Gauge problem in the gravitational self-force: Harmonic gauge approach in the Schwarzschild background

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The metric perturbation induced by a particle in the Schwarzschild background is usually calculated in the Regge-Wheeler (RW) gauge, whereas the gravitational self-force is known to be given by the tail part of the metric perturbation in the harmonic gauge. Thus, to identify the gravitational self-force correctly in a specified gauge, it is necessary to find out a gauge transformation that connects these two gauges. This is called the gauge problem. As a direct approach to solve the gauge problem, we formulate a method to calculate the metric perturbation in the harmonic gauge on the Schwarzschild background. We apply the Fourier-harmonic expansion to the metric perturbation and reduce the problem to the gauge transformation of the Fourier-harmonic coefficients (radial functions) from the RW gauge to the harmonic gauge. We derive a set of decoupled radial equations for the gauge transformation. These equations are found to have a simple second-order form for the odd parity part and the forms of spin $s=0$ and 1 Teukolsky equations for the even parity part. As a by-product, we correct typographical errors in Zerilli's paper and present a set of corrected equations in Appendix A.

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I. INTRODUCTION

A compact object of solar-mass size orbiting a supermassive black hole is one of the promising candidates for the source of gravitational waves. Since the internal structure of such a compact object may be neglected in this situation, we may adopt the black hole perturbation approach with the compact object being regarded as a point particle. In the black hole perturbation approach, we consider the metric perturbation induced by a point particle of mass μ orbiting a black hole of mass M, where $\mu \ll M$. At the lowest-order in the mass ratio (μ/M), the motion of the particle follows a geodesic of the background spacetime. In the next order, however, the particle moves no longer along a geodesic of the background because of its interaction with the self-field.

Although this deviation from a background geodesic is small for $\mu/M \ll 1$ at each instant of time, after a large lapse of time, it accumulates to become non-negligible. For example, a circular orbit will not remain circular but becomes a spiral-in orbit and the orbit eventually plunges into the black hole.

If the time scale of the orbital evolution due to the self-force is sufficiently long compared to the characteristic orbital time, we may adopt the so-called adiabatic approximation in which the orbit is assumed to be instantaneously geodesic with the constants of motion changing very slowly with time. In the Schwarzschild background case, we may assume the orbit to lie on the equatorial plane and the geodesic motion is determined by the energy and (the z-component of) angular momentum of the particle. In this case, the time variation of the energy and angular momentum can be determined from the energy and angular momentum emitted to infinity and absorbed into the black hole horizon by using the conservation law.

However, there are cases when the adiabatic approximation breaks down. For example, in the case of an extremely eccentric orbit or an orbit close to the innermost stable circular orbit, the orbital evolution will not be adiabatic because the stability of the orbit is strongly affected by an infinitesimally small reaction force. Furthermore, in the Kerr background, there is an additional constant of motion, known as the Carter constant. Intuitively, it describes the total orbital angular momentum, but unlike the case of spherical symmetry, it has nothing to do with the Killing vector field of the Kerr geometry. The lack of its relation to the Killing vector makes it impossible to evaluate the time change of the Carter constant from the gravitational waves emitted to infinity and to event horizon, even in the case when the adiabatic approximation is valid. Thus it is in any case necessary to derive the self-force of a particle explicitly.

The gravitational self-force F^{μ} is formally given as

$$\frac{d^2 z^{\alpha}}{d\tau^2}+\Gamma^{\alpha}_{\mu\nu}\frac{dz^{\mu}}{d\tau}\frac{dz^{\nu}}{d\tau}=F^{\alpha},$$

where $\{z^{\alpha}(\tau)\}$ represents the orbit with τ being the proper time measured in the background geometry and $\Gamma^{\alpha}_{\mu\nu}$ is the connection of the background. The self-force arises from the metric perturbation $h_{\mu\nu}$ induced by the particle:

$$\tilde{g}_{\mu\nu}=g_{\mu\nu}+h_{\mu\nu},$$

and it is expressed as

$$F^{\alpha}[h]=-P^{\alpha}_{\beta}\bigg(\bar{h}_{\beta\gamma;\delta}-\frac{1}{2}g_{\beta\gamma}\bar{h}^{\epsilon}{}_{\epsilon;\delta}-\frac{1}{2}\bar{h}_{\gamma\delta;\beta}+\frac{1}{4}g_{\gamma\delta}\bar{h}^{\epsilon}{}_{\epsilon;\beta}\bigg)u^{\gamma}u^{\delta},$$

where $P_{\alpha}{}^{\beta}=\delta_{\alpha}{}^{\beta}+u_{\alpha}u^{\beta}$, $\bar{h}_{\alpha\beta}=h_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}h$ and $u^{\alpha}=dz^{\alpha}/d\tau$.

The metric perturbation diverges at the location of the particle and so does the self-force. Thus the above formal expression is in fact meaningless. Fortunately, however, it is known that the metric perturbation in the vicinity of the orbit can be divided into two parts under the harmonic gauge condition; the direct part which has support only on the past light-cone emanating from the field point x^{μ} and the tail part which has support inside the past light-cone, and the physical self-force is given by the tail part of the metric perturbation which is regular as we let the field point coincide with a point on the orbit; $x^{\mu}\to z^{\mu}(\tau)$ [1,2]. It must be noted that the direct part can be evaluated by local analysis, i.e., only with the knowledge of local geometrical quantities. Therefore the physical self-force can be calculated as

$$\lim_{x\to z(\tau)} F_{\alpha}[h^{\mathrm{tail}}(x)]=\lim_{x\to z(\tau)} (F_{\alpha}[h(x)]-F_{\alpha}[h^{\mathrm{dir}}(x)]).$$

Furthermore, it has been revealed recently by Detweiler and Whiting [3] that the above division of the metric can be slightly modified so that the new direct part, called the *S* part, satisfies the same Einstein equations as the full metric perturbation does, and the new tail part, called the *R* part, satisfies the source-free Einstein equations, and that the *R* part gives the identical, regular self-force as the tail part does. The important point is that the *S* part can be still evaluated locally near the orbit without knowing the global solution. Another important point is that, when the metric perturbation is expanded in spherical harmonics, the self-force consists only of the modes $\ell\geqslant 2$, because the *R* part satisfies the source-free Einstein equations. In practice, however, due to unavoidable errors in the evaluation of the *S* (or *R*) part, the contributions of the $\ell=0$, 1 modes may not be neglected.

When we perform this subtraction, we must evaluate the full self-force and the direct part under the same gauge condition. But the direct part is, by definition, defined only in the harmonic gauge. On the other hand, the full metric perturbation is directly obtainable only by the Regge-Wheeler-Zerilli or Teukolsky formalism [4–7]. Therefore one must find a gauge transformation that brings both the full metric perturbation and the direct part of the metric perturbation to those in the same gauge. This is called the *gauge problem* [18].

For the direct part, methods to obtain it under the harmonic gauge condition were proposed [8–11]. However, it seems extremely difficult to solve the metric perturbation under the harmonic gauge because the metric components couple to each other in a complicated way. This is one of the reasons why the gauge problem is difficult to solve.

Recently, Barack and Ori [12] gave a useful insight into the gauge problem. They proposed an intermediate gauge approach in which only the direct part of the metric in the harmonic gauge is subtracted from the full metric perturbation in the Regge-Wheeler (RW) gauge. They then argued that the gauge-dependence of the self-force is unimportant when averaged over a sufficiently long lapse of time. Using this approach, the gravitational self-force for an orbit plunging into a Schwarzschild black hole was calculated by Barack and Lousto [13]. But they also pointed out that the RW gauge is singular in the sense that the resulting self-force will still have a direction-dependent limit for general orbits. The situation becomes worse in the Kerr background where the only known gauge in which the metric perturbation can be evaluated is the radiation gauge [4], but the metric perturbation becomes ill-defined in the neighborhood of the particle, i.e., the Einstein equations are not satisfied there [12].

In this paper, as a direct approach to the gauge problem, we consider a formalism to calculate the metric perturbation in the harmonic gauge. We focus on the Schwarzschild background. Instead of directly solving the metric perturbation in the harmonic gauge, we consider the metric perturbation in the RW gauge first, and then transform it to the one in the harmonic gauge. Namely, we derive a set of equations for gauge functions that transform the metric perturbation in the RW gauge to the one in the harmonic gauge.

The paper is organized as follows. In Sec. II, we formulate the gauge transformation from the RW gauge to the harmonic gauge. First we decompose the gauge transformation generators into the Fourier-harmonics components. Then the generators are divided into three parts; the odd parity part and the even parity part which is further divided into scalar and divergence free parts. By the above procedure, we find a set of decoupled equations for the gauge functions. In Sec. III, we summarize our formulation and discuss remaining issues. In Appendix A, we recapitulate the equations for the Regge-Wheeler-Zerilli formalism (for $\ell\geqslant 2$) by correcting typographical errors in Zerilli's paper [6]. The field equations for the $\ell=0$, 1 modes are separately presented in Appendix B.

II. FORMULATION

We consider a metric perturbation $h_{\mu\nu}$ in the Schwarzschild background:

$$g_{\mu\nu}dx^{\mu}dx^{\nu}=-f(r)dt^2+f(r)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2),$$

$$f(r)=1-\frac{2M}{r}. \tag{2.1}$$

We express the gauge transformation from the RW gauge to the harmonic gauge as

$$x^{\mu}_{\mathrm{RW}}\to x^{\mu}_{\mathrm{H}}=x^{\mu}_{\mathrm{RW}}+\xi^{\mu}, \tag{2.2}$$

$$h^{\mathrm{RW}}_{\mu\nu}\to h^{\mathrm{H}}_{\mu\nu}=h^{\mathrm{RW}}_{\mu\nu}-\xi_{\mu;\nu}-\xi_{\nu;\mu}, \tag{2.3}$$

where the suffix RW stands for the RW gauge and H for the harmonic gauge.

Substituting Eq. (2.3) into the harmonic gauge condition $\bar{h}^{\mathrm{H}}_{\mu\nu}{}^{;\nu}=0$, we obtain the equations for ξ^{μ}:

$$\xi_{\mu}{}^{;\nu}{}_{;\nu}=\bar{h}^{\mathrm{RW}}_{\mu}{}^{\nu}{}_{;\nu}=h^{\mathrm{RW}}_{\mu}{}^{\nu}{}_{;\nu}-\frac{1}{2}h^{\mathrm{RW}}_{;\mu}. \tag{2.4}$$

Using the static and spherical symmetry of the background, we perform the Fourier-harmonic expansion of the above equation and consider the equations for the expansion coefficients for ξ^{μ} and $h^{\mathrm{RW}}_{\mu\nu}$. We use the tensor harmonics introduced by Zerilli [6] which are recapitulated in Appendix A. Then according to the property under the parity transformation $(\theta,\phi)\rightarrow(\pi-\theta,\phi+\pi)$, we divide ξ^{μ} into the odd and even parity parts,

$$\xi^{\mu}=\xi^{\mu}_{(\mathrm{odd})}+\xi^{\mu}_{(\mathrm{even})}, \tag{2.5}$$

and the even part is further decomposed into the scalar and divergence-free parts,

$$\xi^{\mu}_{(\mathrm{even})}=\xi^{;\mu}+\xi^{\mu}_{(\mathrm{v})}, \tag{2.6}$$

where $\xi^{\mu}_{(\mathrm{v});\mu}=0$.

A. Odd part ($\ell\geqslant 2$)

First, we consider the odd part which has the odd parity $(-1)^{\ell+1}$ under the parity transformation. The gauge transformation generators and the metric perturbation are given in the Fourier-harmonic expanded form as

$$\xi^{(\mathrm{odd})}_{\mu}=\int d\omega\sum_{\ell m} e^{-i\omega t}\Lambda_{\ell m\omega}(r)\times\left\{0,0,\frac{-1}{\sin\theta}\partial_{\phi}Y_{\ell m}(\theta,\phi),\sin\theta\partial_{\theta}Y_{\ell m}(\theta,\phi)\right\}, \tag{2.7}$$

$$h^{(\mathrm{odd})}_{\mu\nu}=\int d\omega\sum_{\ell m}\frac{\sqrt{2\ell(\ell+1)}}{r}e^{-i\omega t}\Bigg[-h_{0\ell m\omega}(r)\boldsymbol{c}^{(0)}_{\ell m\mu\nu}+ih_{1\ell m\omega}(r)\boldsymbol{c}_{\ell m\mu\nu}+\frac{\sqrt{(\ell-1)(\ell+2)}}{2r}\times h_{2\ell m\omega}(r)\boldsymbol{d}_{\ell m\mu\nu}\Bigg], \tag{2.8}$$

where $\boldsymbol{c}^{(0)}_{\ell m\mu\nu},\boldsymbol{c}_{\ell m\mu\nu},\boldsymbol{d}_{\ell m\mu\nu}$ are tensor harmonics with odd parity [6]. By substituting Eqs. (2.7) and (2.8) into Eq. (2.4), we obtain the equation for $\Lambda_{\ell m\omega}(r)$:

$$\mathcal{L}^{(\mathrm{odd})}\Lambda_{\ell m\omega}(r)=2R^{(\mathrm{odd})}_{\ell m\omega}(r)-\frac{16i\pi r^2}{\sqrt{2\ell(\ell+1)(\ell-1)(\ell+2)}}\times D_{\ell m\omega}(r), \tag{2.9}$$

where $R^{(\mathrm{odd})}_{\ell m\omega}(r)$ is the Regge-Wheeler gauge invariant variable, $D_{\ell m\omega}(r)$ is the Fourier-harmonic coefficient of the stress-energy tensor, and the differential operator $\mathcal{L}$ is defined as

$$\mathcal{L}^{(\mathrm{odd})}\equiv f(r)\frac{d^2}{dr^2}+f'(r)\frac{d}{dr}+\left(\frac{\omega^2}{f(r)}-\frac{\ell(\ell+1)}{r^2}\right)=\frac{1}{f(r)}\frac{d^2}{dr^{*2}}+\left(\frac{\omega^2}{f(r)}-\frac{\ell(\ell+1)}{r^2}\right), \tag{2.10}$$

where $r^{*}=r+2M\log(r/2M-1)$ and $'=d/dr$. The form of the differential operator $\mathcal{L}$ is slightly different from the radial part of the scalar d'Alembertian, but Eq. (2.9) may be solved by the standard Green function method. The homogeneous solutions to construct the Green function can be obtained by applying the method developed by Mano *et al.* [14]. Here we note that the retarded causal boundary condition should be imposed for the Green function.

B. Even scalar part ($\ell\geqslant 2$)

Next we consider the even scalar part of ξ^{μ} which has the even parity $(-1)^{\ell}$ and is expressed by a gradient of a scalar function ξ. It is expressed as

$$\xi^{(\mathrm{s})}_{\mu}=\xi_{;\mu},\quad \xi=\int d\omega\sum_{\ell m}\frac{1}{r}\tilde{\xi}_{\ell m\omega}(r)e^{-i\omega t}Y_{\ell m}(\theta,\phi). \tag{2.11}$$

The gauge equation for $\xi^{(\mathrm{s})}_{\mu}$ is derived from $\xi^{(\mathrm{s})}_{\mu;\nu}{}^{;\nu}=J^{(\mathrm{s})}_{;\mu}$, which, because of the vanishing of the background Ricci tensor, gives

$$\xi^{;\nu}{}_{;\nu}=J^{(\mathrm{s})}, \tag{2.12}$$

where the source term $J^{(\mathrm{s})}$ is determined from the equation

$$J^{(\mathrm{s});\mu}{}_{;\mu}=\overline{h}^{\mu\nu}_{\mathrm{RW};\mu\nu}. \tag{2.13}$$

The Fourier-harmonic expanded form of the above equation is derived as follows. First, we take the divergence of the metric perturbation under the RW gauge;

$$\begin{aligned}\overline{h}^{\mathrm{RW};\nu}_{\mu\nu}=\int d\omega\sum_{\ell m} e^{-i\omega t}\Bigg\{&\left[-\frac{2f(r)}{r}H^{\mathrm{RW}}_{1\ell m\omega}(r)-\frac{8\pi i r B^{(0)}_{\ell m\omega}(r)}{\sqrt{\ell(\ell+1)/2}}-\frac{8\pi i\omega r^2F_{\ell m\omega}(r)}{\sqrt{\ell(\ell+1)(\ell-1)(\ell+2)/2}}\right]Y_{\ell m}(\theta,\phi)(\boldsymbol{e}_t)_{\mu}\\&+\left[\frac{2}{r}(H^{\mathrm{RW}}_{2\ell m\omega}(r)-K^{\mathrm{RW}}_{\ell m\omega}(r))+\frac{8\pi r B_{\ell m\omega}(r)}{\sqrt{\ell(\ell+1)/2}}-\frac{8\pi r^2\frac{d}{dr}F_{\ell m\omega}(r)}{\sqrt{\ell(\ell+1)(\ell-1)(\ell+2)/2}}\right]Y_{\ell m}(\theta,\phi)(\boldsymbol{e}_r)_{\mu}\\&+\frac{8\pi r^2}{\sqrt{\ell(\ell+1)(\ell-1)(\ell+2)/2}}F_{\ell m\omega}(r)(\boldsymbol{e}_3)_{\mu}\Bigg\},\end{aligned} \tag{2.14}$$

where $(\boldsymbol{e}_t)_\mu=(-1,0,0,0)$, $(\boldsymbol{e}_r)_\mu=(0,1,0,0)$ and $(\boldsymbol{e}_3)_\mu=(0,0,\partial_\theta Y_{\ell m},\partial_\phi Y_{\ell m})$. Then we take the divergence of the above once more to obtain

$$\overline{h}^{\mathrm{RW};\mu\nu}_{\mu\nu}=\int d\omega\sum_{\ell m} e^{-i\omega t}Y_{\ell m}(\theta,\phi)\Bigg[-\frac{1}{r}\mathcal{L}^{(\mathrm{s})}\tilde{H}_{\ell m\omega}(r)+8\pi\Bigg(\frac{A^{(0)}_{\ell m\omega}(r)}{f(r)}-f(r)A_{\ell m\omega}(r)-\sqrt{2}G^{(\mathrm{s})}_{\ell m\omega}\Bigg)\Bigg] \tag{2.15}$$

where $\tilde{H}_{\ell m\omega}(r)$ is the trace of the metric perturbation given by

$$\frac{1}{r}\tilde{H}_{\ell m\omega}(r)=\frac{H^{\mathrm{RW}}_{0\ell m\omega}(r)-H^{\mathrm{RW}}_{2\ell m\omega}(r)}{2}-K^{\mathrm{RW}}_{\ell m\omega}(r). \tag{2.16}$$

The d'Alembertian of $J^{(\mathrm{s})}$ is expanded as

$$J^{(\mathrm{s});\mu}{}_{;\mu}=\int d\omega\sum_{\ell m} e^{-i\omega t}Y_{\ell m}(\theta,\phi)\frac{1}{r}\mathcal{L}^{(\mathrm{s})}\tilde{J}^{(\mathrm{s})}_{\ell m\omega}(r), \tag{2.17}$$

where $\frac{1}{r}\tilde{J}^{(\mathrm{s})}_{\ell m\omega}$ is the Fourier-harmonic coefficient of $J^{(\mathrm{s})}$ and $\mathcal{L}^{(\mathrm{s})}$ is the radial part of the scalar d'Alembertian,

$$\begin{aligned}\mathcal{L}^{(\mathrm{s})}&\equiv f(r)\frac{d^2}{dr^2}+f'(r)\frac{d}{dr}+\left(\frac{\omega^2}{f(r)}-\frac{\ell(\ell+1)}{r^2}-\frac{f'(r)}{r}\right)\\&=\frac{1}{f(r)}\frac{d^2}{dr^{*2}}+\left(\frac{\omega^2}{f(r)}-\frac{\ell(\ell+1)}{r^2}-\frac{2M}{r^3}\right).\end{aligned} \tag{2.18}$$

Hence Eq. (2.13) becomes

$$\mathcal{L}^{(\mathrm{s})}\tilde{J}^{(\mathrm{s})}_{\ell m\omega}(r)=-\mathcal{L}^{(\mathrm{s})}\tilde{H}_{\ell m\omega}(r)+8\pi r\left(\frac{A^{(0)}_{\ell m\omega}(r)}{f(r)}-f(r)A_{\ell m\omega}(r)-\sqrt{2}G^{(\mathrm{s})}_{\ell m\omega}\right). \tag{2.19}$$

Once the source term is obtained, we obtain from Eq. (2.12) the equation for $\tilde{\xi}_{\ell m\omega}(r)$ as

$$\mathcal{L}^{(\mathrm{s})}\tilde{\xi}_{\ell m\omega}(r)=\tilde{J}^{(\mathrm{s})}_{\ell m\omega}(r). \tag{2.20}$$

Note that this equation as well as Eq. (2.19) are just the radial part of the scalar d'Alembertian, or equivalently the spin $s=0$ Teukolsky equation, and hence can be also solved by the Green function method. The detail analysis of the homogeneous solutions which satisfy the $s=0$ Teukolsky equation is discussed in [14].

C. Even vector part ($\ell \geqslant 2$)

Since the even vector part $\xi^\mu_{(\mathrm{v})}$ satisfies the divergence free condition, $\xi^{(\mathrm{v});\mu}_\mu=0$, this part has two degrees of freedom. Therefore $\xi^\mu_{(\mathrm{v})}$ is expressed in terms of two independent radial functions,

$$\begin{aligned}\xi^{(\mathrm{v})}_\mu=\int d\omega\sum_{\ell m} e^{-i\omega t}\Bigg[&\frac{1}{r}M_{0\ell m\omega}(r)Y_{\ell m}(\theta,\phi)(\boldsymbol{e}_t)_\mu\\&+\frac{1}{r}M_{1\ell m\omega}(r)Y_{\ell m}(\theta,\phi)(\boldsymbol{e}_r)_\mu\\&+\frac{1}{\ell(\ell+1)}\left\{\frac{-i\omega r}{f(r)}M_{0\ell m\omega}(r)+\partial_r(rf(r)M_{1\ell m\omega}(r))\right\}\\&\times(\boldsymbol{e}_3)_\mu\Bigg].\end{aligned} \tag{2.21}$$

To solve the gauge transformation equation (2.4) for this part, we introduce an auxiliary field $F_{\mu\nu}:=\xi^{(\mathrm{v})}_{\nu;\mu}-\xi^{(\mathrm{v})}_{\mu;\nu}$. Then we have

$$F_{\mu\nu}{}^{;\nu}=-J^{(\mathrm{v})}_\mu,\quad J^{(\mathrm{v})}_\mu\equiv\overline{h}^{\mathrm{RW};\nu}_{\mu\nu}-J^{(\mathrm{s})}_{;\mu}, \tag{2.22}$$

where we have again used the fact that the background Ricci tensor vanishes. Note that the Fourier-harmonic expansion of $\overline{h}^{\mathrm{RW};\nu}_{\mu\nu}$ is given by Eq. (2.14). The above equation is the Maxwell equation, so it can be solved by using the Teukolsky formalism [7] for the electromagnetic field ($s=\pm1$). Namely, we introduce the following variables:

$$\phi_0=F_{\mu\nu}l^\mu m^\nu,\quad \phi_2=F_{\mu\nu}m^{*\mu}n^\nu, \tag{2.23}$$

where * denotes the complex conjugate, and l^μ, n^μ, m^μ are the Kinnersley null tetrad defined by

$$l^\mu=\left\{\frac{1}{f(r)},1,0,0\right\},\quad n^\mu=\left\{\frac{1}{2},-\frac{f(r)}{2},0,0\right\},$$
$$m^\mu=\frac{1}{\sqrt{2}r}\left\{0,0,1,\frac{i}{\sin\theta}\right\}. \tag{2.24}$$

The variables ϕ_0 and ϕ_2 satisfy the $s=\pm1$ Teukolsky equation,

$$\mathcal{L}^{(\mathrm{Teuk})}_s\Psi_s=-4\pi r^2T_s, \tag{2.25}$$

where $\Psi_1=\phi_0$ and $\Psi_{-1}=r^2\phi_2$, and $\mathcal{L}^{(\mathrm{Teuk})}_s$ is the Teukolsky differential operator defined as

$$\begin{aligned}\mathcal{L}^{(\mathrm{Teuk})}_s:=&-\frac{r^2}{f(r)}\partial_t^2+2s\left(\frac{M}{f(r)}-r\right)\partial_t\\&+(r^2f(r))^{-s}\partial_r[(r^2f(r))^{s+1}\partial_r]+\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta)\\&+\frac{1}{\sin^2\theta}\partial_\phi^2+\frac{2is\cos\theta}{\sin^2\theta}\partial_\phi-s(s\cot^2\theta-1).\end{aligned} \tag{2.26}$$

The source terms T_s, which are calculated from $J^{(\mathrm{v})}_\mu$, are

$$-4\pi T_1=\frac{1}{\sqrt{2}r}\left(\partial_\theta+\frac{i}{\sin\theta}\partial_\phi\right)[J^{(\mathrm{v})}_\mu l^\mu]-\left(\partial_r-\frac{i\omega}{f(r)}+\frac{3}{r}\right)\times[J^{(\mathrm{v})}_\mu m^\mu], \tag{2.27}$$

$$-4\pi T_{-1}=r^2\left\{-\frac{1}{\sqrt{2}r}\left(\partial_\theta-\frac{i}{\sin\theta}\partial_\phi\right)[J_\mu^{(\mathrm{v})}n^\mu]-\frac{f(r)}{2}\left(\partial_r+\frac{i\omega}{f(r)}+\frac{3}{r}\right)[J_\mu^{(\mathrm{v})}m^{*\mu}]\right\}. \qquad (2.28)$$

The solution of this Teukolsky equation is also analyzed in [14].

Once we obtain ϕ_0 and ϕ_2, the two radial functions for the gauge transformation are calculated from Eq. (2.23) which reads

$$\phi_0=\int d\omega\sum_{\ell m}\ e^{-i\omega t}\frac{-{}_1Y_{\ell m}(\theta,\phi)}{\sqrt{2\ell(\ell+1)}rf(r)}\Bigg[\frac{\ell(\ell+1)}{r}(M_{0\ell m\omega}(r)-f(r)M_{1\ell m\omega}(r))-i\omega\left\{\frac{-i\omega r}{f(r)}M_{0\ell m\omega}(r)+\frac{d}{dr}(rf(r)M_{1\ell m\omega}(r))\right\}+f(r)\frac{d}{dr}\left\{\frac{-i\omega r}{f(r)}M_{0\ell m\omega}(r)+\frac{d}{dr}(rf(r)M_{1\ell m\omega}(r))\right\}\Bigg], \qquad (2.29)$$

$$\phi_2=\int d\omega\sum_{\ell m}\ e^{-i\omega t}\frac{{}_{-1}Y_{\ell m}(\theta,\phi)}{2\sqrt{2\ell(\ell+1)}r}\Bigg[-\frac{\ell(\ell+1)}{r}(M_{0\ell m\omega}(r)+f(r)M_{1\ell m\omega}(r))+i\omega\left\{\frac{-i\omega r}{f(r)}M_{0\ell m\omega}(r)+\frac{d}{dr}(rf(r)M_{1\ell m\omega}(r))\right\}+f(r)\frac{d}{dr}\left\{\frac{-i\omega r}{f(r)}M_{0\ell m\omega}(r)+\frac{d}{dr}(rf(r)M_{1\ell m\omega}(r))\right\}\Bigg], \qquad (2.30)$$

where ${}_sY_{\ell m}(\theta,\phi)$ are the spin-weighted spherical harmonics. Performing the Fourier and spin-weighted spherical harmonic expansion for ϕ_0 and ϕ_2,

$$\phi_0=\int d\omega\sum_{\ell m}\ \tilde{\phi}_{0\ell m\omega}(r)e^{-i\omega t}{}_1Y_{\ell m}(\theta,\phi), \qquad (2.31)$$

$$\phi_2=\int d\omega\sum_{\ell m}\ \tilde{\phi}_{2\ell m\omega}(r)e^{-i\omega t}{}_{-1}Y_{\ell m}(\theta,\phi), \qquad (2.32)$$

we obtain the equations

$$\tilde{\phi}_{0\ell m\omega}(r)+\frac{2}{f(r)}\tilde{\phi}_{2\ell m\omega}(r)=-\frac{\sqrt{2/\ell(\ell+1)}}{rf(r)}\left[\left(\frac{\ell(\ell+1)}{r}-\frac{\omega^2r}{f(r)}\right)M_{0\ell m\omega}(r)-i\omega\frac{d}{dr}(rf(r)M_{1\ell m\omega}(r))\right], \qquad (2.33)$$

$$\tilde{\phi}_{0\ell m\omega}(r)-\frac{2}{f(r)}\tilde{\phi}_{2\ell m\omega}(r)=\frac{\sqrt{2/\ell(\ell+1)}}{rf(r)}\left[\frac{\ell(\ell+1)f(r)}{r}M_{1\ell m\omega}(r)-f(r)\frac{d^2}{dr^2}(rf(r)M_{1\ell m\omega}(r))+i\omega f(r)\frac{d}{dr}\left(\frac{r}{f(r)}M_{0\ell m\omega}(r)\right)\right]. \qquad (2.34)$$

We can eliminate $M_{1\ell m\omega}(r)$ from Eqs. (2.33) and (2.34) to obtain a decoupled equation for $M_{0\ell m\omega}(r)$,

$$\mathcal{L}^{(\mathrm{s})}M_{0\ell m\omega}(r)=-\frac{r^2}{\sqrt{2\ell(\ell+1)}}\Bigg[f(r)^2\frac{d^2}{dr^2}\tilde{\phi}_{0\ell m\omega}(r)+f(r)\left(\frac{2+2f(r)}{r}+f'(r)+i\omega\right)\frac{d}{dr}\tilde{\phi}_{0\ell m\omega}(r)+\frac{rf'(r)-(\ell-1)(\ell+2)f(r)+i\omega r(1+2f(r))}{r^2}\times\tilde{\phi}_{0\ell m\omega}(r)+2f(r)\frac{d^2}{dr^2}\tilde{\phi}_{2\ell m\omega}(r)+2\left(\frac{2+2f(r)}{r}-f'(r)-i\omega\right)\frac{d}{dr}\tilde{\phi}_{2\ell m\omega}(r)-2\frac{rf'(r)+(\ell-1)(\ell+2)+3i\omega r}{r^2}\tilde{\phi}_{2\ell m\omega}(r)\Bigg], \qquad (2.35)$$

$$=\frac{r^2}{\sqrt{2\ell(\ell+1)}}\Bigg[4\pi\left(\frac{2}{r^2}\tilde{T}_{-1\ell m\omega}(r)+f(r)\tilde{T}_{1\ell m\omega}(r)\right)-f(r)(i\omega+f'(r))\left(\frac{d}{dr}+\frac{i\omega}{f(r)}+\frac{1}{rf(r)}\right)\tilde{\phi}_{0\ell m\omega}(r)+2(i\omega-f'(r))\left(\frac{d}{dr}-\frac{i\omega}{f(r)}+\frac{1}{r}\right)\tilde{\phi}_{2\ell m\omega}(r)\Bigg], \qquad (2.36)$$

where $\tilde{T}_{s\ell m\omega}(r)$ is the Fourier-harmonic coefficient of the source term in the Teukolsky equation (2.25). Interestingly, Eq. (2.36) has the same form as the $s=0$ Teukolsky equation. Once we obtain $M_{0\ell m\omega}(r)$ by solving the above equation, $M_{1\ell m\omega}(r)$ is derived from

$$M_{1\ell m\omega}(r)=\frac{-ir^2}{\omega\sqrt{2\ell(\ell+1)}}\left[f(r)\left(\frac{d}{dr}+\frac{i\omega}{f(r)}+\frac{1}{rf(r)}\right)\tilde{\phi}_{0\ell m\omega}(r)+2\left(\frac{d}{dr}-\frac{i\omega}{f(r)}+\frac{1}{r}\right)\tilde{\phi}_{2\ell m\omega}(r)\right]-\frac{i}{\omega}\left(\frac{d}{dr}M_{0\ell m\omega}(r)-\frac{M_{0\ell m\omega}(r)}{r}\right), \tag{2.37}$$

which also follows from Eqs. (2.33) and (2.34).

D. $\ell=0$ mode

The above formalism cannot be applied to the $\ell=0$, 1 modes because some vector and tensor harmonics vanish. (For example, $\boldsymbol{d}_{1m}$ and $\boldsymbol{f}_{1m}$ vanish for $\ell=1$.) Furthermore we cannot use the Regge-Wheeler and Zerilli equation to evaluate the metric perturbations for these modes. So we have to deal with these modes separately. As shown in Appendix B, we may introduce the Zerilli gauge in which we can evaluate the metric perturbation. Then we consider the gauge transformation from the Zerilli gauge to the harmonic gauge for the $\ell=0,1$ modes.

$$\xi_{\mu}^{\mathrm{Z}\to\mathrm{H};\nu}{}_{;\nu}=\overline{h}^{\mathrm{Z}}_{\mu\nu}{}^{;\nu},\quad x_{\mathrm{H}}^{\mu}=x_{\mathrm{Z}}^{\mu}+\xi^{\mu}_{\mathrm{Z}\to\mathrm{H}}\quad (\text{for } \ell=0,1 \text{ modes}). \tag{2.38}$$

At first, we consider the $\ell=0$ mode. The $\ell=0$ mode of the gauge transformation generator is given as the following:

$$\xi_{\mu}^{\mathrm{Z}\to\mathrm{H},\ell=0}=\{-M_0^{\mathrm{Z}\to\mathrm{H}}(t,r)Y_{00}(\Omega),M_1^{\mathrm{Z}\to\mathrm{H}}(t,r)Y_{00}(\Omega),0,0\}. \tag{2.39}$$

Substituting this into Eq. (2.38), we can obtain the gauge transformation equations as

$$\frac{\partial^2M_0^{\mathrm{Z}\to\mathrm{H}}(t,r)}{\partial r^2}+\frac{2}{r}\frac{\partial M_0^{\mathrm{Z}\to\mathrm{H}}}{\partial r}=0, \tag{2.40}$$

$$f(r)\frac{\partial^2M_1^{\mathrm{Z}\to\mathrm{H}}}{\partial r^2}+\frac{2}{r}\frac{\partial M_1^{\mathrm{Z}\to\mathrm{H}}}{\partial r}-\frac{2f(r)}{r^2}M_1^{\mathrm{Z}\to\mathrm{H}}=4\pi r\left(\frac{1}{f(r)^2}A_{00}^{(0)}-A_{00}\right)+\frac{f'(r)}{2f(r)}H_0^{\mathrm{Z}}(t,r)+\frac{2-3rf'(r)}{2rf(r)}H_2^{\mathrm{Z}}(t,r). \tag{2.41}$$

E. Odd part of $\ell=1$ mode

For the odd part of the $\ell=1$ mode, we can give the gauge transformation generator as the following form:

$$\xi_{\mu}^{(\mathrm{odd}),\mathrm{Z}\to\mathrm{H}}=\sum_m\left\{0,0,\frac{-\Lambda_{1m}^{\mathrm{Z}\to\mathrm{H}}(t,r)}{\sin\theta}\partial_\phi Y_{1m}(\theta,\phi),\Lambda_{1m}^{\mathrm{Z}\to\mathrm{H}}(t,r)\sin\theta\,\partial_\theta Y_{1m}(\theta,\phi)\right\}. \tag{2.42}$$

From Eq. (2.38), the gauge transformation equation is now given by

$$\left[-\frac{1}{f(r)}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial r}\left(f(r)\frac{\partial}{\partial r}\right)-\frac{2}{r^2}\right]\Lambda_{1m}^{\mathrm{Z}\to\mathrm{H}}(t,r)=f(r)\frac{\partial h_{1,m}^{\mathrm{Z},\ell=1}(t,r)}{\partial r}+\frac{1+f(r)}{r}h_{1,m}^{\mathrm{Z},\ell=1}(t,r). \tag{2.43}$$

F. Even part of $\ell=1$ mode

For the even part of the $\ell=1$ mode, the gauge transformation generator can be divided into the scalar and vector parts as the case for the $\ell\geqslant2$ modes:

$$\xi_{\mu}^{(\mathrm{even}),\mathrm{Z}\to\mathrm{H}}=\xi^{\mathrm{Z}\to\mathrm{H}}_{;\mu}+\xi_{\mu}^{(\mathrm{v}),\,\mathrm{Z}\to\mathrm{H}},\qquad \xi_{\mu}^{(\mathrm{v}),\mathrm{Z}\to\mathrm{H};\mu}=0, \tag{2.44}$$

where $\xi^{\mathrm{Z}\to\mathrm{H}}$ and $\xi_{\mu}^{(\mathrm{v}),\,\mathrm{Z}\to\mathrm{H}}$ can be expressed in the Fourier-harmonic expanded form:

$$\xi^{\mathrm{Z}\to\mathrm{H}}=\int d\omega\sum_m\frac{1}{r}\tilde{\xi}^{\,\mathrm{Z}\to\mathrm{H}}_{1m\omega}(r)e^{-i\omega t}Y_{1m}(\theta,\phi), \tag{2.45}$$

$$\xi_{\mu}^{(\mathrm{v}),\mathrm{Z}\to\mathrm{H}}=\int d\omega\sum_m e^{-i\omega t}\left[\frac{1}{r}M_{01m\omega}^{\mathrm{Z}\to\mathrm{H}}(r)Y_{1m}(\theta,\phi)(\boldsymbol{e}_t)_\mu+\frac{1}{r}M_{11m\omega}^{\mathrm{Z}\to\mathrm{H}}(r)Y_{1m}(\theta,\phi)(\boldsymbol{e}_r)_\mu+\frac{1}{2}\left\{\frac{-i\omega r}{f(r)}M_{01m\omega}^{\mathrm{Z}\to\mathrm{H}}(r)+\partial_r\big(rf(r)M_{11m\omega}^{\mathrm{Z}\to\mathrm{H}}(r)\big)\right\}(\boldsymbol{e}_3)_\mu\right]. \tag{2.46}$$

By making some changes to Eqs. (2.13) and (2.16), the gauge transformation equations may be derived by using the formalism for the even parity part of the $\ell\geqslant2$ modes. That is, we apply the following replacements to the source term:

$$\overline{h}^{\mathrm{RW};\nu}_{\mu\nu} \to \overline{h}^{\mathrm{Z}}_{\mu\nu}{}^{;\nu} = \int d\omega \sum_m e^{-i\omega t} \tag{2.47}$$

$$\times\Bigg\{\left[-\frac{2f(r)}{r}H_{1,m}^{\mathrm{Z},\ell=1}(r) - \frac{i\omega}{2}\left(H_{0,m}^{\mathrm{Z},\ell=1}(r) - H_{2,m}^{\mathrm{Z},\ell=1}(r)\right) - 8\pi i r B_{1m\omega}^{(0)}(r)\right] Y_{1m}(\theta,\phi)(\boldsymbol{e}_t)_\mu$$
$$+\left[-\frac{r^2}{2}\frac{d}{dr}\left(\frac{1}{r^2}\left(H_{0,m}^{\mathrm{Z},\ell=1}(r) - H_{2,m}^{\mathrm{Z},\ell=1}(r)\right)\right) + 8\pi r B_{1m\omega}(r)\right] Y_{1m}(\theta,\phi)(\boldsymbol{e}_r)_\mu$$
$$+\left[\frac{1}{2}\left(H_{0,m}^{\mathrm{Z},\ell=1}(r) - H_{2,m}^{\mathrm{Z},\ell=1}(r)\right)\right](\boldsymbol{e}_3)_\mu\Bigg\}, \tag{2.48}$$

$$\frac{1}{r}\tilde{H}_{\ell m\omega} \to \frac{1}{r}\tilde{H}_{1m\omega}^{\mathrm{Z},\ell=1} = \frac{H_{0,m}^{\mathrm{Z},\ell=1} - H_{2,m}^{\mathrm{Z},\ell=1}}{2}. \tag{2.49}$$

III. SUMMARY AND DISCUSSION

In this paper, to solve the gauge problem of the gravitational self-force, we have considered the gauge transformation from the Regge-Wheeler gauge to the harmonic gauge and have presented a formalism to obtain the infinitesimal displacement vector of this transformation, ξ^μ. First, we have performed the Fourier-harmonic expansion of ξ^μ and divided it into the odd and even parity parts. The odd part has only one degree of freedom and it turns out that the gauge transformation can be found by solving a single second-order differential equation for the radial function. As for the even parity part, we have further divided it into scalar and vector parts where the scalar part is given by the gradient of a scalar function and the vector part is divergence-free. The scalar part has by definition only one degree of freedom, and we have found that it can be obtained by solving two second-order differential equations consecutively. These two equations are found to be identical to the $s=0$ Teukolsky equation. The vector part has two degrees of freedom, and the gauge transformation equations give equations that are coupled in a complicated way. However, by introducing two auxiliary variables which satisfy the $s=\pm 1$ Teukolsky equations, we have succeeded in deriving a decoupled second-order equation for one of the gauge functions with the source term given by the auxiliary variables. Interestingly, this second-order equation has the same form as the $s=0$ Teukolsky equation. The other gauge function is then simply given by applying a differential operator to the first.

Since all the equations to be solved have the form analogous to or equal to the Regge-Wheeler equation, we can derive analytic expressions for their homogeneous solutions by using the Mano-Suzuki-Takasugi method [14] and construct the Green function from these homogeneous solutions. So we conclude that the gauge transformation can be solved by using the Green function method, and we can construct the metric perturbation in the harmonic gauge. In practice, however, it may not be easy to solve for the gauge transformation since it involves products of Green functions with double integrals. Derivation of the gauge transformation functions in a closed, practically tractable form is left for future study.

Another approach to the gauge problem is to consider the self-force in a gauge different from the harmonic gauge, similar to (but very different in principle from) the intermediate gauge approach proposed by Barack and Ori [12]. Here the recent result by Detweiler and Whiting [3] becomes crucial. Their observation that the *S* part and the *R* part play the identical roles as the direct part and the tail part, respectively, and that the *S* part satisfies the same inhomogeneous Einstein equations as the full metric perturbation enables us to define the *S* part and the *R* part of the metric perturbation unambiguously in an arbitrary gauge as long as the gauge condition is consistent with the Einstein equations. For example, given the *S* part of the metric perturbation in the harmonic gauge, one can perform the gauge transformation of it to the RW gauge and the resulting metric perturbation which satisfies the Einstein equations can be identified as the *S* part of the metric perturbation in the RW gauge. Then, after solving the Regge-Wheeler-Zerilli equations to obtain the full metric perturbation, it is straightforward to derive the *R* part of the metric perturbation in the RW gauge [15]. The calculation of the self-force in the RW gauge in this manner is in progress [16].

Finally, we comment on the self-force in the case of the Kerr background. In the Schwarzschild case, it was possible to use the Regge-Wheeler-Zerilli formalism to obtain the metric perturbation in the RW gauge. However, in the Kerr case, there is no known gauge in which the full metric perturbation can be calculated. The Chrzanowski method [4] based on the Teukolsky formalism can give the metric perturbation in the (ingoing or outgoing) radiation gauge, but only outside the range of radial coordinates the orbit resides in. One possible way to circumvent this difficulty is to consider first the regularization of the Weyl scalar Ψ_4. Given an orbit, Ψ_4 can be calculated by the Teukolsky formalism, and the *S* part of it, Ψ_4^S, can be calculated from the *S* part of the metric perturbation in the harmonic gauge, $h_{\mu\nu}^{\mathrm{S,H}}$,

$$\Psi_4^S = \hat{\Psi}_4[h_{\mu\nu}^{\mathrm{S,H}}], \tag{3.1}$$

where $\hat{\Psi}_4$ is the operator to derive the Weyl scalar from a given metric perturbation. Then the R part of Ψ_4 can be derived by subtracting the S part from the Weyl scalar,

$$\Psi_4^R = \Psi_4 - \Psi_4^S. \tag{3.2}$$

Now Ψ_4^R satisfies the homogeneous Teukolsky equation. Hence using the Chrzanowski method, we may construct the R part of the metric perturbation in the radiation gauge and derive the self-force. Since this procedure involves many derivative operations, the metric perturbation $h_{\mu\nu}^{S,\mathrm{H}}$ has to be evaluated with a sufficiently high accuracy which may be practically a difficult task, if not impossible. The feasibility of this method should surely be investigated.

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APPENDIX A: THE FIELD EQUATION ON THE REGGE-WHEELER GAUGE

In this appendix, we recapitulate the equations for the Regge-Wheeler-Zerilli formalism. In doing so, we correct some minor errors in Zerilli's paper [6]. Here an equation number given as (Z:1) denotes the equation (1) in Zerilli's paper for comparison, and a label $[CRTD]$ to an equation means it is corrected.

We consider the linearized Einstein equations for the perturbed metric,

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu},$$

where $g_{\mu\nu}$ is the background metric. Then the Einstein tensor and the stress-energy tensor up to the linear order can be expanded as

$$G_{\mu\nu}[\tilde{g}_{\mu\nu}] = G_{\mu\nu}[g_{\mu\nu}] + \delta G_{\mu\nu}[h_{\mu\nu}] + O(h^2), \tag{A1}$$

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + \delta T_{\mu\nu}, \tag{A2}$$

where $f_\mu = h_{\mu\alpha}{}^{;\alpha}$ and

$$\begin{aligned}\delta G_{\mu\nu}[h_{\mu\nu}] = & -\frac{1}{2} h_{\mu\nu;\alpha}{}^{;\alpha} + f_{(\mu;\nu)} - R_{\alpha\mu\beta\nu} h^{\alpha\beta} - \frac{1}{2} h_{;\mu;\nu} \\ & + R^{\alpha}{}_{(\mu} h_{\nu)\alpha} - \frac{1}{2} g_{\mu\nu}(f_\lambda{}^{;\lambda} - h_{;\lambda}{}^{;\lambda}) - \frac{1}{2} h_{\mu\nu} R \\ & + \frac{1}{2} g_{\mu\nu} h_{\alpha\beta} R^{\alpha\beta}.\end{aligned} \tag{A3}$$

When the background is Ricci flat, $R_{\mu\nu}^{(\mathrm{b})} = 0$, the above equation is rewritten as

$$\begin{aligned} & -\frac{1}{2} h_{\mu\nu;\alpha}{}^{;\alpha} + f_{(\mu;\nu)} - R_{\alpha\mu\beta\nu} h^{\alpha\beta} - \frac{1}{2} h_{;\mu;\nu} \\ & -\frac{1}{2} g_{\mu\nu}(f_\lambda{}^{;\lambda} - h_{;\lambda}{}^{;\lambda}) = 8\pi \delta T_{\mu\nu}.\end{aligned} \tag{A4}$$

We apply the above to the case of the Schwarzschild background, and expand $h_{\mu\nu}$ [(Z:D2a) and (Z:D2b)] and $\delta T_{\mu\nu}$ in tensor harmonics,

$$\begin{aligned}\boldsymbol{h} = \sum_{\ell m} \Bigg[& f(r) H_{0\ell m}(t,r) \boldsymbol{a}_{\ell m}^{(0)} - i\sqrt{2} H_{1\ell m}(t,r) \boldsymbol{a}_{\ell m}^{(1)} + \frac{1}{f(r)} H_{2\ell m}(t,r) \boldsymbol{a}_{\ell m} - \frac{i}{r}\sqrt{2\ell(\ell+1)} h_{0\ell m}^{(e)}(t,r) \boldsymbol{b}_{\ell m}^{(0)} \\ & + \frac{1}{r}\sqrt{2\ell(\ell+1)} h_{1\ell m}^{(e)}(t,r) \boldsymbol{b}_{\ell m} + \sqrt{\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)} G_{\ell m}(t,r) \boldsymbol{f}_{\ell m} + \left(\sqrt{2} K_{\ell m}(t,r) - \frac{\ell(\ell+1)}{\sqrt{2}} G_{\ell m}(t,r)\right) \boldsymbol{g}_{\ell m} \\ & - \frac{\sqrt{2\ell(\ell+1)}}{r} h_{0\ell m}(t,r) \boldsymbol{c}_{\ell m}^{(0)} + \frac{i\sqrt{2\ell(\ell+1)}}{r} h_{1\ell m}(t,r) \boldsymbol{c}_{\ell m} + \frac{\sqrt{2\ell(\ell+1)(\ell-1)(\ell+2)}}{2r^2} h_{2\ell m}(t,r) \boldsymbol{d}_{\ell m} \Bigg] \quad [CRTD],\end{aligned} \tag{A5}$$

$$\begin{aligned}\delta\boldsymbol{T} = \sum_{\ell m} & [A_{\ell m}^{(0)} \boldsymbol{a}_{\ell m}^{(0)} + A_{\ell m}^{(1)} \boldsymbol{a}_{\ell m}^{(1)} + A_{\ell m} \boldsymbol{a}_{\ell m} + B_{\ell m}^{(0)} \boldsymbol{b}_{\ell m}^{(0)} + B_{\ell m} \boldsymbol{b}_{\ell m} \\ & + Q_{\ell m}^{(0)} \boldsymbol{c}_{\ell m}^{(0)} + Q_{\ell m} \boldsymbol{c}_{\ell m} + D_{\ell m} \boldsymbol{d}_{\ell m} + G_{\ell m}^{(s)} \boldsymbol{g}_{\ell m} + F_{\ell m} \boldsymbol{f}_{\ell m}],\end{aligned} \tag{A6}$$

where we use $h_{0\ell m}^{(e)}$ and $h_{1\ell m}^{(e)}$ for the even part coefficients instead of $h_{0\ell m}^{(m)}$ and $h_{0\ell m}^{(m)}$, respectively, in Zerilli's paper, and the coefficient $G_{\ell m}^{(s)}$ instead of Zerilli's notation $G_{\ell m}$ for the energy-momentum tensor, and $\boldsymbol{a}_{\ell m}^{(0)}$, $\boldsymbol{a}_{\ell m}$, ... are the ten tensor harmonics (Z:A2a-j) defined as

$$\mathbf{a}_{\ell m}{}^{(0)}=\begin{pmatrix} Y_{\ell m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A7}$$

$$\mathbf{a}_{\ell m}{}^{(1)}=(i/\sqrt{2})\begin{pmatrix} 0 & Y_{\ell m} & 0 & 0 \\ Sym & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A8}$$

$$\mathbf{a}_{\ell m}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Y_{\ell m} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A9}$$

$$\mathbf{b}_{\ell m}{}^{(0)}=ir[2\ell(\ell+1)]^{-1/2}\begin{pmatrix} 0 & 0 & (\partial/\partial\theta)Y_{\ell m} & (\partial/\partial\phi)Y_{\ell m} \\ 0 & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \end{pmatrix}, \tag{A10}$$

$$\mathbf{b}_{\ell m}=r[2\ell(\ell+1)]^{-1/2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (\partial/\partial\theta)Y_{\ell m} & (\partial/\partial\phi)Y_{\ell m} \\ 0 & Sym & 0 & 0 \\ 0 & Sym & 0 & 0 \end{pmatrix}, \tag{A11}$$

$$\mathbf{c}_{\ell m}{}^{(0)}=r[2\ell(\ell+1)]^{-1/2}\begin{pmatrix} 0 & 0 & (1/\sin\theta)(\partial/\partial\phi)Y_{\ell m} & -\sin\theta(\partial/\partial\theta)Y_{\ell m} \\ 0 & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \end{pmatrix}, \tag{A12}$$

$$\mathbf{c}_{\ell m}=ir[2\ell(\ell+1)]^{-1/2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (1/\sin\theta)(\partial/\partial\phi)Y_{\ell m} & -\sin\theta(\partial/\partial\theta)Y_{\ell m} \\ 0 & Sym & 0 & 0 \\ 0 & Sym & 0 & 0 \end{pmatrix}, \tag{A13}$$

$$\mathbf{d}_{\ell m}=ir^2[2\ell(\ell+1)(\ell-1)(\ell+2)]^{-1/2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(1/\sin\theta)X_{\ell m} & \sin\theta W_{\ell m} \\ 0 & 0 & Sym & \sin\theta X_{\ell m} \end{pmatrix}, \tag{A14}$$

$$\mathbf{g}_{\ell m}=(r^2/\sqrt{2})\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Y_{\ell m} & 0 \\ 0 & 0 & 0 & \sin^2\theta Y_{\ell m} \end{pmatrix}, \tag{A15}$$

$$\mathbf{f}_{\ell m}=r^2[2\ell(\ell+1)(\ell-1)(\ell+2)]^{-1/2}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&W_{\ell m}&X_{\ell m}\\0&0&Sym&-\sin^2\theta W_{\ell m}\end{pmatrix}\quad[CRTD]. \tag{A16}$$

Here the angular functions $X_{\ell m}$ and $W_{\ell m}$ are given by

$$X_{\ell m}=2\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\theta}-\cot\theta\right)Y_{\ell m}, \tag{A17}$$

$$W_{\ell m}=\left(\frac{\partial^2}{\partial\theta^2}-\cot\theta\frac{\partial}{\partial\theta}-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)Y_{\ell m}. \tag{A18}$$

For a point particle moving along a geodesic, the stress-energy tensor takes the form

$$\begin{aligned}T^{\mu\nu}&=\mu\int_{-\infty}^{+\infty}\delta^{(4)}(x-z(\tau))\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}d\tau\\&=\mu\gamma\frac{dz^\mu}{dt}\frac{dz^\nu}{dt}\frac{\delta(r-R(t))}{r^2}\delta^{(2)}(\Omega-\Omega(t)),\end{aligned} \tag{A19}$$

where the following notation for the particle orbit is used.

$$z^\mu=z^\mu(\tau)=\{T(\tau),R(\tau),\Theta(\tau),\Phi(\tau)\}, \tag{A20}$$

$$\gamma=\frac{dT(\tau)}{d\tau}. \tag{A21}$$

This stress-energy tensor is expressed in terms of the tensor harmonics as given in Table I [corresponding to (Z:Table III)]. Here it is noted that the sign errors in $Q^{(0)}_{\ell m}$ and $Q_{\ell m}$ are corrected.

Substituting Eqs. (A5) and (A6) into Eq. (A4), we obtain the field equations for each harmonic mode. For the odd part which has the odd parity $(-1)^{\ell+1}$, in the RW gauge in which $h_2=0$, the following three equations (Z:C6a-c) are derived:

$$\frac{\partial^2 h_0}{\partial r^2}-\frac{\partial^2 h_1}{\partial t\partial r}-\frac{2}{r}\frac{\partial h_1}{\partial t}+\left[\frac{4M}{r^2}-\frac{\ell(\ell+1)}{r}\right]\frac{h_0}{r-2M}=\frac{8\pi}{\sqrt{\ell(\ell+1)/2}}\frac{r^2}{r-2M}Q^{(0)}_{\ell m}\quad[CRTD], \tag{A22}$$

$$\frac{\partial^2 h_1}{\partial t^2}-\frac{\partial^2 h_0}{\partial t\partial r}+\frac{2}{r}\frac{\partial h_0}{\partial t}+\frac{(\ell-1)(\ell+2)(r-2M)}{r^3}h_1=-\frac{8\pi i(r-2M)}{\sqrt{\ell(\ell+1)/2}}Q_{\ell m}, \tag{A23}$$

$$\frac{\partial}{\partial r}\left[\left(1-\frac{2M}{r}\right)h_1\right]-\frac{r}{r-2M}\frac{\partial h_0}{\partial t}=-\frac{8\pi i r^2}{\sqrt{\ell(\ell+1)(\ell-1)(\ell+2)/2}}D_{\ell m}, \tag{A24}$$

where Zerilli's sign errors of the source terms are corrected.

For the even part which has the even parity $(-1)^\ell$, we have seven equations (Z:C7a-g) in the RW gauge in which $h_0^{(e)}=h_1^{(e)}=G=0$.

$$\begin{aligned}&\left(1-\frac{2M}{r}\right)^2\frac{\partial^2 K}{\partial r^2}+\frac{1}{r}\left(1-\frac{2M}{r}\right)\left(3-\frac{5M}{r}\right)\frac{\partial K}{\partial r}-\frac{1}{r}\left(1-\frac{2M}{r}\right)^2\frac{\partial H_2}{\partial r}-\frac{1}{r^2}\left(1-\frac{2M}{r}\right)(H_2-K)-\frac{\ell(\ell+1)}{2r^2}\left(1-\frac{2M}{r}\right)(H_2+K)\\&\quad=-8\pi A^{(0)}_{\ell m},\end{aligned} \tag{A25}$$

$$\frac{\partial}{\partial t}\left[\frac{\partial K}{\partial r}+\frac{1}{r}(K-H_2)-\frac{M}{r(r-2M)}K\right]-\frac{\ell(\ell+1)}{2r^2}H_1=-4\sqrt{2}\pi i A^{(1)}_{\ell m}, \tag{A26}$$

$$\left(\frac{r}{r-2M}\right)^2\frac{\partial^2 K}{\partial t^2}-\frac{r-M}{r(r-2M)}\frac{\partial K}{\partial r}-\frac{2}{r-2M}\frac{\partial H_1}{\partial t}+\frac{1}{r}\frac{\partial H_0}{\partial r}+\frac{1}{r(r-2M)}(H_2-K)+\frac{\ell(\ell+1)}{2r(r-2M)}(K-H_0)=-8\pi A_{\ell m}, \tag{A27}$$

TABLE I. Stress-energy tensor in terms of tensor harmonics.

Description	Dependence of "driving term" on r and t	Tensor harmonic
Even	$A_{\ell m}(r,t)=\mu\gamma\left(\frac{dR}{dt}\right)^2(r-2M)^{-2}\delta(r-R(t))Y^*_{\ell m}(\Omega(t))$	$\boldsymbol{a}_{\ell m}(\theta,\phi)$
Even	$A^{(0)}_{\ell m}=\mu\gamma\left(1-\frac{2M}{r}\right)^2 r^{-2}\delta(r-R(t))Y^*_{\ell m}(\Omega(t))$	$\boldsymbol{a}^{(0)}_{\ell m}(\theta,\phi)$
Even	$A^{(1)}_{\ell m}=\sqrt{2}i\mu\gamma\frac{dR}{dt}r^{-2}\delta(r-R(t))Y^*_{\ell m}(\Omega(t))$	$\boldsymbol{a}^{(1)}_{\ell m}(\theta,\phi)$
Even	$B^{(0)}_{\ell m}=[\frac{1}{2}\ell(\ell+1)]^{-1/2}i\mu\gamma\left(1-\frac{2M}{r}\right)r^{-1}\delta(r-R(t))dY^*_{\ell m}(\Omega(t))/dt$	$\boldsymbol{b}^{(0)}_{\ell m}(\theta,\phi)$
Even	$B_{\ell m}=[\frac{1}{2}\ell(\ell+1)]^{-1/2}\mu\gamma(r-2M)^{-1}\frac{dR}{dt}\delta(r-R(t))dY^*_{\ell m}(\Omega(t))/dt$	$\boldsymbol{b}_{\ell m}(\theta,\phi)$
Odd	$Q^{(0)}_{\ell m}=-[\frac{1}{2}\ell(\ell+1)]^{-1/2}\mu\gamma\left(1-\frac{2M}{r}\right)r^{-1}\delta(r-R(t))\left[\frac{1}{\sin\Theta}\frac{\partial Y^*_{\ell m}}{\partial\Phi}\frac{d\Theta}{dt}-\sin\Theta\frac{\partial Y^*_{\ell m}}{\partial\Theta}\frac{d\Phi}{dt}\right]$	$\boldsymbol{c}^{(0)}_{\ell m}(\theta,\phi)$
Odd	$Q_{\ell m}=-[\frac{1}{2}\ell(\ell+1)]^{-1/2}i\mu\gamma\frac{dR}{dt}(r-2M)^{-1}\delta(r-R(t))\left[\frac{1}{\sin\Theta}\frac{\partial Y^*_{\ell m}}{\partial\Phi}\frac{d\Theta}{dt}-\sin\Theta\frac{\partial Y^*_{\ell m}}{\partial\Theta}\frac{d\Phi}{dt}\right]$	$\boldsymbol{c}_{\ell m}(\theta,\phi)$
Odd	$D_{\ell m}=-[\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)]^{-1/2}i\mu\gamma\delta(r-R(t))\left(\frac{1}{2}\left[\left(\frac{d\Theta}{dt}\right)^2-\sin^2\Theta\left(\frac{d\Phi}{dt}\right)^2\right]\frac{1}{\sin\Theta}X^*_{\ell m}[\Omega(t)]-\sin\Theta\frac{d\Phi}{dt}\frac{d\Theta}{dt}W^*_{\ell m}[\Omega(t)]\right)$	$\boldsymbol{d}_{\ell m}(\theta,\phi)$
Even	$F_{\ell m}=[\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)]^{-1/2}\mu\gamma\delta(r-R(t))\left(\frac{d\Phi}{dt}\frac{d\Theta}{dt}X^*_{\ell m}[\Omega(t)]+\frac{1}{2}\left[\left(\frac{d\Theta}{dt}\right)^2-\sin^2\Theta\left(\frac{d\Phi}{dt}\right)^2\right]W^*_{\ell m}[\Omega(t)]\right)$	$\boldsymbol{f}_{\ell m}(\theta,\phi)$
Even	$G^{(s)}_{\ell m}=\frac{\mu\gamma}{\sqrt{2}}\delta(r-R(t))\left[\left(\frac{d\Theta}{dt}\right)^2+\sin^2\Theta\left(\frac{d\Phi}{dt}\right)^2\right]Y^*_{\ell m}(\Omega(t))$	$\boldsymbol{g}_{\ell m}(\theta,\phi)$

$$\frac{\partial}{\partial r}\left[\left(1-\frac{2M}{r}\right)H_1\right]-\frac{\partial}{\partial t}(H_2+K)=\frac{8\pi ir}{\sqrt{\ell(\ell+1)/2}}B^{(0)}_{\ell m}, \tag{A28}$$

$$-\frac{\partial H_1}{\partial t}+\left(1-\frac{2M}{r}\right)\frac{\partial}{\partial r}(H_0-K)+\frac{2M}{r^2}H_0+\frac{1}{r}\left(1-\frac{M}{r}\right)(H_2-H_0)=\frac{8\pi(r-2M)}{\sqrt{\ell(\ell+1)/2}}B_{\ell m}, \tag{A29}$$

$$-\frac{r}{r-2M}\frac{\partial^2 K}{\partial t^2}+\left(1-\frac{2M}{r}\right)\frac{\partial^2 K}{\partial r^2}+\frac{2}{r}\left(1-\frac{M}{r}\right)\frac{\partial K}{\partial r}-\frac{r}{r-2M}\frac{\partial^2 H_2}{\partial t^2}+2\frac{\partial^2 H_1}{\partial t\partial r}-\left(1-\frac{2M}{r}\right)\frac{\partial^2 H_0}{\partial r^2}$$
$$+\frac{2(r-M)}{r(r-2M)}\frac{\partial H_1}{\partial t}-\frac{1}{r}\left(1-\frac{M}{r}\right)\frac{\partial H_2}{\partial r}-\frac{r+M}{r^2}\frac{\partial H_0}{\partial r}+\frac{\ell(\ell+1)}{2r^2}(H_0-H_2)=8\sqrt{2}\pi G^{(s)}_{\ell m}, \tag{A30}$$

$$\frac{H_0-H_2}{2}=\frac{8\pi r^2 F_{\ell m}}{\sqrt{\ell(\ell+1)(\ell-1)(\ell+2)/2}}, \tag{A31}$$

where we have corrected the sign errors of the source terms in Zerilli's paper as the odd parity part.

We now consider the Fourier transform of the above field equations. The Fourier coefficients are defined, for example, as

$$h_{0\ell m\omega}(r)=\int_{-\infty}^{+\infty}dt\,h_{0\ell m}(t,r)e^{i\omega t}. \tag{A32}$$

Then we derive the Regge-Wheeler-Zerilli equations and construct the metric perturbation under the RW gauge condition.

For the odd part, a new radial function $R_{\ell m\omega}^{(\mathrm{odd})}(r)$ is introduced, in terms of which the two radial functions $h_{0\ell m\omega}$ and $h_{1\ell m\omega}$ for the metric perturbation are expressed as

$$h_{1\ell m\omega}=\frac{r^2}{r-2M}R_{\ell m\omega}^{(\mathrm{odd})}, \tag{A33}$$

$$h_{0\ell m\omega}=\frac{i}{\omega}\frac{d}{dr^*}(rR_{\ell m\omega}^{(\mathrm{odd})})-\frac{8\pi r(r-2M)}{\omega\left[\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)\right]^{1/2}}D_{\ell m\omega}. \tag{A34}$$

The new radial function satisfies the Regge-Wheeler equation (Z:11),

$$\frac{d^2R_{\ell m\omega}{}^{(\mathrm{odd})}}{dr^{*2}}+[\omega^2-V_\ell^{(\mathrm{odd})}(r)]R_{\ell m\omega}^{(\mathrm{odd})}=\frac{8\pi i}{\left[\frac{1}{2}l(\ell+1)(\ell-1)(\ell+2)\right]^{1/2}}\frac{r-2M}{r^2}\times\left(-r^2\frac{d}{dr}\left[\left(1-\frac{2M}{r}\right)D_{\ell m\omega}\right]+(r-2M)[(\ell-1)(\ell+2)]^{1/2}Q_{\ell m\omega}\right)\quad [CRTD], \tag{A35}$$

where $r^*=r+2M\log(r/2M-1)$ and

$$V_\ell^{(\mathrm{odd})}(r)=\left(1-\frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2}-\frac{6M}{r^3}\right). \tag{A36}$$

For the even part, a new radial function $R_{\ell m\omega}^{(\mathrm{even})}(r)$ is introduced, in terms of which the four radial functions (Z:13–16) are expressed as

$$K_{\ell m\omega}=\frac{\lambda(\lambda+1)r^2+3\lambda Mr+6M^2}{r^2(\lambda r+3M)}R_{\ell m\omega}^{(\mathrm{even})}+\frac{r-2M}{r}\frac{dR_{\ell m\omega}^{(\mathrm{even})}}{dr}-\frac{r(r-2M)}{\lambda r+3M}\tilde{C}_{1\ell m\omega}+\frac{i(r-2M)^2}{r(\lambda r+3M)}\tilde{C}_{2\ell m\omega}\quad [CRTD], \tag{A37}$$

$$H_{1\ell m\omega}=-i\omega\frac{\lambda r^2-3\lambda Mr-3M^2}{(r-2M)(\lambda r+3M)}R_{\ell m\omega}^{(\mathrm{even})}-i\omega r\frac{dR_{\ell m\omega}^{(\mathrm{even})}}{dr}+\frac{i\omega r^3}{\lambda r+3M}\tilde{C}_{1\ell m\omega}+\frac{\omega r(r-2M)}{\lambda r+3M}\tilde{C}_{2\ell m\omega}\quad [CRTD], \tag{A38}$$

$$H_{0\ell m\omega}=\frac{\lambda r(r-2M)-\omega^2r^4+M(r-3M)}{(r-2M)(\lambda r+3M)}K_{\ell m\omega}+\frac{M(\lambda+1)-\omega^2r^3}{i\omega r(\lambda r+3M)}H_{1\ell m\omega}+\tilde{B}_{\ell m\omega}\quad [CRTD], \tag{A39}$$

$$H_{2\ell m\omega}=H_{0\ell m\omega}-16\pi r^2\left[\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)\right]^{-1/2}\times F_{\ell m\omega}\quad [CRTD], \tag{A40}$$

where we have introduced the symbol λ for

$$\lambda=\frac{1}{2}(\ell-1)(\ell+2), \tag{A41}$$

and the local source terms [(Z:17), (Z:20) and (Z:21)] by

$$\tilde{B}_{\ell m\omega}=\frac{8\pi r^2(r-2M)}{\lambda r+3M}\left\{A_{\ell m\omega}+\left[\frac{1}{2}\ell(\ell+1)\right]^{-1/2}B_{\ell m\omega}\right\}-4\pi\sqrt{\frac{2}{\lambda r+3M}}\frac{Mr}{\omega}A_{\ell m\omega}^{(1)}, \tag{A42}$$

$$\tilde{C}_{1\ell m\omega}=\frac{8\pi}{\sqrt{2}\omega}A_{\ell m\omega}^{(1)}+\frac{1}{r}\tilde{B}_{\ell m\omega}-16\pi r\left[\frac{1}{2}\ell(\ell+1)(\ell-1)\times(\ell+2)\right]^{-1/2}F_{\ell m\omega}\quad [CRTD], \tag{A43}$$

$$\tilde{C}_{2\ell m\omega}=-\frac{8\pi r^2}{i\omega}\frac{\left[\frac{1}{2}\ell(\ell+1)\right]^{-1/2}}{r-2M}B_{\ell m\omega}^{(0)}-\frac{ir}{r-2M}\tilde{B}_{\ell m\omega}+\frac{16\pi ir^3}{r-2M}\left[\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)\right]^{-1/2}\times F_{\ell m\omega}\quad [CRTD]. \tag{A44}$$

We note that the above radial functions for the metric perturbation have the local source terms which have the δ-function behavior at the particle location. The new radial function obeys the wave equation,

$$\frac{d^2R_{\ell m\omega}{}^{(\mathrm{even})}}{dr^{*2}}+[\omega^2-V_\ell^{(\mathrm{even})}(r)]R_{\ell m\omega}^{(\mathrm{even})}=S_{\ell m\omega}, \tag{A45}$$

where

$$V_\ell^{(\mathrm{even})}(r)=\left(1-\frac{2M}{r}\right)\times\frac{2\lambda^2(\lambda+1)r^3+6\lambda^2Mr^2+18\lambda M^2r+18M^3}{r^3(\lambda r+3M)^2}, \tag{A46}$$

and the source term is

$$S_{\ell m\omega}=-i\frac{r-2M}{r}\frac{d}{dr}\left[\frac{(r-2M)^2}{r(\lambda r+3M)}\left(\frac{ir^2}{r-2M}\tilde{C}_{1\ell m\omega}+\tilde{C}_{2\ell m\omega}\right)\right]+i\frac{(r-2M)^2}{r(\lambda r+3M)^2}\times\left[\frac{\lambda(\lambda+1)r^2+3\lambda Mr+6M^2}{r^2}\tilde{C}_{2\ell m\omega}+i\frac{\lambda r^2-3\lambda Mr-3M^2}{r-2M}\tilde{C}_{1\ell m\omega}\right]. \tag{A47}$$

The above equation (A45) is called the Zerilli equation. The Zerilli equation can be transformed to the Regge-Wheeler equation by the Chandrasekhar transformation [17]. So we may focus only on the Regge-Wheeler equation if desired. The Regge-Wheeler homogeneous solutions are discussed in detail by Mano *et al.* [14]. Using their method, one can construct the retarded Green function to solve the inhomogeneous Regge-Wheeler equation. Then the metric perturbation in the RW gauge is obtained from Eqs. (A33) and (A34) for the odd part and from Eqs. (A37)–(A40) for the even part.

APPENDIX B: THE FIELD EQUATION FOR THE NONRADIATIVE MODES

The above formalism is applicable only to the $\ell \geqslant 2$ modes. Therefore we have to deal with the $\ell=0,1$ (nonradiative) modes separately. In this appendix, the field equation for these modes are given. (This problem is considered for the point particle case in Appendix G of Zerilli's paper [6].)

1. $\ell=0$ mode

First, we consider the $\ell=0$ mode. The $\ell=0$ mode of the metric perturbation and the gauge transformation generators are given, respectively, as

$$\boldsymbol{h}_{00}=f(r)H_0(t,r)\boldsymbol{a}_{00}^{(0)}-\sqrt{2}iH_1(t,r)\boldsymbol{a}_{00}^{(1)}+\frac{1}{f(r)}\times H_2(t,r)\boldsymbol{a}_{00}+\sqrt{2}K(t,r)\boldsymbol{g}_{00}, \tag{B1}$$

$$\xi_{\mu}\ell=0=\{-M_0(t,r)Y_{00}(\Omega),M_1(t,r)Y_{00}(\Omega),0,0\}. \tag{B2}$$

We can choose $M_0(t,r)$, $M_1(t,r)$ so that $H_1(t,r)=K(t,r)=0$. We call this gauge the Zerilli gauge. In this gauge, the field equations are given as follows:

$$\frac{\partial H_2^{\rm Z}(t,r)}{\partial r}+\frac{1}{rf(r)}H_2^{\rm Z}(t,r)=\frac{8\pi r}{f(r)^2}A_{00}^{(0)}, \tag{B3}$$

$$\frac{\partial H_0^{\rm Z}(t,r)}{\partial r}+\frac{1}{rf(r)}H_2^{\rm Z}(t,r)=-8\pi rA_{00}, \tag{B4}$$

where the superscript Z stands for the Zerilli gauge.

2. Odd part of $\ell=1$ mode

For the odd part of the $\ell=1$ mode, the metric perturbation and the gauge transformation generator are given, respectively, as

$$\boldsymbol{h}_{1m}^{(\rm odd)}=\frac{2i}{r}(ih_{0,m}^{\ell=1}(t,r)\boldsymbol{c}_{1m}^{(0)}+h_{1,m}^{\ell=1}(t,r)\boldsymbol{c}_{1m}), \tag{B5}$$

$$\xi_{\mu}^{(\rm odd),\ell=1}=\sum_m\left\{0,0,\frac{-\Lambda_{1m}(t,r)}{\sin\theta}\partial_\phi Y_{1m}(\theta,\phi),\Lambda_{1m}\times(t,r)\sin\theta\partial_\theta Y_{1m}(\theta,\phi)\right\}. \tag{B6}$$

Here we can choose $\Lambda_{1m}(t,r)$ so that $h_{0,m}^{\ell=1}(t,r)=0$. Then the field equations become

$$\frac{\partial^2}{\partial t\partial r}[r^2h_{1,m}^{{\rm Z},\ell=1}]=-\frac{8\pi r^3}{f(r)}Q_{1m}^{(0)}, \tag{B7}$$

$$\frac{\partial^2 h_{1,m}^{{\rm Z},\ell=1}}{\partial t^2}=-8\pi irf(r)Q_{1m}. \tag{B8}$$

3. Even part of $\ell=1$ mode

For the even part of the $\ell=1$ mode, the metric perturbation and the gauge transformation generator may be given in the following form:

$$\boldsymbol{h}_{1m}^{(\rm even)}=f(r)H_{0,m}^{\ell=1}(t,r)\boldsymbol{a}_{1m}^{(0)}-\sqrt{2}iH_{1,m}^{\ell=1}(t,r)\boldsymbol{a}_{1m}^{(1)}+\frac{1}{f(r)}H_{2,m}^{\ell=1}(t,r)\boldsymbol{a}_{1m}-\frac{2i}{r}h_{0,m}^{({\rm e}),\ell=1}(t,r)\boldsymbol{b}_{1m}^{(0)}+\frac{2}{r}h_{0,m}^{({\rm e}),\ell=1}(t,r)\boldsymbol{b}_{1m}+\sqrt{2}K_m^{\ell=1}(t,r)\boldsymbol{g}_{1m}, \tag{B9}$$

$$\xi_{\mu}^{(\rm even),\ell=1}=\sum_m\{-M_{0,m}^{\ell=1}(t,r)Y_{1m},M_{1,m}^{\ell=1}(t,r)Y_{1m},\times M_{2,m}^{\ell=1}(t,r)\partial_\theta Y_{1m},M_{2,m}^{\ell=1}(t,r)\partial_\phi Y_{1m}\}. \tag{B10}$$

Choosing a gauge in which $h_{0,m}^{({\rm e}),\ell=1}(t,r)=h_{1,m}^{({\rm e}),\ell=1}(t,r)=K_m^{\ell=1}(t,r)=0$, the field equations for the even part of the $\ell=1$ mode are

$$rf(r)\frac{\partial H_{2,m}^{{\rm Z},\ell=1}(t,r)}{\partial r}+2H_{2,m}^{{\rm Z},\ell=1}(t,r)=\frac{8\pi r^2}{f(r)}A_{1m}^{(0)}, \tag{B11}$$

$$H_{1,m}^{{\rm Z},\ell=1}(t,r)+r\frac{\partial H_{2,m}^{{\rm Z},\ell=1}}{\partial t}=4\sqrt{2}\pi ir^2A_{1m}^{(1)}, \tag{B12}$$

$$rf(r)\frac{\partial H^{\mathrm{Z},\ell=1}_{0,m}}{\partial r}-2r\frac{\partial H^{\mathrm{Z},\ell=1}_{1,m}}{\partial t}-H^{\mathrm{Z},\ell=1}_{0,m}(t,r)+H^{\mathrm{Z},\ell=1}_{2,m}(t,r)$$
$$=-8\pi r^2 f(r)A_{1m}, \tag{B13}$$

$$-\frac{\partial}{\partial r}[f(r)H^{\mathrm{Z},\ell=1}_{1,m}(t,r)]+\frac{\partial H^{\mathrm{Z},\ell=1}_{2,m}(t,r)}{\partial t}=-8\pi i r B^{(0)}_{1m}, \tag{B14}$$

$$2rf(r)\frac{\partial H^{\mathrm{Z},\ell=1}_{0,m}(t,r)}{\partial r}-2r\frac{\partial H^{\mathrm{Z},\ell=1}_{1,m}(t,r)}{\partial t}$$
$$+(1-3f(r))H^{\mathrm{Z},\ell=1}_{0,m}(t,r) \tag{B15}$$
$$+(1+f(r))H^{\mathrm{Z},\ell=1}_{2,m}(t,r)=16\pi r^2 f(r)B_{1m}, \tag{B16}$$

$$r^2 f(r)\frac{\partial^2 H^{\mathrm{Z},\ell=1}_{0,m}(t,r)}{\partial r^2}+\frac{3r-rf(r)}{2}\frac{\partial H^{\mathrm{Z},\ell=1}_{0,m}(t,r)}{\partial r}$$
$$-H^{\mathrm{Z},\ell=1}_{0,m}(t,r)-2r^2\frac{\partial^2 H^{\mathrm{Z},\ell=1}_{1,m}(t,r)}{\partial t\partial r}$$
$$-\frac{r(1+f(r))}{f(r)}\frac{\partial H^{\mathrm{Z},\ell=1}_{1,m}(t,r)}{\partial t}+\frac{r^2}{f(r)}\frac{\partial^2 H^{\mathrm{Z},\ell=1}_{2,m}(t,r)}{\partial t^2}$$
$$+\frac{r(1+f(r))}{2}\frac{\partial H^{\mathrm{Z},\ell=1}_{2,m}(t,r)}{\partial r}+H^{\mathrm{Z},\ell=1}_{2,m}(t,r)$$
$$=-8\sqrt{2}\pi r^2 G^{(\mathrm{s})}_{1m}. \tag{B17}$$

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[18] Furthermore, the gravitational self-force is, because of the equivalence principle, a gauge-variant notion. To give a genuinely physical meaning to it, one must solve the second order metric perturbation completely. This is, however, beyond the scope of the present paper.