Relaxation in conformal field theory, Hawking-Page transition, and quasinormal or normal modes

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We study the process of relaxation back to thermal equilibrium in $(1+1)$-dimensional conformal field theory at finite temperature. When the size of the system is much larger than the inverse temperature, perturbations decay exponentially with time. On the other hand, when the inverse temperature is large, the relaxation is oscillatory with the characteristic period set by the size of the system. We then analyze the intermediate regime in two specific models: namely, free fermions, and a strongly coupled large k conformal field theory which is dual to string theory on $(2+1)$-dimensional anti–de Sitter spacetime. In the latter case, there is a sharp transition between the two regimes in the $k=\infty$ limit, which is a manifestation of the gravitational Hawking-Page phase transition. In particular, we establish a direct connection between quasinormal and normal modes of the gravity system, and the decaying and oscillating behavior of the conformal field theory.

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I. INTRODUCTION

An important problem in finite temperature field theory and statistical mechanics is to study the response of a system in thermal equilibrium to a generic perturbation. In particular, one is typically interested in understanding the process of relaxation back to thermal equilibrium. For small perturbations, this is well described by linear response theory, and the time evolution of the relaxation is determined by the retarded correlation function of the perturbation [1–3]. Generally, in the presence of interactions this problem is addressed within finite temperature perturbation theory. A special role is played by scale invariant theories, where the zero temperature 2-point functions are uniquely determined (up to a normalization) by scale invariance. However, finite temperature introduces a new scale and consequently the conformal Ward identities no longer determine the Green's functions completely. However, significant progress can be made if the conformal field theory (CFT) has a dual formulation in terms of gravity (or string theory) on anti–de Sitter space (AdS) spacetime. Indeed, the AdS/CFT [4–7] duality predicts that the retarded CFT correlation functions are in 1-1 correspondence with Green's functions on anti–de Sitter space with appropriate boundary conditions [8–12]. Furthermore, the poles of the retarded CFT correlators are given by the quasinormal modes in AdS [8,11]. This correspondence was verified explicitly in the high temperature regime of the two-dimensional CFT, dual to supergravity on AdS_3 [11]. Other calculations of quasinormal modes in anti–de Sitter backgrounds have appeared in [13].

The purpose of the present paper is to analyze finite volume effects for a quantum conformal field theory living in one space-like dimension (closed to form a circle of length L). In two limiting cases, when the dimensionless parameter LT is infinite or zero, conformal invariance completely determines the correlation function, independent of the details of the theory in question. The behavior of the retarded correlation functions in these two cases is qualitatively different. In the first case, the perturbation decays exponentially with characteristic time proportional to the inverse temperature. In the second case, we have oscillatory behavior with a characteristic period determined by L. Our main purpose here is to study the relaxation process in the intermediate regime when LT changes from zero to infinity.

After describing the qualitative behavior based on general arguments, we analyze quantitatively the case of a noninteracting theory, and contrast its behavior with the strongly coupled large k CFT dual to supergravity on AdS_3 (see also [14,15]). (The parameter k plays the role of N in the usual terminology of large N CFT dual to AdS gravity.) In the latter case, we have an explicit expression for the 2-point function at finite temperature and finite volume. We can thus analyze the linear response of the CFT. In the limit $k=\infty$, there is a sharp transition between a regime of exponential decay and oscillation. This is a manifestation of the gravitational Hawking-Page phase transition between the Bañados-Teitelboim-Zanelli (BTZ) black hole and thermal AdS. Moreover, we then establish a direct connection between the behavior of the linear response and the behavior of the corresponding bulk perturbations. In particular, we show that the regime of exponential decay is governed by the quasinormal modes of the BTZ black hole, while the regime of os-

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cillation is governed by the normal modes of thermal AdS. Thus, the behavior of the bulk AdS perturbation is mirrored precisely by the behavior of the linear response of the CFT. Finally, we speculate on the expected behavior of the strongly coupled CFT at finite values of k.

II. LINEAR RESPONSE THEORY

We consider a system initially in thermal equilibrium, and apply a small perturbation. The main goal of linear response theory is to study the change in the expectation value of an operator $\mathcal{O}(x,t)$ as a result of this perturbation. The total Hamiltonian of the system takes the form $H'(t)=H+H_{\rm ext}(t)$, where H is the unperturbed Hamiltonian, and $H_{\rm ext}(t)$ couples an external field to the system, with the assumption that $H_{\rm ext}=0$ for $t<t_0$. The change in the ensemble average of $\mathcal{O}$ is given by

$$\delta\langle\mathcal{O}(x,t)\rangle=i\int_{t_0}^{t}dt'\,{\rm Tr}\{\hat{\rho}[H_{\rm ext}(t'),\mathcal{O}(x,t)]\},\qquad(1)$$

where $\hat{\rho}$ is the unperturbed thermal density matrix. If we take the perturbation to be

$$H_{\rm ext}(t)=\int dx J(x,t)\mathcal{O}(x,t),\qquad(2)$$

where $J(x,t)$ is the external source, then Eq. (1) takes the form

$$\delta\langle\mathcal{O}(x,t)\rangle=\int_{-\infty}^{\infty}dt'\int dx'\quad J(x',t')D^R(x,t;x',t'),\qquad(3)$$

where

$$D^R(x,t;x',t')=-i\,\theta(t-t')\,{\rm Tr}\{\hat{\rho}[\mathcal{O}(x,t),\mathcal{O}(x',t')]\}\qquad(4)$$

is the retarded propagator.

For the particular case of an instantaneous perturbation of the form

$$J(x,t)=\delta(t)e^{ikx},\qquad(5)$$

one finds

$$\delta\langle\mathcal{O}(x,t)\rangle=e^{ikx}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}D^R(\omega,k).\qquad(6)$$

In the following, it will be useful to recall the Lehmann representation of the retarded propagator

$$D^R(\omega,k)=\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\,\frac{\rho(\omega',k)}{\omega-\omega'+i\epsilon},\qquad(7)$$

where the spectral function $\rho(\omega,k)$ is the Fourier transform of the commutator $\rho(x,t;x',t')=\langle[\mathcal{O}(x,t),\mathcal{O}(x',t')]\rangle$.

From general arguments, we know that $D^R(\omega,k)$ is an analytic function in the upper half ω plane. If the energy levels of the system are discrete and the ensemble sum converges, then we conclude that $D^R(\omega,k)$ is a meromorphic function of ω in the lower half plane, with simple poles on the real axis. Correspondingly, as can be seen from Eq. (6), the linear response will show oscillatory behavior. Indeed, as shown in [16–19], the energy spectrum for a system with finite entropy is discrete, and will in general exhibit the phenomenon of Poincaré recurrence. If the energy levels are continuous, the properties of $D^R(\omega,k)$ can be more complicated. At zero temperature, the retarded Green's function will generally have poles and cuts on the real axis, corresponding to stable states and multi-particle states, respectively. Furthermore, there can also be poles in the lower half ω-plane corresponding to resonances. The distance of the poles from the real line then determines the decay time of such a resonance. At finite temperature, $D^R(\omega,k)$ is a meromorphic function in the lower half plane. Equation (6) then shows that the characteristic times for the thermalization of an instantaneous perturbation is determined by the imaginary part of the poles of $D^R(\omega,k)$ in the complex ω plane. For a generic interacting theory, the location of these poles will depend non-trivially on the coupling constants, and concrete results are known only in specific limits, where perturbation theory is applicable.

III. CONFORMAL FIELD THEORY

To see how these general results are realized in $(1+1)$-dimensional conformal field theory, we first calculate the correlation functions on a torus with periods L and $\beta=T^{-1}$. The correlation function in real time, t, is then obtained by the analytic continuation $\tau=it$, where τ is imaginary time. When the size of one direction on the torus is taken to infinity, the torus becomes a cylinder. Using Cardy's result [20], the correlation function on the cylinder is obtained from the correlator on the plane by the conformal mapping[1] $w=(l/2\pi)\ln z$, $\overline{w}=(\overline{l}/2\pi)\ln\overline{z}$, where w and z are complex coordinates on the cylinder and the plane, respectively. The 2-point function on the cylinder is then given by [20]

$$\langle\mathcal{O}(w,\overline{w})\mathcal{O}(w',\overline{w}')\rangle=\frac{(\pi/l)^{2h}(\pi/\overline{l})^{2\overline{h}}}{[\sinh(\pi/l)(w-w')]^{2h}[\sinh(\pi/\overline{l})(\overline{w}-\overline{w}')]^{2\overline{h}}},\qquad(8)$$

where $(h,\overline{h})$ are the conformal weights of $\mathcal{O}$.

Suppose now that the temperature is finite while the size L is taken to infinity; then $l,\overline{l}=\beta(1\pm\mu)$, where μ is the chemical potential and $w=\sigma+i\tau$. After analytic continuation $\tau=it$, we have $w=\sigma-t$ and $\overline{w}=\sigma+t$. We then obtain the real time correlation function

[1]Note that in two-dimensional CFT the left and right sectors are independent. In particular, they can be defined on cylinders with different radii.

$$D_{\pm}(t,\sigma)=\langle\mathcal{O}(t\mp i\epsilon,\sigma)\mathcal{O}(0,0)\rangle$$
$$=\frac{(\pi T_R)^{2h}(\pi T_L)^{2\bar{h}}}{[\sinh \pi T_R(\sigma-t\pm i\epsilon)]^{2h}[\sinh \pi T_L(\sigma+t\mp i\epsilon)]^{2\bar{h}}}, \tag{9}$$

where $T_R=1/l$ and $T_L=1/\bar{l}$ are the effective right and left temperatures. Clearly, the correlation function decays exponentially for large time t; for example, $D_{\pm}(t,0)\sim e^{-2\pi T(h+\bar{h})t}$, when $T_L=T_R=T$.

For $\{h,\bar{h}\}\in\frac{1}{2}\mathbf{N}$, the Fourier transform of the commutator, $\rho=(D_+-D_-)$, can be evaluated by the method of residues leading to

$$\rho(\omega,k)\propto\left|\Gamma\left(h+i\frac{p_-}{2\pi T_R}\right)\Gamma\left(\bar{h}+i\frac{p_+}{2\pi T_L}\right)\right|^2, \tag{10}$$

where $p_{\pm}=\frac{1}{2}(\omega\mp k)$. From Eq. (10), we see that $\rho(\omega,k)$ has an infinite set of simple poles on either side of the real line. The poles of the Green's function $D^R(\omega,k)$ are then obtained by restricting this set to the lower half plane. These are given by [11]

$$\omega=k-4\pi i T_L(n+\bar{h}),$$
$$\omega=-k-4\pi i T_R(n+h),\quad n\in\mathbf{N}. \tag{11}$$

On the other hand, if the size L is kept finite and the inverse temperature $\beta=T^{-1}$ is taken to infinity then $l=L$. The direction of the imaginary axis can be chosen in the σ direction, $w=\tau+i\sigma$. The analytic continuation to real time now results in $w=i(t+\sigma)$ and $\bar{w}=i(t-\sigma)$. Thus, we find that Eq. (8) leads to the real time correlation function

$$D_{\pm}(t,\sigma)=\langle\mathcal{O}(t\mp i\epsilon,\sigma)\mathcal{O}(0,0)\rangle$$
$$=\frac{(\pi/iL)^{2(h+\bar{h})}}{\left[\sin\frac{\pi}{L}(t\mp i\epsilon+\sigma)\right]^{2h}\left[\sin\frac{\pi}{L}(t\mp i\epsilon-\sigma)\right]^{2\bar{h}}}. \tag{12}$$

This clearly shows the expected oscillatory behavior in real time. In the following, we specialize to the subset of operators with $h=\bar{h}\in\mathbf{N}/2$. To isolate the pole structure of the retarded Green's function, we again consider the Fourier transform of the commutator

$$\rho(\omega,k)=\int_{-\infty}^{\infty}dt\int_0^L d\sigma\; e^{i\omega t-ik\sigma}(D_+-D_-), \tag{13}$$

with $k=(2\pi/L)m$. We first consider the case with $h=\bar{h}=1/2$. Then ρ takes the simple form

$$\rho(\omega,k)\propto\sum_{n\in\mathbf{Z}}\delta\left(\frac{\omega L}{4\pi}-\frac{|m|}{2}-h-n\right)\left[\theta\left(\frac{\omega L}{2\pi}-|m|\right)-\theta\left(-\frac{\omega L}{2\pi}-|m|\right)\right]. \tag{14}$$

Thus, we find the poles

$$\omega=\frac{2\pi}{L}|m|+\frac{4\pi}{L}(n+h),$$
$$\omega=-\frac{2\pi}{L}|m|-\frac{4\pi}{L}(n+h), \tag{15}$$

with $m\in\mathbf{Z}$ and $n\in\mathbf{N}$. The spectral function for higher values of h takes a form similar to Eq. (14), with additional prefactors designed such that the pole structure is given by Eq. (15).

The considerations so far are quite general, and rely only on the conformal properties of the operators driving the perturbation. The behavior of the correlation functions (9) and (12) in the two limiting cases is thus universal, and in agreement with general expectations. However, the behavior in the intermediate regime, when both L and T are finite, cannot be derived entirely from conformal symmetry. In order to understand this regime, we first consider the 2-point function of free fermions on the torus [21]. This takes the form

$$\langle\psi(w)\psi(0)\rangle_{\nu}=\frac{\theta_{\nu}\left(\frac{w}{\beta}\middle|i\frac{L}{\beta}\right)\partial_w\theta_1\left(0\middle|i\frac{L}{\beta}\right)}{\theta_{\nu}\left(0\middle|i\frac{L}{\beta}\right)\theta_1\left(\frac{w}{\beta}\middle|i\frac{L}{\beta}\right)}, \tag{16}$$

where $w=\tau+i\sigma$ and ν characterizes the boundary conditions for $\psi(w)$. For finite temperature boundary conditions, we have $\nu=3,4$. Using the properties of θ functions, it is then easy to see that Eq. (16) is invariant under shifts $w\to w+\beta$ and $w\to w+iL$.

The real time correlation function is obtained from Eq. (16) by the substitution $w=i(t+\sigma)$. The resulting real time correlator is thus a periodic function of t with period L. Zeros of the theta function $\theta_1(w/\beta|iL/\beta)$ are located at [21] $w=m\beta+inL$, where m,n are integers. Therefore, for real time t, the correlation function (16) is a sequence of singular peaks located at $(t+\sigma)=nL$. In the limit $L/\beta\to\infty$, the correlation function (16) approaches (9) [actually the left-moving part of (9) with $h=1/2$] which exponentially decays with time. In the opposite limit, when $L/\beta\to 0$, it approaches the oscillating function (12). This is in agreement with our discussion using Cardy's arguments. It is important to observe that the exponential decay of the function in the limit of large L is consistent with periodicity of the real time correlation function (16) with period L. In order to see this, we note that the correction to the leading behavior is governed by $e^{-\pi L/\beta}e^{\pi(t+\sigma)/\beta}$. Thus, the limit of large L is meaningful only for times much smaller than L. For such t, the exponentially decaying function is a good description of the correlation function. However, for t approaching L the correction terms become important, and the time periodicity of the correlation function (16) is restored.

IV. STRONG COUPLING REGIME

At strong coupling, conformal symmetry is not enough to determine the linear response at finite volume. We are therefore unable to obtain exact results in the general case. However, progress can be made in some special cases. Here, we consider the supersymmetric conformal field theory dual to string theory on AdS_3. This theory describes the low energy excitations of a large number of D1- and D5-branes [4,7,15]. It can be interpreted as a gas of strings that wind around a circle of length L with target space T^4. The individual strings can be simply or multiply wound such that the total winding number is $k=c/6$, where $c\gg 1$ is the central charge. The parameter k plays the role of N in the usual terminology of large N CFT dual to AdS gravity. In order to obtain information about the correlation functions in the strong coupling regime of this theory, we can appeal to the AdS/CFT correspondence. According to this duality, each supergravity perturbation $\Phi_{(m,s)}$ of mass m and spin s propagating on AdS_3 has a corresponding operator $\mathcal{O}_{(h,\bar{h})}$ in the dual conformal field theory. This operator is characterized by conformal weights $(h,\bar{h})$, with $h+\bar{h}=\Delta$, $h-\bar{h}=s$, and Δ is determined in terms of the mass of the field. The CFT correlators are then determined in terms of the corresponding bulk Green's functions. In addition, there is a correspondence between (quasi)normal modes in the gravitational background and the poles of the retarded Green's functions in the conformal field theory [8].

According to the original prescription [6], each AdS space which asymptotically approaches the given two-dimensional manifold should contribute to the calculation, and one thus has to sum over all such spaces. In the case of interest, the two-manifold is a torus (τ,σ), where β and L are the respective periods. There is an $SL(2,\mathbf{Z})$ family of AdS_3 spaces which approach the torus asymptotically [15,22]. For the purpose of understanding the Hawking-Page phase transition, it suffices to consider the BTZ black hole and thermal AdS space, corresponding to anti–de Sitter space filled with thermal radiation. Both spaces can be represented [23] as a quotient of three-dimensional hyperbolic space H^3, with line element

$$ds^2=\frac{1}{y^2}(dzd\bar{z}+dy^2),\ y>0. \tag{17}$$

The BTZ black hole (for simplicity we consider only the non-rotating BTZ black hole) has inverse temperature $\beta=2\pi/r_+$, where r_+ is the horizon radius. Also, $z\sim e^{2\pi w/\beta}$ and $y\sim e^{2\pi\sigma/\beta}$, where $w=\sigma+i\tau$. Thus, the orbifold identification is given by

$$z\rightarrow e^{2\pi L/\beta}z,\quad y\rightarrow e^{2\pi L/\beta}y. \tag{18}$$

For the thermal AdS at the same temperature β^{-1}, we have $z\sim e^{2\pi w/L}$ and $y\sim e^{2\pi\tau/L}$, where $w=\tau+i\sigma$. In this case, the identification is

$$z\rightarrow e^{2\pi\beta/L}z,\quad y\rightarrow e^{2\pi\beta/L}y. \tag{19}$$

In both cases, the boundary of the three-dimensional space is a torus with periods L and β. In particular, the two configurations (thermal AdS and the BTZ black hole) are T-dual to each other, and are obtained by the interchange of the coordinates $\tau\leftrightarrow\sigma$ on the torus.

The correlation function of the dual operators then contains the sum of two contributions [14] as

$$\langle\mathcal{O}(w,\bar{w})\mathcal{O}(w',\bar{w}')\rangle_{\text{Torus}} = e^{-S_{\text{BTZ}}}\langle\mathcal{O}\ \ \mathcal{O}'\rangle_{\text{BTZ}}+e^{-S_{\text{AdS}}}\langle\mathcal{O}\ \ \mathcal{O}'\rangle_{\text{AdS}}, \tag{20}$$

where

$$S_{\text{BTZ}}=-k\pi\frac{L}{\beta},\quad S_{\text{AdS}}=-k\pi\frac{\beta}{L}, \tag{21}$$

are the Euclidean actions of the BTZ black hole and thermal Ads_3, respectively [15]. We see that these contributions are dual to each other under interchange of β and L. Using the AdS/CFT correspondence, the correlation functions on the torus were computed in [24,25] for the case of a scalar field $(h=\bar{h})$. For the BTZ background, the result takes the form [24]

$$\langle\mathcal{O}(w,\bar{w})\mathcal{O}(0,0)\rangle_{\text{BTZ}} = \sum_{n\in\mathbf{Z}}\frac{1}{\sinh\left[\frac{\pi}{\beta}(w+nL)\right]^{2h}\sinh\left[\frac{\pi}{\beta}(\bar{w}+nL)\right]^{2h}}, \tag{22}$$

where $w=\sigma+i\tau$. Note that Eq. (22) takes the form of the strip expression (8) summed over images to make it doubly periodic. On the supergravity side, the justification to sum over images comes simply from the fact that the correlator solves a Green's function equation. From the CFT point of view, this result is non-trivial. Indeed, for a generic CFT, summing over images does not produce the correct finite volume correlator. For example, the free field correlator (16) does not have this form.

Expression (20) is the result for the correlation function in the strong coupling regime. Although each term in the sum (20) is not modular invariant, the sum over the full $SL(2,\mathbf{Z})$ family does have this property [15,22]. Depending on the ratio L/β, one of the two terms in Eq. (20) dominates [15]. For high temperature (L/β is large) the BTZ is dominating, while at low temperature (L/β is small) the thermal AdS is dominant. The transition between the two regimes occurs at $\beta=L$. In terms of the gravitational physics, this corresponds to the Hawking-Page phase transition [26]. This is a sharp transition in the limit $k=\infty$, which is the case when the supergravity description is valid. In this limit, the BTZ black hole is the sole contribution for $L>\beta$, while thermal AdS is the only term which contributes for $L<\beta$.

The two terms in Eq. (20) have drastically different behavior as functions of real time. After the analytic continuation $\tau=it$, the BTZ contribution (22) produces the correlator

$$\langle \mathcal{O}(t,\sigma)\mathcal{O}(0,0)\rangle_{\mathrm{BTZ}} = \sum_{n\in\mathbf{Z}} \frac{1}{\left[\sinh\frac{\pi}{\beta}(\sigma-t+nL)\right]^{2h}\left[\sinh\frac{\pi}{\beta}(t+\sigma+nL)\right]^{2\bar{h}}}, \tag{23}$$

which exponentially decays with time. Furthermore, in the limit $L/\beta\to\infty$, only the $n=0$ term in Eq. (23) contributes, and we thus recover the universal behavior (9).

It is clear that the Fourier transform of the spectral function for Eq. (23) is again given by Eq. (10), with k restricted to the discrete values $(2\pi/L)m$. Thus, the poles of D^R are given by Eq. (11) with $k=(2\pi/L)m$. At first sight, this appears to be in contradiction with the general behavior discussed in Sec. II. Indeed, for finite volume we expect a discrete energy spectrum and consequently that the poles of D^R should lie on the real axis. The resolution of this puzzle lies in the peculiar properties of the CFT under consideration [7]. For $L/\beta>1$, the typical configuration consists of a relatively small number of multiply wound strings so that the effective volume relevant for the energy gap is $L_{\mathrm{eff}}\simeq kL\to\infty$. Of course, this explanation immediately raises another puzzle: If the effective volume of the theory is infinite, how come that the correlation function is periodic in σ with period L. The reason for this lies in the fact that the operator $\mathcal{O}$ does not distinguish between simply wound and multiply wound strings. Consequently, $\mathcal{O}$ still sees a finite volume.

For $h\neq\bar{h}$, which corresponds to fields of non-zero spin in AdS_3, the finite temperature Green's functions have not been worked out in the literature. However, it was shown in [11] that there is a 1-1 correspondence between the poles of the retarded finite temperature Green's function and the quasinormal modes for fields of spin $s=h-\bar{h}$ in the BTZ background. Quasi-normal modes are solutions to the wave equations which are purely ingoing at the horizon and subject to the boundary condition that the current vanishes at infinity.[2] Only a discrete set of modes satisfying these boundary conditions is possible, and the frequencies are shown in [11] to be identical with Eq. (11). In [11], the correspondence was shown for $L/\beta=\infty$. However, the above discussion shows that this is in fact valid for $L/\beta>1$. Indeed, one finds that the bulk perturbation decays via these quasinormal modes in precisely the same way as the linear response of the conformal field theory given by Eq. (6).

The question then arises as to the behavior for $L/\beta<1$. On the gravity side, Eq. (20) implies that the thermal AdS Green's function gives the relevant contribution to the real time correlation function. The result for the thermal AdS is obtained from Eq. (22) under the interchange $\tau\leftrightarrow\sigma$ and $\beta\leftrightarrow L$. Hence, we have [25]

$$\langle \mathcal{O}(t,\sigma)\mathcal{O}(0,0)\rangle_{\mathrm{AdS}} = \sum_{n\in\mathbf{Z}} \frac{1}{\left[\sin\frac{\pi}{L}(t+\sigma+i\beta n)\right]^{2h}\left[\sin\frac{\pi}{L}(t-\sigma+i\beta n)\right]^{2\bar{h}}}, \tag{24}$$

which is periodic in t with period L. Clearly, it represents a periodic sequence of singular peaks at $t\pm\sigma=nL$. In the limit of infinite β/L, only the $n=0$ term contributes and Eq. (24) approaches the expression (12). The Hawking-Page transition is thus a transition between oscillatory behavior at low temperature and exponentially decaying behavior at high temperature. From the CFT point of view, this behavior is explained by the fact that in the low temperature phase the generic configuration is given by simply wound strings so that the effective volume is finite and consequently the energy spectrum is discrete in agreement with general argument.

For $h\in\frac{1}{2}\mathbf{N}$, the Fourier transform of the spectral function for the finite temperature AdS Green's function (24) is again given by Eq. (14), with simple poles given by Eq. (15). On the AdS side, the set of frequencies (15) is identical to the set of normalizable modes [27]: these modes are regular at the origin of AdS_3 and are normalizable at infinity. As a result, the oscillating behavior of the bulk perturbation is mirrored by the oscillating behavior of the linear response. In [27], the normalizable modes were obtained for a scalar field $h=\bar{h}$. However, as shown in [28], higher spin s equations of motion in AdS_3 can be reduced to that of massive scalar fields for any s. We thus expect that the correspondence between normalizable modes and the poles of $D^R(\omega,k)$ in the low temperature phase is, in fact, valid for any $s=h-\bar{h}$.

V. DISCUSSION

As pointed out in [16–19], the energy spectrum for a system with finite entropy S is discrete, with level spacing of the order e^{-S}. As a consequence, the system will necessarily show Poincaré recurrence. These recurrences will occur on a time scale of the order $t\sim e^S$. Indeed, as we saw in Sec. II, for a system with discrete spectrum, the retarded propagator has simple poles on the real axis. Therefore, the linear response will exhibit oscillatory behavior. In general, however, this oscillatory behavior will be quite complicated. Typically, one expects the correlation function for a system with finite entropy to be a quasiperiodic function, with incommensurate frequencies. Poincaré recurrence ensures that the evolution is unitary with no loss of information. This is exactly the behavior seen in the free fermion correlation function (16): at finite L it is oscillating with period L.

Based on these remarks, it is important to understand the behavior observed in the strongly coupled CFT. In the strict $k=\infty$ limit, the sole contribution to the correlator is given by the BTZ black hole. The decay of this correlator is due to the fact that the effective volume $L_{\mathrm{eff}}\simeq kL$ is infinite, and thus the spectrum is continuous. This explains why we found complex poles corresponding to quasinormal modes of the

[2]Originally, the quasinormal modes were required to satisfy Dirichlet conditions at asymptotic infinity [10]. However, as shown in [11], this leads to problems for $h\leqslant 1$ which can be resolved by requiring the vanishing of the current.

black hole. However, for finite k the correlator must ultimately exhibit Poincaré recurrence.

In the present paper, we considered the inclusion of the contribution from thermal AdS. We found that this gives a periodic contribution, whose frequencies are the same as the normal modes of AdS. While the addition of this contribution does prevent the decay of the correlator at late times [14], it is not sufficient to produce the Poincaré recurrences at finite k. One could consider the inclusion of the remaining members of the $SL(2,\mathbf{Z})$ family of solutions with torus boundary. These contributions are necessary in order to ensure modular invariance. However, they will be parametrically of the same order as the thermal AdS contribution. The total correlator will still include the decaying BTZ part with complex frequencies. It seems that in order to see the discreteness of the energy spectrum on the CFT side, one will need to include finite k corrections to the correlator. In this way we expect that the $k=\infty$ (BTZ) contribution will be dressed by $1/k$ corrections, so that the correlator at finite k will no longer be a decaying function of time. One can see how this may happen by recalling the case of free fermions. The correlation function (16) is decaying when $L=\infty$ and is periodic at finite L. A somewhat similar behavior is expected in the strongly coupled case. As a result, the Poncaré recurrence time will become finite and set by $L_{\rm eff}$. This problem certainly warrants further investigation. In effect, there is no fundamental difference between the free and interacting case. The free system is periodic in time, while the interacting system should exhibit Poincaré recurrences as a quasiperiodic function. In both cases, the evolution will be unitary with no loss of information, as expected for a system in finite volume.

We should also stress that the conformal field theory dual to supergravity on AdS_3 is very special. For a generic CFT, we do not expect a phase transition even at strong coupling. It would be interesting, therefore, to find more examples of interacting CFT's in various dimensions, where explicit nonperturbative results can be obtained. In this respect, the duality between the $O(N)$ sigma model in three-dimensions and fields of even spin in AdS_4 might be of interest [29]. Note also that while the explicit computation of finite temperature Green's functions in AdS_d, $d>3$, is generally not possible, the quasinormal modes can nevertheless be obtained numerically [10]. In this way, one can obtain quantitative, non-perturbative, information about thermal Green's functions also in higher dimensions. For $d>3$, the high temperature phase is an AdS-Schwarzschild black hole whereas thermal AdS is dominant at low temperature. This Hawking-Page transition is then related to linear response theory in the dual CFT via quasinormal and normal modes in the two backgrounds.

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