All static spherically symmetric perfect-fluid solutions of Einstein's equations

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An algorithm based on the choice of a single monotone function (subject to boundary conditions) is presented which generates all regular static spherically symmetric perfect-fluid solutions of Einstein's equations. For physically relevant solutions the generating functions must be restricted by nontrivial integral-differential inequalities. Nonetheless, the algorithm is demonstrated here by the construction of an infinite number of previously unknown physically interesting exact solutions.

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Exact solutions of Einstein's field equations provide a route to the physical understanding (and discovery) of relativistic phenomena, a convenient basis from which perturbation methods can proceed, and a check on numerical approximations. Here we look at static spherically symmetric perfect-fluid solutions. Unfortunately, even for this simple type, very few solutions are in fact known, and of these few pass even elementary tests of physical relevance [1]. In this paper, an algorithm based on the choice of a single monotone function (subject to boundary conditions) is presented which generates all regular static spherically symmetric perfect-fluid solutions of Einstein's equations. We are interested only in physically relevant solutions here, and so the algorithm must be supplemented by physical considerations [2]. These additional conditions limit the generating functions allowed by way of nontrivial integral-differential inequalities. The details of how to choose physically relevant generating functions (beyond trial and error) are, at present, not known. Nonetheless, the robustness of the algorithm is demonstrated here by the construction of an infinite number of previously unknown physically interesting exact solutions.

To set the notation, consider a spherically symmetric spacetime $\mathcal{M}$ [3]

$$ds^2_{\mathcal{M}} = ds^2_{\Sigma} + R^2 d\Omega^2, \tag{1}$$

where $d\Omega^2$ is the metric of a unit sphere $[d\theta^2 + \sin^2(\theta) d\phi^2]$ and $R = R(x^1, x^2)$ where the coordinates on the Lorentzian two-space Σ are labeled as x^1 and x^2. Consider a flow (a congruence of unit timelike vectors u^α) tangent to an open region of Σ and write n^α as the normal to u^α in the tangent space of Σ. Both u^α and n^α are uniquely determined. We suppose that Eq. (1) is generated by a fluid subject to the condition $G^\beta_\alpha u^\alpha n_\beta = 0$ where G^β_α is the Einstein tensor (see [4]). Let $G \equiv G^\alpha_\alpha$, $G1 \equiv G^\beta_\alpha u^\alpha u_\beta$, and $G2 \equiv G^\beta_\alpha n^\alpha n_\beta$. In the static case it follows that the flow is shear-free and that

$$G + G1 = 3G2 \tag{2}$$

is a necessary and sufficient condition for Eq. (1) to represent a perfect fluid [5].

First consider $\mathcal{M}$ in "curvature" coordinates:

$$ds^2_{\mathcal{M}} = \frac{dr^2}{1 - 2m(r)/r} + r^2 d\Omega^2 - e^{2\Phi(r)} dt^2. \tag{3}$$

Writing out Eq. (2) [6], we obtain an expression involving $\Phi(r)$ and $m(r)$ with derivatives to order 2 in $\Phi(r)$ and to order 1 in $m(r)$. Viewing Eq. (2) as a differential equation in $\Phi(r)$, given $m(r)$, we obtain a Riccati equation in the first derivative of $\Phi(r)$. However, viewing Eq. (2) as a differential equation in $m(r)$, given $\Phi(r)$, we obtain a linear equation of first order [7]. As a consequence, we have the following algorithm for constructing all possible spherically symmetric perfect-fluid solutions of Einstein's equations.

Given $\Phi(r)$ (sufficiently smooth and subject to boundary conditions explained below)

$$m(r) = \frac{\int b(r) \exp\left(\int a(r) dr\right) dr + \mathcal{C}}{\exp\left(\int a(r) dr\right)}, \tag{4}$$

where

$$a(r) \equiv \frac{2r^2[\Phi''(r) + \Phi'(r)^2] - 3r\Phi'(r) - 3}{r[r\Phi'(r) + 1]} \tag{5}$$

and

$$b(r) \equiv \frac{r\{r[\Phi''(r) + \Phi'(r)^2] - \Phi'(r)\}}{r\Phi'(r) + 1} \tag{6}$$

where the prime indicates d/dr and $\mathcal{C}$ is a constant. The generating function associated with any known solution is of course immediately obvious following the algorithm.

Interior boundary conditions on $\Phi(r)$ are set by the requirement that all invariants polynomial in the Riemann tensor are finite at the origin. In this case there are but three independent invariants [8] and these are expressed here in terms of the physical variables; the energy density

$$\rho = \frac{G1}{8\pi} = \frac{m'(r)}{4\pi r^2} \geqslant 0 \tag{7}$$

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and the isotropic pressure

$$p=\frac{G2}{8\pi}=\frac{r\Phi'(r)[r-2m(r)]-m(r)}{4\pi r^3}\geqslant 0. \tag{8}$$

Note that the inequalities in Eq. (7) and (8) are to be viewed as imposed restrictions on $\Phi(r)$. At the center of symmetry $(r=0)$, the regularity of the Ricci invariants requires that $\rho(0)$ and $p(0)$ be finite. The regularity of the Weyl invariant requires that $m(r)$ is C^3 at $r=0$ with $m(0)=m(0)'=m(0)''=0$ and $m(0)'''=8\pi\rho(0)$ [9]. In summary, for a static spherically symmetric perfect fluid, finite $\rho(0)$ and $p(0)$ guarantee the regularity of all Riemann invariants at the center of symmetry. $\Phi(0)$ is a finite constant (set by the scale of t) and it follows from Eq. (8) that $\Phi'(0)=0$ and $\Phi''(0)=(4\pi/3)[3p(0)+\rho(0)]>0$. Since $\rho\geqslant 0$ and continuous and since $p(0)>0$ and finite, it follows from Eq. (2) that $r>2m(r)$ [10]. With $r>2m(r)$ for $r>0$ it also follows from Eq. (8) for $p(r)>0$ that $\Phi'(r)\neq 0$ for $r>0$. As a result, the source function $\Phi(r)$ must be a monotone increasing function with a regular minimum at $r=0$. Exterior boundary conditions on $\Phi(r)$ exist only for isolated spheres, and these conditions are set by junction conditions [11]. The necessary and sufficient condition that $\mathcal{M}$ have a regular boundary surface with a Schwarzschild vacuum exterior at $r=R>0$ is given by $p(r=R)=0$. Setting $m(r=R)\equiv M$ it follows that $\Phi'(r=R)=M/R(R-2M)$.

Each source function $\Phi(r)$ that is a monotone increasing function with a regular minimum at $r=0$ necessarily gives, via Eq. (4), a static spherically symmetric perfect-fluid solution of Einstein's equations that is regular at $r=0$. Exact solutions, in the present context, can be viewed as those for which Eq. (4) can be evaluated without recourse to numerical methods. The number of source functions $\Phi(r)$ for which Eq. (4) can be evaluated exactly is infinite. It should be noted, however, that the generation of an exact solution does not necessarily mean that the equation $p(r=R)=0$ can be solved exactly. The algorithm presented here is now demonstrated by the construction of an infinite number of previously unknown but physically interesting exact solutions of Einstein's equations.

Let

$$\Phi(r)=\frac{1}{2}N\ln\left(1+\frac{r^2}{\alpha}\right), \tag{9}$$

where N is an integer $\geqslant 1$ and α is a constant >0. The function (9) is monotone increasing with a regular minimum at $r=0$. With the source function (9), Eq. (4) can be evaluated exactly for any N. Whereas Eq. (9) generates a "class" of solutions, the metric [in particular $m(r)$] looks quite distinct, and the physical properties are quite distinct, for each value of N. Previously, only for $N=1,\dots,5$ were solutions known, having been arrived at by various methods, and one solution which is the first term in the Taylor expansion of Eq. (9) [12]. [These solutions, with $N=1,\dots,5$, in fact constitute half of all the previously known physically interesting solutions (of this type) in curvature coordinates.] For $N\geqslant 5$ the solutions are acceptable on physical grounds and even exhibit a monotonically decreasing subluminal adiabatic sound speed [13].

It is, perhaps, worth noting here that the foregoing discussion in curvature coordinates can be transformed directly into Bondi radiation coordinates [14].

Now consider the "isotropic coordinates"

$$ds^2_{\mathcal{M}}=e^{2B(r)}(dr^2+r^2d\Omega^2)-e^{2[\Psi(r)-B(r)]}dt^2. \tag{10}$$

Unlike curvature coordinates, the isotropic form (10) does not offer an immediate invariant physical interpretation of the functions $\Psi(r)$ and $B(r)$ [15]. However, as we now show, the coordinates offer a simplified algorithm for constructing perfect fluid solutions. Writing out Eq. (2) we now obtain an expression involving $\Psi(r)$ and $B(r)$ with derivatives to order 2 in $\Psi(r)$ and to order 1 in $B(r)$. Viewing Eq. (2) as a differential equation in $\Psi(r)$, given $B(r)$, we again obtain a Riccati equation in the first derivative of $\Psi(r)$. However, viewing Eq. (2) as a differential equation in $B(r)$, given $\Psi(r)$, we obtain an equation solvable simply by quadrature. As a consequence, we have the following simplified algorithm for constructing all possible spherically symmetric perfect fluid solutions of Einstein's equations in isotropic coordinates.

Given $\Psi(r)$ (sufficiently smooth and subject to boundary conditions explained below)

$$B(r)=\Psi(r)+\int c(r)dr+\mathcal{C}, \tag{11}$$

where

$$c(r)\equiv\frac{\epsilon}{\sqrt{2}}\sqrt{[\Psi'(r)]^2-\Psi''(r)+\Psi'(r)/r} \tag{12}$$

with $\epsilon=\pm 1$, the prime indicating d/dr, and $\mathcal{C}$ a constant. Recently, Rahman and Visser [16] also presented an algorithm for constructing spherically symmetric perfect-fluid solutions in isotropic coordinates. The source function $\Psi(r)$ used here is related to the source function $z(r)$ used by Rahman and Visser as follows:

$$\Psi(r)=2\int\frac{rz(r)}{1-z(r)r^2}dr. \tag{13}$$

The two algorithms differ fundamentally in the sense that only one integration is used in the present procedure as opposed to two distinct integrations used in the Rahman-Visser procedure. The Rahman-Visser procedure was motivated by the requirement that the metric be manifestly real *ab initio*. The reality of the integral (11) is discussed below.

Interior boundary conditions on $\Psi(r)$ are set exactly as in the case of curvature coordinates. We now have the energy density and pressure in the form

$$\rho=\frac{G1}{8\pi}=\frac{-1}{8\pi e^{2B(r)}}\left(2B''(r)+\frac{4B'(r)}{r}+[B'(r)]^2\right)\geqslant 0 \tag{14}$$

and

$$p=\frac{G2}{8\pi}=\frac{-1}{8\pi e^{2B(r)}}\left(-B'(r)\Psi'(r)+[B'(r)]^2-2\frac{\Psi'(r)}{r}\right)\geqslant 0. \tag{15}$$

$\Psi(0)$ is a finite constant (set by the scale of t) and it follows from Eq. (15) that $\Psi'(0)=0$ and from Eq. (11) that $B'(0)=0$. With $p(r)\geqslant 0$ it follows that the source function $\Psi(r)$ must be a monotone increasing function with a regular minimum at $r=0$ and $\Psi''(0)=4\pi e^{2B(0)}p(0)$. Exterior boundary conditions on $\Psi(r)$ are set as in curvature coordinates. The regularity of $\rho(0)$ requires $B'(0)=0$ and with $\rho(r)\geqslant 0$ it follows that $B(r)$ must be a monotone decreasing function with a regular maximum at $r=0$ and $B''(0)=-4\pi e^{2B(0)}\rho(0)$. The limits $-2/r<B'(r)<0$ guarantee the positivity of the effective gravitational mass. To examine the reality of the metric, consider the function $F(r)\equiv[\Psi'(r)]^2-\Psi''(r)+\Psi'(r)/r$. Now $F(0)=0$, $F'(0)=0$, and $F''(0)>0$, so $F(r)$ has a local minimum at $r=0$. Now suppose that $F(r)=0$ for $r>0$. Then condition (2) requires $B'=\Psi'$ so we have already passed through a region with $\rho<0$ before the reality of the metric breaks down (in agreement with known theorems [10]).

In parallel to the algorithm in curvature coordinates, each source function $\Psi(r)$ that is a smooth monotone increasing function with a regular minimum at $r=0$ necessarily gives, via Eq. (11), a static spherically symmetric perfect-fluid solution of Einstein's equations that is regular at $r=0$. Exact solutions are again those for which Eq. (11) can be evaluated without recourse to numerical methods. Physical considerations must guide the choice of $\Psi(r)$. In isotropic coordinates the ratios of invariants and differential invariants can be obtained directly from the source function $\Psi(r)$ via differentiation. You do not need $B(r)$ and in particular you do not need to integrate. For example, the functions $p(r)/\rho(r)$ and $p'(r)/\rho'(r)$ follow directly without integration. Of course, neither $p(r)$ nor $\rho(r)$ follows without integration. In curvature coordinates you cannot get these ratios without integration, starting from the source function $\Phi(r)$.

To demonstrate the algorithm in isotropic coordinates, let

$$\Psi(r)=\alpha\ln\frac{f(r)}{g(r)}, \tag{16}$$

where α is a constant >0. Of course it is not difficult to find functions $f(r)$ and $g(r)$ so that Eq. (16) is monotone increasing with a regular minimum at $r=0$. Nor indeed is it difficult to find such functions for which $B(r)$ can be evaluated exactly. For example, let $g(r)=(\delta+\epsilon r^2)^{\zeta}$ and $f(r)=\delta^{\zeta}+\gamma r^2$ with δ, ϵ, γ, and ζ constants such that $\delta>0$ and $\delta^{1-\zeta}\gamma>\zeta\epsilon$. This class of solutions includes a number of known solutions including the Schwarzschild interior solution and the Rahman-Visser general quadratic ansatz. Any solution in isotropic coordinates can be immediately recovered and generalized following the algorithm presented [17].

An algorithm based on the choice of a single monotone function (subject to boundary conditions) has been presented which generates all regular static spherically symmetric perfect-fluid solutions of Einstein's equations. In all cases the choice of generating function must be guided by physical considerations. These additional conditions limit the generating functions allowed by way of nontrivial integral-differential inequalities. The details of how to choose physically relevant generating functions (beyond trial and error) are, at present, not known. Moreover, the resultant equation of state is a by-product of the algorithm and cannot be set *a priori*. Despite these reservations, the algorithm has been demonstrated by the construction of an infinite number of previously unknown physically interesting exact solutions [18]. It is a curious fact of history that over half a century ago Wyman [19] pointed out that the algorithm presented here was possible and yet, despite the voluminous literature on the subject [1], the algorithm appears not to have been followed up.

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[1] See M. S. R. Delgaty and K. Lake, Comput. Phys. Commun. **115**, 395 (1998).

[2] The conditions used in [1] were (i) isotropy of the pressure (otherwise any metric is a "solution"), (ii) regularity at the origin, (iii) positivity of the pressure and energy density at the origin, (iv) vanishing of the pressure at a finite boundary, (v) monotone decrease of the energy density to the boundary, and (vi) subluminal adiabatic sound speed. In addition to these, a monotone decrease in the subluminal adiabatic sound speed is desirable.

[3] We use geometrical units throughout. The "curvature coordinates" used in Eq. (3) have the advantage that the metric functions have a clear invariant physical interpretation (but see also [14] below). The function $m(r)$ is the effective gravitational mass. See W. C. Hernandez and C. W. Misner, Astrophys. J. **143**, 452 (1965); E. Poisson and W. Israel, Phys. Rev. D **41**, 1796 (1990); T. Zannias, *ibid.* **41**, 3252 (1990); S. Hayward, *ibid.* **53**, 1938 (1996). Whereas $\Phi(r)$ is (in the weak field limit) the "Newtonian" potential, $re^{-\Phi(r)}$ is the effective potential for null geodesics [see, for example, M. Ishak, L. Chamandy, N. Neary, and K. Lake, Phys. Rev. D **64**, 024005 (2001)].

[4] K. Lake, gr-qc/0209063.

[5] One can take the view that the Tolman-Oppenheimer-Volkoff equation is a consequence of the invariant statement (2).

[6] Explicitly, condition (2) in the static case in curvature coordi-

nates reduces to the Walker pressure isotropy condition $G^r_r = G^\theta_\theta$ [see A. G. Walker, Q. J. Math. **6**, 81 (1935)], which is

$$\left[\frac{d^2}{dr^2}\Phi(r)+\left(\frac{d}{dr}\Phi(r)\right)^2\right]r^2[r-2m(r)]-r\left(\frac{d}{dr}\Phi(r)\right)\left[\left(\frac{d}{dr}m(r)\right)r+r-3m(r)\right]+3m(r)-\left(\frac{d}{dr}m(r)\right)r=0.$$

[7] The problem has also been reduced to a linear equation of first order by A. S. Berger, R. Hojman, and J. Santamarina, J. Math. Phys. **28**, 2949 (1987). Recently G. Fodor (gr-qc/0011040) has reduced the problem to an algebraic one with integration required only for one metric function but not the physical variables ρ and p.

[8] D. Pollney, N. Pelavas, P. Musgrave, and K. Lake, Comput. Phys. Commun. **115**, 381 (1998).

[9] It follows from Eqs. (2) and (3) that the necessary and sufficient condition for conformal flatness for $r>0$ is given by $m(r)=cr^3$, which gives, uniquely, the Schwarzschild interior solution. See also H. A. Buchdahl, Am. J. Phys. **39**, 158 (1971).

[10] See T. W. Baumgarte and A. D. Rendall, Class. Quantum Grav. **10**, 327 (1993); M. Mars, M. Mercè Martín-Prats, and J. M. M. Senovilla, Phys. Lett. A **218**, 147 (1996).

[11] See, for example, P. Musgrave and K. Lake, Class. Quantum Grav. **13**, 1885 (1996). At an interior boundary surface p, but not ρ, must be continuous. Discontinuities in ρ are associated with phase transitions, which we do not consider here. For a discussion of interior phase transitions see, for example, L. Lindblom, Phys. Rev. D **58**, 024008 (1998).

[12] In terms of the classification given in [1] the solutions are Tolman IV for $N=1$, Heint IIa for $N=3$, Durg IV for $N=4$, and Durg V=D-F for $N=5$. If $\Phi(r)$ is taken to be the first term in the Taylor expansion of Eq. (9), the solution is known as Kuch2 III. The case $N=2$ gives $m(r)=\mathcal{C}r^3/(3r^2+\alpha)^{2/3}$ which is usually dismissed under the erroneous assumption that $\mathcal{C}=0$.

[13] N. Neary, J. Lattimer, and K. Lake (in preparation).

[14] These were first discussed (in the spherically symmetric case) by H. Bondi, Proc. R. Soc. London **A281**, 39 (1964) and are a generalization of the well known Eddington-Finkelstein coordinates for the Schwarzschild vacuum. The algorithm presented is equally at home in curvature and radiation coordinates. Writing

$$dv=dt\pm\frac{e^{-\Phi(r)}}{\sqrt{1-2m(r)/r}}dr$$

[+ for advanced (ingoing) v and − for retarded (outgoing) v] it follows that Eq. (3) takes the form

$$ds^2=\pm 2\frac{e^{\Phi(r)}}{\sqrt{1-2m(r)/r}}dv\,dr+r^2d\Omega^2-e^{2\Phi(r)}dv^2.$$

The form of condition (2) (given above in [6]) remains unchanged, as do the functional forms and physical meanings of Φ, m, ρ, and p.

[15] We proceed here in isotropic coordinates *ab initio* without coordinate transformations. Now $re^{2B(r)-\Psi(r)}$ is the effective potential for null geodesics and the effective gravitational mass is given by $m(r)=-B'(r)[B'(r)r+2]e^{B(r)}r^2/2$ where the prime indicates d/dr.

[16] S. Rahman and M. Visser, Class. Quantum Grav. **19**, 935 (2002).

[17] In the terminology of [1], as regards physically interesting solutions, the P-S2 solution follows from the stated form of $\Psi(r)$. Similarly, the choices $g(r)=\cosh(\beta+\gamma r^2)$ and $f(r)=\sinh(\beta+\gamma r^2)$ with β and γ positive constants immediately gives Gold III.

[18] It is also of interest to note that seven of the eleven previously known solutions of this type are special cases resulting from the two generating functions considered here.

[19] M. Wyman, Phys. Rev. **75**, 1930 (1949).

[20] This is a package which runs within MAPLE. It is entirely distinct from packages distributed with MAPLE and must be obtained independently. The GRTENSORII software and documentation is distributed freely on the World Wide Web from the address http://grtensor.org