving

$$|\phi(k,x)|^2 - |\psi_+(k,x)|^2 - |\psi_-(k,x)|^2$$

$$= |\psi_+(k,x)|^2 - |\psi_-(k,x)|^2$$

$$= \frac{1}{2\mathcal{N}^2} \left(\frac{9}{\cosh^4(mx/2)} - 8 \left(\frac{\omega}{m} \right)^2 \frac{1}{\cosh^2(mx/2)} \right).$$

(31)

For the bosonic and fermionic densities to satisfy Eq. (6) one requires

$$\phi_0^2(x) + \phi_B^2(x) + 2 \int_0^\Lambda \frac{dk}{2\pi} |\phi(k,x)|^2$$

$$= \frac{1}{2} \phi_0^2(x) + \left(\frac{1}{2} \phi_B^2(x) + \frac{m}{8 \cosh^2(mx/2)} \right)$$

$$+ 2 \int_0^{\Lambda + \Delta\Lambda} \frac{dk}{2\pi} \{ |\psi_+(k,x)|^2 + |\psi_-(k,x)|^2 \}. \quad (32)$$

The factor $\frac{1}{2}$ in $\frac{1}{2}\phi_0^2(x)$ comes from the mode expansion $\psi_+(x,t) = c_0 \phi_0(x,t) + \cdots$ with $\{c_0, c_0\} = 1$.[7] This $\frac{1}{2}$ is the analogue for Majorana fermions of the fractional fermion charge discovered by Jackiw and Rebbi for Dirac fermions [11]. The two terms in parentheses give the ψ_+ and ψ_- contributions of the bound state: $\psi_{B+}^2 = \frac{1}{2}\phi_B^2$ and $|\psi_{B-}|^2 = m/8 \cosh^2(mx/2)$. We obtain

$$\frac{1}{2} \phi_0^2(x) + \psi_{B+}^2(x) - |\psi_{B-}|^2(x) + 2 \int_0^\Lambda \frac{dk}{2\pi} (|\psi_+(\Lambda,x)|^2$$

$$- |\psi_-(\Lambda,x)|^2) = \frac{\Delta\Lambda(x)}{\pi} (|\psi_+(k,x)|^2 + |\psi_-(k,x)|^2).$$

(33)

Using the completeness relation, and taking the large k limit $|\psi_+(k,x)|^2 + |\psi_-(k,x)|^2 \to 1$, one finds

$$\frac{\Delta\Lambda(x)}{\pi} = -2 \int_\Lambda^\infty \frac{dk}{2\pi} (|\psi_+(k,x)|^2 - |\psi_-(k,x)|^2). \quad (34)$$

As we are interested only in the $1/\Lambda$ term, the calculation is easy. From Eq. (31) we find

$$\frac{\Delta\Lambda(x)}{\pi} = \frac{m^2}{4\pi\Lambda} \frac{1}{\cosh^2(mx/2)}.$$

(35)

With this result in hand, we compute the difference in energy densities for the SUSY kink

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[7]Obtained in [2] from Dirac quantization, this factor equivalently can be deduced from the completeness relation for solutions of the single-particle Dirac equation $\phi_0^2(x) + 2[|\psi_{B+}|^2 + |\psi_{B-}|^2 + \int (dk/2\pi)(|\psi_+|^2 + |\psi_-|^2 - 1)] = 0$, where $|\psi_-(k,x)|^2 = 1/2 - m^2/(8 \cosh^2(mx/2)(\omega^2 - \omega_B^2))$. The relative factor of two between the first term and the later terms in the completeness relation follows from the fact that, if one sums over a complete set of solutions of the Dirac equation, all nonzero frequencies lead to equal contributions from positive and from negative frequency, while the zero mode contributes only once.

$$\epsilon_{\mathrm{Cas,b}}(x)-\epsilon_{\mathrm{Cas,f}}(x)=\frac{1}{2}\,\omega_B(\phi_B^2(x)-\psi_B^\dagger(x)\psi_B(x))$$

$$+2\int_0^\Lambda\frac{dk}{2\pi}(|\psi_+(k,x)|^2$$

$$-|\psi_-(k,x)|^2)\frac{1}{2}\,\omega-\frac{\Delta\Lambda(x)}{\pi}\frac{1}{2}\Lambda$$

$$+\Delta M_{\mathrm{SUSY}}(x).\qquad(36)$$

The counterterm in the SUSY case,

$$\Delta M_{\mathrm{SUSY}}(x)=\frac{m^2}{2}\,\frac{1}{\cosh^2(mx/2)}\int_0^\Lambda\frac{dk}{2\pi}\frac{1}{\omega},\qquad(37)$$

is a factor 1/3 smaller, but still nonvanishing. Again, the counterterm removes the logarithmic divergence in the integral, and with Eq. (17) one finds

$$\epsilon_{\mathrm{Cas,SUSY}}(x)=\epsilon_{\mathrm{b}}(x)-\epsilon_{\mathrm{f}}(x)$$

$$=\left(1-\frac{2}{3}\right)\frac{1}{2}\,\omega_B(\phi_B^2(x)-\psi_B^\dagger(x)\psi_B(x))$$

$$-\frac{m^2}{8\pi}\frac{1}{\cosh^2(mx/2)},\qquad(38)$$

where the last term, the contribution from $\Delta\Lambda$, agrees with the central charge density anomaly Eq. (3.38) in [7]. Integration over x yields the one-loop correction to the mass of the SUSY kink

$$M_{\mathrm{SUSY}}^{(1)}=\lim_{X\to\infty}\int_{-X}^X dx\,\frac{d}{dx}\left[\frac{\sqrt{3}m}{48}[\tanh^3(mx/2)-\tanh(mx/2)]\right.$$

$$\left.-\frac{m}{4\pi}\tanh(mx/2)\right]$$

$$=\frac{1}{6}\,\omega_B(1-1)-\frac{m}{2\pi}$$

$$=-\frac{m}{2\pi},\qquad(39)$$

which of course is the accepted answer. Note that the nonanomalous contributions from the bosons and the fermions do not cancel locally, but in the integral they do: $\frac{1}{6}\omega_B(1-1)=0$. For explicit expressions later it is helpful to rewrite the first part in the bracket of Eq. (39) as

$$\epsilon_{\mathrm{Cas,SUSY}}(\text{non-anom})$$

$$=\frac{d}{dx}\frac{m}{16\sqrt{3}}[-\tanh(mx/2)/\cosh^2(mx/2)].$$

$$(40)$$

As in the bosonic case we must add the missing term in the bosonic Casimir energy density $\Delta\epsilon_{\mathrm{Cas}}$ given in Eqs. (22),

(23), as well as include the shift for the SUSY case $\phi_1(x)$ in the background field. We compute this ϕ_1, again using the second-order field equation for Φ, which now has an additional contribution from fermions:

$$\partial_x^2\phi_1-\left(\frac{1}{2}U^2\right)''\phi_1=\left\{\frac{1}{2!}\left(\frac{1}{2}U^2\right)'''\langle\eta^2\rangle-\frac{1}{2}\Delta m_b^2\phi_{\mathrm{kink}}\right\}$$

$$+\left\{\frac{1}{2}U''\langle\bar\psi\psi\rangle-\frac{1}{2}\Delta m_f^2\phi_{\mathrm{kink}}\right\},\quad(41)$$

with

$$\delta m_F^2=-4\lambda\int_{-\infty}^\infty\frac{dk}{2\pi}\frac{1}{2\omega}$$

and

$$\langle\bar\psi\psi\rangle_r=(\partial_x+2m\tanh(mx/2))\langle\eta^2(x)\rangle_r$$

$$=\int_{-\infty}^\infty\frac{m\,dk}{2\pi\omega}[6m^2\tanh(mx/2)/$$

$$(16(k^2+m^2/4)\cosh^2(mx/2))]$$

$$-(\sqrt{3}m/4)\tanh(mx/2)/\cosh^2(mx/2).\qquad(42)$$

The last term comes from the bound state, and a term with $(k^2+m^2/4)$ in the numerator of the integrand has been canceled by the fermionic part of the mass counterterm. Again using Eq. (17), we find that the fermionic contributions to ϕ_1 are only proportional to $1/\cosh^2(mx/2)$, and not $1/\cosh^4(mx/2)$, so that the net coefficient of the $1/\cosh^2(mx/2)$ term is a factor 2/3 smaller than in the bosonic case. The final result for ϕ_1 reads

$$\phi_{1,\mathrm{SUSY}}=\frac{\lambda}{m^2}\left[\left(\frac{1}{2\sqrt{3}}+\frac{3}{4\pi}\right)\frac{1}{\cosh^2(mx/2)}\right.$$

$$\left.-\frac{1}{2\sqrt{3}}(m\partial_m+2\lambda\partial_\lambda)\right]\phi_{\mathrm{kink}}.\qquad(43)$$

Note that in Eq. (43) the first term is the same as in the bosonic case (as mentioned earlier), while the second term is smaller by a factor 2/3. This is the same result found in [7] using a first order differential equation based on SUSY considerations. Iterating the SUSY relation $\langle\partial_x\phi+U\rangle=0$, one confirms that the second-order and first-order approaches are consistent with each other.[8] For the SUSY energy density we then find full agreement with the central charge density of [7], after restoring a missing factor of $\frac{1}{2}$ in the first line of Eq. (5.21) in that work, kindly pointed out to us by the authors.

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[8]The details are as follows. From $\langle\partial_x\phi+U\rangle=0$ we have $\partial_x\phi_1+U'\phi_1+(1/2!)U''\langle\eta^2\rangle+\sqrt{\lambda/2}(-\Delta\mu^2/\lambda)=0$. Differentiating with respect to x, using $\partial_x\phi_{\mathrm{kink}}=-U$, and eliminating $\partial_x\phi_1$ yields an equation for $\partial_x^2\phi_1$. That this is equivalent to Eq. (41) follows from the identity $\langle\eta(\partial_x+U')\eta+((\partial_x+U')\eta)\eta\rangle=i\langle\psi_+\psi_- -\psi_-\psi_+\rangle$, which in turn is a consequence of $\psi_-=(i/\omega)(\partial_x\tanh(mx/2))\psi_+$ and $\psi_+=\eta\sqrt{\omega}$.

Relation 4:

$$\mathcal{E}(x) = U^2(\bar{\lambda}_s, \bar{m}_s, \phi_{kink}(\bar{\lambda}_s, \bar{m}_s, x))\left(1 + \frac{\bar{\lambda}_s}{2\sqrt{3}\bar{m}_s^2}\right)$$

$$+ \epsilon_{Cas,SUSY}(x) + \Delta\epsilon_{Cas}(x)$$

$$+ \frac{\bar{m}_s}{4}\left(\frac{3}{4\pi} + \frac{1}{2\sqrt{3}}\right)\partial_x\left(\frac{\tanh(\bar{m}_s x/2)}{\cosh^4(\bar{m}_s x/2)}\right). \quad (44)$$

Here we use $\bar{m}_s = m(1 - \lambda/2\sqrt{3}m^2)$ and $\bar{\lambda}_s = \lambda(1 - \lambda/\sqrt{3}m^2)$. Thus the rescaling of the classical density involves a shift 2/3 as big as for the bosonic case, while $\epsilon_{Cas,SUSY}$ includes an anomaly 1/3 as big as in the bosonic case and a different finite-energy contribution, as explained above. Meanwhile, the last two terms, from $\Delta\epsilon_{Cas}$ and from the nonrescaling term in ϕ_{1r}, are the same as in the bosonic case.[9]

CENTRAL CHARGE DENSITY

We now compute the anomalous contribution to the density $\zeta(x)$. Before regularization one has $\mathcal{H}(x) = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + \frac{1}{2}U^2 + i/2(\psi_+\psi_+' + \psi_-\psi_-') - iU'\psi_+\psi_-$, and $\zeta(x) = U\partial_x\phi$ (note that [7] have the opposite sign convention for ζ). Using the equal-time anticommutators of the fermionic fields $\psi_+(x)$ and $\psi_-(x)$ and the definition

$$j_{\pm} = (-(\partial\!\!\!/\phi + U)\gamma^0\psi_{\pm}, \quad (45)$$

one obtains

$$\{Q_{\pm}, j_{\pm}(y)\} = 2\mathcal{H}(y) \pm 2\zeta(y),$$

$$j_{\pm} = \dot{\phi}\psi_{\pm} + (\phi' \pm U)\psi_{\mp}, \quad Q_{\pm} = \int_{-L/2}^{L/2} j_{\pm} dx, \quad (46)$$

$$\zeta(y) = \int dx\left[\frac{1}{2}(\{\psi_+(x), \psi_+(y)\} - \{\psi_-(x), \psi_-(y)\})\right.$$

$$\times\left(\frac{1}{2}\dot{\phi}(x)\dot{\phi}(y) - \frac{1}{2}\phi'(x)\phi'(y) + \frac{1}{2}U(x)U(y)\right)$$

$$+ \frac{1}{2}(\{\psi_+(x), \psi_+(y)\} + \{\psi_-(x), \psi_-(y)\})$$

$$\left.\times\left(\frac{1}{2}\phi'(x)U(y) + \frac{1}{2}U(x)\phi'(y)\right)\right]. \quad (47)$$

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[9]For comparison with [7] note that our coupling λ is equal to $2\lambda^2$ in their formulation. Also, they give the integral of the density from $-x$ to x, while we write the pieces of the quantum correction to the density as local derivatives, so that our expression for the function being differentiated is half theirs for the integral. The specific terms in their equation (5.21) are related to ours as follows: The first line in Eq. (5.21) is simply the integral of what we call U^2. The first term in the second line is the anomaly. In the final bracket, the first term receives equal contributions from ϵ_{Cas} and $\Delta\epsilon_{Cas}$. The remaining piece receives equal contributions from $\Delta\epsilon_{Cas}$ and from the nonrescaling part of ϕ_1, the shift in the classical field.

This is an exact result; all terms with bosonic commutators cancel. The anticommutators $\{\psi_+(x), \psi_-(y)\}$ and $\{\psi_+(y), \psi_-(x)\}$ all vanish, and also the first line in $\zeta(y)$ vanishes, while the second line would seem to give

$$\zeta(y) = \int_{-\infty}^{+\infty} dx\,\delta(x-y)\left[\frac{1}{2}\langle\eta(x)\eta(x)\rangle U''\phi'(x)\right.$$

$$\left.+ \langle\eta'(x)\eta(x)\rangle U'(x)\right]$$

$$= \int_{-\infty}^{+\infty} dx\,\delta(x-y)\frac{\partial}{\partial x}\left[\frac{1}{2}\langle\eta(x)\eta(x)\rangle U'(x)\right], \quad (48)$$

$$\int_{-\infty}^{\infty} dy\,\zeta(y) = \frac{1}{2}\langle\eta(x)\eta(x)\rangle U'(x)\Big|_{-\infty}^{\infty}. \quad (49)$$

This is the expression obtained in [3]. Below we show that with appropriate care (i.e., not setting $x = y$ too soon), there is an extra term—the anomaly. The naive result in Eq. (49) contains a free field propagator for η, because at $x = \pm\infty$ the effects of the kink disappear, and, adding the counterterm to the central charge due to mass renormalization, all quantum corrections to the central charge would seem to vanish. In the approach of [7], on the other hand, the central charge contains a naive term $\phi'U$ and an explicit correction term which is also a total derivative and proportional to $1/M^2$. Because their η propagator contains an extra regulating factor $(k^2 + M^2)^{-1}$, the contribution in Eq. (49) now cancels even after regularization, but because the correction term contains two extra derivatives (to balance the factor $1/M^2$) it yields an extra contribution proportional to M^2/M^2, which is the anomaly.

In our case we start from Eq. (47), but without extra terms as in [7]. We keep $x \neq y$ in Eq. (47), giving

$$\zeta(y) = \int dx\,\delta(x-y)\left[U''(x)\phi'_{kink}(y)\frac{1}{4}\langle\eta^2(x)\rangle\right.$$

$$\left.+ \frac{1}{2}\langle\eta'(y)\eta(x)\rangle U'(x) + (x\leftrightarrow y) + \Delta\mu^2 \text{ term}\right]. \quad (50)$$

We now show that the result is still a total derivative, but instead of the total derivative in Eq. (49), rather a total derivative with an extra term. The crucial point is that one cannot replace $\delta(x-y)\langle\eta'(x)\eta(y)\rangle$ by $\frac{1}{2}\delta(x-y)\partial_x\langle\eta^2(x)\rangle$ because there is a singularity in $\langle\eta'(x)\eta(y)\rangle$ as x tends to y. Setting $\delta(x-y)[\langle\eta'(x)\eta(y)\rangle - (1/2)\partial_x\langle\eta(x)^2\rangle] = 0$ would mean that all terms vanish as in [3]. However, the singularity which invalidates this equality yields the anomaly

$$\int_{-\infty}^{\infty}\zeta(x)dx = \frac{W''(\phi)}{4\pi}\Big|_{-\infty}^{\infty}, \quad (51)$$

where $W'(\phi) = U$. Hence $M^{(1)} = -Z^{(1)}$ in agreement with the invariance of the background under Q_+ (which corresponds to the SUSY transformation with parameter ϵ_-). Let us see this explicitly.

The identity we need is

$$\delta(x-y)\langle \eta'(x)\eta(y)\rangle(f(x)-f(y)) = -\frac{1}{2\pi}\delta(x-y)f'(x),$$
(52)

where f is any smooth function of x. The proof of this identity follows from $\langle \eta(x)\eta(y)\rangle = -(1/2\pi)\ln|x-y|+A(x,y)$, where A is a smooth, symmetric function, therefore involving only nonnegative even powers of $(x-y)$, as can be seen from Eqs. (7)–(9). The actual calculation of the anomaly is now very simple. Expanding all contributions in terms of $x-y$, and using $\delta(x-y)(x-y)\Delta\mu^2 = 0$ after regularization of $\Delta\mu^2$, while $\delta(x-y)(x-y)\partial_x\langle \eta^2(x)\rangle = 0$ does not need regularization because $\partial_x\langle \eta^2(x)\rangle$ is finite, we have from Eq. (52)

$$\int_{-\infty}^{\infty}\zeta(y)dy = \int_{-\infty}^{\infty}dxdy\left[\frac{1}{4\pi}\delta(x-y)U''(\phi)\phi'(x)\right] = \frac{m}{2\pi}.$$
(53)

Here we used

$$U = \sqrt{\lambda/2}(\phi^2-\mu^2/\lambda), \quad \phi = (\mu/\sqrt{\lambda})\tanh(\mu x/\sqrt{2}),$$

and $m = \mu\sqrt{2}$. Again we have the accepted result, and we see that point-splitting regularization yields the same extra term in the central charge as does higher-derivative regularization.

Thus we have shown in a simple way that the term $\langle \eta'(y)\eta(x)\rangle U'(x)$ produces the anomaly if one does not set $x=y$ too soon. In [7] a more complicated but also more powerful regularization scheme was used to prove this. Our observation pinpoints the place where naive methods missed the anomaly. As discussed extensively in the previous sections, one must add to the anomalous part the various contributions to the non-anomalous part of the central charge density. This works exactly as in [7], and of course is completely unaffected by choice of regularization method.

In view of our emphasis on LMR for energy density, it is reasonable to ask why we do not attempt to apply it to central charge density. Looking at Eq. (50), one sees that the expression to be regulated, the bilocal correlator in $\eta(x)$ and $\eta'(y)$, which clearly is not determined by insisting that the regulated sum $\text{Im}\langle \eta(x)\dot{\eta}(x)\rangle$ is unchanged between vacuum and kink backgrounds (the entire content of the LMR prescription). Thus LMR may be applied as a condition on the expressions in the central charge density, but is not sufficient to regulate them.

FOUNDATIONS AND CONCLUSIONS

Finally we comment on the physical basis for LMR. In Planck's original formulation of quantum physics [12], the number of degrees of freedom is defined by the available volume in phase space. To fix the total number of modes while introducing a background potential affecting the fluc-tuations is simply to conserve the total phase space available. The work of Einstein [13] and Debye [14] on crystal vibration contributions to heat capacity introduced the concept of a local density of degrees of freedom, codifying a notion already found in Boltzmann's lectures on gas theory [15]. As was true for their work, in a lattice approach the number of degrees of freedom per unit volume evidently does not change when interactions are introduced, and the local mode density should be equal to this number of degrees of freedom.

Also point splitting methods clarify the meaning of LMR. Consider the bosonic local mode density regulated by point splitting

$$\rho(k,x) = \int_{-\infty}^{\infty}dy\,\phi^*\left(k,x-\frac{y}{2}\right)\phi\left(k,x+\frac{y}{2}\right)f(y),$$
(54)

where $f(y)$ is a function sharply peaked around $y=0$, with $\int dy f(y) = 1$. For large k, the JWKB approximation for $\phi(k,x)$ is

$$\phi(k,x) = e^{ikx}e^{-i\int^x dx'\,V(x')/2k},$$
(55)

where

$$V(x) = U(\phi(x))U''(\phi(x))+(U')^2(\phi(x))$$
$$-(U')^2(\phi(|x|\to\infty)).$$

Substituting this expression into $\rho(k,x)$ one finds, for the integrand of Eq. (54),

$$e^{iky}e^{-i\int_{x-y/2}^{x+y/2}[V(x')/2k]dx'}f(y) \simeq e^{i(k-V(x)/2k)y}f(y).$$
(56)

In the trivial sector $\rho(k,x) = \tilde{f}(k)$, where $\tilde{f}(k)$ is the Fourier transform of $f(x)$, but in the kink sector one finds a modification $\rho(k,x) = \tilde{f}(k-V(x)/2k)$. The energy density therefore contains a term $\delta\epsilon(x) = \int \delta\rho(k,x)\frac{1}{2}\omega(dk/2\pi)$, and expanding $\tilde{f}$ we find[10]

$$\delta\epsilon(x) = -\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{V(x)}{2k}\frac{\omega}{2}\frac{\partial}{\partial k}\tilde{f}(k) = -V(x)/8\pi.$$
(57)

For the bosonic kink, this is the anomaly in Eq. (16), as one may readily check by direct substitution.

From the JWKB form for the wave function at high energies it follows that the quantity $\Delta\Lambda(x)$, and hence the anomaly, depends on x only through the potential felt by the fluctuations. This in turn implies that the local anomaly in the energy density is determined at each x by the value of the classical background field $\phi(x)$, as stated for the central charge density of the supersymmetric case in [7].

While the above discussion shows that the LMR result for the bosonic kink follows from point-splitting, it is possible to

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[10]The narrower $f(x)$, the wider the range in k contributing to Eq. (57), so that in the limit the contribution of any finite range around $k=0$ becomes negligible. This justifies our use of the JWKB approximation, which is valid for $k^2 \gg m^2, V$.

make a much stronger statement, that point-splitting implies LMR for a scalar field in an arbitrary background potential in one space dimension. The same JWKB approximation used to determine the shift in effective wave number due to the potential V can be used also to compute the modulation of the mode density versus energy (or equivalently, versus asymptotic wave number k) in the region of nonzero potential, compared to the asymptotic density far away. This is a standard calculation, obtaining the wave function and hence the density to one higher order in $1/k$ than required for the shift in effective wave number. There is a simple physical mnemonic for the result of the calculation. Treating this system as a Schrödinger problem with ''Hamiltonian'' ω^2, the modulating factor is simply the ratio of the asymptotic classical velocity to the local velocity:

$$\rho(x)/\rho(\infty) = v/v(x) = k/\sqrt{k^2 - V(x)}, \qquad (58)$$

or

$$\delta\rho/\rho \sim V(x)/2k^2. \qquad (59)$$

Integrating the above expression from a nominal sharp cutoff Λ to ∞, we see that to have the same integrated mode number density above the cutoff in the presence of a potential $V(x)$ as in a trivial background, we must shift the cutoff by

$$\delta\Lambda = V(x)/2\Lambda, \qquad (60)$$

exactly the amount implied by point-splitting as found in the discussion leading to Eq. (57) above.

The equality between the wave number shift $\delta k(x)$ and the integral over the density shift is reminiscent of an unsubtracted dispersion relation. Possibly the implementation of LMR in higher dimensions would require the equivalent of subtracted dispersion relations to compensate for the increasing divergence of energy density with cutoff.

We have seen that point-splitting, a regularization scheme which is local but not necessarily useful beyond one-loop order, gives the same anomaly in the central charge found in [7] with higher-derivative regularization, and also implies LMR for the bosonic energy density with arbitrary background potential. As explained above, the converse is not true: LMR does not contain the full content of point-splitting regularization. Nevertheless, it is appealing that it captures in one line the above sequence of equations required to obtain the anomaly from point-splitting regularization. Thus, as stated already in the Introduction, LMR is a simple, easily-used tool for a special but extremely important application, the computation of local one-loop energy densities.

For the supersymmetric case, one gains insight into the requirement of equal bosonic and fermionic mode densities by considering the $N=2$ theory, where there is an Abelian charge density which should be invariant under supersymmetry. That requirement automatically imposes the constraint represented by LMR.

While all of the above are appealing arguments, the accepted criterion for determining the validity of a regulation procedure is to insert regulators into the action in such a way that all relevant symmetries are satisfied at the regulator level, and then to deduce consequences for specific quantities. Thus in the present case a definitive check on the validity of LMR would be to use, for example, higher-derivative regulation (which obeys supersymmetry), and check that this scheme implies LMR. This important analysis remains to be done. As an alternative, one might be able to prove that point-splitting preserves SUSY in models with solitons, and then extend our deduction of LMR from point-splitting in the bosonic case to the SUSY case. This is something to which we intend to return in the future.

To summarize, LMR permits one to isolate and then compute directly the anomalous contribution to the energy density of the bosonic or SUSY kink. Expressed most conservatively, LMR at least gives a simple interpretation of the anomaly as the shift in energy density required to equalize mode densities. In fact as we have just seen, at least for Bose fields in one space dimension with arbitrary scalar background potential, LMR follows from point-splitting regularization of the energy density. In addition we found remarkable phase space factorization identities for the non-anomalous contributions to the energy density, which might hold for all reflectionless potentials. These non-anomalous contributions are independent of the regularization method (though sensitive to renormalization conditions because they only are convergent after subtraction of the mass counterterm). Elsewhere [16] we compute the divergent energy density at the boundary of the kink with supersymmetric boundary conditions, and obtain an analytic expression for the anomaly near the boundary, which in the limit when the regulator energy goes to infinity becomes a delta-function contribution just at the boundary, in agreement with expectations of [7]. It would be interesting to explore local mode regularization further, comparing with complete regularization schemes and studying solitons in higher dimensions such as the magnetic monopole, and also explore phase space factorization, seeking a theoretical basis as well as additional examples.

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