Existence of spinning solitons in gauge field theory

Mikhail S. Volkov*

Laboratoire de Mathématiques et Physique Théorique, Université de Tours, Parc de Grandmont, 37200 Tours, France

Erik Wöhnert[†]

Theoretisch-Physikalisches Institut, Friedrich Schiller Universität Jena, Fröbelstieg 1, 07743 Jena, Germany

We study the existence of classical soliton solutions with intrinsic angular momentum in Yang-Mills-Higgs theory with a compact gauge group $\mathcal{G}$ in $(3+1)$-dimensional Minkowski space. We show that for symmetric gauge fields the Noether charges corresponding to rigid spatial symmetries, as the angular momentum, can be expressed in terms of surface integrals. Using this result, we demonstrate in the case of $\mathcal{G}=SU(2)$ the nonexistence of stationary and axially symmetric spinning excitations for all known topological solitons in the one-soliton sector, that is, for 't Hooft–Polyakov monopoles, Julia-Zee dyons, sphalerons, and also vortices.

I. INTRODUCTION

The existence of globally regular soliton solutions with a nonvanishing angular momentum in classical field theory is an interesting open issue, which has recently been addressed in a number of publications [1–7]. Up to now, such spinning solutions in Minkowski space have been found only in the theory of a self-interacting complex scalar field (Q balls) [5].[1] For these solutions the energy-momentum tensor is stationary and axially symmetric, while the angular momentum $J \sim \omega N$ is generated by the rotating phase of the scalar field $\Phi = \phi(r, \vartheta) e^{-i\omega t + iN\varphi}$.

It is natural to wonder whether rotating solitons can also exist in gauge field theories with spontaneously broken symmetries. For stationary, axially symmetric systems the rotating phase of the Higgs field can be gauged away.[2] A nonzero angular momentum could then be supported only by the Poynting vector of the gauge field, and in fact such solutions can indeed be obtained, as for example rotating [3,4] monopole-antimonopole pairs [8–10]. However, the rotation is then rather associated with the orbital motion in a many-body system. The real challenge is to construct rotating solutions in the *one*-soliton sector, where the rotation would indeed be associated with spinning excitations of an individual object. For some strange reason, up to now such spinning solitons have been found only in anti–de Sitter space [6,7], while their possible existence in Minkowski space remains rather obscure. In fact, the results obtained so far in this area have all been negative. For example, it has been shown that 't Hooft–Polyakov monopoles and Julia-Zee dy-

ons in Minkowski space cannot rotate slowly [3]. The same conclusion holds for gravitating monopoles and sphalerons [12]. In addition, it was noticed in [4] that for axially symmetric deformations of Julia-Zee dyons the angular momentum can be represented as a flux integral, a fact that was used in [4] to argue that dyons cannot rotate rapidly either.

In this paper we study the existence of spinning solitons in the context of Yang-Mills-Higgs (YMH) theory for an arbitrary compact gauge group $\mathcal{G}$ in $(3+1)$-dimensional Minkowski space. First of all, we analyze the observation of [4] that the angular momentum of Julia-Zee dyons can be expressed as a flux integral. It is natural to wonder why such a representation of the angular momentum exists at all and whether it can be generalized to other models. Usually, conserved quantities associated with Poincaré symmetries in Minkowski space, such as, for example, the energy, are given by volume integrals and not surface integrals. We therefore study the relationship between conservation laws, spacetime, and gauge symmetries, and what we find is the following. For *symmetric* gauge fields, the action of a *rigid* spacetime symmetry generated by a Killing vector X is equivalent to that of a *local* gauge symmetry generated by a Lie algebra valued function W_X. It is then a consequence of the Bianchi identities imposed by the local gauge symmetry that the Noether current for the global Poincaré symmetry is essentially a total divergence. In the case of *spatial* symmetries, the Noether charge can then be expressed by a surface integral

$$\Theta_X = \oint \langle (A_X - W_X) F^{0k} \rangle dS_k, \qquad (1)$$

where A_X is the X projection of the gauge field. In the case of spatial rotations $X = \partial/\partial\varphi$, this gives the conserved and gauge invariant angular momentum.

Making use of this representation of the angular momentum, we then systematically study the fields in the asymptotic region near spatial infinity, looking for field modes that could give a contribution to the surface integral. In this way we show that for 't Hooft–Polyakov monopoles and Julia-Zee dyons there are no stationary, axially symmetric deformations giving a nonzero contribution to the angular

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*Electronic address: volkov@phys.univ-tours.fr

[†]Electronic address: pew@tpi.uni-jena.de

[1]In curved space similar rotating solutions are known for a self-gravitating scalar field (boson stars) [1,2].

[2]It is not excluded that the action could be invariant under time translations and axial rotations while the fields are not stationary and axially symmetric. In such a case it would not be possible to gauge away the rotating phases. Such a possibility, however, is beyond the scope of our present consideration.

momentum. We then carry out a similar analysis for sphalerons and also for vortices—with the same conclusion. As a result, we in fact show the absence of stationary and axially symmetric spinning excitations in the one-soliton sector for all known topological solitons with gauge group $\mathcal{G} = SU(2)$. The still remaining possibilities of constructing rotating solutions can then be related only either to studying solutions with higher gauge groups or to considering fields that are not manifestly stationary or axially symmetric.

II. YANG-MILLS-HIGGS THEORY

The theory under consideration is a Yang-Mills-Higgs theory with compact gauge group $\mathcal{G}$ defined by the action

$$S_{\mathrm{YMH}} = \int \mathcal{L} d^4 x, \qquad (2)$$

where

$$\mathcal{L} = -\frac{1}{4} \langle F_{\mu\nu} F^{\mu\nu} \rangle + \frac{1}{2} (\mathcal{D}_\mu \Phi)^\dagger \mathcal{D}^\mu \Phi - \frac{\lambda}{4} (\Phi^\dagger \Phi - 1)^2. \qquad (3)$$

Here, the gauge field strength $F_{\mu\nu} \equiv T_a F^a_{\ \mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ with the gauge field $A_\mu \equiv T_a A^a_\mu$. The anti-Hermitian gauge group generators T_a $(a = 1, 2, \ldots, \dim \mathcal{G})$ satisfy the relations

$$[T_a, T_b] = f_{abc} T_c, \quad \mathrm{tr}(T_a T_b) = K \delta_{ab}. \qquad (4)$$

The invariant scalar product in the Lie algebra is defined as $\langle AB \rangle = (1/K) \mathrm{tr}(AB)$. The Higgs field Φ is a vector in the representation space of $\mathcal{G}$ where the generators T_a act; this space can be complex or real. $\mathcal{D}_\mu \Phi = (\partial_\mu + A_\mu)\Phi$ is the covariant derivative of the Higgs field. The units are chosen such that the gauge coupling constant and the vacuum value of the Higgs field are equal to 1. Spacetime indices are lifted with the Minkowski metric.

Below, we will consider two important particular cases corresponding to $\mathcal{G} = SU(2)$. The Higgs field Φ then will be chosen to be either in the real triplet representation, in which case

$$(T_a)_{ik} = -\varepsilon_{aik}, \qquad (5)$$

or in the complex doublet representation with

$$T_a = \frac{\tau_a}{2i}, \qquad (6)$$

where τ_a are the Pauli matrices.

The action is invariant under gauge transformations

$$A_\mu \rightarrow U(A_\mu + \partial_\mu) U^{-1}, \quad \Phi \rightarrow U\Phi, \qquad (7)$$

where U is a $\mathcal{G}$ valued function. Varying the action with respect to the gauge and Higgs fields gives the equations of motion

$$D_\sigma F^{\sigma\mu} = \frac{1}{2} [\Phi^\dagger T_a \mathcal{D}^\mu \Phi - (\mathcal{D}^\mu \Phi)^\dagger T_a \Phi] T_a, \qquad (8)$$

$$\mathcal{D}_\mu \mathcal{D}^\mu \Phi = -\lambda (\Phi^\dagger \Phi - 1)\Phi, \qquad (9)$$

where $D_\mu = \partial_\mu + [A_\mu, \ ]$ is the covariant derivative in the adjoint representation.

In what follows, we will consider stationary, axially symmetric fields subject to the symmetry conditions [13]

$$\mathcal{L}_{\xi_m} A_\mu = D_\mu W_m, \quad \mathcal{L}_{\xi_m} \Phi = -W_m \Phi, \quad m = t, \varphi. \qquad (10)$$

Here, $\mathcal{L}_{\xi_m}$ are the Lie derivatives along the two Killing vectors $\xi_t = \partial_t$ and $\xi_\varphi = \partial_\varphi$, while W_m are compensating Lie algebra valued functions. The general solution of these equations is well known: since the two Killing vectors commute, there exists a gauge where $W_m = 0$. Therefore, the symmetry conditions in this gauge require simply the independence from t and φ. As a result, the most general solution is

$$A_\mu = T_a A^a_\mu(\rho, z) dx^\mu, \quad \Phi = \Phi(\rho, z). \qquad (11)$$

The regularity on the symmetry axis requires that

$$A_\varphi(0, z) = f(z) T, \qquad (12)$$

where T is an element of the Cartan subalgebra of the Lie algebra of $\mathcal{G}$ and $f(z)$ is a bounded function. Passing to a new gauge with the gauge transformation $U = e^{-\varphi f(z) T}$ will then send $A_\varphi(0, z)$ to zero.

III. NOETHER CHARGES AS FLUX INTEGRALS

Conserved quantities in field theory are determined by Noether charges corresponding to global symmetries of the action. These charges can be expressed as volume integrals of the local charge densities. In some cases, such as, for example, for the electric charge, these volume integrals can be further transformed to surface integrals. The reason for this is as follows (see [14] for a discussion). Electric charge is conserved owing to the invariance under global phase rotations. In gauge field theory, this symmetry is a special case of the local gauge invariance. The local gauge invariance leads to the existence of identity relations between the field equations (Bianchi identities) and implies the identical conservation of Noether's currents, since they can be represented as divergences of antisymmetric quantities (sometimes called superpotentials)

$$\Theta^\mu = \partial_\sigma [\omega(x) \mathcal{F}^{\sigma\mu}]. \qquad (13)$$

Here, $\omega(x)$ is the parameter of local gauge transformations, the case of global phase rotations corresponding to constant ω's. Since Θ^0 is a total divergence, the Noether charge can be expressed as a surface integral.

The procedure described above is very well illustrated in the context of general relativity, where the conserved energy, momentum, and angular momentum are given by flux integrals. This can be traced back to the fact that the Poincaré symmetries are a special case of general spacetime diffeo-

morphisms. For theories in Minkowski space, on the other hand, there is no diffeomorphism invariance, and so Poincaré symmetries are not related to any local symmetries. As a result, the energy, for example, cannot be expressed as a flux integral. However, for *symmetric* gauge fields some of the spacetime symmetries can be equivalent to local gauge symmetries in the sense that the result of Poincaré transformations can be compensated by gauge transformations. As a result, the corresponding Noether charges will have essentially the same structure as in Eq. (13), and the Noether charges can be expressed as flux integrals. We will now show how this works in the context of YMH theory.

It is well known [15,16] that in the presence of gauge invariance spacetime symmetries must be combined with the internal gauge symmetries in order to give conserved and *gauge invariant* charges via Noether's procedure. If X^μ is a Killing vector of the system[3] then the corresponding conserved and gauge invariant Noether current is

$$\Theta^\mu = \sum_B \frac{\partial \mathcal{L}}{\partial(\partial_\mu u^B)} \delta u^B - X^\mu \mathcal{L}. \tag{14}$$

Here, u^B collectively denotes the fields $(A_\mu, \Phi, \Phi^\dagger)$, and the variations δu^B include the part generated by X^μ plus another part due to an infinitesimal gauge transformation generated by a Lie algebra valued function W:

$$\delta u^B = \mathcal{L}_X u^B - \delta_W u^B. \tag{15}$$

Here, the Lie derivatives are

$$\mathcal{L}_X A_\mu = X^\alpha \partial_\alpha A_\mu + A^\alpha \partial_\alpha X_\mu, \quad \mathcal{L}_X \Phi = X^\alpha \partial_\alpha \Phi, \tag{16}$$

while the gauge variations are given by

$$\delta_W A_\mu = D_\mu W, \quad \delta_W \Phi = -W\Phi. \tag{17}$$

The function W is determined by the requirement that the variations δu^B transform under gauge transformations *covariantly*, thus ensuring the gauge invariance of the Noether current. Using the identity [15,16]

$$\mathcal{L}_X A_\mu = X^\alpha F_{\alpha\mu} + D_\mu(X^\alpha A_\alpha), \tag{18}$$

one obtains

$$\delta A_\mu = X^\alpha F_{\alpha\mu} + D_\mu(X^\alpha A_\alpha - W),$$

$$\delta \Phi = X^\alpha \mathcal{D}_\alpha \Phi - (X^\alpha A_\alpha - W), \tag{19}$$

which shows that the transformation law for W must be

$$W \rightarrow U(W + X^\sigma \partial_\sigma)U^{-1}, \tag{20}$$

since then $(X^\alpha A_\alpha - W)$ transforms covariantly. Having this in mind and inserting Eqs. (15)–(17) into Eq. (14), one obtains after straightforward calculations

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[3]Thus, one has $\partial_\mu X_\nu + \partial_\nu X_\mu = 0$.

$$\Theta^\mu = X^\alpha T_\alpha^\mu + \partial_\sigma \langle (X^\nu A_\nu - W) F^{\sigma\mu} \rangle. \tag{21}$$

Here the tensor

$$T_\nu^\mu = -\langle F^{\mu\sigma} F_{\nu\sigma} \rangle + \frac{1}{2}[(\mathcal{D}^\mu \Phi)^\dagger \mathcal{D}_\nu \Phi + (\mathcal{D}_\nu \Phi)^\dagger \mathcal{D}^\mu \Phi] - \delta_\nu^\mu \mathcal{L} \tag{22}$$

coincides with the metrical energy-momentum tensor obtained by varying the action with respect to the spacetime metric. This tensor is symmetric and divergence-free, $\partial_\mu T^{\mu\nu} = 0$.

The Noether current (21) is conserved and gauge invariant. However, it is not yet completely defined, since W is not uniquely determined by the condition (20). This reflects the well-known ambiguity in the definition of Noether currents, as they can always be changed by adding the divergence of an antisymmetric tensor. The way to uniquely define the Noether currents (see, for example, [15–17]) is dictated by the agreement with the general relativity (GR), since they should coincide with the conserved currents obtained from the metrical energy-momentum tensor. The *canonical* Noether energy-momentum tensor will then be symmetric and will coincide with the metrical one. All this is achieved if only one chooses [15,16]

$$W = X^\alpha A_\alpha \tag{23}$$

[notice that this transforms according to Eq. (20)] in order to get rid of the second term on the right in Eq. (21). The Noether current then becomes

$$\Theta^\mu = X^\alpha T_\alpha^\mu. \tag{24}$$

This coincides with the standard GR current and leads to the conserved charge expressed by the volume integral over the three-space,

$$\Theta_X = \int X^\alpha T_\alpha^0 d^3 x. \tag{25}$$

This formula reproduces the known result for the conserved and gauge invariant Noether charge associated with a rigid Poincaré symmetry X^μ [15].

Let us now repeat the calculation above by assuming that the symmetry generated by X^μ is not only a symmetry of the *action*, but also a symmetry of the *fields*, in the sense that there exists a Lie algebra valued function W_X such that

$$\mathcal{L}_X A_\mu = D_\mu W_X, \quad \mathcal{L}_X \Phi = -W_X \Phi. \tag{26}$$

Substituting this into Eq. (15) and using Eq. (17) gives

$$\delta A_\mu = -D_\mu \Psi_X, \quad \delta \Phi = \Psi_X \Phi, \tag{27}$$

where $\Psi_X = W - W_X = X^\alpha A_\alpha - W_X$. Therefore, the field variations generated by X^μ can in this case be viewed as *pure gauge* variations. Inserting Eq. (27) into Eq. (14) gives

$$\Theta^{\mu}=\langle F^{\mu\alpha}D_{\alpha}\Psi_{X}\rangle+\frac{1}{2}(\mathcal{D}^{\mu}\Phi)^{\dagger}\Psi_{X}\Phi-\frac{1}{2}\Phi^{\dagger}\Psi_{X}\mathcal{D}^{\mu}\Phi$$

$$-X^{\mu}\mathcal{L}, \tag{28}$$

and using the equations of motion (8) this reduces to

$$\Theta^{\mu}=-\partial_{\alpha}\langle\Psi_{X}F^{\alpha\mu}\rangle-X^{\mu}\mathcal{L}. \tag{29}$$

This almost has the structure of an identically conserved current, if it were not for the last term. This term is the remnant of the fact that the symmetries under consideration, although closely related to gauge symmetries, are actually spacetime symmetries. Now, if the vector X^{μ} is *spacelike*, as is the case for strictly spatial translations and rotations, then there exist reference frames where the temporal component X^{0} vanishes. As a result, Θ^{0} is a total divergence and its integral over the spatial hypersurface can be transformed into a surface integral (provided that there is no contribution from the inner boundary). The conserved and gauge invariant Noether charge is then given by the flux integral over a closed two-surface at spatial infinity:

$$\Theta_{X}=-\oint\langle\Psi_{X}F^{k0}\rangle dS_{k}. \tag{30}$$

This is the main result of this section. It shows that the Noether charges associated with *rigid* spatial symmetries can be represented as flux integrals when the fields under consideration are *symmetric*.

It is instructive to see how the general Noether current (24) assumes the special form (29) when the symmetry conditions (26) are imposed. One has

$$\Theta^{\mu}=X^{\alpha}T^{\mu}_{\alpha}=-X^{\alpha}\langle F^{\mu\sigma}F_{\alpha\sigma}\rangle+\frac{1}{2}X^{\alpha}[(\mathcal{D}^{\mu}\Phi)^{\dagger}\mathcal{D}_{\alpha}\Phi$$

$$+(\mathcal{D}_{\alpha}\Phi)^{\dagger}\mathcal{D}^{\mu}\Phi]-X^{\mu}\mathcal{L}. \tag{31}$$

Using Eqs. (27), (19), and (23), one obtains

$$F_{\sigma\mu}X^{\mu}=D_{\sigma}\Psi_{X}, \quad X^{\mu}\mathcal{D}_{\mu}\Phi=\Psi_{X}\Phi. \tag{32}$$

As a result, the first term in Eq. (31) can be transformed as

$$-X^{\alpha}\langle F^{\mu\sigma}F_{\alpha\sigma}\rangle=-\langle F^{\sigma\mu}D_{\sigma}\Psi_{X}\rangle$$

$$=-\langle D_{\sigma}(F^{\sigma\mu}\Psi_{X})\rangle+\langle\Psi_{X}D_{\sigma}F^{\sigma\mu}\rangle$$

$$=-\partial_{\sigma}\langle\Psi_{X}F^{\sigma\mu}\rangle$$

$$+\frac{1}{2}[\Phi^{\dagger}\Psi_{X}\mathcal{D}^{\mu}\Phi-(\mathcal{D}^{\mu}\Phi)^{\dagger}\Psi_{X}\Phi], \tag{33}$$

where the equations of motion have been used. Inserting this into Eq. (31) and using Eq. (32) the terms containing the Higgs field exactly cancel, giving

$$\Theta^{\mu}=X^{\alpha}T^{\mu}_{\alpha}=-\partial_{\sigma}\langle\Psi_{X}F^{\sigma\mu}\rangle-X^{\mu}\mathcal{L}, \tag{34}$$

which coincides with Eq. (29).

IV. CALCULATION OF THE ANGULAR MOMENTUM

Let us now choose $X=\partial_{\varphi}$ in Eqs. (25),(30). This gives the conserved and gauge invariant angular momentum

$$J=\int T^{0}_{\varphi}d^{3}x=-\oint\langle(A_{\varphi}-W_{\varphi})F^{k0}\rangle dS_{k}. \tag{35}$$

Here the second equality on the right applies for fields subject to the symmetry conditions (10), W_{φ} being the compensating parameter in these conditions. In addition, one has to make sure that, when transforming the volume integral into the surface integral, the contribution of the inner boundary is zero. This can be checked in the gauge (11), where $W_{\varphi}=0$ while A_{φ} given by Eq. (12) is finite at the origin, so that the integral over a small surface enclosing the origin would be nonzero only if the electric field was $\sim1/r^{2}$. This, however, would imply that the total energy is infinite.

The surface integral structure of J shows that only the asymptotic long-range tails of the fields can contribute to the angular momentum. In order to calculate this integral, it suffices therefore to analyze the asymptotics of the fields near spatial infinity, where the problem reduces to studying the linearized field equations. More precisely, let (A_{μ},Φ) be a given static soliton solution with $J=0$. We consider all possible axial deformation of this solution with the only condition that, asymptotically, the deformed configurations approach the initial static solution, such that they will belong to the same topological sector. Therefore, the deformed configurations can be described by $(A_{\mu}+\psi_{\mu},\Phi+\phi)$, where the deformations (ψ_{μ},ϕ) can be arbitrary, with the only condition that they vanish as $r\to\infty$. As a result, in the asymptotic region the deformations satisfy the YMH equations linearized around the (A_{μ},Φ) background:

$$D_{\sigma}D^{\sigma}\psi_{\mu}-D_{\mu}D_{\sigma}\psi^{\sigma}+2[F_{\mu\sigma},\psi^{\sigma}]-\mathcal{M}_{ab}\psi^{a}_{\mu}T_{b}$$

$$=\frac{1}{2}\{\phi^{\dagger}T_{a}\mathcal{D}_{\mu}\Phi-(\mathcal{D}_{\mu}\Phi)^{\dagger}T_{a}\phi+\Phi^{\dagger}T_{a}\mathcal{D}_{\mu}\phi$$

$$-(\mathcal{D}_{\mu}\phi)^{\dagger}T_{a}\Phi\}T_{a}, \tag{36}$$

$$\mathcal{D}_{\sigma}\mathcal{D}^{\sigma}\phi+D_{\sigma}\psi^{\sigma}\Phi+2\psi_{\sigma}\mathcal{D}^{\sigma}\Phi$$

$$=-\lambda\{(\Phi^{\dagger}\Phi-1)\phi+(\Phi^{\dagger}\phi+\phi^{\dagger}\Phi)\Phi\}, \tag{37}$$

where the mass matrix is

$$\mathcal{M}_{ab}=\frac{1}{2}\Phi^{\dagger}(T_{a}T_{b}+T_{b}T_{a})\Phi. \tag{38}$$

Our strategy now is to solve these linearized equations in the asymptotic region to see if there are modes giving a nonvanishing contribution to the flux integral (35). We shall study axial deformations of all known topological solutions for the gauge group $\mathcal{G}=SU(2)$: 't Hooft–Polyakov monopoles and Julia-Zee dyons, sphalerons, and also vortices.

A. 't Hooft–Polyakov monopoles and Julia-Zee dyons

These are spherically symmetric solutions of YMH theory with $\mathcal{G}=SU(2)$ and the Higgs field in the real triplet representation [18–20]. The gauge group generators T_a are chosen according to Eq. (5), $(T_a)_{ik}=-\varepsilon_{aik}$. The mass matrix (38) has one zero eigenvalue corresponding to a massless gauge boson. Hence, there are long-range gauge field modes that may give a nonzero contribution to Eq. (35).

Static, spherically symmetric YMH fields are characterized in this case by the following gauge connection and the Higgs field [passing in the gauge (11) to spherical coordinates]:

$$A=\Omega(r)T_3 dt+w(r)(-T_2 d\vartheta+T_1 \sin\vartheta d\varphi)$$

$$+T_3 \cos\vartheta d\varphi, \tag{39}$$

$$\Phi^k=\delta_3^k \Phi(r). \tag{40}$$

The field equations (8) and (9) reduce to

$$(r^2\Omega')'=2w^2\Omega, \tag{41}$$

$$(r^2\Phi')'=2w^2\Phi+\lambda r^2(\Phi^2-1)\Phi, \tag{42}$$

$$r^2w''=w(w^2-1)+r^2(\Phi^2-\Omega^2)w. \tag{43}$$

The 't Hooft–Polyakov monopoles ($\Omega=0$) and Julia-Zee dyons ($\Omega\neq 0$) are solutions of this system that are regular at the origin, corresponding to $\Omega(0)=\Phi(0)=0$ and $w(0)=1$, while for large r they approach exponentially fast (for $\lambda\neq 0$) the asymptotic values

$$\Omega=\Sigma+\frac{Q}{r}, \quad \Phi=1, \quad w=0, \tag{44}$$

with constant Q,Σ. These solutions have finite energy, electric charge Q, and unit magnetic charge. For non-zero values of the self-coupling λ these solutions can be obtained numerically. For $\lambda=0$ the Higgs field is massless and has a long-range Coulomb tail: $\Phi=1+O(1/r)$ as $r\rightarrow\infty$. In this case, the solution is known analytically [21]:

$$\Omega=\Sigma\Phi, \quad \Phi=\coth Cr-\frac{1}{Cr}, \quad w=\frac{Cr}{\sinh Cr}, \tag{45}$$

with $C=\sqrt{1-\Sigma^2}$.

We would now like to study all possible axial deformations of these solutions in the asymptotic region by solving the linearized equations (36) and (37). The first step is to carry out a multipole decomposition of perturbations to identify the most general modes corresponding to axial deformations of the background solutions. Since the backgrounds are spherically symmetric, the angular quantum number j is conserved and perturbations for different values of j decouple from each other. It is convenient to introduce the basis of complex one-forms

$$\theta^0=dt, \quad \theta^1=dr, \quad \theta^2=\frac{r}{\sqrt{2}}(d\vartheta-i\sin\vartheta d\varphi), \quad \theta^3=(\theta^2)^*, \tag{46}$$

whose nonzero scalar products $\theta^{\alpha\beta}\equiv(\theta^\alpha,\theta^\beta)$ are $\theta^{00}=-\theta^{11}=-\theta^{23}=1$. In addition, one introduces the new Lie algebra basis $L_1=T_1+iT_2$, $L_2=T_1-iT_2$, $L_3=T_3$. The perturbations are then expanded as

$$\psi_\mu dx^\mu=L_a\psi_\alpha^a\theta^\alpha, \quad \phi^a=\langle L_a T_b\rangle f^b. \tag{47}$$

A complete separation of the angular variables in the perturbation equations (36) and (37) is achieved by making the following ansatz:

$$\psi_\alpha^a=Z_\alpha^a(r)_s Y_{jm}(\vartheta,\varphi), \quad f^a=U^a(r)_\sigma Y_{jm}(\vartheta,\varphi). \tag{48}$$

Here, $_s Y_{jm}(\vartheta,\varphi)$ are the spin-weighted spherical harmonics [22]. The quantum numbers j,m are the same for all values of the indices a,α, while the values of the spin weights $s=s(a,\alpha)$ and $\sigma=\sigma(a)$ are determined by direct inspection of Eqs. (36) and (37) using the properties of the spin-weighted harmonics [22].

Within the multipole decomposition obtained, we specialize to the dipole ($j=1$) and axially symmetric ($m=0$) sector. The most general perturbations in this case are described by (passing back to the standard basis)

$$\psi=\left(T_1\frac{Z_1(r)}{r}\sin\vartheta+T_3\frac{Z_2(r)}{r}\cos\vartheta\right)dt+T_2 Z_3(r)\sin\vartheta dr$$

$$+T_2 Z_5(r)\cos\vartheta d\vartheta+[-T_1 Z_5(r)\cos\vartheta$$

$$+T_3 Z_4(r)\sin\vartheta]\sin\vartheta d\varphi,$$

$$\phi^k=\delta_1^k\frac{U_1(r)}{r}\sin\vartheta+\delta_3^k\frac{U_2(r)}{r}\cos\vartheta. \tag{49}$$

This ansatz has a residual $U(1)$ gauge symmetry generated by the infinitesimal gauge transformations (7) with $U=\exp(-L)$,

$$\psi\rightarrow\psi+dL+[A,L], \quad \phi\rightarrow\phi-L\Phi, \tag{50}$$

where $L=\alpha(r)T_2\sin\vartheta$. This symmetry does not change the values of Z_2 and U_2, while

$$Z_1\rightarrow Z_1-r\Omega\alpha, \quad Z_3\rightarrow Z_3+\alpha', \quad Z_4\rightarrow Z_4+w\alpha,$$

$$Z_5\rightarrow Z_5+\alpha, \quad U_1\rightarrow U_1-r\alpha\Phi, \tag{51}$$

which can be used to impose the gauge condition[4] $Z_3=0$. Inserting now the ansatz (49) into the perturbation equations (36) and (37), the angular dependence decouples and we obtain a system of radial equations for the amplitudes Z_1,Z_2,Z_4,Z_5,U_1,U_2 which is listed in the Appendix. Inserting the ansatz into the angular momentum integral (35) gives [we are working in the gauge (11) where $W_\varphi=0$]

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[4]There remains one pure gauge mode generated by constant α.

$$J = \lim_{r\to\infty} r^2 \oint \langle (A_\varphi + \psi_\varphi)(A_0 + \psi_0)' \rangle \sin\vartheta\, d\vartheta\, d\varphi$$

$$= \frac{4\pi}{3} \lim_{r\to\infty} r^2 \left[2w\left(\frac{Z_1}{r}\right)' + \left(\frac{Z_2}{r}\right)' + 2\Omega' Z_4 \right]. \qquad (52)$$

Since the background amplitudes approach their asymptotic values (for large r) exponentially fast, we can replace Ω, Φ, w by their asymptotics (44). This gives

$$J = \frac{4\pi}{3} \lim_{r\to\infty} r^2 \left[\left(\frac{Z_2}{r}\right)' - \frac{2Q}{r^2} Z_4 \right]. \qquad (53)$$

The asymptotic behavior of the amplitudes Z_2 and Z_4 is determined from the radial equations (A2) and (A3), which in the asymptotic region reduce to

$$\left(-\frac{d^2}{dr^2} + \frac{2}{r^2} \right) Z_2 = 0, \quad \left(-\frac{d^2}{dr^2} + \frac{2}{r^2} \right) Z_4 = 0. \qquad (54)$$

Solutions that are regular at infinity are

$$Z_2 \sim \frac{1}{r}, \quad Z_4 \sim \frac{1}{r}. \qquad (55)$$

Inserting these into Eq. (53) finally gives[5]

$$J = 0. \qquad (56)$$

In fact, in order to ensure a nonzero value of J, the amplitudes Z_2, Z_4 should approach nonzero constant values at infinity, which is not the case. The conclusion is that there are no stationary, axial deformations of the 't Hooft–Polyakov monopoles and Julia-Zee dyons that would support a nonzero angular momentum. The same is true for higher (quadrupole, etc.) multipole deformations, since all of them decay at infinity even faster than the dipole ones. This conclusion did not require smallness of deformations for all r, the only requirement having been that deformed configurations must approach the spherically symmetric solutions for $r\to\infty$.[6]

B. Sphalerons and vortices

Sphalerons are spherically symmetric solutions of a YMH theory with $\mathcal{G} = SU(2)$ and the Higgs field in the complex doublet representation [23,24]. The gauge group generators T_a are thus chosen according to Eq. (6), $(T_a) = (1/2i)\tau^a$. In the simplest case [23], static and spherically symmetric

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[5]The same result is obtained for $\lambda = 0$, in which case all perturbation equations can be solved exactly [3].

[6]The rotational excitations of monopoles were also studied in Ref. [3]; this work, however, used the *volume* integral representation of the angular momentum. In view of this, it was necessary to assume the perturbative regime of rotational deformations *everywhere*, thus restricting consideration to the case of *slow* rotation. In our analysis, on the other hand, the rotation is not assumed to be slow.

YMH fields are characterized by the following purely magnetic gauge connection and Higgs field:

$$A = w(r)(-T_2 d\vartheta + T_1 \sin\vartheta\, d\varphi) + T_3 \cos\vartheta\, d\varphi, \quad (57)$$

$$\Phi^k = \delta_1^k \Phi(r). \qquad (58)$$

The field equations (8) and (9) reduce to

$$(r^2\Phi')' = \frac{1}{2}(w+1)^2\Phi + \lambda r^2(\Phi^2 - 1)\Phi, \qquad (59)$$

$$r^2 w'' = w(w^2 - 1) + \frac{r^2}{2}\Phi^2(w+1). \qquad (60)$$

Sphalerons are solutions of this system which are regular at the origin $[\Phi(0) = 0, \; w(0) = 1]$ and approach the asymptotic values

$$w = -1, \quad \Phi = 1 \qquad (61)$$

for large r exponentially fast. The crucial point now is that all deformations of these background solutions also approach zero exponentially fast. This is a manifestation of the fact that the gauge symmetry of the vacuum (61) is broken completely, since all eigenvalues of the mass matrix (38) are nonzero. As a result, there are no long-range solutions of the linearized field equations, and the angular momentum integral is zero. The only subtlety is the limit $\lambda \to 0$, since then the Higgs field becomes long range. However, as the background fields are purely magnetic, the equations for the most general dipole, axially symmetric gauge field perturbations do not contain any Higgs field perturbations.[7] The relevant perturbation equations, therefore, contain only massive amplitudes. Thus, their solutions approach zero exponentially fast. The conclusion[8] is that there are no stationary and axially symmetric spinning excitations of sphalerons.

To complete our considerations, we also want to consider the YMH vortices. It is known that the Abelian Nielsen-Olesen vortex [25] does not admit spinning generalizations within the original YMH theory with $\mathcal{G} = U(1)$ [20]. However, it is not excluded that such generalizations may exist within a YMH theory with a larger gauge group $\mathcal{G}$. Let us restrict consideration to cylindrically symmetric, i.e., z-independent, YMH fields. Then one can straightforwardly obtain from Eq. (35) the angular momentum per unit length z,

$$J = -\oint \langle (A_\varphi - W_\varphi) F_{0\rho} \rangle dl, \qquad (62)$$

where the integration is over a circle of radius $\rho \to \infty$ in a plane of constant z. For spinning excitations that asymptotically approach the Nielsen-Olesen vortex, both A_φ and W_φ stay finite as $\rho \to \infty$, and so the integral will be nonzero if

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[7]The same thing happens for the dyons, since Eqs. (A1) and (A2) decouple from the rest in the purely magnetic limit $\Omega \to 0$.

[8]This conclusion also applies to the deformed sphalerons of [24].

only $F_{0\rho} \sim 1/\rho$. However, this would imply that the energy is divergent. The conclusion is that there are no axially symmetric, spinning excitations of the Nielsen-Olesen vortex within YMH theory[9] for a compact gauge group $\mathcal{G}$.

V. CONCLUDING REMARKS

Summarizing our results, we have shown that none of the "canonical" topological solitons of the $\mathcal{G} = SU(2)$ YMH theory admit spinning excitations in the stationary and axisymmetric one-soliton sector. Although not completely eliminating all spinning solitons in gauge field theory, this conclusion renders their existence somewhat less probable. Therefore, we would like to list the remaining possibilities for constructing spinning solutions (if they exist at all). First, one can try to consider YMH theories with $\mathcal{G} > SU(2)$, which might work in the case of monopoles or dyons. The pattern of symmetry breaking can be quite different for higher gauge groups and for different representations of the Higgs field. If there remain several massless gauge group generators after symmetry breaking, then there is a better chance to have long-range modes giving a contribution to the angular momentum surface integral.[10]

The other possibility is to consider YMH systems that are *not* symmetric under the combined action of axial rotations and gauge transformations, while their action *is* symmetric. The angular momentum then will still be conserved, but it will be given by a volume integral. Thus, it may receive contributions also from short-range field modes.

Finally, we would like to make some remarks on the non-existence of rotating monopoles. First, it should be emphasized that monopoles do not rotate only within classical theory. Quantum monopoles, on the other hand, *do* have angular momentum associated with the fermionic zero modes [28]; this effect, however, disappears in the classical limit. For example, supersymmetric monopoles are conjectured to be dual to the elementary particles with spin (Monteon-Olive duality), thus implying that monopoles themselves have a spin. However, this spin is carried by the fermionic superpartners of monopoles and not by the bosonic monopole configurations.

Second, it is well known that the angular momentum of an electric charge moving around a magnetic monopole contains an extra term that can be interpreted as the angular momentum of the field [16]. At first glance, this disagrees with our conclusion that the angular momentum of the monopole field is zero. However, this extra term does not in fact relate to the monopole alone, but to the system of both charges, one of which is electric and the other magnetic.

Even when these charges are at rest, the angular momentum of the total field $\int \vec{r} \times (\vec{E} \times \vec{B}) d^3x$ does not vanish. However, if the electric field $\vec{E}$ of the electric charge is zero (no charge), the contribution of the magnetic charge alone will be zero.

We would also like to emphasize once again that our results apply only within the *one-soliton* sector, thus showing the absence of *spinning* excitations of isolated solitons. Outside this sector one can have solutions with $J \neq 0$ describing *orbital* motions of solitons. Such solutions are explicitly known in the case of rotating monopole-antimonopoles pairs [3,4,8].[11] It is also not excluded that in many-soliton systems, as for example in soliton scatterings, solitons might develop some kind of spinlike deformation due to their mutual polarization. However, such deformations will tend to zero in the limit of infinite separation of solitons.

APPENDIX

In this appendix we list the full system of radial equations describing the most general stationary, axially symmetric excitations of the Julia-Zee dyons. These equations are obtained by putting Eq. (49) (with $Z_3 = 0$) into the field equations (36) and (37),

$$0 = \left(-\frac{d^2}{dr^2} + \frac{w^2+1}{r^2} + \Phi^2 \right) Z_1 - \frac{2w}{r^2} Z_2 + \frac{\Omega}{r} (Z_5 - w Z_4)$$
$$- \Omega \Phi U_1, \tag{A1}$$

$$0 = \left(-\frac{d^2}{dr^2} + 2\frac{w^2+1}{r^2} \right) Z_2 - \frac{4w}{r^2} Z_1 - \frac{4w\Omega}{r} Z_5, \tag{A2}$$

$$0 = \left(-\frac{d^2}{dr^2} + \frac{w^2+2}{r^2} \right) Z_4 - \frac{3w}{r^2} Z_5 + \frac{w}{r} (\Omega Z_1 - \Phi U_1), \tag{A3}$$

$$0 = \left(-\frac{d^2}{dr^2} + \frac{3w^2}{r^2} + \Phi^2 - \Omega^2 \right) Z_5 - \frac{3w}{r^2} Z_4$$
$$+ \frac{\Omega}{r} (2w Z_2 - Z_1) + \frac{\Phi}{r} (U_1 - 2w U_2), \tag{A4}$$

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[9]Spinning vortices can exist in generalized YMH theories including the Chern-Simons term [26,27].

[10]In the Einstein-Yang-Mills theory, for example, where the symmetry is not broken at all, there exist static solitons whose linear axial deformations do support a nonzero angular momentum [11]. It is, however, unclear at present whether these linear rotational modes can be promoted to spinning solutions also at the nonlinear level [4].

[11]However, axially symmetric dyons with higher values of topological charge [29] do not rotate [4].

$$0 = \left(-r\Omega \frac{d}{dr} + (r\Omega)' \right) Z_1 - \frac{dZ_5}{dr} + \left(-w\frac{d}{dr} + w' \right) Z_4$$

$$+ \left(r\Phi \frac{d}{dr} - (r\Phi)' \right) U_1, \tag{A5}$$

$$0 = \left(-\frac{d^2}{dr^2} + \frac{w^2+1}{r^2} - \Omega^2 + \lambda(\Phi^2-1) \right) U_1 - \frac{2w}{r^2} U_2$$

$$+ \frac{\Phi}{r}(Z_5 - wZ_4) + \Omega\Phi Z_1, \tag{A6}$$

$$0 = \left(-\frac{d^2}{dr^2} + 2\frac{w^2+1}{r^2} + \lambda(3\Phi^2-1) \right) U_2 - \frac{4w}{r^2} U_1$$

$$-\frac{4w\Phi}{r} Z_5. \tag{A7}$$

It is instructive to verify that for $\lambda = 0$ these equations admit

a global symmetry: if $\{Z_1(r), Z_2(r), Z_4(r),$ $Z_5(r), U_1(r), U_2(r)\}$ is a solution for the purely magnetic background $\{\Omega(r) = 0, \Phi(r), w(r)\}$, then

$$Z_1^\gamma(r) = Z_1(\gamma r) + \sqrt{1-\gamma^2} U_1(\gamma r),$$

$$Z_2^\gamma(r) = Z_2(\gamma r) + \sqrt{1-\gamma^2} U_2(\gamma r),$$

$$U_1^\gamma(r) = U_1(\gamma r) + \sqrt{1-\gamma^2} Z_1(\gamma r),$$

$$U_2^\gamma(r) = U_2(\gamma r) + \sqrt{1-\gamma^2} Z_2(\gamma r),$$

$$Z_4^\gamma(r) = \gamma Z_4(\gamma r), \quad Z_5^\gamma(r) = \gamma Z_5(\gamma r)$$

is a solution corresponding to a ''rotated'' background characterized by

$$\Omega^\gamma(r) = \sqrt{1-\gamma^2}\Phi(\gamma r), \quad \Phi^\gamma(r) = \Phi(\gamma r),$$

$$w^\gamma(r) = w(\gamma r).$$

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