

Pressure of hot QCD up to $g^6 \ln(1/g)$

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The free energy density, or pressure, of QCD has at high temperatures an expansion in the coupling constant g , known so far up to order g^5 . We compute here the last contribution which can be determined perturbatively, $g^6 \ln(1/g)$, by summing together results for the 4-loop vacuum energy densities of two different three-dimensional effective field theories. We also demonstrate that the inclusion of the new perturbative $g^6 \ln(1/g)$ terms, once they are summed together with the so far unknown perturbative and nonperturbative g^6 terms, could potentially extend the applicability of the coupling constant series down to surprisingly low temperatures.

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I. INTRODUCTION

Because of asymptotic freedom, the properties of QCD might be expected to be perturbatively computable in various “extreme” limits, such as high virtuality, high baryon density, or high temperature. We concentrate here on the last of these circumstances, that is, temperatures T larger than a few hundred MeV.

The physics observable we consider is the pressure, or minus the free energy density, of the QCD plasma. Potential phenomenological applications include the expansion rate of the early Universe after it has settled into the standard model vacuum, as well as the properties of the apparently ideal hydrodynamic expansion observed in on-going heavy ion collision experiments, just shortly after impact.

In these environments, it turns out that the naive expectation concerning the validity of perturbation theory is too optimistic. Indeed, even assuming an arbitrarily weak coupling constant g , perturbation theory can only be worked out to a finite order in it, before the serious infrared problems of finite temperature field theory deny further analytic progress [1,2]. For the pressure, the problem is met at the 4-loop order, or $\mathcal{O}(g^6)$.

This leads to the interesting situation that there is a definite limit to how far perturbation theory needs to be pushed. So far, there are known loop contributions at orders $\mathcal{O}(g^2)$ [3], $\mathcal{O}(g^3)$ [4], $\mathcal{O}(g^4 \ln(1/g))$ [5], $\mathcal{O}(g^4)$ [6], and $\mathcal{O}(g^5)$ [7]. There is also an all-orders numerical result available for a theory with an asymptotically large number of fermion flavors [8]. The purpose of the present paper is to collect together results from two accompanying papers [9,10], allowing us to determine analytically the last remaining perturbative contribution, $\mathcal{O}(g^6 \ln(1/g))$, for the physical QCD.

It must be understood that even if computed up to such a high order, the perturbative expansion could well converge only very slowly, requiring perhaps something like $T \gg \text{TeV}$, to make any sense at all [7,11,12]. With one further coefficient available, we can to some extent now reinspect this issue. To do so we actually also need to assume something about the unknown $\mathcal{O}(g^6)$ term, since the numerical factor

inside the logarithm in $\mathcal{O}(g^6 \ln(1/g))$ remains otherwise undetermined. Therefore, our conclusions on this point remain on a conjectural level, but turn out to show nevertheless a somewhat interesting pattern, which is why we would like to include them in this presentation.

Finally, it should be stressed that even if the perturbative expansion as such were to remain numerically useless at realistic temperatures, these multiloop computations are still worthwhile: the infrared problems of finite temperature QCD can be isolated to a three-dimensional (3D) effective field theory [13] and studied nonperturbatively there with simple lattice simulations [14]. However, to convert the results from 3D lattice regularization to 3D continuum regularization, and from the 3D continuum theory to the original four-dimensional (4D) physical theory, still necessitates a number of perturbative “matching” computations. Both of these steps are very closely related to what we do here, although we discuss explicitly only the latter one.

II. THE BASIC SETTING

We start by reviewing briefly how it is believed that the properties of QCD at a finite temperature T can be reduced to a number of perturbatively computable matching coefficients, as well as some remaining contributions from a series of effective field theories [13]. Our presentation follows mostly that in [11], but there are a few significant differences.

The underlying theory is finite temperature QCD with the gauge group $SU(N_c)$, and N_f flavors of massless quarks. In dimensional regularization the bare Euclidean Lagrangian reads, before gauge fixing,

$$S_{\text{QCD}} = \int_0^{\beta\hbar} d\tau \int d^d x \mathcal{L}_{\text{QCD}}, \quad (2.1)$$

$$\mathcal{L}_{\text{QCD}} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \bar{\psi} \gamma_\mu D_\mu \psi, \quad (2.2)$$

where $\beta = T^{-1}$, $d = 3 - 2\epsilon$, $\mu, \nu = 0, \dots, d$, $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$, $D_\mu = \partial_\mu - ig A_\mu$, $A_\mu = A_\mu^a T^a$, $\gamma_\mu^\dagger = \gamma_\mu$, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, and ψ carries Dirac, color, and flavor indices.

Denoting the generators of the adjoint representation by $(F^a)_{bc} = -if^{abc}$, we define the usual group theory factors:

$$C_A \delta_{ab} = [F^c F^c]_{ab}, \quad C_F \delta_{ij} = [T^a T^a]_{ij}, \quad (2.3)$$

$$T_A \delta^{ab} = \text{Tr } F^a F^b, \quad T_F \delta^{ab} = \text{Tr } T^a T^b, \quad (2.4)$$

$$d_A = \delta^{aa} = N_c^2 - 1, \quad d_F = \delta_{ii} = T_F d_A / C_F. \quad (2.5)$$

Obviously $T_A = C_A$. For the standard normalization, with N_f quark flavors, $C_A = N_c$, $C_F = (N_c^2 - 1)/(2N_c)$, $T_A = N_c$, $T_F = N_f/2$, $d_A = N_c^2 - 1$, $d_F = N_c N_f$.

We use dimensional regularization throughout this paper. The spatial part of each momentum integration measure is written as

$$\int_p \equiv \int \frac{d^d p}{(2\pi)^d} = \mu^{-2\epsilon} \left[\bar{\mu}^{-2\epsilon} \left(\frac{e^\gamma}{4\pi} \right)^\epsilon \int \frac{d^d p}{(2\pi)^d} \right], \quad (2.6)$$

where $\mu = \bar{\mu} (e^\gamma/4\pi)^{1/2}$, and the expression in square brackets has integer dimensionality. From now on we always assume implicitly that the factor $\mu^{-2\epsilon}$ is attached to some relevant coupling constant, so that the 4D g^2 is dimensionless, while the dimensionalities of $g_E^2, \lambda_E^{(1)}, \lambda_E^{(2)}$ and g_M^2 , to be introduced presently, are GeV.

The basic quantity of interest to us here is minus the free energy density $f_{\text{QCD}}(T)$, or the pressure $p_{\text{QCD}}(T)$, defined by

$$p_{\text{QCD}}(T) \equiv \lim_{V \rightarrow \infty} \frac{T}{V} \ln \int \mathcal{D}A_\mu^a \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\frac{1}{\hbar} S_{\text{QCD}}\right), \quad (2.7)$$

where V denotes the d -dimensional volume. Boundary conditions over the compact time-like direction are periodic for bosons and anti-periodic for fermions. Moreover, we assume $p_{\text{QCD}}(T)$ renormalized such that it vanishes at $T=0$. To simplify the notation, we do not show the infinite volume limit explicitly in the following.

At high temperatures and a small coupling, there are parametrically three different mass scales in the problem, $\sim 2\pi T, gT, g^2 T$ [13]. All the effects of the hard mass scale $\sim 2\pi T$ can be accounted for by a method called dimensional reduction [13,15]. Specifically,

$$p_{\text{QCD}}(T) \equiv p_E(T) + \frac{T}{V} \ln \int \mathcal{D}A_k^a \mathcal{D}A_0^a \exp(-S_E), \quad (2.8)$$

$$S_E = \int d^d x \mathcal{L}_E, \quad (2.9)$$

$$\mathcal{L}_E = \frac{1}{2} \text{Tr } F_{kl}^2 + \text{Tr } [D_k, A_0]^2 + m_E^2 \text{Tr } A_0^2 + \lambda_E^{(1)} (\text{Tr } A_0^2)^2 + \lambda_E^{(2)} \text{Tr } A_0^4 + \dots \quad (2.10)$$

Here $k = 1, \dots, d$, $F_{kl} = (i/g_E)[D_k, D_l]$, $D_k = \partial_k - ig_E A_k$, and we have used the shorthand notation $A_k = A_k^a \bar{T}^a, A_0 = A_0^a \bar{T}^a$, where \bar{T}^a are Hermitean generators of $\text{SU}(N_c)$ normalized such that $\text{Tr } \bar{T}^a \bar{T}^b = \delta^{ab}/2$. Note that the quartic couplings $\lambda_E^{(1)}, \lambda_E^{(2)}$ are linearly independent only for $N_c \geq 4$.

The relation in Eq. (2.8) contains five different matching coefficients, $p_E, m_E^2, g_E^2, \lambda_E^{(1)}, \lambda_E^{(2)}$. We are interested in the expression for $p_{\text{QCD}}(T)$ up to order $\mathcal{O}(g^6 T^4)$. They will then have to be determined to some sufficient depths, as we will specify later on. Let us here note that the leading order magnitudes are $p_E \sim T^4$, $m_E^2 \sim g^2 T^2$, $g_E^2 \sim g^2 T$, $\lambda_E^{(1)} \sim g^4 T$, $\lambda_E^{(2)} \sim g^4 T$.

Apart from the operators shown explicitly in Eq. (2.10), there are of course also higher order ones in \mathcal{L}_E . The lowest such operators have been classified in [16]. Their general structure is that one must add at least two powers of D_k or gA_0 to the basic structures in Eq. (2.10). Since higher order operators are generated through interactions with the scales that have been integrated out, $\sim 2\pi T$, they must also contain an explicit factor of at least g^2 . For dimensional reasons, the schematic structure is thus

$$\delta \mathcal{L}_E \sim g^2 \frac{D_k D_l}{(2\pi T)^2} \mathcal{L}_E. \quad (2.11)$$

To estimate the largest possible contributions such operators could give, let us assume the most conservative possibility that the only dynamical scale in the effective theory is $\sim gT$. By dimensional analysis, we then obtain a contribution

$$\frac{\delta p_{\text{QCD}}(T)}{T} \sim \delta \mathcal{L}_E \sim g^2 \frac{(gT)^2}{(2\pi T)^2} (gT)^3 \sim g^7 T^3. \quad (2.12)$$

Therefore, all higher dimensional operators can be omitted from the action in Eq. (2.10), if we are only interested in computing $p_{\text{QCD}}(T)$ up to order $\mathcal{O}(g^6 T^4)$.

The theory in Eq. (2.10) contains still two dynamical scales, $gT, g^2 T$. All the effects of the ‘‘color-electric’’ scale, gT , can be accounted for by integrating out A_0 [13]. Specifically,

$$\begin{aligned} \frac{T}{V} \ln \int \mathcal{D}A_k^a \mathcal{D}A_0^a \exp(-S_E) &\equiv p_M(T) \\ &+ \frac{T}{V} \ln \int \mathcal{D}A_k^a \exp(-S_M), \end{aligned} \quad (2.13)$$

$$S_M = \int d^d x \mathcal{L}_M, \quad (2.14)$$

$$\mathcal{L}_M = \frac{1}{2} \text{Tr } F_{kl}^2 + \dots, \quad (2.15)$$

where $F_{kl} = (i/g_M)[D_k, D_l]$, $D_k = \partial_k - ig_M A_k$, and $A_k = A_k^a \bar{T}^a$.

The relation in Eq. (2.13) contains two matching coefficients, p_M, g_M^2 , which again have to be determined to sufficient depths. At leading order, $p_M \sim m_E^3 T$, $g_M^2 \sim g_E^2$. In addi-

tion, there are also higher order operators in Eq. (2.15). The lowest ones can be obtained by imagining again that we apply at least two covariant derivatives to Eq. (2.15), together with at least one factor g_E^2 brought in by the interactions with the massive modes. This leads to an operator

$$\delta\mathcal{L}_M \sim g_E^2 \frac{D_k D_l}{m_E^3} \mathcal{L}_M. \quad (2.16)$$

The only dynamical scale in the effective theory being $\sim g^2 T$, dimensional analysis indicates that we then obtain a contribution of the order

$$\frac{\delta p_{\text{QCD}}(T)}{T} \sim \delta\mathcal{L}_M \sim g_E^2 \frac{(g^2 T)^2}{m_E^3} (g^2 T)^3 \sim g^9 T^3. \quad (2.17)$$

Therefore, higher dimensional operators can again be omitted, if we are only interested in the order $\mathcal{O}(g^6 T^4)$ for $p_{\text{QCD}}(T)$.

After the two reduction steps, there still remains a contribution from the scale $g^2 T$:

$$p_G(T) \equiv \frac{T}{V} \ln \int \mathcal{D}A_k^a \exp(-S_M), \quad (2.18)$$

with S_M in Eqs. (2.14), (2.15). Since \mathcal{L}_M only has one parameter, and it is dimensionful, the contribution is of the form

$$p_G(T) \sim T g_M^6. \quad (2.19)$$

The coefficient of this contribution is, however, non-perturbative [1,2].

In the following sections, we proceed in the opposite direction with regard to the presentation above, from the ‘‘bottom’’ scale $g^2 T$, producing $p_G(T)$, through the ‘‘middle’’ scale gT , producing $p_M(T)$, back to the ‘‘top’’ scale $2\pi T$, producing $p_E(T)$. We collect on the way all contributions up to order $g^6 T^4$ to obtain $p_{\text{QCD}}(T) = p_E(T) + p_M(T) + p_G(T)$.

III. CONTRIBUTIONS FROM THE SCALE $g^2 T$

The contribution to $p_{\text{QCD}}(T)$ from the scale $p \sim g^2 T$ is obtained by using the theory \mathcal{L}_M in Eq. (2.15) in order to compute $p_G(T)$, as defined by Eq. (2.18). As is well known [1,2], the computation involves infrared divergent integrals, starting at the 4-loop level. This is a reflection of the fact that \mathcal{L}_M defines a confining field theory. Therefore, $p_G(T)$ cannot be evaluated in perturbation theory.

What can be evaluated, however, is the logarithmic ultraviolet divergence contained in $p_G(T)$. For dimensional reasons, the nonperturbative answer would have to be of the form

$$\frac{p_G(T)}{T\mu^{-2\epsilon}} = d_A C_A^3 \frac{g_M^6}{(4\pi)^4} \left[\alpha_G \left(\frac{1}{\epsilon} + 8 \ln \frac{\bar{\mu}}{2m_M} \right) + \beta_G + \mathcal{O}(\epsilon) \right], \quad (3.1)$$

where $m_M \equiv C_A g_M^2$. Now, because of the super-renormalizability of \mathcal{L}_M , the coefficient α_G can be computed in 4-loop perturbation theory, even if the constant part β_G cannot [29].

Of course, if we just carry out the 4-loop computation in strict dimensional regularization, then the result vanishes, because there are no perturbative mass scales in the problem. This means that ultraviolet and infrared divergences (erroneously) cancel against each other. Therefore, we have to be more careful in order to determine α_G .

To regulate the infrared divergences we introduce by hand a mass scale, m_G^2 , into the gauge field (and ghost) propagators. This computation is described in detail in [9]. Individual diagrams contain then higher order poles, like $1/\epsilon^2$, as well as a polynomial of degree up to nine in the gauge parameter ξ . However, terms of both of these types cancel in the final result, which serves as a nice check of the procedure.

As a result, we obtain

$$\frac{p_G(T)}{T\mu^{-2\epsilon}} \approx d_A C_A^3 \frac{g_M^6}{(4\pi)^4} \left[\alpha_G \left(\frac{1}{\epsilon} + 8 \ln \frac{\bar{\mu}}{2m_G} \right) + \tilde{\beta}_G(\xi) + \mathcal{O}(\epsilon) \right], \quad (3.2)$$

where ‘‘ \approx ’’ is used to denote that only the coefficient α_G multiplying $1/\epsilon$ is physically meaningful, as it contains the desired gauge independent ultraviolet divergence, defined in Eq. (3.1). The value of the coefficient, obtained by extensive use of techniques of symbolic computation (implemented [17] in FORM [18]), is [9]

$$\alpha_G = \frac{43}{96} - \frac{157}{6144} \pi^2 \approx 0.195715. \quad (3.3)$$

On the contrary, the constant part $\tilde{\beta}_G(\xi)$ depends on the gauge parameter ξ , because the introduction of m_G^2 breaks gauge invariance, and has nothing to do with β_G in Eq. (3.1).

IV. CONTRIBUTIONS FROM THE SCALE gT

We next proceed to include the contribution from the scale gT , contained in $p_M(T)$, as defined by Eq. (2.13). By construction, Eq. (2.13) assumes that all the infrared divergences of the expression on the left-hand side are contained in $p_G(T)$, defined in Eq. (2.18), and determined in Eq. (3.1). Therefore, if we compute the functional integral $(T/V) \ln[\int \mathcal{D}A_i^a \mathcal{D}A_0^a \exp(-S_E)]$ using strict dimensional regularization (i.e., without introducing by hand any mass m_G for the gauge field A_i), whereby $p_G(T)$ vanishes due to the cancellation between infrared and ultraviolet divergences mentioned above, we are guaranteed to obtain just the infrared insensitive matching coefficient $p_M(T)$. This is exactly the computation we need, and carry out in [10,19]. It may be mentioned that we have checked explicitly the infrared in-

sensitivity of the result, by giving an equal mass to both A_0 and A_i in the 4-loop expression for the functional integral, and then subtracting the graphs responsible for $p_G(T)$, with the same infrared regularization. This result is also independent of the gauge parameter.

Keeping terms up to order $\mathcal{O}(g^6 T^4)$, the full outcome for $p_M(T)$ is

$$\begin{aligned} \frac{p_M(T)}{T\mu^{-2\epsilon}} = & \frac{1}{(4\pi)} d_A m_E^3 \left[\frac{1}{3} + \mathcal{O}(\epsilon) \right] \\ & + \frac{1}{(4\pi)^2} d_A C_A g_E^2 m_E^2 \left[-\frac{1}{4\epsilon} - \frac{3}{4} - \ln \frac{\bar{\mu}}{2m_E} + \mathcal{O}(\epsilon) \right] \\ & + \frac{1}{(4\pi)^3} d_A C_A^2 g_E^4 m_E \\ & \times \left[-\frac{89}{24} - \frac{1}{6} \pi^2 + \frac{11}{6} \ln 2 + \mathcal{O}(\epsilon) \right] \\ & + \frac{1}{(4\pi)^4} d_A C_A^3 g_E^6 \left[\alpha_M \left(\frac{1}{\epsilon} + 8 \ln \frac{\bar{\mu}}{2m_E} \right) + \beta_M + \mathcal{O}(\epsilon) \right] \\ & + \frac{1}{(4\pi)^2} d_A (d_A + 2) \lambda_E^{(1)} m_E^2 \left[-\frac{1}{4} + \mathcal{O}(\epsilon) \right] \\ & + \frac{1}{(4\pi)^2} d_A \frac{2d_A - 1}{N_c} \lambda_E^{(2)} m_E^2 \left[-\frac{1}{4} + \mathcal{O}(\epsilon) \right], \quad (4.1) \end{aligned}$$

where [10]

$$\alpha_M = \frac{43}{32} - \frac{491}{6144} \pi^2 \approx 0.555017. \quad (4.2)$$

The finite constant β_M can be expressed in terms of a number of finite coefficients related to 4-loop vacuum scalar integrals [10], but we do not need it here.

In addition to $p_M(T)$, we also need to specify the effective parameter g_M^2 appearing in \mathcal{L}_M , to complete contributions from the scale gT . It is of the form

$$g_M^2 = g_E^2 [1 + \mathcal{O}(g_E^2/m_E)], \quad (4.3)$$

where the next-to-leading order correction is known (see, e.g., [20]), but not needed here.

V. CONTRIBUTIONS FROM THE SCALE $2\pi T$

The contributions from the scale $2\pi T$ are contained in the expressions for the parameters of the previous effective theories, as well as in $p_E(T)$. We write these as

$$\begin{aligned} \mu^{2\epsilon} p_E(T) = & T^4 \left(\alpha_{E1} + g^2 [\alpha_{E2} + \mathcal{O}(\epsilon)] + \frac{g^4}{(4\pi)^2} [\alpha_{E3} + \mathcal{O}(\epsilon)] \right. \\ & \left. + \frac{g^6}{(4\pi)^4} [\beta_{E1} + \mathcal{O}(\epsilon)] + \mathcal{O}(g^8) \right), \quad (5.1) \end{aligned}$$

$$\begin{aligned} m_E^2 = & T^2 \left(g^2 [\alpha_{E4} + \alpha_{E5} \epsilon + \mathcal{O}(\epsilon^2)] \right. \\ & \left. + \frac{g^4}{(4\pi)^2} [\alpha_{E6} + \beta_{E2} \epsilon + \mathcal{O}(\epsilon^2)] + \mathcal{O}(g^6) \right), \quad (5.2) \end{aligned}$$

$$g_E^2 = T \left(g^2 + \frac{g^4}{(4\pi)^2} [\alpha_{E7} + \beta_{E3} \epsilon + \mathcal{O}(\epsilon^2)] + \mathcal{O}(g^6) \right), \quad (5.3)$$

$$\lambda_E^{(1)} = T \left(\frac{g^4}{(4\pi)^2} [\beta_{E4} + \mathcal{O}(\epsilon)] + \mathcal{O}(g^6) \right), \quad (5.4)$$

$$\lambda_E^{(2)} = T \left[\frac{g^4}{(4\pi)^2} [\beta_{E5} + \mathcal{O}(\epsilon)] + \mathcal{O}(g^6) \right], \quad (5.5)$$

where g^2 is the renormalized coupling. We have named explicitly (α_E, β_E) the coefficients needed up to order $\mathcal{O}(g^6)$. The actual values for those needed at order $\mathcal{O}[g^6 \ln(1/g)]$, denoted by α_E , are given in Appendix A. The additional coefficients needed at the full order $\mathcal{O}(g^6)$ are denoted by β_E ; some of these are also known (for β_{E4}, β_{E5} , e.g., see [21]). The rest of the terms contribute only beyond $\mathcal{O}(g^6)$.

The expression for $p_E(T)$ is simply the functional integral in Eq. (2.7), calculated to the 4-loop level in the modified minimal subtraction ($\overline{\text{MS}}$) scheme, but without any resummations. The only physical scale entering is thus $2\pi T$. The calculation has so far been carried out only to three loops [6,11] so that β_{E1} is not known. Even when performed with the fully renormalized theory, the results in general contain uncanceled $1/\epsilon$ poles, as explicitly seen in the 3-loop expression in Eq. (A3) for α_{E3} . These only cancel when a physical fully resummed quantity is evaluated, i.e., in the sum $p_{\text{QCD}} = p_E + p_M + p_G$. Similarly, $m_E^2, g_E^2, \lambda_E^{(i)}$ can be obtained, for instance, from suitable 2-, 3-, and 4-point functions, respectively.

VI. THE COMPLETE RESULT

Combining now the results of Secs. III, IV, V and expanding in g , we arrive at

$$\begin{aligned}
\frac{p_{\text{QCD}}(T)}{T^4 \mu^{-2\epsilon}} &= \frac{p_{\text{E}}(T) + p_{\text{M}}(T) + p_{\text{G}}(T)}{T^4 \mu^{-2\epsilon}} \\
&= g^0 \{ \alpha_{\text{E1}} \} + g^2 \{ \alpha_{\text{E2}} \} + \frac{g^3}{(4\pi)} \left\{ \frac{d_A}{3} \alpha_{\text{E4}}^{3/2} \right\} + \frac{g^4}{(4\pi)^2} \left\{ \alpha_{\text{E3}} - d_A C_A \left[\alpha_{\text{E4}} \left(\frac{1}{4\epsilon} + \frac{3}{4} + \ln \frac{\bar{\mu}}{2gT\alpha_{\text{E4}}^{1/2}} \right) + \frac{1}{4} \alpha_{\text{E5}} \right] \right\} \\
&\quad + \frac{g^5}{(4\pi)^3} \left\{ d_A \alpha_{\text{E4}}^{1/2} \left[\frac{1}{2} \alpha_{\text{E6}} - C_A^2 \left(\frac{89}{24} + \frac{\pi^2}{6} - \frac{11}{6} \ln 2 \right) \right] \right\} + \frac{g^6}{(4\pi)^4} \left\{ \beta_{\text{E1}} - \frac{1}{4} d_A \alpha_{\text{E4}} \left[(d_A + 2) \beta_{\text{E4}} + \frac{2d_A - 1}{N_c} \beta_{\text{E5}} \right] \right. \\
&\quad \left. - d_A C_A \left[\frac{1}{4} (\alpha_{\text{E6}} + \alpha_{\text{E5}} \alpha_{\text{E7}} + 3 \alpha_{\text{E4}} \alpha_{\text{E7}} + \beta_{\text{E2}} + \alpha_{\text{E4}} \beta_{\text{E3}}) + (\alpha_{\text{E6}} + \alpha_{\text{E4}} \alpha_{\text{E7}}) \left(\frac{1}{4\epsilon} + \ln \frac{\bar{\mu}}{2gT\alpha_{\text{E4}}^{1/2}} \right) \right] \right. \\
&\quad \left. + d_A C_A^3 \left[\beta_{\text{M}} + \beta_{\text{G}} + \alpha_{\text{M}} \left(\frac{1}{\epsilon} + 8 \ln \frac{\bar{\mu}}{2gT\alpha_{\text{E4}}^{1/2}} \right) + \alpha_{\text{G}} \left(\frac{1}{\epsilon} + 8 \ln \frac{\bar{\mu}}{2g^2 T C_A} \right) \right] \right\} + \mathcal{O}(g^7) + \mathcal{O}(\epsilon). \tag{6.1}
\end{aligned}$$

Utilizing the expressions in Appendix A, the terms up to order $\mathcal{O}(g^5)$ reproduce the known result in [7].

For the contribution at order $\mathcal{O}(g^4)$, the $1/\epsilon$ divergence in α_{E3} [cf. Eq. (A3)] and the $1/\epsilon$ divergence from $p_{\text{M}}(T)$, shown explicitly in Eq. (6.1), cancel. This must happen since $p_{\text{QCD}}(T)$ is a physical quantity. The associated $\bar{\mu}$'s also cancel, but a physical effect $\ln[m_{\text{E}}/(2\pi T)] \sim \ln(g\alpha_{\text{E4}}^{1/2})$ remains [5].

For the contribution at order $\mathcal{O}(g^6)$, a number of unknown coefficients remain (the β_{E} 's, β_{M} , β_{G}), but a similar cancellation is guaranteed to take place. In addition, the result must be scale independent to the order it has been computed. The first point can be achieved by β_{E1} (the other β_{E} 's are finite), so that it has to have the structure

$$\beta_{\text{E1}} \equiv d_A C_A (\alpha_{\text{E6}} + \alpha_{\text{E4}} \alpha_{\text{E7}}) \frac{1}{4\epsilon} - d_A C_A^3 (\alpha_{\text{M}} + \alpha_{\text{G}}) \frac{1}{\epsilon} + \beta_{\text{E6}}, \tag{6.2}$$

where β_{E6} does not contain any $1/\epsilon$ poles. The latter point can be achieved by adding and subtracting $\ln[\bar{\mu}/(2\pi T)]$'s, such that $\bar{\mu}$ gets effectively replaced by $2\pi T$ in the logarithms visible in the $\mathcal{O}(g^6)$ term in Eq. (6.1). The $\ln[\bar{\mu}/(2\pi T)]$'s left over, together with those coming from the β_{E} 's, serve to cancel the effects from the 2-loop running of $g^2(\bar{\mu})$ and 1-loop running of $g^4(\bar{\mu})$ in the lower order contributions, without introducing large logarithms.

This general information is enough to fix the contributions of order $\mathcal{O}(g^6 \ln(1/g))$ to $p_{\text{QCD}}(T)$. Indeed, after inserting Eq. (6.2) and reorganizing the logarithms appearing in the β_{E} 's as mentioned, there remains a logarithmic 4-loop term,

$$\begin{aligned}
\frac{p_{\text{QCD}}(T)}{T^4 \mu^{-2\epsilon}} \Big|_{g^6 \ln(1/g)} &= g^6 \frac{d_A C_A}{(4\pi)^4} \{ (\alpha_{\text{E6}} + \alpha_{\text{E4}} \alpha_{\text{E7}}) \ln(g\alpha_{\text{E4}}^{1/2}) \\
&\quad - 8 C_A^2 [\alpha_{\text{M}} \ln(g\alpha_{\text{E4}}^{1/2}) + 2\alpha_{\text{G}} \ln(gC_A^{1/2})] \}, \tag{6.3}
\end{aligned}$$

where α_{E4} is in Eq. (A4), α_{E6} is in Eq. (A6), α_{E7} is in Eq. (A7), α_{M} is in Eq. (4.2), and α_{G} is in Eq. (3.3). Note that there are logarithms of two types, with different non-analytic dependences on group theory factors inside them. Equation (6.3) is our main result.

Following [7,11], let us finally insert $N_c = 3$, and give also the numerical values for the various coefficients, for an arbitrary N_f . We obtain

$$p_{\text{QCD}}(T) = \frac{8\pi^2}{45} T^4 \left[\sum_{i=0}^6 p_i \left(\frac{\alpha_s(\bar{\mu})}{\pi} \right)^{i/2} \right], \tag{6.4}$$

where

$$p_0 = 1 + \frac{21}{32} N_f, \tag{6.5}$$

$$p_1 = 0, \tag{6.6}$$

$$p_2 = -\frac{15}{4} \left(1 + \frac{5}{12} N_f \right), \tag{6.7}$$

$$p_3 = 30 \left(1 + \frac{1}{6} N_f \right)^{3/2}, \tag{6.8}$$

$$\begin{aligned}
p_4 &= 237.2 + 15.96 N_f - 0.4150 N_f^2 \\
&\quad + \frac{135}{2} \left(1 + \frac{1}{6} N_f \right) \ln \left[\frac{\alpha_s}{\pi} \left(1 + \frac{1}{6} N_f \right) \right] \\
&\quad - \frac{165}{8} \left(1 + \frac{5}{12} N_f \right) \left(1 - \frac{2}{33} N_f \right) \ln \frac{\bar{\mu}}{2\pi T}, \tag{6.9}
\end{aligned}$$

$$p_5 = \left(1 + \frac{1}{6}N_f\right)^{1/2} \left[-799.1 - 21.96N_f - 1.926N_f^2 + \frac{495}{2} \left(1 + \frac{1}{6}N_f\right) \left(1 - \frac{2}{33}N_f\right) \ln \frac{\bar{\mu}}{2\pi T} \right], \quad (6.10)$$

$$p_6 = \left[-659.2 - 65.89N_f - 7.653N_f^2 + \frac{1485}{2} \left(1 + \frac{1}{6}N_f\right) \times \left(1 - \frac{2}{33}N_f\right) \ln \frac{\bar{\mu}}{2\pi T} \right] \ln \left[\frac{\alpha_s}{\pi} \left(1 + \frac{1}{6}N_f\right) \right] - 475.6 \ln \frac{\alpha_s}{\pi} + q_a(N_f) \ln^2 \frac{\bar{\mu}}{2\pi T} + q_b(N_f) \ln \frac{\bar{\mu}}{2\pi T} + q_c(N_f), \quad (6.11)$$

where $q_a(N_f)$, $q_b(N_f)$, $q_c(N_f)$ are α_s -independent polynomials in N_f . Two of them, $q_a(N_f)$, $q_b(N_f)$, can already be written down because they just cancel the $\bar{\mu}$ dependence arising from the terms of orders $\alpha_s(\bar{\mu})$, $\alpha_s^2(\bar{\mu})$:

$$q_a(N_f) = -\frac{1815}{16} \left(1 + \frac{5}{12}N_f\right) \left(1 - \frac{2}{33}N_f\right)^2, \quad (6.12)$$

$$q_b(N_f) = 2932.9 + 42.83N_f - 16.48N_f^2 + 0.2767N_f^3. \quad (6.13)$$

The third one, $q_c(N_f)$, remains, however, unknown.

VII. THE NUMERICAL CONVERGENCE

This section is devoted to a numerical discussion of the result. Since the $\mathcal{O}(g^6 \ln(1/g))$ term cannot be given an unambiguous numerical meaning until the $\mathcal{O}(g^6)$ term is specified, we have to present the result for various choices of the latter. In the relevant range of $T/\Lambda_{\overline{\text{MS}}}$ the outcome will de-

pend sensitively, even qualitatively, on this uncomputed term. One choice will be seen to agree with 4D lattice data down to about $T/\Lambda_{\overline{\text{MS}}} \sim 2 \dots 3$. Since, however, dimensional reduction, that is, an effective description of QCD via the theory in Eq. (2.10), is known to break down at about this point, and we have only kept a finite number of terms in the expansion following from Eq. (2.10), this cannot really be considered a prediction, even if the eventual computation of the $\mathcal{O}(g^6)$ term gave just the appropriate value. It is just an observation that a smooth transition from the domain of validity of our results to a domain of different approximations should be possible.

A standard procedure in the discussion of perturbative results would be to take the expansion in Eq. (6.4) and to study whether its scale dependence is reduced when further orders of perturbation theory are included. As is well known since [6], this fails for the pressure, unless $T \gg \Lambda_{\overline{\text{MS}}}$. Related to this, the numerical convergence of the perturbative expansion is known to be quite poor for any fixed scale choice, at least for temperatures below the electroweak scale [7,11,12]. The new term we have computed does not change this general pattern. But the culprit is known: it is $p_M(T) + p_G(T)$ emerging from the 3D sector of the theory, where the expansion parameter is only $g_E^2/(\pi m_E) \sim g/\pi$. In contrast, for $p_E(T)$ as well as for, say, jet physics, the expansion parameter is α_s/π , and there are good reasons to expect numerical convergence to be much better.

For these reasons, we will only discuss the sensitivity of the result on the so far unknown $\mathcal{O}(g^6)$ coefficient, as well as the slow convergence of the 3D sector, in the following. For simplicity, we only consider the case $N_c=3$, $N_f=0$ here.

As in [14], the actual form we choose for plotting contains $p_M(T) + p_G(T)$ [Eqs. (4.1)+(3.1)] in an “unexpanded” form, that is, with m_E , g_E^2 inserted from Eqs. (5.2), (5.3), and g_M^2 from Eq. (4.3). This means that we are effectively summing up higher orders: the $\mathcal{O}(g^3)$ term is really $\mathcal{O}(g^2 + g^4)^{3/2}$, while the $\mathcal{O}(g^6 \ln(1/g))$ term contains a

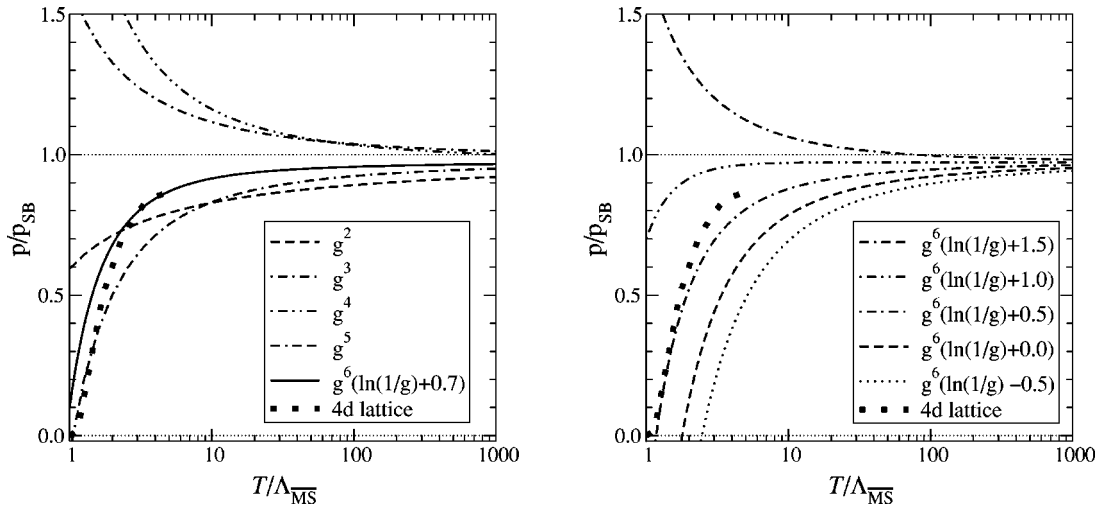


FIG. 1. Left: perturbative results at various orders (the precise meanings thereof are explained in Sec. VII), including $\mathcal{O}(g^6)$ for an optimal constant, normalized to the non-interacting Stefan-Boltzmann value p_{SB} . Right: the dependence of the $\mathcal{O}(g^6)$ result on the (not yet computed) constant, which contains both perturbative and nonperturbative contributions. The 4D lattice results are from [22].

resummed coefficient, being then effectively $\mathcal{O}((g^2 + g^4)^3 \ln(1/g))$. We proceed in this way because then a comparison with numerical determinations [14] of the slowly convergent part $p_M(T) + p_G(T)$ is more straightforward, and also because the resummations carried out reduce the $\bar{\mu}$ dependence of the outcome. However, we have checked that the practical conclusions remain the same even if we plot directly the expression in Eqs. (6.4)–(6.11) (but with a larger scale dependence).

To be specific, the genuine $\mathcal{O}(g^6 \ln(1/g) + g^6)$ contribution, which collects the effects from all the terms involving the β_E 's, β_M , β_G , α_M , and α_G in Eq. (6.1), is now written in the form (specific for $N_c=3, N_f=0$, where $m_E/g_E^2 \sim 1/g$),

$$\delta \left[\frac{p_{\text{QCD}}(T)}{T \mu^{-2\epsilon}} \right]_{g^6 \ln(1/g)} \equiv 8d_A C_A^3 \frac{g_E^6}{(4\pi)^4} \times \left[(\alpha_M + 2\alpha_G) \ln \frac{m_E}{g_E^2} + \delta \right], \quad (7.1)$$

while the remaining $\mathcal{O}(g^6)$ terms of Eq. (6.1) are contained in the resummed lower order contributions. The results are shown in Fig. 1 for various values of δ . The power of g labelling the curves indicates the leading magnitude of the highest order resummed contribution appearing. The scale is chosen as $\bar{\mu} \approx 6.7T$, as suggested by the next-to-leading order expression for g_E^2 [12]. We observe that for a specific value of δ , the curve extrapolates well to 4D lattice data.

While Fig. 1 looks tempting, the question still remains whether the good match to 4D lattice data with a specific value of the constant is simply a coincidence. This issue can be fully settled only once the constant is actually computed. However, we can already inspect how the slowly convergent part of the pressure, $p_M + p_G$, really behaves.

The different finite terms in $(p_M + p_G)/(Tg_E^6)$ are plotted in Fig. 2. The $\lambda_E^{(i)}$ contributions are negligible. The results depend then essentially only on m_E^2/g_E^4 , which for $N_c=3, N_f=0$ is $m_E^2/g_E^4 \approx 0.32 \log_{10}(T/\Lambda_{\overline{\text{MS}}}) + 0.29$. We observe that the leading 1-loop term $\mathcal{O}(g^3)$ is dominant for $T/\Lambda_{\overline{\text{MS}}} \gtrsim 10$, the 3-loop term $\mathcal{O}(g^5)$ is rather big, bigger in absolute value than the 2-loop term $\mathcal{O}(g^4)$ within the T -range of the figure, while the 4-loop term is always very small. Therefore, while it is quite possible that there is again a big ‘‘odd’’ $\mathcal{O}(g^7)$ contribution, it is perhaps not completely outrageous either to hope that the convergence could also already be reasonable, once the full $\mathcal{O}(g^6)$ contribution is included. If this were the case, then all higher order contributions would have to sum up to a small number.

Finally, it is perhaps interesting to remark that at the time of the numerical lattice Monte Carlo study in Ref. [14], nothing was known about the coefficient β_{E1} , which was therefore set to zero [cf. Eq. (4) in [14]], while the part $p_M(T) + p_G(T)$ was determined nonperturbatively. But this means

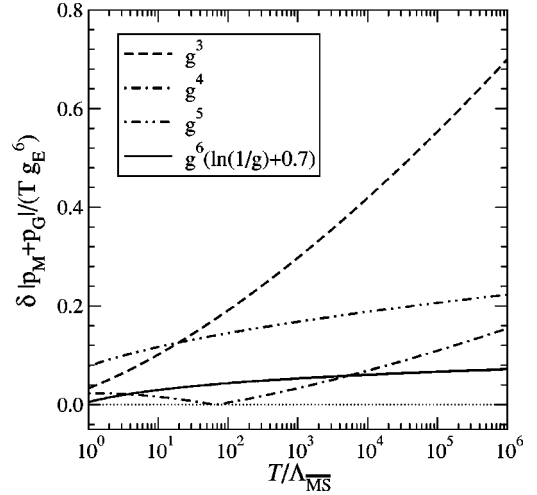


FIG. 2. The absolute values of the various terms of the slowly convergent expansion for $p_M(T) + p_G(T)$, normalized by Tg_E^6 .

that a logarithmic term coming from the scale $2\pi T$, $\sim -g^6(\alpha_M + \alpha_G) \ln[\bar{\mu}/(2\pi T)]$, was missed. With the scale choice $\bar{\mu} \equiv \bar{\mu}_E = g_E^2$ within results obtained with \mathcal{L}_E , this converted to a missing $\mathcal{O}(g^6 \ln(1/g))$ contribution $g^6(2\alpha_M + 2\alpha_G) \ln(1/g)$. With the same scale choice the nonperturbative part, on the other hand, contributed $-g^6 \alpha_M \ln(1/g)$ and led to the wrong curvature of the pressure seen at small $T/\Lambda_{\overline{\text{MS}}}$. Adding the missing part, which now has been computed, leads to a total of $g^6(\alpha_M + 2\alpha_G) \ln(1/g)$, with the opposite sign and the correct (i.e., the one seen in 4D lattice measurements) curvature in Fig. 1 (for small values of δ). Therefore the $\mathcal{O}(g^6 \ln(1/g))$ terms are indeed physically very relevant.

VIII. CONCLUSIONS

We have addressed in this paper the 4-loop logarithmic contributions to the pressure of hot QCD. Physical (regularization independent) logarithms can only arise from a ratio of two scales. Since there are three parametrically different scales in the system, $2\pi T, gT, g^2T$, there are then various types of perturbatively computable logarithms in the 4-loop expression for the pressure:

- (1) Logarithms of the type $g^6 \ln[(2\pi T)/(g^2 T)]$. The coefficient of these is computed in [9], and given in Eq. (3.3).
- (2) Logarithms of the type $g^6 \ln[(2\pi T)/(gT)]$. The coefficient of these is computed in [10], and given in Eq. (4.2).
- (3) Logarithms related to the running of the coupling constant in the 3-loop expression of order $\mathcal{O}(g^4 \ln[(2\pi T)/(gT)])$. Their $\overline{\text{MS}}$ coefficient can be seen in the first term in Eq. (6.3), but it depends on the scheme, and can in principle even be chosen to vanish.

Logarithms of the first and second types can be written in many ways: it may be more intuitive, for instance, to reorganize them as

$$\begin{aligned}
& g^6 \alpha_G \ln \left(\frac{2\pi T}{g^2 T} \right) + g^6 \alpha_M \ln \left(\frac{2\pi T}{gT} \right) \\
& = g^6 (\alpha_M + \alpha_G) \ln \left(\frac{2\pi T}{gT} \right) + g^6 \alpha_G \ln \left(\frac{gT}{g^2 T} \right).
\end{aligned} \tag{8.1}$$

The existence of three kinds of logarithms is somewhat specific to non-Abelian gauge theory. In QED, in particular, none of the logarithms appear. This is due to the fact that the effective theories we have used for their computation, Eqs. (2.10), (2.15), are non-interacting [apart from a term $\sim A_0^4$ in Eq. (2.10), which does not lead to logarithms]. Therefore we have nothing to add to the known $\mathcal{O}(g^5)$ QED result obtained in [23]. In the ϕ^4 scalar theory, on the other hand, there is a logarithm of the second type, and also one somewhat analogous to the third type. Their coefficients were already computed in [24].

There are interesting checks that can be made on the various logarithms mentioned, using methods completely different from those employed here. For instance, logarithms of the first and second types could in principle be seen with 3D lattice Monte Carlo methods [25,26], as well as with stochastic perturbation theory [27]. A very interesting analytical check would be to compute the 4-loop free energy directly in 4D in strict dimensional regularization, but without any resummation. By definition, this computation produces the coefficient β_{E1} in Eq. (5.1) [11], and one check is that the result must contain the $1/\epsilon$ divergences shown in Eq. (6.2).

To complete the free energy from the current level $\mathcal{O}(g^6 \ln(1/g))$ to the full level $\mathcal{O}(g^6)$ would require significantly more work than the computation presented here. More specifically, there are contributions from all the scales in the problem, ranging from $2\pi T$ (the coefficients $\beta_{E1}, \dots, \beta_{E5}$), through gT (the coefficient β_M), down to the non-perturbative scale $g^2 T$ (the coefficient β_G). This then requires carrying out 4-loop finite temperature sum-integrals, 4-loop vacuum integrals in $d=3-2\epsilon$, 4-loop vacuum integrals in 3D lattice regularization, and lattice simulations of the pure 3D gauge theory in Eq. (2.15). Nevertheless, given the potentially important combined effect of all these contributions, as indicated by Fig. 1, such computations would clearly be well motivated.

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APPENDIX: MATCHING COEFFICIENTS

In Eqs. (5.1)–(5.5) we have defined a number of matching coefficients, the α_E 's and β_E 's. For the α_E 's, the following

expressions can be extracted from [11,15,28]:

$$\alpha_{E1} = \frac{\pi^2}{180} (4d_A + 7d_F), \tag{A1}$$

$$\alpha_{E2} = -\frac{d_A}{144} \left(C_A + \frac{5}{2} T_F \right), \tag{A2}$$

$$\begin{aligned}
\alpha_{E3} = & \frac{d_A}{144} \left[C_A^2 \left(\frac{12}{\epsilon} + \frac{194}{3} \ln \frac{\bar{\mu}}{4\pi T} + \frac{116}{5} + 4\gamma + \frac{220}{3} \frac{\zeta'(-1)}{\zeta(-1)} \right. \right. \\
& \left. \left. - \frac{38}{3} \frac{\zeta'(-3)}{\zeta(-3)} \right) \right. \\
& + C_A T_F \left(\frac{12}{\epsilon} + \frac{169}{3} \ln \frac{\bar{\mu}}{4\pi T} + \frac{1121}{60} - \frac{157}{5} \ln 2 + 8\gamma \right. \\
& \left. + \frac{146}{3} \frac{\zeta'(-1)}{\zeta(-1)} - \frac{1}{3} \frac{\zeta'(-3)}{\zeta(-3)} \right) \\
& + T_F^2 \left(\frac{20}{3} \ln \frac{\bar{\mu}}{4\pi T} + \frac{1}{3} - \frac{88}{5} \ln 2 + 4\gamma + \frac{16}{3} \frac{\zeta'(-1)}{\zeta(-1)} \right. \\
& \left. \left. - \frac{8}{3} \frac{\zeta'(-3)}{\zeta(-3)} \right) + C_F T_F \left(\frac{105}{4} - 24 \ln 2 \right) \right], \tag{A3}
\end{aligned}$$

$$\alpha_{E4} = \frac{1}{3} (C_A + T_F), \tag{A4}$$

$$\begin{aligned}
\alpha_{E5} = & \frac{2}{3} \left[C_A \left(\ln \frac{\bar{\mu}}{4\pi T} + \frac{\zeta'(-1)}{\zeta(-1)} \right) \right. \\
& \left. + T_F \left(\ln \frac{\bar{\mu}}{4\pi T} + \frac{1}{2} - \ln 2 + \frac{\zeta'(-1)}{\zeta(-1)} \right) \right], \tag{A5}
\end{aligned}$$

$$\begin{aligned}
\alpha_{E6} = & C_A^2 \left(\frac{22}{9} \ln \frac{\bar{\mu} e^\gamma}{4\pi T} + \frac{5}{9} \right) + C_A T_F \left(\frac{14}{9} \ln \frac{\bar{\mu} e^\gamma}{4\pi T} - \frac{16}{9} \ln 2 + 1 \right) \\
& + T_F^2 \left(-\frac{8}{9} \ln \frac{\bar{\mu} e^\gamma}{4\pi T} - \frac{16}{9} \ln 2 + \frac{4}{9} \right) - 2 C_F T_F, \tag{A6}
\end{aligned}$$

$$\alpha_{E7} = C_A \left(\frac{22}{3} \ln \frac{\bar{\mu} e^\gamma}{4\pi T} + \frac{1}{3} \right) - T_F \left(\frac{8}{3} \ln \frac{\bar{\mu} e^\gamma}{4\pi T} + \frac{16}{3} \ln 2 \right). \tag{A7}$$

Note that with our notation, the 1-loop running of the renormalized coupling constant goes as

$$g^2(\bar{\mu}) = g^2(\bar{\mu}_0) - \frac{2}{3} (11C_A - 4T_F) \frac{g^4(\bar{\mu}_0)}{(4\pi)^2} \ln \frac{\bar{\mu}}{\bar{\mu}_0}. \tag{A8}$$

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