From $N=1$ to $N=2$ supersymmetries in $2+1$ dimensions

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Starting from $N=1$ scalar and vector supermultiplets in $2+1$ dimensions, we construct superfields which constitute Lagrangians invariant under $N=2$ supersymmetries. We first recover the $N=2$ supersymmetric Abelian-Higgs model and then the $N=2$ pure super Yang-Mills model. The conditions for this elevation are consistent with previous results found by other authors.

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$N=2$ supersymmetry in $2+1$ dimensions had been studied [1] in order to investigate the supersymmetrization of the instantons effects which lead to a linear confinement [2]. Since then, systematic studies of $N=2$ supersymmetry in $2+1$ dimensions have been done in [3] where exact results were obtained, such as superpotentials and topologies of moduli spaces in various cases.

These exact results can be derived since $N=2$ supersymmetry in $2+1$ dimensions can be obtained by dimensional reduction of $N=1$ supersymmetry in $3+1$ dimensions [4] and thus has similar properties, such as non-renormalization theorems. These theorems are not present for $N=1$ supersymmetry in $2+1$ dimensions and it is thus interesting to study its elevation to $N=2$. In this context, it was shown that the presence of topologically conserved currents leads to a centrally extended $N=2$ supersymmetry, the central charge of the superalebra being the topological charge [5]. In [6], the $N=1$ supersymmetric Abelian-Higgs model was considered and it was shown that the on-shell Lagrangian can be extended to the $N=2$ Abelian-Higgs model if a relation is imposed between the gauge coupling and the Higgs self-coupling. Such a condition is in general expected in a $N=2$ invariant theory built out of $N=1$ Lagrangians [7]. Another example of extension was given in [8] where $N=1$ supersymmetries of composite operators was elevated to a $N=2$ Abelian model, up to irrelevant operators. In this work, the coupling of matter to the gauge field was obtained with higher order composites, simulating the dynamical generation of the $N=2$ supersymmetry that would occur after an appropriate functional integration of a gauge field coupled to the original $N=1$ supermultiplets.

We propose here another illustration of the supersymmetry extension, with the superfield construction of $N=2$ Lagrangians in terms of $N=1$ scalar and vector superfields. The $N=1$ superspace in $2+1$ dimensions contains only one real two-component Grassmann coordinate θ and the invariant actions are integrals over superspace which involve $\int d^2\theta$. The Lagrangians are constructed out of superfields which mix the original $N=1$ superfields in such a way that a $N=1$ supersymmetric transformation on the original superfields leaves the $N=2$ Lagrangians invariant.

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We first consider the pure $U(1)$ case and then add matter so as to construct the Abelian Higgs model, using a Fayet-Iliopoulos term. We finally make a superfield construction for the $N=2$ pure Yang-Mills Lagrangian. As will be seen, the present contruction exhibits naturally the conditions found in [5] and [6] for the elevation of a $N=1$ to a $N=2$ supersymmetry.

The gamma matrices are given by $\gamma^0=\sigma^2, \gamma^1=i\sigma^1, \gamma^2=i\sigma^3$, where $\sigma^1,\sigma^2,\sigma^3$ are the Pauli matrices, such that $g^{\mu\nu}=\mathrm{diag}(1,-1,-1)$ and $[\gamma^\mu,\gamma^\nu]=-2i\epsilon^{\mu\nu\rho}\gamma_\rho$. We have the following usual properties, valid for any 2-component complex spinors η,ζ:

$$\eta\zeta=\eta^\alpha\zeta_\alpha=\zeta\eta \quad \text{and} \quad \eta\gamma^\mu\zeta=-\zeta\gamma^\mu\eta. \tag{1}$$

The 2-component real spinor θ, Grassmann coordinate in the superspace, satisfies the properties

$$\begin{aligned}
&(\theta\eta)(\theta\zeta)=-\frac{1}{2}\theta^2(\eta\zeta)\\
&\theta\gamma^\mu\theta=0\\
&\theta\gamma^\mu\gamma^\nu\theta=-\theta^2 g^{\mu\nu}\\
&\theta\gamma^\mu\gamma^\nu\gamma^\rho\theta=-i\theta^2\epsilon^{\mu\nu\rho}\\
&\theta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\theta=-\theta^2(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}),\\
&\int d^2\theta\ \theta^2=1.
\end{aligned} \tag{2}$$

In $2+1$ dimensions, the $N=1$ scalar superfield and the $N=1$ vector superfield in the Wess-Zumino gauge are respectively given by

$$\begin{aligned}
\Phi&=\rho+(\theta\xi)+\frac{1}{2}\theta^2 D\\
V_\alpha&=i(\not{A}\theta)_\alpha+\frac{1}{2}\theta^2\chi_\alpha,
\end{aligned} \tag{3}$$

where all the fields are real. To form an $N=2$ supermultiplet, we define the complex gaugino $\lambda=\xi+i\chi$. The two fermionic degrees of freedom then balance the two bosonic ones, since A_μ has one degree of freedom [9]. The (complex) scalar superfield G containing these degrees of freedom is

$$G=\Phi+iD^\alpha V_\alpha$$
$$=\rho+(\theta\lambda)+\frac{1}{2}\theta^2 D^2+i\partial_\mu A_\nu(\theta\gamma^\mu\gamma^\nu\theta), \quad (4)$$

where the superderivative is $D_\alpha=\partial_\alpha+i(\partial\!\!\!/\theta)_\alpha$ [11]. We will see that the elevation of the supersymmetry is possible under a gauge condition which affects the superfield G and the relevant fundamental superfield is, actually,

$$D^\beta G=-\lambda^\beta-D\theta^\beta-i\partial_\mu A_\nu(\gamma^\nu\gamma^\mu\theta)^\beta+i(\partial\!\!\!/\rho\theta)^\beta$$
$$+i(\theta\partial_\mu\lambda)(\gamma^\mu\theta)^\beta. \quad (5)$$

With the properties (2), it is easy to see that

$$\int d^2\theta D^\beta G D_\beta \overline{G}=-\frac{1}{2}F^{\mu\nu}F_{\mu\nu}+i\overline{\lambda}\partial\!\!\!/\lambda+\partial_\mu\rho\partial^\mu\rho+D^2$$
$$+(\partial^\mu A_\mu)^2+\text{surface term}, \quad (6)$$

where the surface term is $\partial_\mu(\overline{\lambda}\gamma^\mu\lambda)$ and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. If the gauge condition $\partial^\mu A_\mu=0$ is imposed, we find then the $N=2$ Abelian gauge kinetic term. This gauge condition was found in [5] where the authors explain that they need to choose a gauge in which the vector superfield satisfies $D^\alpha V_\alpha=0$ so as to construct a superfield containing a topological current and two supercurrents which are at the origin of the $N=2$ structure. This condition implies then for the gauge field component that $\partial^\mu A_\mu=0$, where A_μ is given the role of the topologically conserved current. It is then natural that we find here the same condition, which should be independent of the dynamics. Indeed, it was explicitly shown for the CP^1 model in [5] and will be found again for the non-Abelian dynamics in the present article. Note here that, with the condition $D^\alpha V_\alpha=0$, the superfield G reduces to Φ, showing that the fundamental superfield is actually $D^\beta G$, which is not affected by this gauge condition, as can be seen with Eq. (5).

Disregarding the surface term, the expected $N=2$ Lagrangian is then expressed in terms of the original $N=1$ superfields as follows:

$$\mathcal{L}_{gauge}=\frac{1}{2}\int d^2\theta D^\beta G D_\beta \overline{G}$$
$$=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{i}{2}\overline{\lambda}\partial\!\!\!/\lambda+\frac{1}{2}\partial_\mu\rho\partial^\mu\rho+\frac{1}{2}D^2. \quad (7)$$

Matter is included with a complex $N=1$ scalar superfield Q:

$$Q=\phi+(\theta\psi)+\frac{1}{2}\theta^2 F. \quad (8)$$

So as to avoid the generation of parity violating terms in the quantum corrections, we can introduce an even number of superfields [3], but we do not consider this problem here. The interested reader can find a review of supersymmetric Chern-Simons theories in [10]. We remind that a $N=1$ scalar superfield in 2+1 dimensions cannot be chiral: since θ is real, the chirality condition $D^\alpha Q=0$ would constraint the space-time dependence of the component fields ϕ,ψ,F [11].

The derivatives of the fields are obtained with the highest component of $D^\alpha Q D_\alpha \overline{Q}$ which reads

$$D^\alpha Q D_\alpha \overline{Q}|_{\theta^2}=\theta^2(\partial_\mu\phi\partial^\mu\phi+i\overline{\psi}\partial\!\!\!/\psi+F\overline{F}+\text{surface term}), \quad (9)$$

where the surface term is $\partial_\mu(\overline{\psi}\partial^\mu\psi)$. The coupling to the gauge multiplet is obtained with the highest components of the following superfields:

$$D^\alpha Q V_\alpha \overline{Q}|_{\theta^2}=-\frac{1}{2}\theta^2(\overline{\phi}(\psi\chi)-\overline{\phi}A_\mu\partial^\mu\phi+i\overline{\psi}A\!\!\!/\psi)$$

$$Q\Phi\overline{Q}|_{\theta^2}=\frac{1}{2}\theta^2(\phi\overline{\phi}D-\phi\overline{\psi}\xi-\overline{\phi}\psi\xi+\rho\phi\overline{F}$$
$$+\rho\overline{\phi}F-\rho\overline{\psi}\psi) \quad (10)$$

$$QV^\alpha V_\alpha \overline{Q}|_{\theta^2}=-\theta^2\phi\overline{\phi}A^\mu A_\mu,$$

such that the matter Lagrangian is

$$\mathcal{L}_{matter}=\frac{1}{2}\int d^2\theta\{(D^\alpha-igV^\alpha)Q(D_\alpha+igV_\alpha)\overline{Q}$$
$$+2gQ\Phi\overline{Q}\}$$
$$=\frac{i}{2}\overline{\psi}D\!\!\!\!/\psi+\frac{1}{2}D_\mu\phi D^\mu\overline{\phi}-\frac{g}{2}(\phi\overline{\overline{\psi}\lambda}+\overline{\phi}\psi\lambda)$$
$$-\frac{g}{2}\rho\overline{\psi}\psi+\frac{g}{2}\phi\overline{\phi}D+\frac{1}{2}F\overline{F}+\frac{g}{2}\rho\phi\overline{F}$$
$$+\frac{g}{2}\rho\overline{\phi}F, \quad (11)$$

where g is a dimensionfull gauge coupling and $D_\mu=\partial_\mu+igA_\mu$. The Lagrangian (11) was found in [4] as a consequence of the dimensional reduction of a $N=1$ theory in 3+1 dimensions. It was also derived in [12] where the $N=2$ Lagrangian is expressed with $N=1$ superfields. In both these works, the authors start from $N=2$, and do not elevate an initial $N=1$ Lagrangian to $N=2$; hence they do not find any constraint. The reader can find in [13] a discussion of the relation between $N=1$, $N=2$ and $N=4$ supersymmetries in 1+1, 2+1 and 3+1 dimensions.

We can recover the scalar interactions if we write the Lagrangians (7) and (11) on-shell. We write for this the equations of motion of the auxiliary fields D and F:

$$\overline{F}+g\rho\overline{\phi}=0$$
$$D+\frac{g}{2}\phi\overline{\phi}=0, \tag{12}$$

such that the terms depending on the auxiliary fields lead to the following potential:

$$(\mathcal{L}_{gauge}+\mathcal{L}_{matter})_{pot}=\frac{1}{2}D^2+\frac{g}{2}\phi\overline{\phi}D+\frac{1}{2}F\overline{F}+\frac{g}{2}\rho\overline{\phi}F+\frac{g}{2}\rho\phi\overline{F}$$
$$=-\frac{g^2}{2}\rho^2\phi\overline{\phi}-\frac{g^2}{8}(\phi\overline{\phi})^2. \tag{13}$$

The Abelian Higgs model is obtained by adding a Fayet-Iliopoulos term which in the present context is

$$\mathcal{L}_{F.I.}=-\frac{g}{2}\phi_0^2\int d^2\theta(G+\overline{G})=-g\phi_0^2\int d^2\theta\Phi$$
$$=-\frac{g}{2}\phi_0^2 D, \tag{14}$$

where ϕ_0 is a real parameter. The addition of this term to the Lagrangian leads to the following equation of motion for the auxiliary field D:

$$D+\frac{g}{2}\phi\overline{\phi}-\frac{g}{2}\phi_0^2=0, \tag{15}$$

such that we obtain the expected gauge-symmetry breaking potential

$$(\mathcal{L}_{gauge}+\mathcal{L}_{matter}+\mathcal{L}_{F.I.})_{pot}=-\frac{g^2}{2}\rho^2\phi\overline{\phi}-\frac{g^2}{8}(\phi\overline{\phi}-\phi_0^2)^2. \tag{16}$$

Note that the Higgs self-coupling is $g^2/8$, what was found in [6] as a consistency condition for the elevation of the $N=1$ on-shell Lagrangian to $N=2$. The result (16) shows that the moduli space contains a Higgs branch only, where the vacuum expectation values of the scalar fields satisfy

$$\langle\phi\overline{\phi}\rangle=\phi_0^2 \quad \text{and} \quad \langle\rho\rangle=0. \tag{17}$$

The extension to a non-Abelian gauge group necessitates the introduction of quadratic superfields to generate the interactions. We will consider $SU(N)$ dynamics, with structure constants f^{abc} and coupling constant g. A non-Abelian supermultiplet contains gauginos and scalars in the adjoint representation, so that the starting point is the set of scalar and vector $N=1$ superfields

$$\Phi^a=\rho^a+(\theta\xi^a)+\frac{1}{2}\theta^2 D^a$$
$$V_\alpha^a=i(\not{A}^a\theta)_\alpha+\frac{1}{2}\theta^2\chi_\alpha^a, \tag{18}$$

where $a=1,\ldots,N^2-1$ is the gauge indice. We then introduce the complex superfields

$$G^a=\Phi^a+iD^\alpha V_\alpha^a, \tag{19}$$

and, as in the Abelian case, the derivatives of the component fields are obtained with the term $D^\beta G^a D_\beta\overline{G}^a$, provided that the gauge condition $\partial^\mu A_\mu^a=0$ holds, which shows again that the fundamental superfield is actually $D^\beta G^a$ and not G^a. To generate the interactions of the superpartners, we will add to $D^\beta G^a$ linear combinations of the following two superfields

$$G^b V^{c\beta}, \qquad D^\beta(V^{b\alpha}V_\alpha^c), \tag{20}$$

and the remaining terms for the covariant derivatives are obtained with the products

$$f^{abc}D^\beta\overline{G}^a G^b V_\beta^c\big|_{\theta^2}=\theta^2 f^{abc}\bigg(-\rho^b\xi^a\chi^c+\frac{1}{2}(i\overline{\lambda}^b\not{A}^c\lambda^a+\text{c.c.})-2\rho^b\partial^\mu\rho^a A_\mu^c\bigg)$$
$$f^{abc}f^{ade}G^b V^{c\beta}\overline{G}^d V_\beta^e\big|_{\theta^2}=f^{abc}f^{ade}\theta^2\rho^b\rho^d A_\mu^c A^{e\mu}. \tag{21}$$

The term (21) also generates the Yukawa interactions since

$$2\rho^b\xi^a\chi^c=\rho^b(i\overline{\lambda}^a\lambda^c+\text{c.c.}). \tag{22}$$

The non-Abelian gauge kinetic term is obtained with the products

$$f^{abc}D^\beta\overline{G}^a D_\beta(V^{b\alpha}V_\alpha^c)\big|_{\theta^2}=2f^{abc}(\partial_\mu A_\nu^a)A_\rho^b A_\sigma^c(\theta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\theta)$$
$$f^{abc}f^{ade}D^\beta(V^{b\alpha}V_\alpha^c)D_\beta(V^{b\alpha}V_\alpha^c)\big|_{\theta^2}=f^{abc}f^{ade}A_\mu^b A_\nu^c A_\rho^d A_\sigma^e(\theta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\theta), \tag{23}$$

since we have, using the properties (2) and $f^{abc}+f^{acb}=0$,

$$f^{abc}\int d^2\theta(\partial_\mu A_\nu^a)A_\rho^b A_\sigma^c(\theta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\theta)=f^{abc}(\partial^\mu A_\mu^a)A_\nu^b A^{c\nu}+f^{abc}A_\mu^b A_\nu^c(\partial^\mu A^{a\nu}-\partial^\nu A^{a\mu}), \tag{24}$$

and

$$f^{abc}f^{ade}\int d^2\theta A^b_\mu A^c_\nu A^d_\rho A^e_\sigma(\theta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\theta)$$
$$=2f^{abc}f^{ade}A^b_\mu A^{d\mu}A^c_\nu A^{e\nu}. \tag{25}$$

With the gauge condition $\partial^\mu A^a_\mu=0$, the first term in the right-hand side of Eq. (24) vanishes and only the expected term remains. Gathering these results, we find that the extension to an $N=2$ pure super-Yang-Mills Lagrangian is given by

$$\mathcal{L}_{Y.M.}=\frac{1}{2}\int d^2\theta\left|D^\beta G^a+gf^{abc}\left(G^bV^{c\beta}+\frac{i}{2}D^\beta(V^{b\alpha}V^c_\alpha)\right)\right|^2$$
$$=-\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}+\frac{i}{2}\overline{\lambda^a}\not{D}\lambda^a+\frac{1}{2}D^\mu\rho^aD_\mu\rho^a$$
$$-\frac{g}{2}f^{abc}(i\rho^b\overline{\lambda^a}\lambda^c+\text{c.c.})+\frac{1}{2}D^aD^a, \tag{26}$$

where $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu$ and $D_\mu(\cdots)^a=\partial_\mu(\cdots)^a+gf^{abc}A^b_\mu(\cdots)^c$.

To conclude, let us stress the central point of these results. Whereas the elevation of a $N=1$ to a $N=2$ supersymmetry was shown explicitely for the CP^1 model in [5] and for the Abelian Higgs model in [6], we do not start here with any specific dynamics but instead build directly $N=2$ off-shell Lagrangians with $N=1$ superfields. This allows us to generate different dynamics and we generalize the elevation to a $N=2$ non-Abelian theory. Clearly, one could consider with the same method other $N=2$ dynamics.

Finally, this work might be used in the context of effective models for high-temperature (planar) superconductivity [14], where the initial $N=1$ supermultiplets are built out of composites of spinons and holons in the spin-charge separation framework.

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