Unconstrained $SU(2)$ Yang-Mills theory with a topological term in the long-wavelength approximation

A. M. Khvedelidze,[1,2] D. M. Mladenov,[3] H.-P. Pavel,[3,4] and G. Röpke[4]
[1]*A. Razmadze Mathematical Institute, Tbilisi, 380093, Georgia*
[2]*Laboratory of Information Technologies, Joint Institute for Nuclear Research, 141980 Dubna, Russia*
[3]*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia*
[4]*Fachbereich Physik der Universität Rostock, D-18051 Rostock, Germany*
(Received 22 February 2002; revised manuscript received 10 February 2003; published 22 May 2003)

The Hamiltonian reduction of $SU(2)$ Yang-Mills theory for an arbitrary θ angle to an unconstrained nonlocal theory of a self-interacting positive definite symmetric 3×3 matrix field $S(x)$ is performed. It is shown that, after exact projection to a reduced phase space, the density of the Pontryagin index remains a pure divergence, proving the θ independence of the unconstrained theory obtained. An expansion of the nonlocal kinetic part of the Hamiltonian in powers of the inverse coupling constant and truncation to lowest order, however, lead to violation of the θ independence of the theory. In order to maintain this property on the level of the local approximate theory, a modified expansion in the inverse coupling constant is suggested, which for a vanishing θ angle coincides with the original expansion. The corresponding approximate Lagrangian up to second order in derivatives is obtained, and the explicit form of the unconstrained analogue of the Chern-Simons current linear in derivatives is given. Finally, for the case of degenerate field configurations $S(x)$ with $\mathrm{rank}\|S\|=1$, a nonlinear σ-type model is obtained, with the Pontryagin topological term reducing to the Hopf invariant of the mapping from the three-sphere $\mathbb{S}^3$ to the unit two-sphere $\mathbb{S}^2$ in the Whitehead form.

DOI: 10.1103/PhysRevD.67.105013 PACS number(s): 11.15.Me, 11.10.Ef, 12.38.Aw

I. INTRODUCTION

For a complete understanding of the low-energy quantum phenomena of Yang-Mills theory, it is necessary to have a nonperturbative, gauge invariant description of the underlying classical theory including the θ-dependent Pontryagin term [1–4]. Several representations of Yang-Mills theory in terms of local gauge invariant fields have been proposed [5–24] in recent decades, implementing the Gauss law as a generator of small gauge transformations. However, in dealing with such local gauge invariant fields special consideration is needed when the topological term is included, since it is the four-divergence of a current changing under large gauge transformations. In particular, the consistency of constrained and unconstrained formulations of gauge theories with topological term requires us to verify that, after projection to the reduced phase space, the classical equations of motion for the unconstrained variables remain θ independent.[1] Furthermore, the question of which trace the large gauge transformation with a nontrivial Pontryagin topological index leaves on the local gauge invariant fields has to be addressed.

Having this in mind, in the present paper we extend our approach [22,27,28] to constructing the unconstrained form of $SU(2)$ Yang-Mills theory to the case when the topological term is included in the classical action. We generalize the Hamiltonian reduction of classical $SU(2)$ Yang-Mills field theory to arbitrary θ angle by reformulating the original degenerate Yang-Mills theory as a nonlocal theory of a self-interacting positive definite symmetric 3×3 matrix field. The consistency of the Hamiltonian reduction in the presence of the Pontryagin term is demonstrated by constructing the canonical transformation, well defined on the reduced phase space, that eliminates the θ dependence of the classical equations of motion for the unconstrained variables.

With the aim of obtaining a practical form of the nonlocal unconstrained Hamiltonian, we perform an expansion in powers of the inverse coupling constant, equivalent to an expansion in the number of spatial derivatives. We find that a straightforward application of the derivative expansion violates the principle of θ independence of the classical observables. To cure this problem, we propose to exploit the property of chromoelectromagnetic duality of pure Yang-Mills theory, symmetry under the exchange of the chromoelectric and -magnetic fields. The electric and magnetic fields are subject to dual constraints, the Gauss law and Bianchi identity, and only when both are satisfied are the classical equations of motion θ independent. Thus any approximation in resolving the Gauss law constraints should be consistent with the Bianchi identity. We show how to use the Bianchi identity to rearrange the derivative expansion in such a way that the θ independence is restored to all orders on the classical level.

In order to have a representation of the gauge invariant degrees of freedom suitable for a study of the low-energy phase of Yang-Mills theory, we perform a principal-axes transformation of the symmetric tensor field and obtain the unconstrained Hamiltonian in terms of the principal-axes variables in the lowest order in $1/g$. Carrying out an inverse Legendre transformation to the corresponding unconstrained

[1]The question of consistency of the elimination of redundant variables in theories containing both constraints and pure divergencies, the so-called "divergence problem," was analyzed for the first time in the context of the canonical reduction of general relativity by Dirac [25] and by Arnowitt, Deser, and Misner [26].

Lagrangian, we find the explicit form of the unconstrained analogue of the Chern-Simons current, linear in the derivatives.

Finally, we consider the case of degenerate symmetric field configurations S with $\mathrm{rank}\|S(x)\|=1$. We find a nonlinear classical theory of a three-dimensional unit-vector **n** field interacting with a scalar field. Using typical boundary conditions for the unit-vector field at spatial infinity, the Pontryagin topological charge density reduces to the Abelian Chern-Simons invariant density [4]. We discuss its relation to the Hopf number of the mapping from the three-sphere $\mathbb{S}^3$ to the unit two-sphere $\mathbb{S}^2$ in the Whitehead representation [29]. The Abelian Chern-Simons invariant is known from different areas in physics, in fluid mechanics as "fluid helicity," in plasma physics and magnetohydrodynamics as "magnetic helicity" [30–33]. In the context of four-dimensional Yang-Mills theory a connection between non-Abelian vacuum configurations and certain Abelian fields with nonvanishing helicity established already in [34,35].

The paper is organized as follows. In Sec. II the θ independence of classical Yang-Mills theory in the framework of the constrained Hamiltonian formulation is revised. Section III is devoted to the derivation of unconstrained $SU(2)$ Yang-Mills theory for arbitrary θ angle. The consistency of our reduction procedure is demonstrated by explicitly quoting the canonical transformation, which removes the θ dependence from the unconstrained classical theory. In Sec. IV the unconstrained Hamiltonian up to order $o(1/g)$ is obtained. Section V presents the long-wavelength classical Hamiltonian in terms of principal-axes variables. The corresponding Lagrangian up to second order in derivatives, and the unconstrained analogue of the Chern-Simons current, linear in the derivatives, are obtained. In Sec. VI the unconstrained action for degenerate field configurations is considered. Section VII finally gives our conclusions. Several more technical details are presented in the Appendixes A, B, C, and D. Appendix A summarizes our notation and definitions, Appendix B is devoted to the question of the existence of the "symmetric gauge," in Appendix C the proof of the θ dependence of the "naive" $1/g$ approximation is given, and Appendix D contains some technical details for the representation of the unconstrained theory in terms of principal-axes variables.

II. CONSTRAINED HAMILTONIAN FORMULATION

Yang-Mills gauge fields are classified topologically by the value of the Pontryagin index[2]

$$p_1=-\frac{1}{8\pi^2}\int \mathrm{tr}\, F\wedge F. \tag{1}$$

Its density, the so-called topological charge density $Q=-(1/8\pi^2)\,\mathrm{tr}\, F\wedge F$, being locally exact $Q=dC$, can be added to the conventional Yang-Mills Lagrangian with arbitrary parameter θ:

$$\mathcal{L}=-\frac{1}{g^2}\,\mathrm{tr}\, F\wedge{}^*F-\frac{\theta}{8\pi^2 g^2}\,\mathrm{tr}\, F\wedge F, \tag{2}$$

without changing the classical equations of motion. In the Hamiltonian formulation, this shifts the canonical momenta, conjugated to the field variables A_{ai},

$$\Pi_{ai}=\frac{\partial\mathcal{L}}{\partial\dot{A}_{ai}}=\dot{A}_{ai}-(D_i(A))_{ac}A_{c0}+\frac{\theta}{8\pi^2}B_{ai}, \tag{3}$$

by the magnetic field $(\theta/8\pi^2)B_{ai}$. As a result, the total Hamiltonian [36,37] of Yang-Mills theory with the θ angle, as a functional of canonical variables (A_{a0},Π_a) and (A_{ai},Π_{ai}) obeying the Poisson bracket relations

$$\{A_{ai}(t,\vec{x}),\Pi_{bj}(t,\vec{y})\}=\delta_{ab}\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}), \tag{4}$$

$$\{A_{a0}(t,\vec{x}),\Pi_{b}(t,\vec{y})\}=\delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}) \tag{5}$$

takes the form

$$H_T=\int d^3x\Bigg[\frac{1}{2}\Bigg(\Pi_{ai}-\frac{\theta}{8\pi^2}B_{ai}\Bigg)^2+\frac{1}{2}B_{ai}^2 - A_{a0}(D_i(A))_{ac}\Pi_{ci}+\lambda_a\Pi_a\Bigg]. \tag{6}$$

Here, the linear combination of three primary constraints

$$\Pi_a(x)=0 \tag{7}$$

with arbitrary functions $\lambda_a(x)$ and the secondary constraints, the non-Abelian Gauss law

$$(D_i(A))_{ac}\Pi_{ci}=0, \tag{8}$$

reflect the gauge invariance of the theory.

Based on the representation (6) for the total Hamiltonian, one can immediately verify that classical theories with different value of the θ angle are equivalent. Performing the canonical transformation

$$A_{ai}(x)\mapsto A_{ai}(x),$$

$$\Pi_{bj}(x)\mapsto E_{bj}:=\Pi_{bj}(x)-\frac{\theta}{8\pi^2}B_{bj}(x) \tag{9}$$

to the new variables A_{ai} and E_{bj}, and using the Bianchi identity

$$(D_i(A))_{ab}B_{bi}(A)=0, \tag{10}$$

[2]The necessary notation and definitions for $SU(2)$ Yang-Mills theory used in the text have been collected in Appendix A.

one can then see that the θ dependence completely disappears from the Hamiltonian (6). Note that the canonical transformation (9) can be represented in the form

$$E_{ai}=\Pi_{ai}-\theta\frac{\delta}{\delta A_{ai}}W[A], \tag{11}$$

where $W[A]$ denotes the winding number functional

$$W[A]=\int d^3xK^0[A] \tag{12}$$

constructed from the zero component of the Chern-Simons current

$$K^{\mu}[A]=-\frac{1}{16\pi^2}\varepsilon^{\mu\alpha\beta\gamma}\,\mathrm{tr}\left(F_{\alpha\beta}A_{\gamma}-\frac{2}{3}A_{\alpha}A_{\beta}A_{\gamma}\right). \tag{13}$$

The question now arises whether, after reduction of Yang-Mills theory including the topological term to the unconstrained system, a transformation analogous to Eq. (9) can be found that correspondingly eliminates any θ dependence on the reduced level, proving the consistency of the Hamiltonian reduction.

III. UNCONSTRAINED HAMILTONIAN FORMULATION

A. Hamiltonian reduction for arbitrary θ angle

In order to derive the unconstrained form of $SU(2)$ Yang Mills theory with the θ angle we follow the method developed in [22]. We perform the point transformation

$$A_{ai}(q,S)=O_{ak}(q)S_{ki}+\frac{1}{2g}\varepsilon_{abc}(\partial_i O(q)O^T(q))_{bc} \tag{14}$$

from the gauge fields $A_{ai}(x)$ to the new set of three fields $q_j(x), j=1,2,3$, parametrizing an orthogonal 3×3 matrix $O(q)$ and the six fields $S_{ik}(x)=S_{ki}(x), i,k=1,2,3$, collected in the positive definite symmetric 3×3 matrix $S(x)$.[3] Equation (14) can be seen as a gauge transformation to the new field configuration $S(x)$ which satisfies the "symmetric gauge" condition

$$\chi_a(S):=\varepsilon_{abc}S_{bc}=0. \tag{15}$$

The complete analysis of the existence and uniqueness of this gauge, i.e., whether any gauge potential A_{ai} can be made symmetric by a unique gauge transformation, is a complex mathematical problem. Here we shall consider the transformation (14) in a region where the uniqueness and regularity of the change of coordinates can be guaranteed. In Appendix B, we prove the existence and uniqueness of the symmetric gauge for the case of a nondegenerate matrix A using the inverse coupling constant expansion. Furthermore, as an illustration of the obstruction of the uniqueness of the symmetric gauge fixing (the appearance of Gribov copies) for degenerate matrices A, the Wu-Yang monopole configuration is considered. Although it is antisymmetric in space and color indices, it can be brought into the symmetric form, but there exist two gauge transformations by which this can be achieved. The case of a degenerate matrix field S, $\det\|S\|=0$, will be discussed for the special situation $\mathrm{rank}\|S\|=1$ in Sec. VI.

The transformation (14) induces a point canonical transformation linear in the new momenta $P_{ik}(x)$ and $p_i(x)$, conjugated with $S_{ik}(x)$ and $q_i(x)$, respectively. Their expressions in terms of the old variables $(A_{ai}(x),\Pi_{ai}(x))$ can be obtained from the requirement of the canonical invariance of the symplectic one-form

$$\sum_{i,a=1}^{3}\Pi_{ai}\dot{A}_{ai}dt=\sum_{i,j=1}^{3}P_{ij}\dot{S}_{ij}dt+\sum_{i=1}^{3}p_i\dot{q}_i dt \tag{16}$$

with the fundamental brackets

$$\{S_{ij}(t,\vec{x}),P_{kl}(t,\vec{y})\}=\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\delta^{(3)}(\vec{x}-\vec{y}), \tag{17}$$

$$\{q_i(t,\vec{x}),p_j(t,\vec{y})\}=\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}) \tag{18}$$

for the new canonical pairs $(S_{ij}(x),P_{ij}(x))$ and $(q_i(x),p_i(x))$. The brackets (17) account for the second-class symmetry constraints $S_{ij}=S_{ji}$ and $P_{ij}=P_{ji}$ and therefore are Dirac brackets. As a result, we obtain the expression

$$\Pi_{ai}=O_{ak}(q)[P_{ki}+g\varepsilon_{kin}{}^*D^{-1}_{nm}(S)(\mathcal{S}_m-\Omega^{-1}_{jm}p_j)] \tag{19}$$

for the old momenta Π_{ai} in terms of the new canonical variables (for a detailed derivation see [22]). Here ${}^*D^{-1}_{mn}(S)$ denotes the inverse of the differential matrix operator[4]

$${}^*D_{mn}(S)=\varepsilon_{njc}(D_j(S))_{mc}, \tag{20}$$

the vector $\mathcal{S}$ is defined as

$$\mathcal{S}_m=\frac{1}{g}(D_j(S))_{mn}P_{nj}, \tag{21}$$

and the matrix Ω^{-1} is the inverse of

$$\Omega_{ni}(q):=-\frac{1}{2}\varepsilon_{nbc}\left(O^T(q)\frac{\partial O(q)}{\partial q_i}\right)_{bc}. \tag{22}$$

Here we would like to comment on the geometrical meaning of the above expressions. The vector $\mathcal{S}$ coincides up to divergence with the spin density part of the Noetherian an-

[3]It is necessary to note that a decomposition similar to Eq. (14) was used in [11] as a generalization of the well-known polar decomposition valid for arbitrary quadratic matrices.

[4]Note that the operator ${}^*D_{mn}(S)$ corresponds in the conventional gauge-fixing method to the so-called Faddeev-Popov (FP) operator, the matrix of Poisson brackets between the Gauss law constraint (8) and the symmetric gauge (15), $\{(D_i(S))_{mc}\Pi_{ci}(x),\chi_n(y)\}={}^*D_{mn}(S)\delta^3(x-y)$.

gular momentum after projection to the surface given by the Gauss law constraints. Furthermore, the matrix Ω^{-1} defines the main geometrical structures on the $SO(3,R)$ group manifold, namely, the three left-invariant Killing vector fields $\eta_a := \Omega^{-1}_{ja}\partial/\partial q_j$ obeying the $so(3)$ algebra $[\eta_a,\eta_b] = \epsilon_{abc}\eta_c$, and the invariant metric $g := -\mathrm{tr}(O^T dO O^T dO) = (1/2)(\Omega^T\Omega)_{ij} dq_i dq_j$ as the standard metric on S^3. Since $\det\Omega$ is proportional to the Haar measure on $SO(3,R)$ $\sqrt{\det\|g\|} = |\det\|\Omega(q)\||$, and it is expected to vanish at certain coordinate singularities (see also, e.g., discussion in Chap. 8 of [38]). In deriving the expression (19) we shall here limit ourselves to the region where the matrix Ω is invertible.

The main advantage of introducing the variables S_{ij} and q_i is that they Abelianize the non-Abelian Gauss law constraints (8). In terms of the new variables the Gauss law constraints

$$gO_{as}(q)\Omega^{-1}_{is}(q)p_i = 0 \tag{23}$$

depend only on (q_i, p_i), showing that the variables (S_{ij}, P_{ij}) are gauge-invariant, physical fields. Hence, assuming $\det\Omega(q)\neq 0$ in Eqs. (19) and (23), the reduced Hamiltonian, defined as the projection of the total Hamiltonian onto the constraint shell, can be obtained from Eq. (6) by imposing the equivalent set of Abelian constraints

$$p_i = 0. \tag{24}$$

Due to gauge invariance, the reduced Hamiltonian is independent of the coordinates q_i canonically conjugated to p_i and is hence a function of the unconstrained gauge-invariant variables S_{ij} and P_{ij} only

$$H = \int d^3x\left[\frac{1}{2}\left(P_{ai} - \frac{\theta}{8\pi^2}B^{(+)}_{ai}(S)\right)^2 + \left(P_a - \frac{\theta}{8\pi^2}B^{(-)}_a(S)\right)^2 + \frac{1}{2}V(S)\right]. \tag{25}$$

Here the P_a denotes the nonlocal functional, according to Eq. (19) defined as the solution of the system of differential equations

$$^*D_{ks}(S)P_s = (D_j(S))_{kn}P_{nj}. \tag{26}$$

The nonlocal second term in the Hamiltonian (25) therefore stems from the antisymmetric part of the Π_{ai}, which remains after implementing Gauss's law $p_a = 0$, in terms of the physical P_{ai}. Hence this term contains FP^{-2} [see Eq. (26)], and is the analogue of the well-known nonlocal part of the Hamiltonian in the Coulomb gauge (see, e.g., [9]).

Furthermore,

$$B^{(+)}_{ai}(S) := \frac{1}{2}[B_{ai}(S) + B_{ia}(S)], \quad B^{(-)}_a(S) := \frac{1}{2}\varepsilon_{abc}B_{bc}(S) \tag{27}$$

denote the symmetric and antisymmetric parts of the reduced chromomagnetic field

$$B_{ai}(S) = \varepsilon_{ijk}\left(\partial_j S_{ak} + \frac{g}{2}\varepsilon_{abc}S_{bj}S_{ck}\right). \tag{28}$$

It is the same functional of the symmetric field S as the original $B_{ai}(A)$, since the chromomagnetic field transforms homogeneously under the change of coordinates (14). Finally, the potential $V(S)$ is the square of the reduced magnetic field (28),

$$V(S)d^3x = B^2_{ai}(S)d^3x = \frac{1}{2}\mathrm{tr}\, {}^*F^{(3)}\wedge F^{(3)}, \tag{29}$$

with the curvature two-form in three-dimensional Euclidean space

$$F^{(3)} = dS + S\wedge S, \tag{30}$$

in terms of the symmetric one-form

$$S = g\tau_k S_{kl}dx_l, \quad k,l = 1,2,3, \tag{31}$$

whose six components depend on the time variable as an external parameter. The reduced chromomagnetic field (28) is given in terms of the dual field strength $^*F^{(3)}$ as $B_{ai}(S) = \frac{1}{2}\varepsilon_{ijk}F^{(3)}_{ajk}$.

B. Canonical equivalence of unconstrained theories with different θ angles

For the original degenerate action in terms of the A_μ fields the equivalence of classical theories with arbitrary values of θ angle has been reviewed in Sec. II. Let us now examine the same problem for the unconstrained theory derived considering the analogue of the canonical transformation (9) after projection onto the constraint surface,

$$S_{ai}(x) \mapsto S_{ai}(x),$$

$$P_{bj}(x) \mapsto \mathcal{E}_{bj}(x) := P_{bj}(x) - \frac{\theta}{8\pi^2}B^{(+)}_{bj}(x). \tag{32}$$

One can easily check that this transformation to new variables S_{ai} and $\mathcal{E}_{bj}$ is canonical with respect to the Dirac brackets (17). In terms of the new variables S_{ai} and $\mathcal{E}_{bj}$ the Hamiltonian (25) can be written as

$$H = \int d^3x\left[\frac{1}{2}\mathcal{E}^2_{ai} + \mathcal{E}^2_a + \frac{1}{2}V(S)\right], \tag{33}$$

with $\mathcal{E}_a$ defined as

$$\mathcal{E}_a := P_a - \frac{\theta}{8\pi^2}B^{(-)}_a. \tag{34}$$

Now, if P_a is a solution of Eq. (26), then $\mathcal{E}_a$ is a solution of the same equation

$$^*D_{ks}(S)\mathcal{E}_s = (D_j(S))_{kn}\mathcal{E}_{nj} \tag{35}$$

with the replacement $P_{ai}\mapsto\mathcal{E}_{ai}$, since the reduced field B_{ai} satisfies the Bianchi identity

$$(D_i(S))_{ab}B_{bi}(S)=0. \tag{36}$$

Hence we arrive at the same unconstrained Hamiltonian system (33) and (35) with vanishing θ angle. Note that after the elimination of the three unphysical fields $q_j(x)$ the projected canonical transformation (32) that removes the θ dependence from the Hamiltonian can be written as

$$\mathcal{E}_{bj}(x)=P_{bj}(x)-\theta\frac{\delta}{\delta S_{bj}}W[S], \tag{37}$$

which is of the same form as Eq. (11) with the nine gauge fields $A_{ik}(x)$ replaced by the six unconstrained fields $S_{ik}(x)$.

In summary, the exact projection to a reduced phase space leads to an unconstrained system whose equations of motion are consistent with the original degenerate theory in the sense that they are θ independent. Thus if our consideration is restricted only to the classical level of the exact nonlocal unconstrained theory, the generalization to arbitrary θ angle can be avoided.[5] However, in order to work with such a complicated nonlocal Hamiltonian it is necessary to make approximations, such as, for example, expansion in the number of spatial derivatives, which we shall carry out in the next section. For these one has to check that this approximation is free of the "divergence problem," that is, all terms in the corresponding truncated action containing the θ angle can be collected into a four-divergence and all dependence on θ disappears from the classical equations of motion.

IV. EXPANSION OF THE UNCONSTRAINED HAMILTONIAN IN 1/g

Let us now consider the regime when the unconstrained fields are slowly varying in space-time and expand the nonlocal part of the kinetic term in the unconstrained Hamiltonian (25) as a series of terms with increasing powers of the inverse coupling constant $1/g$, equivalent to an expansion in the number of spatial derivatives of field and momentum. Our expansion is purely formal and we shall not study the question of its convergence in this work. We shall see that for nonvanishing θ angle a straightforward expansion in $1/g$ leads to the above mentioned "divergence problem," and suggest an improved form of the expansion in $1/g$ of the unconstrained Hamiltonian exploiting the Bianchi identity.

A. Divergence problem in lowest-order approximation

According to [22], the nonlocal funtional P_a in the unconstrained Hamiltonian (33), defined as a solution of the system of linear differential equations (26), can formally be expanded in powers of $1/g$. The vector P_a is then given as a sum of terms containing an increasing number of spatial derivatives of field and momentum:

$$P_s(S,P)=\sum_{n=0}^{\infty}(1/g)^n a_s^{(n)}(S,P). \tag{38}$$

The zeroth-order term is

$$a_s^{(0)}=\gamma_{sk}^{-1}\varepsilon_{klm}(PS)_{lm}, \tag{39}$$

with $\gamma_{ik}:=S_{ik}-\delta_{ik}\operatorname{tr}S$, and the first-order term is determined as

$$a_s^{(1)}=-\gamma_{sl}^{-1}[(\operatorname{rot}\vec{a}^{(0)})_l+\partial_k P_{kl}] \tag{40}$$

from the zeroth-order term. The higher terms are then obtained by the simple recurrence relations

$$a_s^{(n+1)}=-\gamma_{sl}^{-1}(\operatorname{rot}\vec{a}^{(n)})_l. \tag{41}$$

Inserting these expressions into Eq. (25) we obtain the corresponding expansion of the unconstrained Hamiltonian as a series in higher and higher numbers of derivatives.

Let us check whether the truncation of the expansion (38) to lowest order is consistent with θ independence, that is, whether all θ-dependent terms can be collected into four-divergence after Legendre transformation to the corresponding Lagrangian. In $o(1/g)$ approximation (39), the Hamiltonian reads[6]

$$H^{(2)}=\int d^3x\left[\frac{1}{2}\operatorname{tr}\left(P-\frac{\theta}{8\pi^2}B^{(+)}\right)^2 +\left(a_s^{(0)}(S,P)-\frac{\theta}{8\pi^2}B_s^{(-)}\right)^2+\frac{1}{2}V(S)\right], \tag{42}$$

where $B^{(+)}$ and $B^{(-)}$ denote the symmetric and antisymmetric parts of the chromomagnetic field, defined in Eq. (27).

After inverse Legendre transformation of the Hamiltonian (42), the θ-dependent terms in the corresponding Lagrangian cannot be collected into a total four-divergence, as is shown in Appendix C, and therefore contribute to the unconstrained equations of motion. Hence, on applying a straightforward derivative expansion to the Yang-Mills theory with a topological term after projection to a reduced phase space, we face the "divergence problem" discussed above.

B. Improved 1/g expansion using the Bianchi identity

In order to avoid the "divergence problem" one can proceed as follows. Let us consider additionally to the differential equation (26), which determines the nonlocal term P_a, the Bianchi identity (36) as an equation for determination of the antisymmetric part $B_s^{(-)}$ of the chromomagnetic field

[5] The extension of the proof of θ independence to quantum theory requires showing the unitarity of the operator corresponding to the transformation (32).

[6] When all spatial derivatives of the fields and momenta are neglected, Yang-Mills theory reduces to the so-called Yang-Mills mechanics and its θ independence has been shown in [27].

$$ {}^{*}D_{ks}(S)B_s^{(-)}=(D_i(S))_{kl}B_{li}^{(+)} \tag{43}$$

in terms of its symmetric part $B_{bc}^{(+)}$. The complete analogy of this equation with Eq. (26) expresses the duality of the chromoelectric and chromagnetic fields on the unconstrained level. Hence one can write

$$ {}^{*}D_{ks}(S)\left[P_s-\frac{\theta}{8\pi^2}B_s^{(-)}\right]=(D_i(S))_{kl}\left[P_{li}-\frac{\theta}{8\pi^2}B_{li}^{(+)}\right]. \tag{44}$$

Using the same type of spatial derivative expansion as before in Eqs. (39)–(41), we obtain

$$ P_s-\frac{\theta}{8\pi^2}B_s^{(-)}=\sum_{n=0}^{\infty}(1/g)^n a_s^{(n)}\left(S,P-\frac{\theta}{8\pi^2}B^{(+)}\right). \tag{45}$$

In this way we achieve a form of the derivative expansion such that the unconstrained Hamiltonian is a functional of the field combination $P_{ai}-(\theta/8\pi^2)B_{ai}^{(+)}$,

$$ H=\int d^3x\left\{\frac{1}{2}\left(P_{ai}-\frac{\theta}{8\pi^2}B_{ai}^{(+)}\right)^2 + \left[\sum_{n=0}^{\infty}(1/g)^n a_i^{(n)}\left(S,P-\frac{\theta}{8\pi^2}B^{(+)}\right)\right]^2+\frac{1}{2}V(S)\right\}, \tag{46}$$

explicitly showing the chromoelectromagnetic duality on the reduced level and hence free of the "divergence problem." To obtain the unconstrained Hamiltonian up to leading order $o(1/g)$, only the lowest term $a_s^{(0)}[S,P-(\theta/8\pi^2)B^{(+)}]$ in the sum in Eq. (46) has to be taken into account, so that

$$ H^{(2)}=\frac{1}{2}\int d^3x\left\{\mathrm{tr}\left(P-\frac{\theta}{8\pi^2}B^{(+)}\right)^2-\frac{1}{\det^2\gamma} \times\mathrm{tr}\left(\gamma\left[S,P-\frac{\theta}{8\pi^2}B^{(+)}\right]\gamma\right)^2+V(S)\right\}. \tag{47}$$

The advantage of this Hamiltonian compared with Eq. (42), derived before, is that the classical equations of motion following from Eq. (47) are θ independent. In order to obtain a transparent form of the corresponding surface term in the unconstrained action, it is useful to perform a principal-axes transformation of the symmetric matrix field $S(x)$.

V. LONG-WAVELENGTH APPROXIMATION TO REDUCED THEORY

In this section we shall first rewrite the unconstrained Hamiltonian (47) in terms of principal-axes variables of the symmetric tensor field S_{ij}. The corresponding second-order Lagrangian $L^{(2)}$ is then obtained via Legendre transformation and the form of the corresponding unconstrained total divergence derived in an explicit way.

A. Hamiltonian in terms of principal-axes variables

In [22] it was shown that the field $S_{ij}(x)$ transforms as a second-rank tensor under spatial rotations. This can be used to explicitly separate the rotational degrees of freedom from the scalars in the Hamiltonian (47). Following [22], we introduce the principal-axes representation of the symmetric 3×3 matrix field $S(x)$,

$$ S(x)=R^T[\chi(x)]\begin{pmatrix}\phi_1(x) & 0 & 0\\ 0 & \phi_2(x) & 0\\ 0 & 0 & \phi_3(x)\end{pmatrix}R[\chi(x)]. \tag{48}$$

The Jacobian of this transformation is

$$ J\left(\frac{S_{ij}[\phi,\chi]}{\phi_k,\chi_l}\right)\propto\prod_{i\neq j}|\phi_i(x)-\phi_j(x)|, \tag{49}$$

and thus Eq. (48) can be used as a definition of the new configuration variables, the three diagonal fields ϕ_1,ϕ_2,ϕ_3 and the three angular fields χ_1,χ_2,χ_3, only if all eigenvalues of the matrix S are different. To have uniqueness of the inverse transformation we assume here that

$$ 0<\phi_1(x)<\phi_2(x)<\phi_3(x). \tag{50}$$

The variables ϕ_i in the principal-axes transformation (48) parametrize the orbits of the action of a group element $g\in SO(3,\mathbb{R})$ on symmetric matrices $S\rightarrow S'=gSg^{-1}$. The configuration (50) belongs to the so-called principal orbit class, whereas all orbits with coinciding eigenvalues of the matrix S are singular orbits [39]. In order to parametrize configurations belonging to a singular stratum one should in principle use a decomposition of the S field different from the above principal-axes transformation (48). Alternatively, one can consider the singular orbits as the boundary of the principal-orbit-type stratum and study the corresponding dynamics using a certain limiting procedure.[7] In this section we shall limit ourselves to the consideration of the dynamics on the principal orbits and leave the important case of the singular orbits expected to contain interesting physics for future studies.

The momenta π_i and p_{χ_i}, canonically conjugate to the diagonal elements ϕ_i and χ_i, can be found using the condition of the canonical invariance of the symplectic one-form

$$ \sum_{i,j=1}^{3}P_{ij}\dot{S}_{ij}dt=\sum_{i=1}^{3}\pi_i\dot{\phi}_i dt+\sum_{i=1}^{3}p_{\chi_i}\dot{\chi}_i dt. \tag{51}$$

The original physical momenta P_{ik}, expressed in terms of the new canonical variables, read

[7] The relation between an explicit parametrization of the singular strata and their description as a certain limit of the principal orbit stratum has been studied recently in [40] investigating the geodesic motion on the $GL(n,R)$ group manifold.

$$P(x)=R^T(x)\sum_{s=1}^{3}\left(\pi_s(x)\overline{\alpha}_s+\frac{1}{2}\mathcal{P}_s(x)\alpha_s\right)R(x). \quad (52)$$

Here $\overline{\alpha}_i$ and α_i denote the diagonal and off-diagonal basis elements for symmetric matrices with the orthogonality relations $\mathrm{tr}(\overline{\alpha}_i\overline{\alpha}_j)=\delta_{ij}$, $\mathrm{tr}(\alpha_i\alpha_j)=2\delta_{ij}$, $\mathrm{tr}(\overline{\alpha}_i\alpha_j)=0$, and

$$\mathcal{P}_i(x)=-\frac{\xi_i(x)}{\phi_j(x)-\phi_k(x)} \quad (\text{cyclic permutations } i\neq j\neq k). \quad (53)$$

The ξ_i are the three $SO(3,\mathbb{R})$ right-invariant Killing vector fields, satisfying locally the "intrinsic frame" angular momentum brackets $\{\xi_i(x),\xi_j(y)\}=-\epsilon_{ijk}\xi_k(x)\delta(x-y)$, and are given in terms of the angles χ_i and their conjugated momenta p_{χ_i} via[8]

$$\xi_i=\sum_{j=1}^{3} M^{-1}_{ji}p_{\chi_j}, \quad (54)$$

where the matrix M is

$$M_{ji}:=-\frac{1}{2}\sum_{a,b=1}^{3}\varepsilon_{jab}\left(\frac{\partial R}{\partial\chi_i}R^T\right)_{ab}. \quad (55)$$

In terms of the principal-axes variables (48), the $o(1/g)$ Hamiltonian (47) can be written in the form (for technical details see Appendix D)

$$H^{(2)}=\frac{1}{2}\int d^3x\left[\sum_{i=1}^{3}\left(\pi_i-\frac{\theta}{8\pi^2}\beta_i\right)^2+\sum_{cyclic}^{i,j,k}k_i\left(\xi_i+\frac{\theta}{8\pi^2}(\phi_j-\phi_k)b_i\right)^2+V(\phi,\chi)\right], \quad (56)$$

with the diagonal components β_i and the off-diagonal components b_i of the the symmetric part of the chromomagnetic field (see Appendix D)

$$\beta_i=g\phi_j\phi_k-(\phi_i-\phi_j)\Gamma_{ikj}+(\phi_i-\phi_k)\Gamma_{ijk} \quad (\text{cyclic permutations } i\neq j\neq k), \quad (57)$$

$$b_i=X_i(\phi_j-\phi_k)-(\phi_i-\phi_j)\Gamma_{ijj}+(\phi_i-\phi_k)\Gamma_{ikk}, \quad (\text{cyclic permutations } i\neq j\neq k), \quad (58)$$

the abbreviations

$$k_i:=\frac{\phi_j^2+\phi_k^2}{(\phi_j^2-\phi_k^2)^2} \quad (\text{cyclic permutations } i\neq j\neq k), \quad (59)$$

and the potential V, defined in Eq. (29) and rewritten in the principal-axes variables as (see [22] and the errata [23])

$$V(\phi,\chi)=\sum_{i\neq j}^{3}[(\phi_i-\phi_j)\Gamma_{iij}-X_j\phi_i]^2+\sum_{cyclic}^{i,j,k}[(\phi_i-\phi_k)\Gamma_{ijk}-(\phi_i-\phi_k)\Gamma_{ikj}-g\phi_j\phi_k]^2. \quad (60)$$

The dependence on the angular variables χ_i in Eqs. (57), (58), and (60) has been collected into the vector fields

$$X_i:=\sum_{j=1}^{3}R_{ij}\partial_j, \quad (61)$$

and the components of the connection one-form Γ

$$\Gamma_{aib}:=(X_iRR^T)_{ab}. \quad (62)$$

We see that through the principal-axes transformation of the symmetric tensor field S, the highest order g^2 terms in the Hamiltonian (56), which are proportional to the spatially homogeneous part V_{hom} of the potential (60),

$$V_{\mathrm{hom}}=g^2(\phi_1^2\phi_2^2+\phi_2^2\phi_3^2+\phi_3^2\phi_1^2), \quad (63)$$

depend only on the diagonal fields ϕ_i, while the rotational degrees of freedom χ_i and their canonically conjugate momenta p_{χ_i} appear in the unconstrained Hamiltonian (56) only via the Killing vector fields ξ_i, the connection Γ, and the vectors X_i.

The transformation (32), rewritten in terms of angular and scalar variables,

$$\pi_i\mapsto\pi_i+\frac{\theta}{8\pi^2}\beta_i, \quad \phi_i\mapsto\phi_i,$$

$$\xi_i\mapsto\xi_i-\frac{\theta}{8\pi^2}(\phi_j-\phi_k)b_i, \quad (64)$$

excludes the θ dependence from the Hamiltonian (56), reducing it to the zero θ angle expression [22]

$$H^{(2)}=\frac{1}{2}\int d^3x\left[\sum_{i=1}^{3}\pi_i^2+\sum_{cyclic}^{i,j,k}\xi_i^2\frac{\phi_j^2+\phi_k^2}{(\phi_j^2-\phi_k^2)^2}+V(\phi,\chi)\right]. \quad (65)$$

B. Second-order unconstrained Lagrangian

We are now ready to derive the Lagrangian up to second order in derivatives corresponding to the Hamiltonian (56). Carrying out the inverse Legendre transformation,

$$\dot{\phi}_i=\pi_i-\frac{\theta}{8\pi^2}\beta_i, \quad (66)$$

$$\dot{\chi}_i=\sum_{j=1}^{3}G_{ij}\left(p_{\chi_j}-\frac{\theta}{8\pi^2}\sum_{cyclic}^{a,b,c}M^T_{ja}(\phi_b-\phi_c)b_a\right), \quad (67)$$

[8] In terms of the Euler angles $\chi_i=(\alpha,\beta,\gamma)$ the three right-invariant Killing vector fields ξ_i read $\xi_1=\sin\gamma p_\alpha+(\cos\gamma/\sin\alpha)p_\beta+\cos\gamma\cot\alpha p_\gamma$, $\xi_2=\cos\gamma p_\alpha-(\sin\gamma/\sin\alpha)p_\beta-\sin\gamma\cot\alpha p_\gamma$, and $\xi_3=p_\gamma$.

with the matrix M given in Eq. (55) and the 3×3 matrix G,

$$G=M^{-1}kM^{-1T}, \tag{68}$$

similar to the diagonal matrix $k=\text{diag}\|k_1,k_2,k_3\|$ with entries k_i of Eq. (59), we arrive at the second-order Lagrangian

$$L^{(2)}(\phi,\chi)=\frac{1}{2}\int d^3x\left[\sum_{i=1}^{3}\dot{\phi}_i^2+\sum_{i,j=1}^{3}\dot{\chi}_iG_{ij}^{-1}\dot{\chi}_j-V(\phi,\chi)\right] -\theta\int d^3xQ^{(2)}(\phi,\chi), \tag{69}$$

with all θ dependence gathered in the reduced topological charge density

$$Q^{(2)}=\frac{1}{8\pi^2}\sum_{i=1}^{3}\left(\dot{\phi}_i\beta_i+\sum_{cyclic}^{a,b,c}\dot{\chi}_iM_{ia}^T(\phi_b-\phi_c)b_a\right). \tag{70}$$

The $Q^{(2)}$ in the effective Lagrangian (69) can be represented as the divergence

$$Q^{(2)}=\partial^\mu K_\mu^{(2)} \tag{71}$$

of the four-vector $K_\mu^{(2)}=(K_0^{(2)},K_i^{(2)})$, with the components

$$K_0^{(2)}=\frac{1}{16\pi^2}\sum_{cyclic}^{a,b,c}\left[(\phi_a-\phi_b)^2\Gamma_{acb}-\frac{2}{3}g\,\phi_a\phi_b\phi_c\right], \tag{72}$$

$$K_i^{(2)}=\frac{1}{16\pi^2}\sum_{cyclic}^{a,b,c}R_{ia}^T(\phi_b-\phi_c)^2\Gamma_{b0c}, \tag{73}$$

with the space components of Γ given in Eq. (62), and the time components correspondingly defined as

$$\Gamma_{a0b}=(\dot{R}R^T)_{ab}. \tag{74}$$

This completes our construction of the second-order Lagrangian with all θ contributions gathered in a total differential (70) (see also Appendix D). We have found the unconstrained analogue of the Chern-Simons current $K_\mu^{(2)}$, linear in the derivatives. Under the assumption that the vector part $K_i^{(2)}$ vanishes at spatial infinity, the unconstrained form of the Pontryagin index p_1 can be represented as the difference of the two surface integrals

$$W_\pm=\int d^3xK_0^{(2)}(t\to\pm\infty,\vec{x}), \tag{75}$$

which are the winding number functional (12) for the physical field S in terms of principal-axes variables (48) at $t\to\pm\infty$, respectively, since $K_0^{(2)}(\phi,\chi)$ of Eq. (72) coincides with the full $K_0[S[\phi,\chi]]$ of Eq. (13). In the next section we shall show how for certain field configurations it reduces to the Hopf number of the mapping from the three-sphere $\mathbb{S}^3$ to the unit two-sphere $\mathbb{S}^2$.

VI. UNCONSTRAINED THEORY FOR DEGENERATE CONFIGURATIONS

The previous study was restricted to consideration of the domain of configuration space with $\det\|S\|\neq0$, where the change of variables (14) is well defined. In this section we would like to discuss the dynamics on the special degenerate stratum (DS) with $\text{rank}\|S\|=1$, corresponding to the case of two eigenvalues of the matrix S vanishing. To investigate the dynamics on degenerate orbits it is in principle necessary to use a decomposition of the gauge potential different from our representation (14) and the corresponding subsequent principal-axes transformation (48). Instead of this, we shall use here the fact that the degenerate orbits can be regarded as the boundary of the nondegenerate ones and find the corresponding dynamics by taking the corresponding limit from the nondegenerate orbits. Assuming the validity of such an approach we shall analyze the limit when two eigenvalues of the symmetric matrix S tend to zero.[9] Due to the cyclic symmetry under permutation of the diagonal fields it is enough to choose one singular configuration

$$\phi_1(x)=\phi_2(x)=0 \quad \text{and} \quad \phi_3(x) \text{ arbitrary.} \tag{76}$$

Note that for the configuration (76) the spatially homogeneous part (63) of the square of the magnetic field vanishes and the potential term in the Lagrangian (69) reduces to the expression

$$\begin{aligned}V=\phi_3^2[&(\Gamma_{213})^2+(\Gamma_{223})^2+(\Gamma_{233})^2+(\Gamma_{311})^2+(\Gamma_{321})^2\\&+(\Gamma_{331})^2+(\Gamma_{3[12]})^2]+[(X_1\phi_3)^2+(X_2\phi_3)^2]\\&+2\phi_3[\Gamma_{331}X_1\phi_3+\Gamma_{332}X_2\phi_3],\end{aligned} \tag{77}$$

which can be rewritten as [22,23]

$$V=(\nabla\phi_3)^2+\phi_3^2[(\partial_i\mathbf{n})^2+(\mathbf{n}\cdot\text{rot}\,\mathbf{n})^2]-(\mathbf{n}\cdot\boldsymbol{\nabla}\phi_3)^2 +([\mathbf{n}\times\text{rot}\,\mathbf{n}]\cdot\boldsymbol{\nabla}\phi_3^2), \tag{78}$$

introducing the unit vector

$$n_i(x):=R_{3i}[\chi(x)]. \tag{79}$$

Hence the unconstrained second-order Lagrangian corresponding to the degenerate stratum with $\text{rank}\|S(x)\|=1$ takes the form of the nonlinear σ-model type Lagrangian

[9] It can easily be checked that the degenerate stratum with $\text{rank}\|S\|=1$ is dynamically invariant. Furthermore, it is obvious from the representation (65) of the unconstrained Hamiltonian that it is necessary to have $\xi_k\to0$ for some fixed k, in order to obtain a finite contribution of the kinetic term to the Hamiltonian in the limit $\phi_i,\phi_j\to0$ for $(i,j\neq k)$.

$$L_{\mathrm{DS}}=\frac{1}{2}\int d^3x[(\partial_\mu\phi_3)^2+\phi_3^2(\partial_\mu\mathbf{n})^2-\phi_3^2(\mathbf{n}\cdot\mathrm{rot}\,\mathbf{n})^2+(\mathbf{n}\cdot\nabla\phi_3)^2-([\mathbf{n}\times\mathrm{rot}\,\mathbf{n}]\cdot\nabla\phi_3^2)]-\theta\int d^3x Q_{\mathrm{DS}} \tag{80}$$

for the unit-vector $\mathbf{n}(x)$ field coupled to the field $\phi_3(x)$. The density of the topological term Q_{DS} in the Lagrangian (80) can be represented as the divergence

$$Q_{\mathrm{DS}}=\partial_\mu K^\mu_{\mathrm{DS}} \tag{81}$$

of the four-vector

$$K^\mu_{\mathrm{DS}}=\frac{1}{16\pi^2}\phi_3^2([\mathbf{n}(x)\cdot\mathrm{rot}\,\mathbf{n}(x)],[\mathbf{n}(x)\times\dot{\mathbf{n}}(x)]). \tag{82}$$

If we impose the usual boundary condition that the field $\mathbf{n}$ becomes time independent at spatial infinity, the contribution from the vector part K^i_{DS} vanishes and the unconstrained form of the Pontryagin topological index p_1 for the degenerate stratum with $\mathrm{rank}\|S\|=1$ can be represented as the difference

$$p_1=n_+-n_- \tag{83}$$

of the surface integrals

$$n_\pm=\frac{1}{16\pi^2}\int d^3x[\mathbf{V}_\pm(\vec{x})\cdot\mathrm{rot}\mathbf{V}_\pm(\vec{x})] \tag{84}$$

of the fields

$$\mathbf{V}_\pm(\vec{x}):=\lim_{t\to\pm\infty}\phi_3(x)\mathbf{n}. \tag{85}$$

We shall show now that the surface integrals (84) are Hopf invariants in the representation of Whitehead [29].

Under the Hopf mapping of a three-sphere to a two-sphere having unit radius, $N:\mathbb{S}^3\to\mathbb{S}^2$, the preimage of a point on $\mathbb{S}^2$ is a closed loop. The number Q_H of times the loops corresponding to two distinct points on $\mathbb{S}^2$ are linked to each other is the so-called Hopf invariant. According to Whitehead [29], this linking number can be represented by the integral

$$Q_H=\frac{1}{32\pi^2}\int_{S^3}w^1\wedge w^2, \tag{86}$$

with the so-called Hopf two-form curvature $w^2=H_{ij}dx^i\wedge dx^j$ given in terms of the map N as

$$H_{ij}=\varepsilon_{abc}N_a(\partial_i N_b)(\partial_j N_c), \tag{87}$$

and the one-form w^1 related to it via $w^2=dw^1$. Since the curvature H_{ij} is divergence-free,

$$\varepsilon_{ijk}\partial_i H_{jk}=0, \tag{88}$$

it can be represented as the rotation

$$H_{ij}=\partial_i\mathcal{A}_j-\partial_j\mathcal{A}_i \tag{89}$$

in terms of some vector field $\mathcal{A}_i(i=1,2,3)$ defined over the whole of $\mathbb{S}^3$. Thus the Hopf invariant takes the form

$$Q_H=\frac{1}{16\pi^2}\int d^3x(\mathcal{A}\cdot\mathrm{rot}\,\mathcal{A}). \tag{90}$$

Therefore, the surface integrals (84) are just Hopf invariants in the Whitehead representation (90) and the unconstrained form of the topological term $Q^{(2)}$ is a three-dimensional Abelian Chern-Simons term [4] with "potential" V_i and the corresponding "magnetic field" rot $\mathbf{V}$. The topological term in the original $SU(2)$ Yang-Mills theory reduces for rank-1 degenerate orbits not to a winding number, but to the linking number Q_H of the field lines.

We would like to end this section with two important open questions to be posed for future investigations. First, it would be very interesting to work out whether the classical unconstrained theory obtained for degenerate field configurations can be used to obtain some effective quantum model relevant to the low energy region of Yang-Mills theory, such as those proposed and discussed recently in [41–44]. Second, due to the noncovariance of the symmetric gauge imposed, the Lorentz transformation properties of the fields ϕ_3 and $\mathbf{n}$ are nonstandard (see, e.g., similar discussions for the case of the Coulomb gauge in electrodynamics [48–50]). A careful investigation is necessary, taking into account surface contributions to the unconstrained form of the generators of the Poincaré group.

VII. CONCLUSIONS AND REMARKS

We have generalized the Hamiltonian reduction of $SU(2)$ Yang-Mills gauge theory to the case of nonvanishing θ angle and shown that there is agreement between the reduced and original constrained equations of motions. We have employed an improved derivative expansion of the nonlocal kinetic term in the unconstrained Hamiltonian obtained and investigated it in the long-wavelength approximation. The corresponding second-order Lagrangian has been constructed, with all θ dependence gathered in the four-divergence of a current, linear in the derivatives, which is the unconstrained analogue of the original Chern-Simons current.

For the degenerate gauge field configurations S with $\mathrm{rank}\|S\|=1$, we have argued that the long-wavelength Lagrangian obtained reduces to a classical theory with an Abelian Chern-Simons term originating from the Pontryagin topological functional. Therefore the topological characteristic of the degenerate configuration is given not by a winding number, but by the linking number of the field lines.

Finally, let us comment on the Poincaré covariance of our unconstrained version of Yang-Mills theory. It is well known that the Hamiltonian formulation of degenerate theories reduced with the help of noncovariant gauges destroy the manifest Poincaré invariance. Our "symmetric" gauge condition (15) is not covariant under standard Lorentz transformations. This, however, does not necessarily violate the Poincaré invariance of our reduced theory. Such a situation can be found in classical electrodynamics. After imposing the Coulomb gauge condition the vector potential ceases to be an ordinary Lorentz vector and transforms nonhomogeneously under Lorentz transformations. The standard Lorentz boosts are compensated by some additional gauge-type transformation depending on the boost parameters and the gauge potential itself (see, e.g., [48–50]). As for the case of the Coulomb gauge in electrodynamics, a thorough analysis of the Poincaré group representation for our reduced theory obtained by imposing the symmetric gauge condition is required. This problem is technically highly difficult and demands special consideration that is beyond the scope of the present article.

ACKNOWLEDGMENTS

We are grateful for discussions with Z. Aouissat, R. Horan, A. Kovner, A. N. Kvinikhidze, M. Lavelle, M. D. Mateev, D. McMullan, P. Schuck, D. V. Shirkov, M. Staudacher, and A. N. Tavkhelidze. A.K. and D.M. would like to thank Professor G. Röpke for his kind hospitality in the group ''Theoretische Vielteilchenphysik'' at the Fachbereich Physik of Rostock University, where part of this work was done. A.K. thanks the Deutsche Forschungsgemeinschaft and H.-P.P. the Bundesministerium fuer Forschung und Technologie for financial support.

APPENDIX A: CONVENTIONS AND NOTATION

In this appendix, we collect the notation and definitions for $SU(2)$ Yang-Mills theory used in the text following [4].

The classical Yang-Mills action of the $su(2)$-valued connection one-form A in four-dimensional Minkowski spacetime with a metric $\eta = \mathrm{diag}\|1,-1,-1,-1\|$ reads

$$I = -\frac{1}{g^2}\int \mathrm{tr}\, F\wedge {}^*F - \frac{\theta}{8\pi^2 g^2}\int \mathrm{tr}\, F\wedge F, \tag{A1}$$

with the curvature two-form

$$F = dA + A\wedge A \tag{A2}$$

and its Hodge dual ${}^*F$. The trace in Eq. (A1) is calculated in the anti-Hermitian $su(2)$ algebra basis $\tau^a = \sigma^a/2i$ with Pauli matrices $\sigma^a, a=1,2,3$, satisfying $[\tau_a,\tau_b] = \varepsilon_{abc}\tau_c$ and $\mathrm{tr}(\tau_a\tau_b) = -\frac{1}{2}\delta_{ab}$.

In the coordinate basis the components of the connection one-form A are

$$A = g\,\tau^a A^a_\mu dx^\mu, \tag{A3}$$

and the components of the curvature 2-form F are

$$F = \frac{1}{2} g\,\tau^a F^a_{\mu\nu} dx^\mu \wedge dx^\nu, \tag{A4}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,\varepsilon^{abc} A^b_\mu A^c_\nu. \tag{A5}$$

Its duals ${}^*F$ are given as

$${}^*F = \frac{1}{2} g\,\tau^a\, {}^*F^a_{\mu\nu} dx^\mu \wedge dx^\nu, \tag{A6}$$

$${}^*F^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F^{a\rho\sigma}, \tag{A7}$$

with a totally antisymmetric Levi-Cività pseudotensor $\varepsilon_{\mu\nu\rho\sigma}$, using the convention

$$\varepsilon^{0123} = -\varepsilon_{0123} = 1. \tag{A8}$$

The θ angle enters the classical action as the coefficient in front of the Pontryagin index density

$$Q = -\frac{1}{8\pi^2}\,\mathrm{tr}\, F\wedge F. \tag{A9}$$

The Pontryagin index density is a closed form $dQ=0$ and thus locally exact

$$Q = dC, \tag{A10}$$

with the Chern three-form

$$C = -\frac{1}{8\pi^2}\,\mathrm{tr}\left(A\wedge dA + \frac{2}{3} A\wedge A\wedge A\right). \tag{A11}$$

The corresponding Chern-Simons current K^μ is a dual of the three-form C,

$$\begin{aligned} K^\mu &= (1/3!)\,\varepsilon^{\mu\nu\rho\sigma} C_{\nu\rho\sigma} \\ &= -\frac{1}{16\pi^2}\varepsilon^{\mu\alpha\beta\gamma}\,\mathrm{tr}\left(F_{\alpha\beta}A_\gamma - \frac{2}{3} A_\alpha A_\beta A_\gamma\right), \end{aligned} \tag{A12}$$

with the notations $A_\mu := g\,\tau^a A^a_\mu$ and $F_{\mu\nu} := g\,\tau^a F^a_{\mu\nu}$. The chromomagnetic field is given by

$$B^a_i = \frac{1}{2}\varepsilon_{ijk} F^a_{jk} = \varepsilon_{ijk}\left(\partial_j A_{ak} + \frac{g}{2}\varepsilon_{abc} A_{bj} A_{ck}\right), \tag{A13}$$

and the covariant derivative in the adjoint representation as

$$(D_i(A))_{ac} = \delta_{ac}\partial_i + g\,\varepsilon_{abc} A_{bi}. \tag{A14}$$

Finally, we frequently use the matrix notation

$$A_{ai} := A^a_i, \quad B_{ai} := B^a_i. \tag{A15}$$

APPENDIX B: ON THE EXISTENCE OF THE "SYMMETRIC GAUGE"

In this appendix we discuss the condition under which the symmetric gauge

$$\chi_a(A) = \varepsilon_{abi} A_{bi}(x) = 0 \tag{B1}$$

exists.

According to the conventional gauge-fixing method (see, e.g., [45]), a gauge $\chi_a(A)=0$ exists if the corresponding equation

$$\chi_a(A^\omega)=0 \tag{B2}$$

in terms of the gauge transformed potential

$$A^\omega_{ai}\tau_a=U^+(\omega)\left(A_{ai}\tau_a+\frac{1}{g}\frac{\partial}{\partial x_i}\right)U(\omega) \tag{B3}$$

has a unique solution for the unknown function $\omega(x)$.[10]

Hence the symmetric gauge (B1) exists if any gauge potential A can be made symmetric by a unique time-independent gauge transformation. The equation that determines the gauge transformation $\omega(x)$ which converts an arbitrary gauge potential $A(x)$ into its symmetric counterpart can be written as a matrix equation

$$O^T(\omega)A-A^TO(\omega)=\frac{1}{g}[\Sigma(\omega)-\Sigma^T(\omega)], \tag{B4}$$

with the orthogonal 3×3 matrix related to the $SU(2)$ group element

$$O_{ab}(\omega)=-2\,\mathrm{tr}[U^+(\omega)\tau_aU(\omega)\tau_b] \tag{B5}$$

and the 3×3 matrix Σ

$$\Sigma_{ai}(\omega):=-\frac{1}{4i}\varepsilon_{amn}\left(O^T(\omega)\frac{\partial O(\omega)}{\partial x_i}\right)_{mn}. \tag{B6}$$

We shall now prove the following theorem.

Theorem. For any nondegenerate matrix A Eq. (B4) admits a unique solution in the form of a $1/g$ expansion

$$O(\omega)=O^{(0)}\left[1+\sum_{n=1}^{\infty}\left(\frac{1}{g}\right)^nX^{(n)}\right]. \tag{B7}$$

Proof. In order to prove the statement, we first note that equating coefficients of equal powers in $1/g$ in the orthogonality condition $O^TO=OO^T=I$ of the matrix O imposes the condition of orthogonality of $O^{(0)}$,

$$O^{(0)T}O^{(0)}=O^{(0)}O^{(0)T}=I, \tag{B8}$$

as well as the conditions

$$X^{(1)}+X^{(1)T}=0,$$

$$X^{(2)}+X^{(2)T}+X^{(1)}X^{(1)T}=0,$$

$$\cdots\quad\cdots,$$

$$X^{(n)}+X^{(n)T}+\sum_{i+j=n}X^{(i)}X^{(j)T}=0,$$

$$\cdots\quad\cdots \tag{B9}$$

for the unknown functions $X^{(n)}$. Furthermore, plugging expansion (B7) into Eq. (B4) and combining the terms of equal powers of $1/g$, we find that the orthogonal matrix $O^{(0)}$ should satisfy Eq. (B4) to leading order in $1/g$,

$$O^{(0)T}A-A^TO^{(0)}=0, \tag{B10}$$

and the $X^{(n)}$ should satisfy the infinite set of equations

$$X^{(1)T}O^{(0)T}A-A^TO^{(0)}X^{(1)}=\Sigma^{(0)}-\Sigma^{(0)},$$

$$\cdots\quad\cdots,$$

$$X^{(n)T}O^{(0)T}A-A^TO^{(0)T}X^{(n)}=\Sigma^{(n-1)}-\Sigma^{(n-1)T},$$

$$\cdots\quad\cdots, \tag{B11}$$

where the corresponding $1/g$ expansion for the matrix $\Sigma(\omega)$

$$\Sigma(\omega)=\sum_{n=0}^{\infty}\left(\frac{1}{g}\right)^n\Sigma^{(n)} \tag{B12}$$

has been used. Note that in the expansion (B12) the nth order term $\Sigma^{(n)}$ is given in terms of $O^{(0)}$ and $X^{(a)}$ with $a=1,\ldots,n-1$.

From the structure of Eqs. (B8)–(B11) one can see that the solution to Eq. (B4) reduces to an algebraic problem. Indeed, the solution to the first, homogeneous equation (B10) is given by the polar decomposition for the arbitrary matrix A,

$$O^{(0)}=AS^{(0)-1},\quad S^{(0)}=\sqrt{AA^T}. \tag{B13}$$

This solution is unique only if $\det\|A\|\neq0$. It follows from the well-known property that the polar decomposition is valid for an arbitrary matrix A, but the orthogonal matrix $O^{(0)}$ is unique only for nondegenerate matrices [47].

To proceed further we use this solution and Eqs. (B9) for unknown X to rewrite the remaining equations (B11) as

$$X^{(1)}S^{(0)}+S^{(0)}X^{(1)}=C^{(0)},$$

$$\cdots\quad\cdots,$$

$$X^{(n)}S^{(0)}+S^{(0)}X^{(n)}=C^{(n-1)},$$

$$\cdots\quad\cdots, \tag{B14}$$

where the nth order coefficient $C^{(n)}$ is given in terms of $O^{(0)}$ and $X^{(1)},X^{(2)},\ldots,X^{(n-1)}$.

Thus, starting from the zeroth-order term, the higher-order terms $X^{(n)}$ are given recursively as solutions of matrix equations of the type $XS^{(0)}+S^{(0)}X=C$ with a known symmetric positive definite matrix $S^{(0)}=\sqrt{AA^T}$ and matrix C, expressed in terms of the preceding $X^{(a)}$, $a=1,\ldots,n-1$. The theory

[10] Here we assume that the second gauge condition $A_{a0}=0$ is satisfied and the function $\omega(x)$ therefore depends only on the space coordinates.

of such algebraic equations is well elaborated (see, e.g., [46,47]). In particular, Theorem 8.5.1 in [46] states that for matrix equations for unknown matrix X of the type $XA + BX = C$, there is a unique solution if and only if the matrices A and $-B$ have no common eigenvalues. Based on this theorem one can conclude that the unique solution to Eqs. (B8)–(B11) and hence to our original problem (B4) exists always for any nondegenerate matrix A.

It is necessary to emphasize that in order to prove the existence and uniqueness of the representation (14) it should be shown additionally to the above Theorem that the corresponding symmetric matrix field S,

$$S(x) = \sum_{n=0}^{\infty} \left(\frac{1}{g}\right)^n S^{(n)}(x), \tag{B15}$$

is sign definite. Above, the positive definiteness has been shown only for the zeroth-order term $S^{(0)} = \sqrt{AA^T}$. The study of this problem, as well as an analogous investigation for the degenerate field configurations A with $\det\|A\| = 0$, are beyond the scope of this appendix and will be discussed in detail elsewhere. Here we limit ourselves to the consideration of a specific example, elucidating the generic picture.

In the case that the matrix A is degenerate, we encounter the problem of Gribov's copies. As an illustration of the nonuniqueness of the gauge transformation that turns a given field configuration A into the corresponding symmetric form, we consider the "degenerate" field

$$A_{a0} = 0, \quad A_{ai} = -\frac{1}{gr}\varepsilon_{aic}\hat{r}_c, \tag{B16}$$

known as the non-Abelian Wu-Yang monopole field, with the unit vector $\hat{r}_a = x_a/r$ and $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Performing the gauge transformation

$$S_{ai}\tau_a = U^+(\omega)\left(A_{ai}\tau_a + \frac{1}{g}\frac{\partial}{\partial x_i}\right)U(\omega), \tag{B17}$$

with $U(\omega) = e^{\omega_a \tau_a}$ parametrized by one time-independent spherical symmetric function

$$\omega_a = f(r)\hat{r}_a, \tag{B18}$$

the Wu-Yang monopole configuration (B16), antisymmetric in space and color indices, can be brought into the "symmetric form"

$$S^{\pm}_{ai} = \pm\frac{\sqrt{3}}{gr}(\delta_{ai} - \hat{r}_a\hat{r}_i), \tag{B19}$$

if the function $f(r)$ is constant and takes four values:

$$f(r) = \begin{cases} \pi/3, \quad 7\pi/3 \quad \text{for} \quad (+), \\ 5\pi/3, \quad 11\pi/3 \quad \text{for} \quad (-). \end{cases} \tag{B20}$$

Here S^+ can be obtained from the Wu-Yang monopole configuration (B16) by applying two different gauge transformations with $f(r) = \pi/3, 7\pi/3$,

$$U_{1,2} = \pm\left(\frac{\sqrt{3}}{2} - \hat{r}\cdot\tau\right), \tag{B21}$$

while the S^- configuration can be reached using $f(r) = 5\pi/3, 11\pi/3$,

$$U_{3,4} = \mp\left(\frac{\sqrt{3}}{2} + \hat{r}\cdot\tau\right). \tag{B22}$$

Here it is in order to make the following comments.

For the above gauge transformations we have $\lim_{r\to\infty} U \neq \pm I$. Thus they are neither small gauge transformations nor large gauge transformations belonging to any integer n-homotopy class [4].

The symmetric configurations (B19) corresponding to the Wu-Yang monopole lie on the stratum of degenerate symmetric matrices with one eigenvalue vanishing and two eigenvalues equal to each other.

The symmetric configurations S^+ and S^- in Eq. (B19) with twofold Gribov degeneracy are related to each other by parity conjugation.

APPENDIX C: PROOF OF θ DEPENDENCE OF THE NAIVE 1/g APPROXIMATION

In this appendix it is shown that straightforward application of expansion of the nonlocal part P_a of the kinetic term in the unconstrained Hamiltonian to zeroth order discussed in Sec. IV A leads to the appearance of θ dependence of the reduced system on the classical level. Expressing the Hamiltonian (42), in terms of the principal-axes variables, defined in Sec. V, and performing an inverse Legendre transformation, one obtains the Lagrangian density

$$\begin{aligned}\mathcal{L}^{(2)}(\phi,\chi) = {}& \frac{1}{2}\left(\sum_{i=1}^{3}\dot{\phi}_i^2 + \sum_{i,j=1}^{3}\dot{\chi}_i G^{-1}_{ij}\dot{\chi}_j - V(\phi,\chi)\right) \\ & - \frac{1}{2}\left(\frac{\theta}{8\pi^2}\right)^2 \sum_{cyclic}^{i,j,k} \frac{\Delta_i^2}{\phi_j^2 + \phi_k^2} - \frac{\theta}{8\pi^2}\sum_{a=1}^{3}\Bigg[\dot{\phi}_a\beta_a \\ & + \sum_{cyclic}^{i,j,k}\dot{\chi}_a M^T_{ai}(\phi_j - \phi_k)\left(b_i + \frac{(\phi_j - \phi_k)}{\phi_j^2 + \phi_k^2}\Delta_i\right)\Bigg],\end{aligned} \tag{C1}$$

denoting the difference

$$\Delta_i = \frac{1}{2}(\phi_j - \phi_k)b_i - (\phi_j + \phi_k)\sum_{s=1}^{3} R_{is}B_s^{(-)} \quad \text{(cyclic permutations } i \neq j \neq k), \tag{C2}$$

with b_i of Eq. (58) and $B_i^{(-)}$ of Eq. (D7), or, explicitly,

$$\Delta_i = -[X_i(\phi_j\phi_k) + (\Gamma_{ijj}+\Gamma_{ikk})\phi_j\phi_k - \phi_i(\phi_j\Gamma_{ikk} + \phi_k\Gamma_{ijj})] \quad (\text{cyclic permutations } i\neq j\neq k). \tag{C3}$$

It easy to convince ourselves that the term proportional to θ^2 is not a surface term. Indeed, considering for simplicity configurations of spatially constant angular variables χ_i and $\phi_1 = \phi_2 = \phi_3 =: \phi$, it reduces to

$$-\left(\frac{\theta}{8\pi^2}\right)^2 \sum_{i=1}^{3} \partial_i\phi\partial_i\phi, \tag{C4}$$

which is not a four-divergence. For $\Delta_i = 0$ the Lagrangian density (C1) reduces to Eq. (69), obtained from the improved Hamiltonian (47), free of the divergence problem.

APPENDIX D: REPRESENTATION OF THE UNCONSTRAINED FIELDS IN THE BASIS OF PRINCIPAL-AXES VARIABLES

Starting from the coordinate basis expression of S in Eq. (31), we observe that the principal-axes transformation (48) corresponds to the representation

$$S = \sum_{a=1}^{3} e_a\phi_a\omega_a, \tag{D1}$$

with the one-forms

$$\omega_i := \sum_{j=1}^{3} R_{ij}[\chi(x)]dx_j, \quad i=1,2,3, \tag{D2}$$

and the $su(2)$ Lie algebra basis

$$e_a := \sum_{b=1}^{3} R_{ab}[\chi(x)]\tau_b, \quad a=1,2,3. \tag{D3}$$

1. Unconstrained magnetic field

The physical chromomagnetic fields $B_{ai}(S)$, given in Eq. (28), can be regarded as the components of the dual $*F^{(3)}$

$$B_{ai}(S) = \frac{1}{2}\sum_{i,j=1}^{3} \varepsilon_{ijk}F^{(3)}_{ajk}$$

of the curvature two-form $F^{(3)}$, defined in terms of the symmetric one-form S in Eq. (30) as

$$F^{(3)} = dS + S\wedge S.$$

In the principal-axes basis the components of the non-Abelian field strength $F^{(3)}$ read

$$F^{(3)}_{aij} = \delta_{aj}X_i\phi_j - \delta_{ai}X_j\phi_i + \phi_i\Gamma_{aji} - \phi_j\Gamma_{aij} + \Gamma_{a[ij]}\phi_a + g\varepsilon_{aij}\phi_i\phi_j \quad (\text{no summation}), \tag{D4}$$

with the components of the connection one-form Γ defined as

$$\Gamma_{aib} := (X_i R R^T)_{ab},$$

and the vector fields

$$X_i := \sum_{j=1}^{3} R_{ij}\,\partial_j$$

dual to the one-forms $\omega_j, \omega_i(X_j) = \delta_{ij}$, and acting on the basis elements e_a as

$$X_i e_a = -\sum_{b=1}^{3} \Gamma_{bia}e_b. \tag{D5}$$

The explicit expressions for the diagonal components β_i and the off-diagonal components b_i of the symmetric part of the chromomagnetic field

$$B^{(+)} = R^T(\chi)\sum_{i=1}^{3}\left(\beta_i\overline{\alpha}_i + \frac{1}{2}b_i\alpha_i\right)R(\chi) \tag{D6}$$

are given in terms of the diagonal fields ϕ_i and the angular fields χ_i in cyclic form

$$\beta_i = g\phi_j\phi_k - (\phi_i-\phi_j)\Gamma_{ikj} + (\phi_i-\phi_k)\Gamma_{ijk} \quad (\text{cyclic permutations } i\neq j\neq k),$$

$$b_i = X_i(\phi_j-\phi_k) - (\phi_i-\phi_j)\Gamma_{ijj} + (\phi_i-\phi_k)\Gamma_{ikk} \quad (\text{cyclic permutations } i\neq j\neq k),$$

and the antisymmetric part $B_i^{(-)}$ of the unconstrained magnetic field is

$$B_i^{(-)} = \frac{1}{2}\sum_{cyclic}^{a,b,c} R^T_{ia}[X_a(\phi_b+\phi_c) + (\phi_b-\phi_a)\Gamma_{abb} + (\phi_c-\phi_a)\Gamma_{acc}]. \tag{D7}$$

2. Unconstrained Chern-Simons three-form

Using the Maurer-Cartan structure equations for the one-forms ω_i

$$d\omega_a = \sum_{c=1}^{3}\Gamma_{a0c}dt\wedge\omega_c + \sum_{b,c=1}^{3}\Gamma_{abc}\omega_b\wedge\omega_c, \tag{D8}$$

with the space components of Γ given in Eq. (62), and the time components correspondingly defined as

$$\Gamma_{a0b}=(\dot{R}R^T)_{ab}\,,$$

Eq. (70) can be written as

$$Q^{(2)}=dC^{(2)} \tag{D9}$$

with the three-form

$$C^{(2)}=\frac{1}{8\pi^2}\sum_{a<b}^{3}(\phi_a-\phi_b)^2\Gamma_{a0b}dt\wedge\omega_a\wedge\omega_b - \frac{3}{8\pi^2}\sum_{cyclic}^{a,b,c}\left[(\phi_a-\phi_b)^2\Gamma_{acb}-\frac{2}{3}\varepsilon_{abc}\phi_1\phi_2\phi_3\right] \times\omega_a\wedge\omega_b\wedge\omega_c\,. \tag{D10}$$

[1] R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37**, 172 (1976).
[2] C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Lett. **63B**, 334 (1976).
[3] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).
[4] R. Jackiw, in *Current Algebra and Anomalies* (World Scientific, Singapore, 1985).
[5] J. Goldstone and R. Jackiw, Phys. Lett. **74B**, 81 (1978).
[6] A. G. Izergin, V. F. Korepin, M. E. Semenov-Tyan-Shanskii, and L. D. Faddeev, Teor. Mat. Fiz. **38**, 3 (1979) [Theor. Math. Phys. **38**, 1 (1979)].
[7] A. Das, M. Kaku, and P. K. Townsend, Nucl. Phys. **B149**, 109 (1979).
[8] M. Creutz, I. J. Muzinich, and T. N. Tudron, Phys. Rev. D **19**, 531 (1979).
[9] N. H. Christ and T. D. Lee, Phys. Rev. D **22**, 939 (1980).
[10] Yu. Simonov, Yad. Fiz. **41**, 1311 (1985) [Sov. J. Nucl. Phys. **41**, 835 (1985)].
[11] Yu. Simonov, Yad. Fiz. **41**, 1601 (1985) [Sov. J. Nucl. Phys. **41**, 1014 (1985)]
[12] V. V. Vlasov, V. A. Matveev, A. N. Tavkhelidze, S. Yu. Khlebnikov, and M. E. Shaposhnikov, Sov. J. Part. Nucl. **18**, 1 (1987).
[13] K. Haller, Phys. Rev. D **36**, 1839 (1987).
[14] R. Anishetty, Phys. Rev. D **44**, 1895 (1991).
[15] E. T. Newman and C. Rovelli, Phys. Rev. Lett. **69**, 1300 (1992).
[16] F. A. Lunev, Mod. Phys. Lett. A **9**, 2281 (1994).
[17] M. Bauer, D. Z. Freedman, and P. E. Haagensen, Nucl. Phys. **B428**, 147 (1994); P. E. Haagensen and K. Johnson, *ibid.* **B439**, 597 (1995); R. Schiappa, *ibid.* **B517**, 462 (1998).
[18] H. Nachbagauer, Phys. Rev. D **52**, 3672 (1995).
[19] G. Chechelashvili, G. Jorjadze, and N. Kiknadze, Teor. Mat. Fiz. **109**, 90 (1996) [Theor. Math. Phys. **109**, 1316 (1997)].
[20] M. Lavelle and D. McMullan, Phys. Rep. **279**, 1 (1997).
[21] R. Horan, M. Lavelle, and D. McMullan, Pramana **51**, 317 (1998).
[22] A. M. Khvedelidze and H.-P. Pavel, Phys. Rev. D **59**, 105017 (1999).
[23] Erratum to [22]: The last line of Eq. (3.42) should be replaced by $(\Gamma^1_{23}\phi_3+\Gamma^1_{32}\phi_2+\Gamma^1_{[23]}\phi_1-g\phi_2\phi_3)^2$. Correspondingly, in formulas (4.6) the term $(\Gamma^3_{[12]}\phi_3)^2$ should be added. Finally, in formulas (4.8), (4.10), and (4.11) the term $(\mathbf{n}\cdot\mathrm{rot}\,\mathbf{n})^2$ should be included. The correct formulas are given in the present work in Eqs. (60), (77), and (78), respectively.
[24] P. Majumdar and H. S. Sharatchandra, Nucl. Phys. B (Proc. Suppl.) **94**, 715 (2001).
[25] P. A. M. Dirac, Proc. R. Soc. London **A246**, 333 (1958); Phys. Rev. **114**, 924 (1959); Phys. Rev. Lett. **2**, 368 (1959).
[26] R. Arnowitt, S. Deser, and C. W. Misner, J. Math. Phys. **1**, 434 (1960).
[27] A. Khvedelidze, H.-P. Pavel, and G. Röpke, Phys. Rev. D **61**, 025017 (2000).
[28] A. M. Khvedelidze, D. M. Mladenov, H.-P. Pavel, and G. Röpke, Eur. Phys. J. C **24**, 137 (2002).
[29] J. H. C. Whitehead, Proc. Natl. Acad. Sci. U.S.A. **33**, 117 (1947).
[30] L. Woltier, Proc. Natl. Acad. Sci. U.S.A. **44**, 489 (1958).
[31] H. Moffat, J. Fluid Mech. **35**, 117 (1969).
[32] E. A. Kuznetsov and A. V. Mikhailov, Phys. Lett. **77A**, 37 (1980).
[33] P. G. Saffman, *Vortex Dynamics* (Cambridge University Press, Cambridge, England, 1992).
[34] R. Jackiw and S. Y. Pi, Phys. Rev. D **61**, 105015 (2000).
[35] R. Jackiw, V. P. Nair, and S. Y. Pi, Phys. Rev. D **62**, 085018 (2000).
[36] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University Press, New York, 1964).
[37] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, NJ, 1992).
[38] M. Creutz, *Quarks, Gluons and Lattices* (Cambridge University Press, Cambridge, England, 1983).
[39] L. O'Raifeartaigh, *Group Structure of Gauge Theories* (Cambridge University Press, Cambridge, England, 1986).
[40] A. M. Khvedelidze and D. M. Mladenov, Phys. Lett. A **299**, 522 (2002).
[41] L. Faddeev and A. J. Niemi, Phys. Rev. Lett. **82**, 1624 (1999).
[42] E. Langmann and A. J. Niemi, Phys. Lett. B **463**, 252 (1999).
[43] P. van Baal and A. Wipf, Phys. Lett. B **515**, 181 (2001).
[44] R. A. Battye and P. Sutcliffe, Proc. R. Soc. London **A455**, 4305 (1999).
[45] L. D. Faddeev and A. A. Slavnov, *Gauge Fields: Introduction to Quantum Theory* (Benjamin-Cummings, New York, 1984).
[46] P. Lancaster, *Theory of Matrices* (Academic Press, New York, 1969).
[47] F. R. Gantmacher, *Matrix Theory* (Chelsea, New York, 1959).
[48] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).
[49] A. J. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Rome, 1976).
[50] H.-P. Pavel and V. N. Pervushin, Int. J. Mod. Phys. A **14**, 2285 (1999).