Five-dimensional black hole and particle solution with a non-Abelian gauge field

Naoya Okuyama
Department of Physics, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan

Kei-ichi Maeda
Department of Physics, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan,
Advanced Research Institute for Science and Engineering, Waseda University, Shinjuku, Tokyo 169-8555, Japan,
and Waseda Institute for Astrophysics, Waseda University, Shinjuku, Tokyo 169-8555, Japan
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We study the five-dimensional Einstein-Yang-Mills system with a cosmological constant. Assuming a spherically symmetric spacetime, we find a new analytic black hole solution, which approaches asymptotically ''quasi-Minkowski,'' ''quasi–anti-de Sitter,'' or ''quasi–de Sitter'' spacetime depending on the sign of the cosmological constant. Since there is no singularity except for the origin that is covered by an event horizon, we regard it as a localized object. This solution corresponds to a magnetically charged black hole. We also present a singularity-free particlelike solution and a nontrivial black hole solution numerically. Those solutions correspond to the Bartnik-McKinnon solution and a colored black hole with a cosmological constant in four dimensions. We analyze their asymptotic behavior, spacetime structures, and thermodynamical properties. We show that there is a set of stable solutions if the cosmological constant is negative.

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I. INTRODUCTION

Recent progress in superstring theory shows that different string theories are connected with each other via dualities, making them unified with M theory in 11 dimensions [1]. This provides a motivation to study a higher-dimensional gravitational theory. String theory also predicts a boundary layer, a *brane*, on which the edges of open strings stand [2]. This suggests a new perspective in cosmology, that is, we are living in a brane world, which is a three-dimensional hypersurface in a higher-dimensional spacetime. In contrast with the already familiar Kaluza-Klein picture in which we live in four-dimensional spacetime with *n*-dimensional compactified ''internal space,'' our world view appears to be changed completely. Particles in the standard model are expected to be confined to the brane, whereas the gravitons propagate in the entire bulk spacetime.

In the brane world cosmological scenario [3], a higher-dimensional black hole solution plays an important role. Our Universe is just a domain wall expanding in the black hole background spacetime [4]. The black hole mass gives a contribution to dark radiation through its tidal force. Hence, a higher-dimensional black hole or a globally regular solution with a cosmological constant is now a very interesting subject. In particular, in the context of the AdS conformal field theory (CFT) correspondence [5] or proposed dS/CFT correspondence [6], since the five-dimensional Einstein gravity with a cosmological constant gives a description of four-dimensional conformal field theory in the large N limit, many authors study such localized objects in five dimensions [7].

However, from the viewpoint of brane cosmology, a black hole solution has a singularity in a bulk spacetime, although it is covered by a horizon. If a string theory or M theory is fundamental, such a singularity should not exist. Then, if we can construct some nonsingular object in the bulk spacetime, it might be a manifestation of singularity avoidance immanent in a fundamental theory. In four dimensions, Bartnik and McKinnon found a particlelike solution as a globally regular spacetime in a spherically symmetric Einstein-Yang-Mills system with SU(2) gauge group [8]. Soon after, a colored black hole solution with a nontrivial non-Abelian structure was also found [9]. These solutions were also extended to those in a system with a cosmological constant [10–12]. From stability analysis, it turns out that solutions with zero or positive cosmological constant are unstable [13], while those with negative cosmological constant are stable [12,14]. Since a negative cosmological constant is naturally expected in a brane world scenario just as in the Randall-Sundrum model [15], the above fact is very interesting. In this paper, then, we study a nontrivial particlelike solution or black hole solution in five dimensions with a cosmological constant.

As for non-Abelian gauge fields in a bulk spacetime, although gauge interactions are confined on a brane and Yang-Mills fields are expected to exist only in the brane, if our five-dimensional spacetime is obtained as an effective theory, this may not be the case. In fact, Lukas *et al.* [16] showed that a U(1) field appears in the effective five-dimensional bulk spacetime, from dimensional reduction of the Hořava-Witten model [1]. We may find non-Abelian gauge fields from some other type of dimensional reduction of a unified theory.

There is another interesting point in discussing non-Abelian gauge fields in the bulk. Using a brane structure, new mechanisms of spontaneous symmetry breaking of gauge interactions have been proposed [17]. In this picture, the present standard model [SU(3)×SU(2)×U(1)] is obtained on the brane assuming some higher-symmetric gauge interactions such as SU(5) in the bulk.

Therefore, in this paper, we assume that a non-Abelian gauge field appears in five-dimensional bulk spacetime. In Sec. II, we first derive the basic equations of a spherically symmetric Einstein-Yang-Mills system in five dimensions.

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With a spherically symmetric ansatz, the gauge potential of the SU(2) Yang-Mills field is be decomposed into "electric" and "magnetic" parts; the derivation is given in Appendix A. There is a nontrivial analytic solution in the case with a "magnetic" field, which corresponds to a magnetically charged black hole in four-dimensions. This analytic solution and its properties are examined in Sec. III. We also present nontrivial particlelike and black hole solutions, which correspond to the Bartnik-McKinnon type and colored black hole type solutions in four dimensions, in Sec. IV. We also analyze their stability in Sec. V. A summary and discussion follow in Sec. VI.

II. BASIC EQUATION

In order to find a black hole and particlelike solution of the five-dimensional Einstein-Yang-Mills system, we first write down the basic equations. The action is given by

$$S=\frac{1}{16\pi}\int d^5x\sqrt{-g_5}\left[\frac{1}{G_5}(R-2\Lambda)-\frac{1}{g^2}\mathrm{Tr}\,\mathbf{F}^2\right], \quad (2.1)$$

where G_5 is a five-dimensional gravitational constant, Λ is a five-dimensional cosmological constant, and g is a gauge coupling constant. Now we assume that the gauge group is SU(2). $\mathbf{F}=F_{\mu\nu}dx^\mu\wedge dx^\nu$ is the field strength of the gauge field, which is described by the vector potential $\mathbf{A}=A_\mu dx^\mu$ as

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-[A_\mu,A_\nu]. \quad (2.2)$$

Defining the five-dimensional Planck mass by $m_5=G_5^{-1/3}$ and the fundamental mass scale of the gauge field by $m_g=g^{-2}$, we introduce a typical length scale of the present system, which is given by

$$\lambda=\left(\frac{m_g}{m_5^3}\right)^{1/2}=\left(\frac{G_5}{g^2}\right)^{1/2}. \quad (2.3)$$

We will normalize the scale length by this λ.

We consider a spherically symmetric five-dimensional spacetime, whose metric is given by

$$ds^2=\lambda^2\left[-f(t,r)e^{-2\delta(t,r)}dt^2+\frac{dr^2}{f(t,r)}+r^2d\Omega_3^2\right], \quad (2.4)$$

where

$$f(t,r)=1-\frac{\mu(t,r)}{r^2}+\epsilon\frac{r^2}{\ell^2}, \quad (2.5)$$

$$d\Omega_3^2=d\psi^2+\sin^2\psi(d\theta^2+\sin^2\theta\, d\varphi^2), \quad (2.6)$$

where we set $\Lambda=-6\epsilon/(\lambda\ell)^2$ with $\epsilon=0$ or ±1, corresponding to the signature of Λ, i.e., $\epsilon=1$, 0, and -1 corresponds to $\Lambda<0$, $\Lambda=0$, and $\Lambda>0$, respectively. Note that t, r, and μ are all dimensionless variables. We shall call μ a "mass" function. ℓ denotes the ratio of the length scale of the cosmological constant to λ.

From Appendix A, we find a generic form of the spherically symmetric SU(2) gauge potential. If we take only the "electric" part of the field, the gauge potential is given by Eq. (A19), which yields the basic equations as

$$\mu'=\frac{2}{3}r^3(A'e^\delta)^2, \quad (2.7)$$

$$\dot{\mu}=0, \quad (2.8)$$

$$\delta'=0, \quad (2.9)$$

$$[(A'e^\delta)^2]^{\cdot}=0, \quad (2.10)$$

$$[(A'e^\delta)^2]'+\frac{6}{r}(A'e^\delta)^2=0, \quad (2.11)$$

where the prime and overdot denote the partial derivatives with respect to r and t, respectively. This equation gives a Reissner-Nordström type solution such as

$$\mu=\mathcal{M}-\frac{2Q^2}{3r^2}, \quad (2.12)$$

$$\delta=0, \quad (2.13)$$

$$A=-\frac{Q}{r^2}. \quad (2.14)$$

This result is the same as the case of four dimensions.

If the "magnetic" part of the gauge field, which is given by Eq. (A22), appears, we find other basic equations as follows. Using the gauge freedom, we set $X=0$, resulting in the gauge potentials as

$$A_t^a=0,\quad A_r^a=0, \quad (2.15)$$

$$A_\psi^a=(0,0,w), \quad (2.16)$$

$$A_\theta^a=(w\sin\psi,-\cos\psi,0), \quad (2.17)$$

$$A_\varphi^a=(-\cos\psi\sin\theta,-w\sin\psi\sin\theta,\cos\theta), \quad (2.18)$$

where we set $\phi=w(r,t)$. With the above ansatz, we find the Einstein equations and Yang-Mills equation of the present system as

$$\mu'=2r\left[fw'^2+f^{-1}e^{2\delta}\dot{w}^2+\frac{(1-w^2)^2}{r^2}\right], \quad (2.19)$$

$$\dot{\mu}=4rfw'\dot{w}, \quad (2.20)$$

$$\delta'=-\frac{2}{r}[w'^2+f^{-2}e^{2\delta}\dot{w}^2], \quad (2.21)$$

and

$$\frac{1}{r}(rfe^{-\delta}w')' + \frac{2}{r^2}e^{-\delta}w(1-w^2) = (f^{-1}e^{\delta}\dot{w})^{\cdot}. \tag{2.22}$$

Equations (2.19)–(2.22) look very similar to those in the case of the four-dimensional Einstein-Yang-Mills system. However, a small difference of the power exponent of r brings a large difference in the behavior of solutions, as we will see later.

III. ANALYTIC SOLUTIONS

Now we look for a "magnetic" type static solution of the system (2.19)–(2.22). Dropping the time derivative terms, we find the basic equations as

$$\mu' = 2r\left[fw'^2 + \frac{(1-w^2)^2}{r^2}\right], \tag{3.1}$$

$$\delta' = -\frac{2}{r}w'^2, \tag{3.2}$$

$$\frac{1}{r}(rfe^{-\delta}w')' + \frac{2}{r^2}e^{-\delta}w(1-w^2) = 0. \tag{3.3}$$

The above differential equations (3.1)–(3.3) have two analytic solutions. One analytic solution is

$$w = \pm 1, \quad \mu = \mathcal{M}, \quad \delta = 0, \tag{3.4}$$

which corresponds to the Schwarzschild or the Schwarzschild–anti de Sitter (or de Sitter) spacetime, whose properties are well known.

Another analytic solution is given by

$$w = 0, \quad \mu = \mathcal{M} + 2\ln r, \quad \delta = 0. \tag{3.5}$$

This solution has a nontrivial geometry. In four-dimensional spacetime, this type of solution describes the Reissner-Nordström type geometry with a magnetic charge. In five-dimensional spacetime, a $2\ln r$ term appears in the mass function μ. Although μ diverges, the metric itself approaches that of well-known symmetric spacetimes for each ϵ, i.e., the Minkowski, de Sitter, and anti–de Sitter spacetimes. We first study the properties of this solution in the following subsections.

A. Asymptotic structure

Since the mass function diverges, we have to analyze carefully the asymptotic behaviors. For the case of $\epsilon = 0$, the Riemann curvature is finite except at $r = 0$ and vanishes at infinity as

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \to 288\left(\frac{\ln r}{r^4}\right)^2. \tag{3.6}$$

For the case of $\epsilon = \pm 1$, the Riemann curvature is also finite everywhere except at $r = 0$ and converges as

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \to \frac{40}{\ell^4} \tag{3.7}$$

as $r \to \infty$. This finite value just comes from the Ricci curvature. The metric form approaches

$$f(r) \to 1 + \epsilon\frac{r^2}{\ell^2} \tag{3.8}$$

as $r \to \infty$. These spherically symmetric and static spacetimes are singular only at $r = 0$, and seem to approach a "maximally symmetric spacetime." Therefore we may recognize the metric as a localized object in such a "maximally symmetric spacetime."

However, we have to analyze the asymptotic behaviors more carefully. The asymptotic flatness condition is mathematically defined using the conformal transformation. We can also extend this formulation to an asymptotically de Sitter (or anti–de Sitter) spacetime as well as to a higher-dimensional spacetime.

In an asymptotically flat spacetime, we can naturally define the mass of an isolated object, which is called the Arnowitt-Deser-Misner (ADM) mass [18]. It is defined by

$$G_5 M_{\rm ADM} = \frac{1}{16\pi}\oint_{I_0} dS_i[\partial_j h^{ij} - \eta^{ij}\partial_j h^k_k] \tag{3.9}$$

in five-dimensional spacetime, where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. dS_i is an infinitesimal surface element of spacelike infinity I_0. For the present nontrivial solution with $\epsilon = 0$, we find

$$G_5 M_{\rm ADM} = \lim_{r\to\infty}\frac{3\pi}{8}\lambda^2(\mathcal{M} + 2\ln r) \tag{3.10}$$

which diverges as $\ln r$. The coefficient $3\pi/8$ appears just because Eq. (2.19) yields

$$\mu = \frac{3\pi}{8}\int dv[-T^0{}_0]. \tag{3.11}$$

For $\epsilon = -1$, if the spacetime is asymptotically de Sitter, we can also introduce a conserved mass, which is called the Abbott-Deser mass defined by [19,20]. If the spacetime is asymptotically de Sitter, $M_{\rm AD} = M_{\rm ADM}$, which diverges again as $\ln r$.

In a five-dimensional asymptotically anti–de Sitter spacetime, we can also define a conserved mass associated with a timelike Killing vector $\vec{\xi}$ at the three-sphere C on conformal infinity $\mathcal{I}$ as [21,22]

$$G_5 M_\xi[C] := -\frac{\lambda^2\ell}{16\pi}\oint_C \mathcal{E}_{\mu\nu}\xi^\mu dS^\nu, \tag{3.12}$$

where $\mathcal{E}_{\mu\nu}$ is the electric part of the Weyl tensor defined by

$$\mathcal{E}_{\mu\nu} := \frac{\ell^2}{\Omega^2}C_{\mu\rho\nu\sigma}n^\rho n^\sigma. \tag{3.13}$$

Ω is a conformal factor and $n_\mu = \nabla_\mu \Omega$. In the case of the Schwarzschild–anti de Sitter spacetime (3.4), this mass gives $\mathcal{M}$. In the nontrivial solution, however, this quantity is calculated on the three-sphere C with radius r as

$$G_5 M_\xi[C] = \frac{3\pi}{8}\lambda^2\left[\mathcal{M} + 2\ln r - \frac{7}{6}\right]. \quad (3.14)$$

It diverges as $\ln r$ as $r \to \infty$.

In any case, the "mass" is not finite, which means that the "total energy" of the system is not finite. Therefore, strictly speaking, we should not regard it as an isolated object. However, there is no singularity except at $r=0$ and the metric form itself approaches either the Minkowski or the de Sitter (anti–de Sitter) one. Hence, we call it a "quasi-isolated" object. We recall that we know a similar "isolated" object, i.e., the four-dimensional self-gravitating global monopole. Its metric is described as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2, \quad (3.15)$$

where $f(r) \equiv 1 - 2m(r)/r \sim 1 - \alpha - 2M/r + O(1/r^2)$. In this case, the mass function $m(r)$ diverges as $M + \alpha r/2 + O(1/r)$ as $r \to \infty$. In fact the ADM mass diverges. Rescaling the time and radial coordinates as $r \to (1-\alpha)^{1/2} r$ and $t \to (1-\alpha)^{-1/2} t$, we can rewrite the metric form as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + (1-\alpha) r^2 d\Omega^2, \quad (3.16)$$

where $f(r) = 1 - 2\tilde{M}/r$ with $\tilde{M} = M(1-\alpha)^{-3/2}$. This spacetime looks asymptotically flat but has a deficit angle α. Nucamendi and Sudarsky showed that this spacetime is asymptotically simple but not asymptotically empty [23]. They called it a "quasiasymptotically flat" spacetime and defined a new mass for a spacetime with a deficit angle, which is a generalization of the ADM mass, using the first law of black hole thermodynamics.

In our case, the mass function diverges as $\ln r$, which is less divergent than the case with a deficit angle (r^2 in five dimensions). Then we can also call such a spacetime a "quasiasymptotically" flat or "quasiasymptotically" de Sitter (anti–de Sitter) spacetime.

B. Spacetime structure: Horizon and singularity

This solution has a horizon, where

$$f(r) = 1 - \frac{\mathcal{M} + 2\ln r}{r^2} + \epsilon \frac{r^2}{\ell^2} = 0. \quad (3.17)$$

We study those horizons and the singularity separately for each value of ϵ.

1. $\epsilon = 0$

In this case, if $\mathcal{M} > 1$, Eq. (3.17) has two roots $r = r_\pm$ ($r_- < r_+$), which correspond to two horizons; r_+ corresponds to an event horizon, while r_- is an inner horizon. A timelike singularity appears at $r=0$. For the case of $\mathcal{M} = 1$, the two horizons become degenerate and the black hole becomes extreme. If $\mathcal{M} < 1$ there is no horizon, so a naked singularity appears.

2. $\epsilon = 1$

This case also has two horizons $r = r_\pm$ ($r_- < r_+$) if $\mathcal{M} > \mathcal{M}_{\rm cr}$. r_+ and r_- are an event horizon and an inner horizon, respectively. The critical mass parameter $\mathcal{M}_{\rm cr}$ is given by the horizon radius of the extreme case ($r_{+\rm cr}$), i.e.,

$$\mathcal{M}_{\rm cr} = \frac{1}{2}(1 + r_{+\rm cr}^2) - 2\ln r_{+\rm cr}, \quad (3.18)$$

where

$$r_{+\rm cr} = \frac{\ell}{2}\left(-1 + \sqrt{1 + \frac{8}{\ell^2}}\right)^{1/2}. \quad (3.19)$$

$\mathcal{M}_{\rm cr}$ is always larger than unity and it approaches 1 as $\ell \to \infty$, which corresponds to the case of $\epsilon = 0$. A timelike singularity appears at $r=0$. For the case of $\mathcal{M} = \mathcal{M}_{\rm cr}$, the black hole is extreme, and for $\mathcal{M} < \mathcal{M}_{\rm cr}$ the horizon disappears.

3. $\epsilon = -1$

If a cosmological constant is positive, we expect a cosmological horizon just as in a de Sitter spacetime. In fact, we always find at least one horizon. If $\ell > 2\sqrt{2}$ and

$$\mathcal{M}_{\rm min} < \mathcal{M} < \mathcal{M}_{\rm max}, \quad (3.20)$$

where $\mathcal{M}_{\rm min} = g(r_{+\rm cr})$ and $\mathcal{M}_{\rm max} = g(r_{-\rm cr})$ with $g(r) = r^2 - 2\ln r - r^4/\ell^2$ and

$$r_{\pm\rm cr} = \frac{\ell}{2}\left(1 \pm \sqrt{1 - \frac{8}{\ell^2}}\right)^{1/2}, \quad (3.21)$$

we find three horizons, r_- ($< r_{-\rm cr}$) $< r_+$ ($< r_{+\rm cr}$) $< r_c$. r_-, r_+, and r_c are an inner, event, and cosmological horizon, respectively. When $\mathcal{M} = \mathcal{M}_{\rm max}$, the inner and event horizons become degenerate ($r_- = r_+$), while if $\mathcal{M} = \mathcal{M}_{\rm min}$, the event and cosmological horizons coincide ($r_+ = r_c$). In the limit of $\ell \to 2\sqrt{2}$, $\mathcal{M}_{\rm min} = \mathcal{M}_{\rm max} = \mathcal{M}_{\rm cr} = 3/2 - \ln 2 \approx 0.80685$, and then the three horizons become degenerate for $\mathcal{M} = \mathcal{M}_{\rm cr}$.

For other cases, we have only one horizon. The singularity at $r=0$ becomes naked. We summarize the types of horizon in Table I.

C. Thermodynamical properties

Next we shall look at the thermodynamical properties. The Hawking temperature is easily calculated from the regularity condition at the event horizon [24]. We find

$$T_{\rm BH} = \frac{1}{2\pi r_+}\left[1 - \frac{1}{r_+^2} + 2\epsilon \frac{r_+^2}{\ell^2}\right]. \quad (3.22)$$

The entropy $S = A/4$ is given by

TABLE I. Types of horizon. I, E, C, and D denote an inner, event, cosmological, and degenerate horizon, respectively. "0" means no horizon. $\mathcal{M}_{\rm cr}$, $\mathcal{M}_{\rm min}$, and $\mathcal{M}_{\rm max}$ are defined in the text.

$\epsilon=0$	$\mathcal{M}<1$	0
	$\mathcal{M}=1$	D
	$\mathcal{M}>1$	I, E
$\epsilon=1$	$\mathcal{M}<\mathcal{M}_{\rm cr}$	0
	$\mathcal{M}=\mathcal{M}_{\rm cr}$	D
	$\mathcal{M}_{\rm cr}<\mathcal{M}$	I, E
$\epsilon=-1,\ \ell>2\sqrt{2}$	$\mathcal{M}<\mathcal{M}_{\rm min}$	I
	$\mathcal{M}=\mathcal{M}_{\rm min}$	I, D
	$\mathcal{M}_{\rm min}<\mathcal{M}<\mathcal{M}_{\rm max}$	I, E, C
	$\mathcal{M}=\mathcal{M}_{\rm max}$	D, C
	$\mathcal{M}>\mathcal{M}_{\rm max}$	C
$\epsilon=-1,\ \ell\leqslant 2\sqrt{2}$		C

$$S=\frac{1}{2}\pi^2 r_+^3, \tag{3.23}$$

because the volume of a unit three-sphere is $2\pi^2$. Since the solution does not satisfy the asymptotically flat or de Sitter (or anti–de Sitter) conditions, we cannot define the gravitational mass. However, if we use the first law of thermodynamics just as in the case of a global monopole with a deficit angle [23], we can define the thermodynamical mass M_T as $dM_T=T\,dS+\Phi\,dQ$. We find

$$M_T=\frac{3\pi}{8}\mathcal{M}, \tag{3.24}$$

where the integration constant is set to zero. This result shows that the mass parameter $\mathcal{M}$ essentially denotes the thermodynamical mass.

From Eq. (3.17), the thermodynamical mass is given by the horizon radius as

$$M_T=\frac{3\pi}{8}\left[r_+^2\left(1+\epsilon\frac{r_+^2}{\ell^2}\right)-2\ln r_+\right]. \tag{3.25}$$

We depict the M_T-r_+ relation in Fig. 1. We find that the horizon radius is larger than that of the electrically charged Reissner-Nordström black hole. We also show the M_T-T_{BH} relation in Fig. 2. From Eqs. (3.22) and (3.24), we find

$$\frac{dT_{\rm BH}}{dM_T}=-\frac{2}{3\pi^2 r_+^3}\frac{1-3/r_+^2-2\epsilon r_+^2/\ell^2}{1-1/r_+^2+2\epsilon r_+^2/\ell^2}, \tag{3.26}$$

which gives a turning point where the specific heat changes its sign. For the case of $\epsilon=0$, the specific heat is positive in $1<r_+<\sqrt{3}$ but becomes negative for $r_+>\sqrt{3}$. [The corresponding critical value for thermodynamical mass is obtained via Eq. (3.25).] For the case of $\epsilon=1$, if $\ell\leqslant 2\sqrt{6}$, the specific heat is always positive. If $\ell>2\sqrt{6}$, the specific heat is positive in $r_{+\rm cr}<r_+<r_{-\rm ch}$ and in $r_+>r_{+\rm ch}$, while it is negative in $r_{-\rm ch}<r_+<r_{+\rm ch}$, where

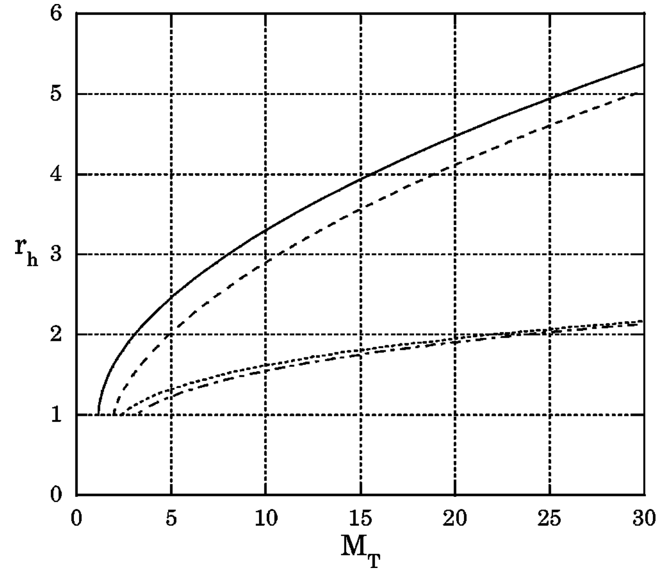

FIG. 1. M_T-r_+ relation. The horizon radius r_+ is depicted in terms of the thermodynamical mass M_T for $\epsilon=0$ and $\epsilon=1$ by the solid and dotted lines, respectively. That for the Reissner-Nordström solution with the same charge for $\epsilon=0$ and $\epsilon=1$ is given by the dashed and dot-dashed lines as reference, respectively.

$$r_{\pm\rm ch}=\frac{\ell}{2}\left(1\pm\sqrt{1-\frac{24}{\ell^2}}\right)^{1/2}. \tag{3.27}$$

For the case of $\epsilon=-1$, the specific heat is positive in $r_{-\rm cr}<r_+<r_{\rm ch}$, while it is negative for $r_{\rm ch}<r_+\leqslant r_{+\rm cr}$, where

$$r_{\rm ch}=\frac{\ell}{2}\left(-1+\sqrt{1+\frac{24}{\ell^2}}\right)^{1/2}. \tag{3.28}$$

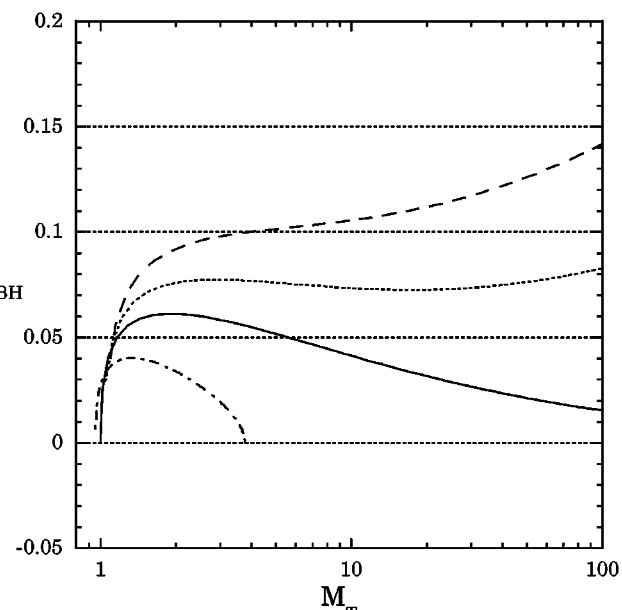

FIG. 2. M_T-$T_{\rm BH}$ relation. The solid line depicts the relation for $\epsilon=0$, while the dotted and dashed lines represent those for $\epsilon=1$ with $\ell=6.0$ and $\ell=4.0$, and the dot-dashed line corresponds to that for $\epsilon=-1$ with $\ell=5.0$.

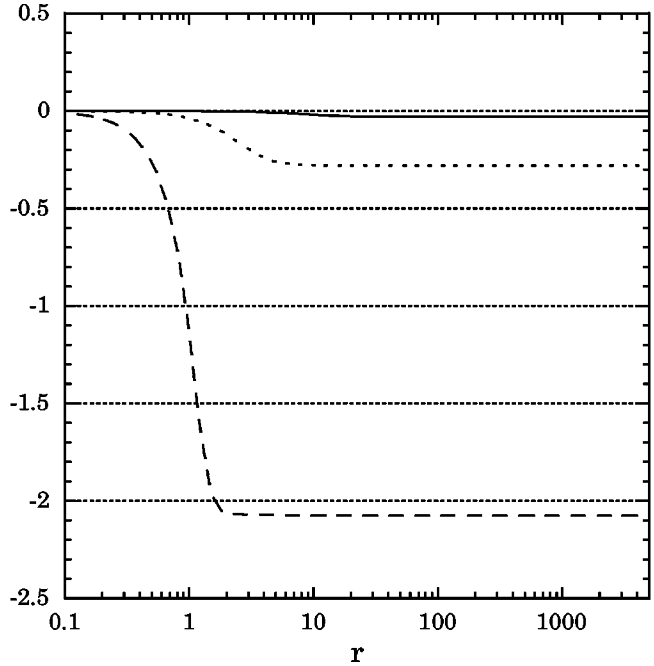

FIG. 3. The metric function $\delta(r)$ for a particlelike solution with $\epsilon=0$. The solid, dotted, and dashed lines depict those for $b=-0.01$, -0.1, and -0.5, respectively.

IV. NUMERICAL SOLUTIONS

Just as in four dimensions [8,9,12,14], we can find the nontrivial structure of a self-gravitating Yang-Mills field. We obtain those solutions numerically. We discuss two cases; a particle solution and a black hole, separately. Here, we analyze only the case of $\epsilon=0$ or 1.

A. Particle solution

In the case of a particle solution, we have to impose regularity at the origin $r=0$. Since Eqs. (2.19)–(2.22) are invariant under the transformation of $w\to -w$, we can set $w(0)>0$ without loss of generality. Expanding μ and w around $r=0$, we find the behavior near the origin as

$$\mu(r)=4b^2r^4+O(r^5), \tag{4.1}$$

$$\delta(r)=-4b^2r^2+\frac{4}{3}b^2\left(\frac{4\epsilon}{\ell^2}-3b-8b^2\right)r^4+O(r^5), \tag{4.2}$$

$$w(r)=1+br^2-\frac{b}{6}\left(\frac{4\epsilon}{\ell^2}-3b-8b^2\right)r^4+O(r^5) \tag{4.3}$$

with one free parameter b. Using this boundary condition, we integrate the basic equations by the Runge-Kutta method.

For the case of $\epsilon=0$, we find solutions whose metrics are regular in the whole spacetime and approach the Minkowski metric as $r\to\infty$ for $b_{\rm min}<b<0$, where $b_{\rm min}\approx -0.635607$. We show the numerical results in Figs. 3–5.

The potential function w is oscillating between ± 1 and the mass function μ is increasing without bound just as a step function. As we show in Appendix B, there is no finite

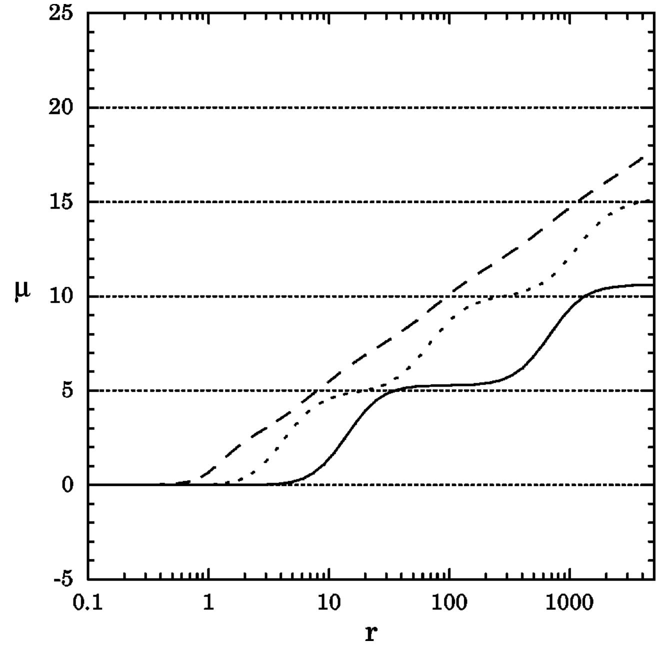

FIG. 4. The mass function $\mu(r)$ for a particlelike solution with $\epsilon=0$. The solid, dotted, and dashed lines depict those for $b=-0.01$, -0.1, and -0.5, respectively.

mass particlelike solution. The mass function increases as $\ln r$ asymptotically just like the analytic solution (3.5). The period of oscillation of w is the same as that of the steps in μ and it is constant in terms of $\ln r$. This behavior is easily understood by solving the basic equations in the asymptotic far region $(r\to\infty)$; the analytic forms are given in Appendix C. We can check that the asymptotic solution is consistent with our numerical solutions. The oscillations of w and the

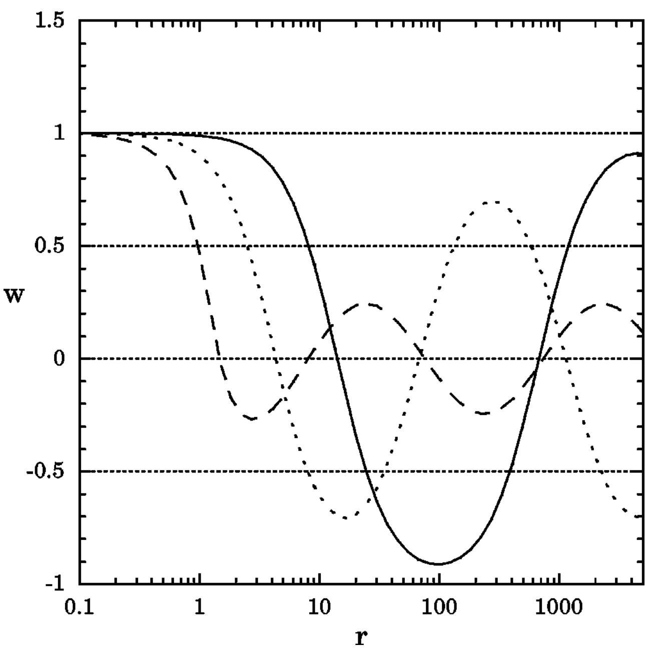

FIG. 5. The potential function $w(r)$ for a particlelike solution with $\epsilon=0$. The solid, dotted, and dashed lines depict those for $b=-0.01$, -0.1, and -0.5, respectively.

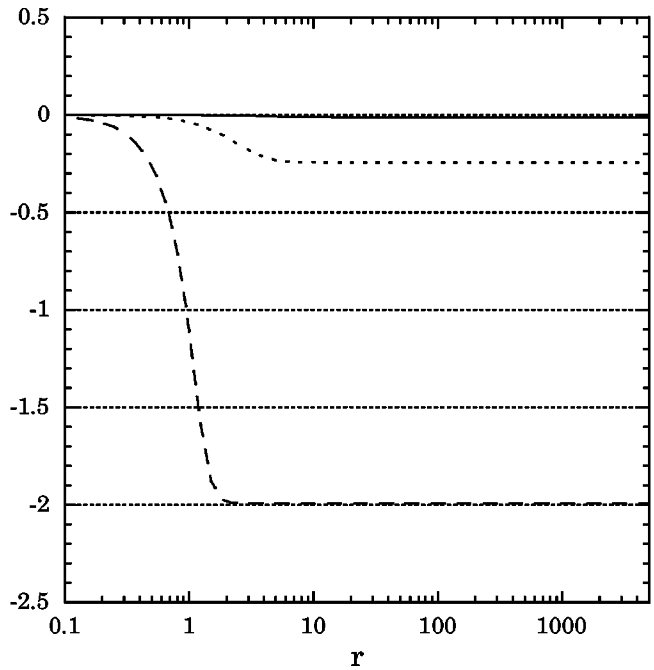

FIG. 6. The metric function $\delta(r)$ for a particlelike solution with $\epsilon=1$. The solid, dotted, and dashed lines depict those for $b=-0.01$, -0.1, and -0.5, respectively. We set $\ell=10$.

periodic steps in μ are caused by an infinite number of instantons (see Appendix C).

For the case of $\epsilon=1$, we also find a regular solution for $b_{\rm min}<b<0$. $b_{\rm min}$ depends on ℓ and decreases as ℓ decreases. For example, $b_{\rm min}\approx-0.644036$ for $\ell=10$, $b_{\rm min}\approx-1.105002$ for $\ell=1$. We show the numerical results in Figs. 6–8.

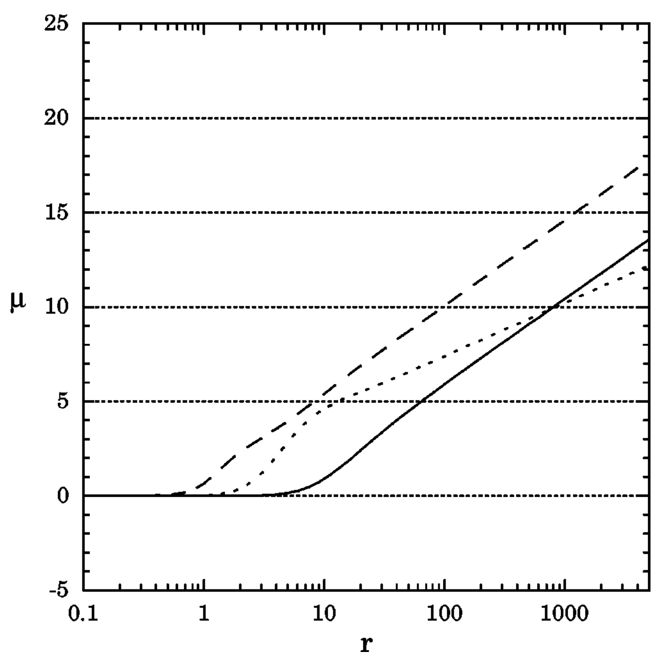

FIG. 7. The mass function $\mu(r)$ for a particlelike solution with $\epsilon=1$. The solid, dotted, and dashed lines depict those for $b=-0.01$, -0.1, and -0.5, respectively. We set $\ell=10$.

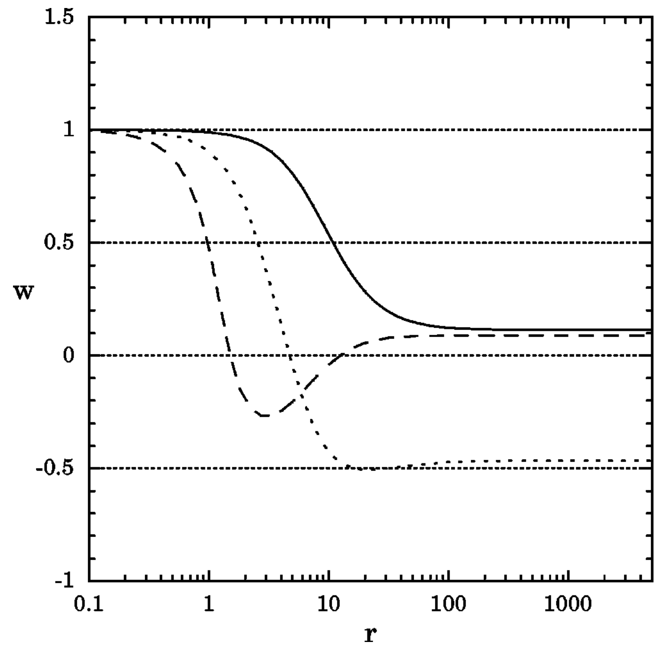

FIG. 8. The potential function $w(r)$ for a particlelike solution with $\epsilon=1$. The solid, dotted, and dashed lines depict those for $b=-0.01$, -0.1, and -0.5, respectively. We set $\ell=10$.

In this case, the potential w does not oscillate and converges to some value w_∞; thus the number of nodes is finite. The mass function increases monotonically as

$$\mu\rightarrow2(1-w_\infty^2)^2\ln r \tag{4.4}$$

as $r\rightarrow\infty$. This behavior is also understood by solving the asymptotic solution, which is given in Appendix C.

B. Black hole solution

Next we show a nontrivial black hole solution. To find a black hole solution, we have to impose a boundary condition at a horizon r_h. The horizon is defined by $f(r_h)=0$, which gives

$$\mu(r_h)=r_h^2\left(1+\epsilon\frac{r_h^2}{\ell^2}\right). \tag{4.5}$$

Here we set $\delta(r_h)=0$. The proper time of the observer at infinity [i.e., $\delta(\infty)=0$] is obtained by the transformation $t'=e^{-\delta(\infty)}t$. From Eq. (2.22), $w'(r_h)$ has to satisfy

$$w'(r_h)=-\frac{w_h(1-w_h^2)}{r_h[1+2\epsilon r_h^2/\ell^2-(1-w_h^2)^2/r_h^2]}, \tag{4.6}$$

where $w_h=w(r_h)$. There is only one free parameter w_h for a given value of r_h. Since Eqs. (2.19)–(2.22) are invariant under the transformation $w\rightarrow-w$, we can set $w_h>0$ without loss of generality.

For the solution with $w_h>1$, we find that the curvature diverges at a finite distance. Then we obtain a numerical solution for $0\leqslant w_h\leqslant1$ for a given r_h. We show the results in Figs. 9–11.

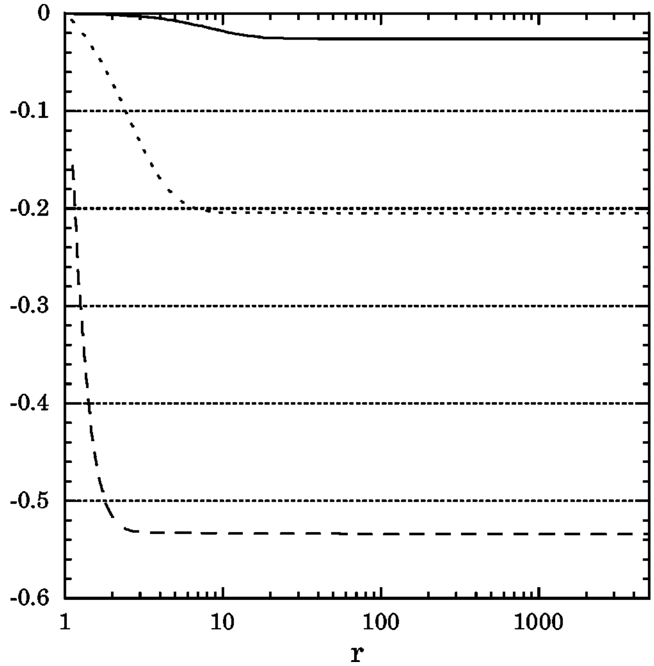

FIG. 9. The metric function $\delta(r)$ for a black hole solution with $\epsilon=0$. The solid, dotted, and dashed lines depict those for $w_h=0.99$, 0.9, and 0.5, respectively.

The asymptotic behavior is similar to that of the particle solution. The potential w oscillates infinitely with a constant period in terms of $\ln r$. For any solutions with $0<w_h<1$, we find that the mass function $\mu(r)$ diverges as $\ln r$ at large distance.

We also show the case with $\epsilon=1$ in Figs. 12–14. This

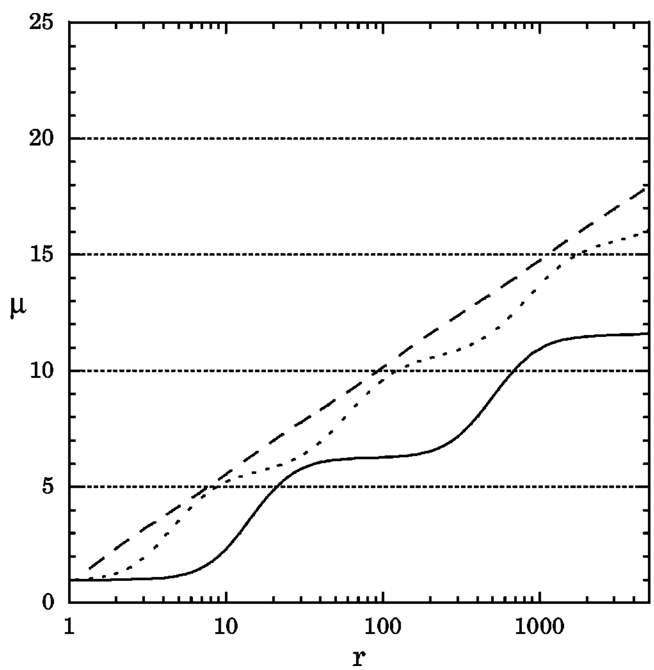

FIG. 10. The mass function $\mu(r)$ for a black hole solution with $\epsilon=0$. The solid, dotted, and dashed lines depict those for $w_h=0.99$, 0.9, and 0.5, respectively.

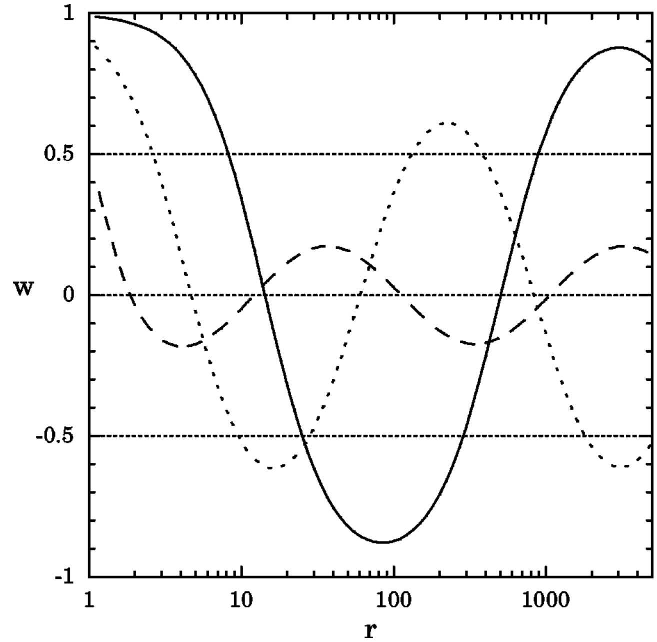

FIG. 11. The potential function $w(r)$ for a black hole solution with $\epsilon=0$. The solid, dotted, and dashed lines depict those for $w_h=0.99$, 0.9, and 0.5, respectively.

also shows similar asymptotic behaviors to those of a particle solution with $\epsilon=1$.

As for the thermodynamical properties, we find the Hawking temperature as

$$T_{\mathrm{BH}}=\frac{e^{\delta(\infty)}}{2\pi r_h}\left[1-\frac{(1-w_h^2)^2}{r_h^2}+2\epsilon\frac{r_h^2}{\ell^2}\right], \tag{4.7}$$

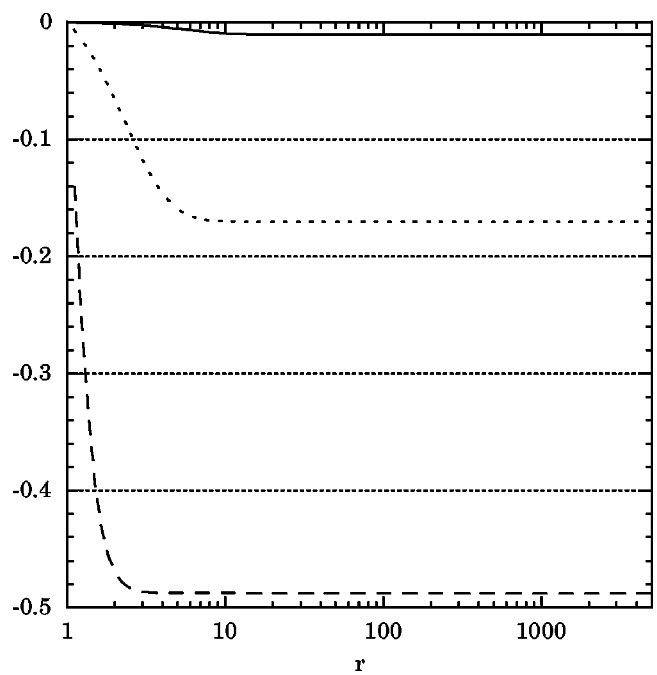

FIG. 12. The metric function $\delta(r)$ for a black hole solution with $\epsilon=1$. The solid, dotted, and dashed lines depict those for $w_h=0.99$, 0.9, and 0.5, respectively. We set $\ell=10$.

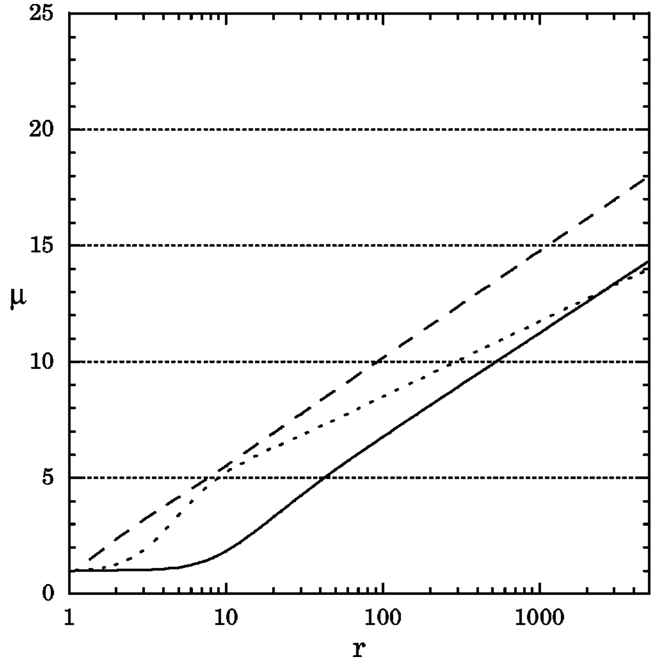

FIG. 13. The mass function $\mu(r)$ for a black hole solution with $\epsilon=1$. The solid, dotted, and dashed lines depict those for $w_h=0.99$, 0.9, and 0.5, respectively. We set $\ell=10$.

where $\delta(\infty)$ comes from our coordinate condition, that is, we set $\delta(r_h)=0$. The thermodynamical mass M_T is found from the first law of black hole thermodynamics, $dM_T=T\,dS+\Phi\,dQ$. In order to calculate M_T, fixing $w_\infty=w(\infty)$, we obtain a black hole solution because the ''global charge'' is proportional to $(1-w_\infty^2)$. The result is shown in Fig. 15.

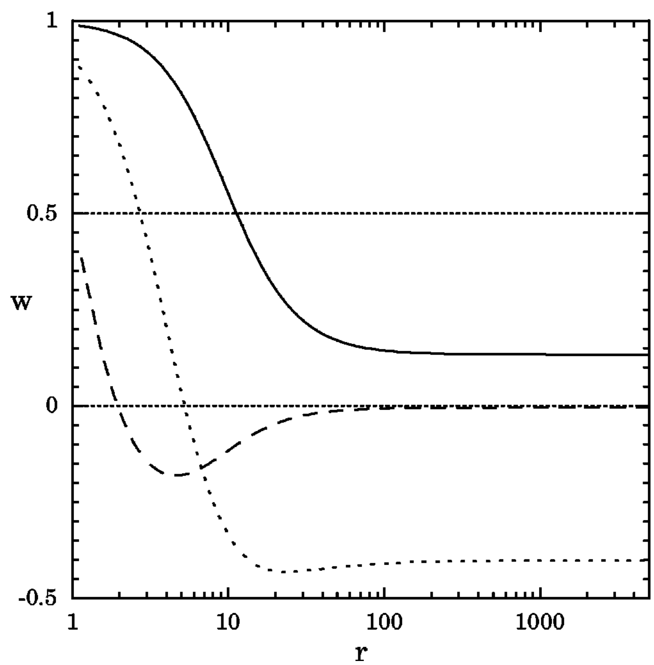

FIG. 14. The potential function $w(r)$ for a black hole solution with $\epsilon=1$. The solid, dotted, and dashed lines depict those for $w_h=0.99$, 0.9, and 0.5, respectively. We set $\ell=10$.

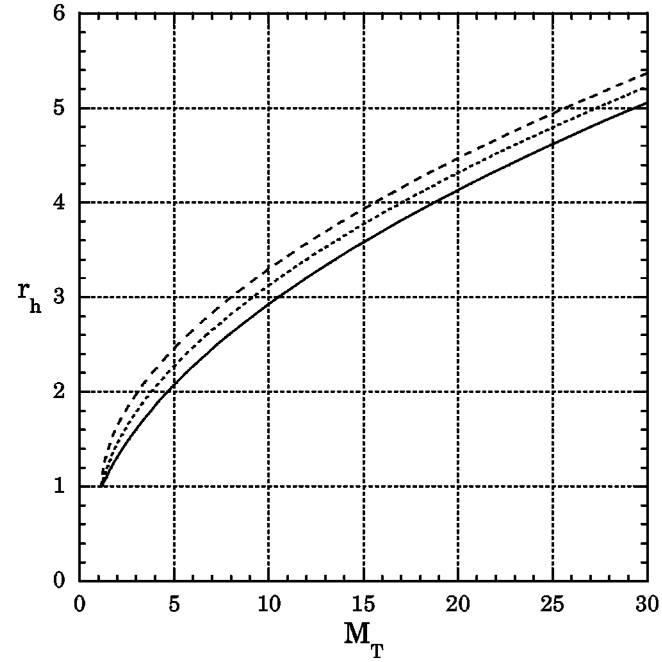

FIG. 15. The r_h-M_T relation. The solid, dotted, and dashed lines depict those for $w_\infty=0.9$, 0.5, and 0.0 (analytic solution).

We numerically confirm that the thermodynamical mass M_T is equal to

$$M_T=\lim_{r\to\infty}\frac{3\pi}{8}[\mu-2(1-w^2)^2\ln r]. \tag{4.8}$$

V. STABILITY

In this section, we analyze the stability of the static solutions obtained above. We perturb the metric and potential as

$$\mu(r,t)=\mu_0(r)+\mu_1(r)e^{i\omega t}, \tag{5.1}$$

$$\delta(r,t)=\delta_0(r)+\delta_1(r)e^{i\omega t}, \tag{5.2}$$

$$w(r,t)=w_0(r)+w_1(r)e^{i\omega t}, \tag{5.3}$$

where $\mu_0(r)$, $\delta_0(r)$, and $w_0(r)$ are those of the static solution obtained in the previous section. Substituting them into the Einstein equations and Yang-Mills equation, we find the perturbation equations as

$$\mu_1'=2r\left[2f_0w_0'w_1'-\frac{w_0'^2}{r^2}\mu_1-\frac{4(1-w_0^2)w_0}{r^2}w_1\right], \tag{5.4}$$

$$\mu_1=4rf_0w_0'w_1, \tag{5.5}$$

$$\delta_1'=-\frac{4}{r}w_0'w_1', \tag{5.6}$$

and

$$-\frac{1}{r^3}(rf_0e^{-\delta_0}w_0')'f_0^{-1}\mu_1-f_0e^{-\delta_0}w_0'\left(\frac{1}{r^2}f_0^{-1}\mu_1+\delta_1\right)'$$
$$+\frac{1}{r}(rf_0e^{-\delta_0}w_1')'+\frac{2}{r^2}e^{-\delta_0}(1-3w_0^2)w_1$$
$$=-\omega^2f_0^{-1}e^{\delta_0}w_1, \quad (5.7)$$

where $f_0=1-\mu_0(r)/r^2+\epsilon r^2/\ell^2$. Equation (5.4) is derived from Eq. (5.5) by differentiation.

We introduce a tortoise coordinate r_* such that

$$\frac{dr_*}{dr}=e^{\delta_0}f_0^{-1} \quad (5.8)$$

and define $\chi=w_1r^{1/2}$. Then, by substituting Eqs. (5.4)–(5.6), Eq. (5.7) turns out to be the single uncoupled equation

$$-\frac{d^2\chi}{dr_*^2}+V(r_*)\chi=\omega^2\chi, \quad (5.9)$$

where

$$V(r_*)=f_0e^{-\delta_0}\left\{\frac{2}{r^2}e^{-\delta_0}(3w_0^2-1)+\frac{r^{-1/2}}{2}(r^{-1/2}f_0e^{-\delta_0})'+\frac{4}{r}[f_0e^{-\delta_0}w_0'^2]'\right\}. \quad (5.10)$$

When $V(r_*)$ is positive definite, we can prove its stability as follows. Multiplying Eq. (5.9) by $\bar{\chi}$ and integrating from $r=r_+$ ($r_*=-\infty$) in the case of a black hole solution or $r=0$ ($r_*=0$) in the case of a particle solution to $r=\infty$ [$r_*=r_{*,\max}(<\infty)$], Eq. (5.9) is written as

$$-\left[\bar{\chi}\frac{d\chi}{dr_*}\right]_{r=r_+(\text{or }0)}^{r=\infty}+\int\left[\left|\frac{d\chi}{dr_*}\right|^2+V(r)|\chi|^2\right]dr_*$$
$$=\omega^2\int|\chi|^2\,dr_*. \quad (5.11)$$

We assume that $w_1\to0$ at infinity [$r\to\infty(r_*\to r_{*,\max})$]. Then $\bar{\chi}\,d\chi/dr_*\to0$. In the case of a black hole, χ must be ingoing at the horizon [$r=r_+$ ($r_*=-\infty$)]. Since the potential V vanishes at the horizon, the ingoing wave condition gives $\chi\sim e^{i\omega r_*}$. If we assume that $\mathrm{Im}\,\omega<0$, then $\bar{\chi}\,d\chi/dr_*\to0$ at the horizon is obtained. Because $V(r)$ is positive definite, Eq. (5.11) implies that the eigenvalue ω is real, that is, $\mathrm{Im}\,\omega=0$, which contradicts the above assumption. Hence, we conclude that $\mathrm{Im}\,\omega\geqslant0$, which means that the present system is stable. In the case of a particle solution, we should impose $w_1=0$ at the origin [$r=0$ ($r_*=0$)]; then we find $\bar{\chi}\,d\chi/dr_*\to0$. If $V(r)$ is positive definite, Eq. (5.11) again implies that the eigenvalue ω is real. Hence, in both cases, we obtain the result that solutions with a positive definite potential $V(r_*)$ are stable.

For the analytic solution (3.5), we find

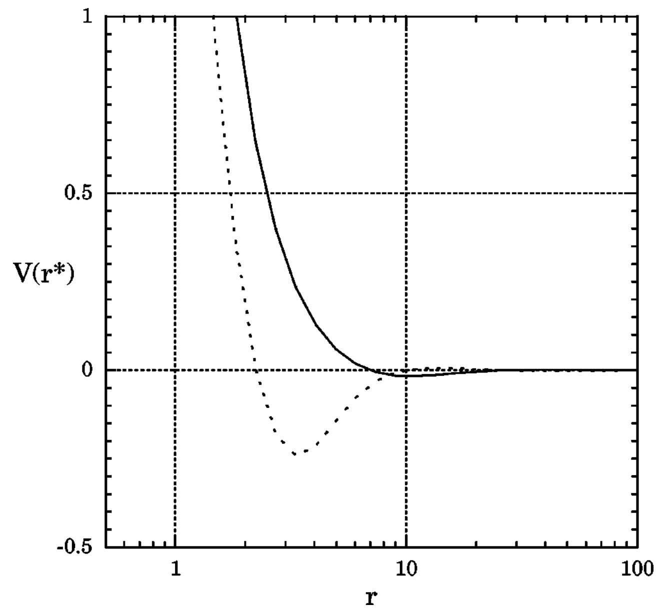

FIG. 16. Potential $V(r_*)$ for $\epsilon=0$. The solid and dotted lines denote those for $b=-0.01$ and $b=-0.1$. There is a negative region [$V(r_*)<0$] for both potentials.

$$V(r_*)=\frac{f_0}{4r^4}\left[5\mathcal{M}-4+10\ln r-9r^2+3\epsilon\frac{r^4}{\ell^2}\right]. \quad (5.12)$$

In the case of $\epsilon=0$ or -1, $V(r_*)$ is negative at large distance r, while, in the case of $\epsilon=1$, we see that $V(r_*)$ is positive definite for sufficiently large $\mathcal{M}$, i.e., for

$$\mathcal{M}>\frac{1}{5}\left(4-10\ln r_p+9r_p^2-3\frac{r_p^4}{\ell^2}\right), \quad (5.13)$$

where $r_p=(9-\sqrt{81-120/\ell^2})\ell/12$ and $\ell>\sqrt{40/27}$.

For the numerical solutions, we also find a positive definite potential $V(r_*)$ only for the case of $\epsilon=1$. For a particle solution, for example, a positive definite potential is found in the parameter range of $-0.010368<b<0$ for $\ell=10$ and $-0.654211<b<0$ for $\ell=1$. We depict some typical potentials in Figs. 16 and 17.

If $V(r_*)$ is positive definite, we conclude that the system is stable; however, we cannot predict anything if $V(r_*)$ is not positive definite. We have to solve Eqs. (5.4)–(5.7) as an eigenvalue problem numerically. We leave this to future work and in this paper we do not discuss it further.

VI. SUMMARY AND DISCUSSION

In this paper we have studied a spherically symmetric Einstein–SU(2)–Yang-Mills system in five dimensions. If we consider only the "electric part" of the Yang-Mills field, we find the five-dimensional Reissner-Nordström black hole solution. As for the "magnetic part" of the Yang-Mills field, apart from a trivial Schwarzschild (Schwarzschild–de Sitter or Schwarzschild–anti de Sitter) solution, we find a nontrivial analytic solution, which corresponds to a magnetically charged black hole. (It turns out to be just the Reissner-

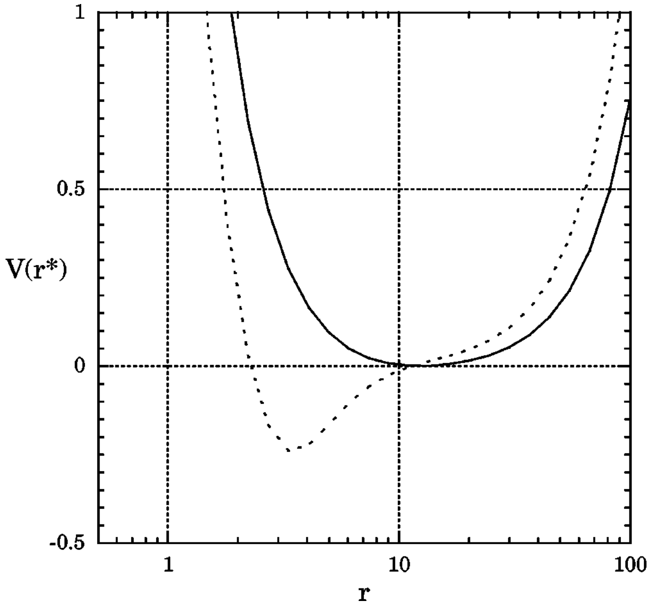

FIG. 17. Potential $V(r_*)$ for $\epsilon=1$. The solid and dotted lines denote those for $b=-0.01$ and $b=-0.1$. There is a negative region $[V(r_*)<0]$ for $b=-0.1$ but the result is positive definite for $b=-0.01$. We set $\ell=10$.

Nordström solution in the four-dimensional case.) This nontrivial solution shows that the gravitational ''mass'' is infinite and the spacetime does not satisfy asymptotically flat, de Sitter, or anti–de Sitter conditions, in contrast to the case of four dimensions. However, its metric approaches either the Minkowski or de Sitter (or anti–de Sitter) metric. We also find that there is no singularity except one at the origin which is covered by a horizon. Hence we call its behavior at infinity ''quasiasymptotically'' flat, de Sitter, or anti–de Sitter and we regard our solution as a localized object. We analyze the spacetime structure and thermodynamical properties. They show that the mass parameter $\mathcal{M}$ in the solution can be regarded as a thermodynamical mass, which satisfies the first law of the black hole thermodynamics.

For the case with zero or negative cosmological constant, we also find numerically particlelike solutions, which have no singularity, and black hole solutions with nontrivial structures of the Yang-Mills field. Although, for both cases, the mass function diverges as $\ln r$, they satisfy ''quasiasymptotically'' flat or anti–de Sitter conditions. If $\Lambda=0$, in contrast to the case of four dimensions, the Yang-Mills field oscillates and has an infinite number of nodes. For a negative cosmological constant, the Yang-Mills field potential settles to some constant, which is similar to that in the four-dimensional case.

From the stability analysis, we find that there is a set of stable solutions if a cosmological constant is negative. This result is very similar to the four-dimensional case, in which the Bartnik-McKinnon solution and a colored black hole are unstable, while those extended to the case with a negative cosmological constant become stable.

Since we find a stable nonsingular solution in the five dimensions, if we apply it to a brane world scenario, we may find some interesting effect on the brane dynamics. We will publish its analysis in a separate paper.

ACKNOWLEDGMENTS

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APPENDIX A: FIVE-DIMENSIONAL SPHERICALLY SYMMETRIC SU(2) GAUGE FIELD

Here we calculate a generic form of the spherically symmetric SU(2) Yang-Mills field in five-dimensional spacetime. In the case of four dimensions, Witten gave its generic form [25], which was called the Witten ansatz and proved by Forgác and Manton [26]. Forgác and Manton showed how to find a generic form of the spherically symmetric Yang-Mills field in arbitrary dimensions. We just follow their method.

Suppose we have some symmetry of spacetime generated by a vector $\vec{\eta}$. A tensor field must be invariant under an infinitesimal transformation generated by $\vec{\eta}$, i.e., the Lie derivative of this tensor field with respect to $\vec{\eta}$ must vanish. However, in the case of a gauge field A_μ, there is gauge freedom by which we can weaken this condition such that there exists an infinitesimal gauge transformation equivalent to a spacetime transformation, that is,

$$\pounds_{\vec{\eta}} A_\mu = D_\mu W \equiv \partial_\mu W - [A_\mu, W] \tag{A1}$$

for some scalar field W [27].

Suppose that a D-dimensional Riemannian manifold M has some spacetime symmetry represented by N-dimensional isometry group S. This isometry group is generated by N Killing vectors $\vec{\xi}_{(n)}$ $(1 \leqslant n \leqslant N)$, with commutation relations given by

$$[\vec{\xi}_{(m)}, \vec{\xi}_{(n)}]^\mu = f_{mnp} \xi^\mu_{(p)}, \tag{A2}$$

where f_{mnp} is a structure constant. We assume that the orbit $X = \{ap \in M | a \in S\}$ for some point $p \in M$ is an N'-dimensional submanifold of M. Then we choose the local coordinate system as

$$x^\mu = (x^i, y^\alpha), \quad 1 \leqslant i \leqslant D - N', \quad 1 \leqslant \alpha \leqslant N' \tag{A3}$$

so that a hypersurface of $x^i = \text{const}$ defines the orbit space X. By Frobenius' theorem, the above Killing vectors are orthogonal to X; then $\xi^\mu_{(n)} = (0, \xi^\alpha_{(n)})$ in this coordinate system.

Because the isometry group is an N-dimensional Lie group, we can define right and left invariant vectors ($\vec{\xi}^R_{(n)}$ and $\vec{\xi}^L_{(n)}$) as

$$\pounds_{\vec{\xi}^R_{(n)}} s = s J^{(n)}, \quad \pounds_{\vec{\xi}^L_{(n)}} s = -J^{(n)} s \tag{A4}$$

for any $s\in S$, where $J^{(n)}$ is the generator of the Lie group S associated with the Killing vector $\vec{\xi}_{(n)}$. Then both $\vec{\xi}^R_{(n)}$ and $\vec{\xi}^L_{(n)}$ have the same commutation relations as those of $\vec{\xi}_{(n)}$. We also define the covariant vector fields $\xi^R_{(n)\hat{\alpha}},\xi^L_{(n)\hat{\alpha}}$ by

$$\xi^R_{(m)\hat{\alpha}}\xi^{R\hat{\alpha}}_{(n)}=\delta_{mn}, \quad \xi^L_{(m)\hat{\alpha}}\xi^{L\hat{\alpha}}_{(n)}=\delta_{mn}. \tag{A5}$$

For a fixed point $q\in X$, $R=\{a\in S|aq=q\}\subset S$ is an invariant subgroup of S with dimension $N-N'$, and the quotient group S/R is diffeomorphic to X. So we can adopt the same coordinates y^α in X for the coset $Rs\in S/R$. We take the other coordinate components to be expressed by $y^\omega(1\leqslant\omega\leqslant N-N')$, corresponding to those of the isotropy group R. If we fix the origin $s_0(y^\alpha)\in Rs(y^\alpha)$ for each coset in a smooth way, then any element s of S is written uniquely with coordinates $y^{\hat{\alpha}}=(y^\omega,y^\alpha)$ as

$$s(y^{\hat{\alpha}})=r(y^\omega)s_0(y^\alpha) \tag{A6}$$

for some $r\in R$. In these coordinates, the right invariant vector $\vec{\xi}^R_{(n)}$ is expressed with Killing vector $\vec{\xi}_{(n)}$ as

$$\xi^{R\hat{\alpha}}_{(n)}=(\xi^{R\omega}_{(n)},\xi^\alpha_{(n)}). \tag{A7}$$

By the above definition, we find a generic form of the gauge potential $A_\mu(x^\mu)$ with a gauge symmetry G and a spacetime symmetry S as

$$A_i(x^i,y^\alpha)=A_i(x^i), \quad A_\alpha(x^i,y^\alpha)=\Phi_n(x^i)\xi^L_{(n)\alpha}(y^\alpha)|_{y^\omega=y^\omega_0}, \tag{A8}$$

where $A_i(x^i)$ and $\Phi_n(x^i)$ satisfy the conditions

$$f_{mnp}\Phi_p+[\Phi_m,\Phi_n]=0(^\forall m,^\forall n>N'),$$
$$\partial_i\Phi_n-[A_i,\Phi_n]=0(^\forall i,^\forall n>N'), \tag{A9}$$

and $y^\omega=y^\omega_0$ is a coordinate component of a unit element of the isotropy group R. Incidentally, W_n in Eq. (A1) are obtained as

$$W_n(x^i,y^\alpha)=-\Phi_m(x^i)\xi^{R\omega}_{(n)}(y^\alpha)\xi^L_{(m)\omega}(y^\alpha)|_{y^\omega=y^\omega_0}. \tag{A10}$$

Applying this formalism, the five-dimensional spherically symmetric SU(2) gauge field, we obtain a generic form of the gauge potential $A_\mu(x^\mu)$. We assume that the isometric group is SO(4). In the coordinate system (2.4), the orbit X is given as $t,r=$const, and then x^μ is divided into $x^i=(t,r)$ and $y^\alpha=(\psi,\theta,\varphi)$.

The Killing vectors are given by

$$\vec{\xi}_{(1)}=(0,0,-\cos\theta,\cot\psi\sin\theta,0),$$
$$\vec{\xi}_{(2)}=\left(0,0,-\sin\theta\cos\varphi,-\cot\psi\cos\theta\cos\varphi,\frac{\cot\psi\sin\varphi}{\sin\theta}\right),$$
$$\vec{\xi}_{(3)}=\left(0,0,-\sin\theta\sin\varphi,-\cot\psi\cos\theta\sin\varphi,-\frac{\cot\psi\cos\varphi}{\sin\theta}\right),$$
$$\vec{\xi}_{(4)}=(0,0,0,-\cos\varphi,\cot\theta\sin\varphi),$$
$$\vec{\xi}_{(5)}=(0,0,0,-\sin\varphi,-\cot\theta\cos\varphi),$$
$$\vec{\xi}_{(6)}=(0,0,0,0,-1), \tag{A11}$$

and the structure constants f_{mnp} are found to be

$$f_{124}=1, \quad f_{135}=1, \quad f_{236}=1, \quad f_{456}=1, \tag{A12}$$

with the other components totally antisymmetrized.

Next we adopt the local coordinate system which satisfies Eq. (A6) in SO(4). It is given as a four-dimensional Euler angle $(\alpha,\beta,\chi,\psi,\theta,\varphi)$ by

$$\begin{aligned}s(\alpha,\beta,\chi,\psi,\theta,\varphi)&=r(\alpha,\beta,\chi)s_0(\psi,\theta,\varphi)\\&=R_{xy}(\alpha)R_{yz}(\beta)R_{xy}(\chi)R_{zu}(\psi)\\&\quad\times R_{yz}(\theta)R_{xy}(\varphi),\end{aligned} \tag{A13}$$

where R_{pq} denotes a rotation matrix of the pq plane. Note that $R_{xy}(\alpha)R_{yz}(\beta)R_{xy}(\chi)$ describes any element of an isotropy group R.

In this coordinate system, the right invariant vector $\vec{\xi}^R_n$ and the covariant left invariant vector $\vec{\xi}^L_n$ are

$$\xi^{R\hat{\alpha}}_{(1)}=\left(-\frac{\sin\chi\sin\theta}{\sin\beta\sin\psi},-\frac{\cos\chi\sin\theta}{\sin\psi},\frac{\cot\beta\sin\chi\sin\theta}{\sin\psi},-\cos\theta,\cot\psi\sin\theta,0\right),$$

$$\begin{aligned}\xi^{R\hat{\alpha}}_{(2)}=\Bigg(&\frac{\cos\chi\sin\varphi+\sin\chi\cos\theta\cos\varphi}{\sin\beta\sin\psi},-\frac{\sin\chi\sin\varphi-\cos\chi\cos\theta\cos\varphi}{\sin\psi},\\&-\frac{\cot\beta\sin\chi\cos\theta\cos\varphi+(\cos\psi\cot\theta+\cot\beta\cos\chi)\sin\varphi}{\sin\psi},\\&-\sin\theta\cos\varphi,-\cot\psi\cos\theta\cos\varphi,\frac{\cot\psi\sin\varphi}{\sin\theta}\Bigg),\end{aligned}$$

$$\xi_{(3)}^{R\hat{\alpha}}=\Bigg(-\frac{\cos\chi\cos\varphi-\sin\chi\cos\theta\sin\varphi}{\sin\beta\sin\psi},\frac{\sin\chi\cos\varphi+\cos\chi\cos\theta\sin\varphi}{\sin\psi},$$
$$-\frac{\cot\beta\sin\chi\cos\theta\sin\varphi-(\cos\psi\cot\theta+\cot\beta\cos\chi)\cos\varphi}{\sin\psi},$$
$$-\sin\theta\sin\varphi,-\cot\psi\cos\theta\sin\varphi,-\frac{\cot\psi\cos\varphi}{\sin\theta}\Bigg),$$
$$\xi_{(4)}^{R\hat{\alpha}}=\Bigg(0,0,-\frac{\sin\varphi}{\sin\theta},0,-\cos\varphi,\cot\theta\sin\varphi\Bigg),$$
$$\xi_{(5)}^{R\hat{\alpha}}=\Bigg(0,0,\frac{\cos\varphi}{\sin\theta},0,-\sin\varphi,-\cot\theta\cos\varphi\Bigg),$$
$$\xi_{(6)}^{R\hat{\alpha}}=(0,0,0,0,0,-1), \tag{A14}$$

and

$$\xi^{L}_{(1)\hat{\alpha}}=(0,0,0,\cos\beta,\sin\beta\cos\chi\sin\psi,\sin\beta\sin\chi\sin\psi\sin\theta),$$
$$\xi^{L}_{(2)\hat{\alpha}}=(0,0,0,-\cos\alpha\sin\beta,-\sin\alpha\sin\chi\sin\psi+\cos\alpha\cos\beta\cos\chi\sin\psi,$$
$$\sin\alpha\cos\chi\sin\psi\sin\theta+\cos\alpha\cos\beta\sin\chi\sin\psi\sin\theta),$$
$$\xi^{L}_{(3)\hat{\alpha}}=(0,0,0,\sin\alpha\sin\beta,-\cos\alpha\sin\chi\sin\psi-\sin\alpha\cos\beta\cos\chi\sin\psi,$$
$$\cos\alpha\cos\chi\sin\psi\sin\theta-\sin\alpha\cos\beta\sin\chi\sin\psi\sin\theta),$$
$$\xi^{L}_{(4)\hat{\alpha}}=(0,\cos\alpha,\sin\alpha\sin\beta,0,\cos\alpha\cos\chi\cos\psi-\sin\alpha\cos\beta\sin\chi\cos\psi,$$
$$\sin\alpha\sin\beta\cos\theta+\cos\alpha\sin\chi\cos\psi\sin\theta+\sin\alpha\cos\beta\cos\chi\cos\psi\sin\theta),$$
$$\xi^{L}_{(5)\hat{\alpha}}=(0,-\sin\alpha,\cos\alpha\sin\beta,0,-\sin\alpha\cos\chi\cos\psi-\cos\alpha\cos\beta\sin\chi\cos\psi,$$
$$\cos\alpha\sin\beta\cos\theta-\sin\alpha\sin\chi\cos\psi\sin\theta+\cos\alpha\cos\beta\cos\chi\cos\psi\sin\theta),$$
$$\xi^{L}_{(6)\hat{\alpha}}=(1,0,\cos\beta,0,\sin\beta\sin\chi\cos\psi,\cos\beta\cos\theta-\sin\beta\cos\chi\cos\psi\sin\theta). \tag{A15}$$

Equations (A9) are given as

$$f_{mnp}\Phi^a_p+\varepsilon^{abc}\Phi^b_m\Phi^c_n=0$$
$$(a=1,2,3;m=1,\ldots,6;n=4,5,6),$$
$$\partial_i\Phi^a_n-\varepsilon^{abc}A^b_i\Phi^c_n=0 \quad (a=1,2,3;i=t,r;n=4,5,6). \tag{A16}$$

This set of equations has two types of solutions; one is the ‘‘electric’’ type and the other is the ‘‘magnetic’’ one. The former type is given by

$$A^a_t=(0,0,A_t),\quad A^a_r=(0,0,A_r),\quad \text{and}\quad \Phi^a_m=0, \tag{A17}$$

leading to the potential form

$$\mathbf{A}=\tau_3(A_t\,dt+A_r\,dr). \tag{A18}$$

Using the gauge freedom, we can set $A_r=0$. The ‘‘electric’’ type of potential is now given by

$$\mathbf{A}=\tau_3A(t,r)dt. \tag{A19}$$

The latter type of solution is given by

$$A^a_t=(0,0,\dot{X}),\quad A^a_r=(0,0,X'),$$
$$\Phi^a_1=(0,0,\phi),\quad \Phi^a_2=\pm(\phi\cos X,\phi\sin X,0),$$
$$\Phi^a_3=(\phi\sin X,-\phi\cos X,0), \tag{A20}$$
$$\Phi^a_4=\pm(\sin X,-\cos X,0),\quad \Phi^a_5=-(\cos X,\sin X,0),$$
$$\Phi^a_6=(0,0,\pm1).$$

We then obtain a general form of A^a_μ as

$$\begin{aligned}\mathbf{A}=&\tau_3(\dot{X}\,dt+X'\,dr+\phi d\psi+\cos\theta\,d\varphi)\\&+\cos\psi[(\tau_1\sin X-\tau_2\cos X)d\theta\\&-(\tau_1\cos X+\tau_2\sin X)\sin\theta\,d\varphi]\\&+\phi\sin\psi[(\tau_1\cos X+\tau_2\sin X)d\theta\\&+(\tau_1\sin X-\tau_2\cos X)\sin\theta\,d\varphi].\end{aligned}\tag{A21}$$

X is not a dynamical variable but it is regarded as a gauge variable. In fact, the field strength $F_{\mu\nu}{}^{a}$ is given by

$$\begin{aligned}\mathbf{F}=&\tau_3[\dot{\phi}\,dt\wedge d\psi+\phi'\,dr\wedge d\psi\\&-(1-\phi^2)(\sin\psi d\theta)\wedge(\sin\psi\sin\theta\,d\varphi)]\\&+(\tau_1\cos X+\tau_2\sin X)[\dot{\phi}\,dt\wedge(\sin\psi\,d\theta)\\&+\phi'\,dr\wedge(\sin\psi\,d\theta)+(1-\phi^2)d\psi\wedge(\sin\psi\sin\theta\,d\varphi)]\\&+(\tau_1\sin X-\tau_2\cos X)[\dot{\phi}\,dt\wedge(\sin\psi\sin\theta\,d\varphi)\\&+\phi'\,dr\wedge(\sin\psi\sin\theta\,d\varphi)-(1-\phi^2)d\psi\wedge(\sin\psi\,d\theta)].\end{aligned}\tag{A22}$$

Rotating the τ_1-τ_2 plane of the interior space by $-X$, the variable X is eliminated. If we choose $X=0$, we find

$$\begin{aligned}\mathbf{A}=&\tau_3(\phi\,d\psi+\cos\theta\,d\varphi)-\cos\psi[\tau_2\,d\theta+\tau_1\sin\theta\,d\varphi]\\&+\phi\sin\psi[\tau_1\,d\theta-\tau_2\sin\theta\,d\varphi],\end{aligned}$$

$$\begin{aligned}\mathbf{F}=&\tau_3[\dot{\phi}\,dt\wedge d\psi+\phi'\,dr\wedge d\psi\\&-(1-\phi^2)(\sin\psi\,d\theta)\wedge(\sin\psi\sin\theta\,d\varphi)]\\&+\tau_1[\dot{\phi}\,dt\wedge(\sin\psi\,d\theta)+\phi'\,dr\wedge(\sin\psi\,d\theta)\\&+(1-\phi^2)d\psi\wedge(\sin\psi\sin\theta\,d\varphi)]\\&-\tau_2[\dot{\phi}\,dt\wedge(\sin\psi\sin\theta\,d\varphi)\\&+\phi'\,dr\wedge(\sin\psi\sin\theta\,d\varphi)\\&-(1-\phi^2)d\psi\wedge(\sin\psi\,d\theta)].\end{aligned}\tag{A23}$$

APPENDIX B: NONEXISTENCE OF FINITE MASS OBJECT ($\Lambda\leqslant 0$)

Here we show that there is no particlelike solution with finite mass if $\Lambda\leqslant 0$ ($\epsilon=0$ or 1).

Introducing the new variable

$$z=2\ln r,\tag{B1}$$

we rewrite the basic equations (3.1) and (3.3) with Eq. (3.2) as

$$\frac{d\mu}{dz}=4f\left(\frac{dw}{dz}\right)^2+(1-w^2)^2,\tag{B2}$$

$$f\frac{d^2w}{dz^2}+\left[e^{-z}\mu+\frac{\epsilon}{\ell^2}e^{z}-e^{-z}(1-w^2)^2\right]\frac{dw}{dz}+\frac{1}{2}w(1-w^2)=0\tag{B3}$$

with

$$f=1-e^{-z}\mu+\frac{\epsilon}{\ell^2}e^{z},\tag{B4}$$

where the function δ is eliminated.

If we turn off gravity, that is, if we consider the Yang-Mills field equation in Minkowski space, we have one basic equation

$$\frac{d^2w}{dz^2}-e^{-z}(1-w^2)^2\frac{dw}{dz}+\frac{1}{2}w(1-w^2)=0.\tag{B5}$$

This is easily integrated as

$$\frac{1}{2}\left(\frac{dw}{dz}\right)^2-\frac{1}{8}(1-w^2)^2=E_0,\tag{B6}$$

where E_0 is an integration constant. Integrating this equation with the boundary condition $w\to\pm 1$ as $z\to-\infty$ ($r\to 0$) and $z\to\infty$ ($r\to\infty$), which implies $E_0=0$, we obtain the solution for w as

$$w=\pm\tanh\frac{z}{2}.\tag{B7}$$

This is exactly the same as the Yang-Mills instanton solution in four-dimensional Euclidean spacetime [28]. If we regard

$$U(z)=-\frac{1}{8}(1-w^2)^2\tag{B8}$$

as a potential, Eq. (B6) just denotes energy conservation. The instanton corresponds to the zero energy solution, in which w varies from ± 1 to ∓ 1 as $z=-\infty\to\infty$.

When we include the effect of gravity, do we still have such a nontrivial structure or not? This is our question. In this appendix, we will show that there is no self-gravitating nontrivial solution with a finite mass energy. To discuss this, we introduce the energy function E by

$$E=\frac{1}{2}f\left(\frac{dw}{dz}\right)^2-\frac{1}{8}(1-w^2)^2.\tag{B9}$$

The basic equations (B2) and (B3) are described by

$$\frac{dE}{dz}=-4e^{-z}\left(\frac{dw}{dz}\right)^2\left[E+\frac{\mu}{8}+\frac{\epsilon}{8\ell^2}e^{2z}\right],\tag{B10}$$

$$\frac{d\mu}{dz}=8E+2(1-w^2)^2.\tag{B11}$$

Since we are interested in a particlelike solution, which must be regular at the origin, we can expand the functions μ and w as

$$\mu=\mu_1 e^{z}+\mu_2 e^{2z}+\mu_3 e^{3z}+\cdots,$$
$$w=1+w_1 e^{z}+w_2 e^{2z}+w_3 e^{3z}+\cdots \tag{B12}$$

as $z\to-\infty(r\to 0)$. Inserting this form into Eqs. (B2) and (B3), we find the expansion coefficients as

$$\mu_1=0,\quad \mu_2=4w_1^2,\quad \mu_3=-\frac{4\epsilon}{\ell^2}w_1^2+\frac{16}{3}w_1^3(1+w_1),$$
$$w_2=-\frac{2\epsilon}{3\ell^2}w_1+\frac{w_1^2}{6}(3+8w_1),$$
$$w_3=\frac{\epsilon^2}{2\ell^4}w_1-\frac{\epsilon}{8\ell^2}w_1^2(5+24w_1)+\frac{w_1^3}{4}(1+8w_1)(1+2w_1), \tag{B13}$$

where w_1 is a free parameter.

Putting those relations into Eqs. (B9) and (B12), we find

$$E=-\frac{1}{6}w_1^2 e^{3z}\left(\frac{\epsilon}{\ell^2}+4w_1^2\right), \tag{B14}$$

$$\mu=4w_1^2 e^{2z}. \tag{B15}$$

For $\epsilon=0$ or 1, $E\to-0$ and $\mu\to+0$ as $z\to-\infty$. The right-hand side of Eq. (B11) is positive definite because

$$8E+2(1-w^2)^2=4f\left(\frac{dw}{dz}\right)^2+(1-w^2)^2, \tag{B16}$$

and $f>0$ should be imposed for a particlelike solution. Hence, the mass function μ is also positive definite.

Next, we analyze the behavior of the solution near infinity $(z\to\infty)$. If the mass function does not diverge, we can expand μ and w as

$$\mu=\mathcal{M}_0+\mathcal{M}_1 e^{-z}+\mathcal{M}_2 e^{-2z}+\mathcal{M}_3 e^{-3z}+\cdots,$$
$$w=-1+\mathcal{W}_1 e^{-z}+\mathcal{W}_2 e^{-2z}+\mathcal{W}_3 e^{-3z}+\cdots, \tag{B17}$$

as $z\to\infty$.

From the basic equations, we find the relations between the expansion coefficients as

$$\mathcal{M}_1=0,\quad \mathcal{M}_2=-4\mathcal{W}_1^2,$$
$$\mathcal{M}_3=\frac{4}{3}\mathcal{W}_1^2(4\mathcal{W}_1-3\mathcal{M}_0),$$
$$\mathcal{W}_2=\frac{2}{3}\mathcal{M}_0\mathcal{W}_1-\frac{1}{2}\mathcal{W}_1^2,$$
$$\mathcal{W}_3=\frac{\mathcal{W}_1}{8}(4\mathcal{M}_0^2-5\mathcal{M}_0\mathcal{W}_1+2\mathcal{W}_1^2) \tag{B18}$$

for $\epsilon=0$, and

$$\mathcal{M}_1=-4\frac{\mathcal{W}_1^2}{\ell^2}\quad \mathcal{M}_2=-\frac{8}{\ell^2}\mathcal{W}_1\mathcal{W}_2,$$
$$\mathcal{M}_3=-\frac{2}{3}\mathcal{W}_1^2(4\mathcal{M}_0-3\mathcal{W}_1),\quad \mathcal{W}_2=0,$$
$$\mathcal{W}_3=\frac{\ell^2}{12}\mathcal{W}_1(4\mathcal{M}_0-3\mathcal{W}_1) \tag{B19}$$

for $\epsilon=1$. Here, $\mathcal{M}_0$ and $\mathcal{W}_1$ are free parameters.

Using those relations, the energy function E near infinity is evaluated as

$$E=\frac{1}{6}e^{-3z}\mathcal{W}_1^2+\cdots\to+0 \tag{B20}$$

for $\epsilon=0$ and

$$E=\frac{1}{2\ell^2}e^{-z}\mathcal{W}_1^2+\cdots\to+0 \tag{B21}$$

for $\epsilon=1$.

Since $E\to-0$ near the origin while $E\to+0$ at infinity, if the solution is regular everywhere, E must vanish at some finite point (z_0) and $dE/dz\geqslant 0$ there. On the other hand, Eq. (B10) yields $dE/dz\leqslant 0$ since $E(z_0)=0$ and $\mu(z_0)>0$. As a result, we have $dE/dz(z_0)=0$. Using Eq. (B10), we then find $dw/dz(z_0)=0$. $E(z_0)=0$ with this equation implies $w(z_0)=\pm 1$. Solving the basic equations (B2) and (B3) with the above initial values at z_0 $[w(z_0)=\pm 1, dw/dz(z_0)=0, \mu(z_0)=\text{positive and finite}]$, we find a trivial solution $[w(z)=\pm 1, \mu(z)=\text{a positive constant}]$. We conclude that there is no nontrivial particlelike solution with a finite mass for $\Lambda\leqslant 0$.

APPENDIX C: ASYMPTOTIC SOLUTION ($\Lambda\leqslant 0$)

We present the asymptotic solution of the present system with $\Lambda\leqslant 0$. As we proved for a particlelike solution in the previous appendix and numerically solved for a more generic case, the mass function μ seems to diverge. Here we solve the basic equations with some ansatz and find the analytic solution in the asymptotically far region.

First, we consider the case of $\epsilon=0$. As our ansatz, we adopt

$$\mu\approx\mathcal{M}_L z, \tag{C1}$$

which is suggested from numerical solutions and also confirmed from the following result. The basic equation for the Yang-Mills field is now written as

$$\frac{d^2w}{dz^2}+e^{-z}\mu\frac{dw}{dz}+\frac{1}{2}w(1-w^2)\approx\frac{d^2w}{dz^2}+\frac{1}{2}w(1-w^2)=0 \tag{C2}$$

as $z\to\infty$. We can integrate this equation as

$$\frac{1}{2}\left(\frac{dw}{dz}\right)^2-\frac{1}{8}(1-w^2)^2=E_0, \tag{C3}$$

where E_0 is an integration constant and denotes the asymptotic value of the energy. E_0 must be negative, otherwise w diverges as $z\to\infty$.

Rewriting Eq. (C3), we find

$$\frac{dw}{dz}=\pm\frac{1}{2}\sqrt{8E_0+(1-w^2)^2}$$
$$=\pm\frac{1}{2}\sqrt{(w_-^2-w^2)^2(w_+^2-w^2)^2}, \tag{C4}$$

where

$$w_\pm=\sqrt{1\pm2\sqrt{-2E_0}}, \tag{C5}$$

which is integrated as

$$w=\pm w_-\mathrm{sn}\left(\frac{w_+}{2}z,k\right), \tag{C6}$$

where $k=w_-/w_+$. w is oscillating in a potential $U(z)=-\frac{1}{8}(1-w^2)^2$ with a negative energy E_0.

In order to check our ansatz, we also solve the mass function with the above solution of w. The mass function μ is obtained by integration of Eq. (B11), that is,

$$\mu=\int dz[8E_0+2(1-w^2)^2]$$
$$=\mu_0z+4w_+w_-^2\int^{w_+z/2}dx[(1-k^2)\mathrm{cn}^2(x,k)$$
$$+k^2\mathrm{cn}^4(x,k)], \tag{C7}$$

where $\mu_0=8E_0+2(1-w_-^2)^2=-8E_0=(1-w_-^2)^2$. The integration of the functions cn^2 and cn^4 is evaluated by the elliptic functions as

$$\int dx\,\mathrm{cn}^2(x,k)$$
$$=\frac{1}{k^2}\left[-(1-k^2)x\right.$$
$$\left.+\frac{E(\sin^{-1}[\mathrm{sn}(x,k)],k)\times[1-k^2\,\mathrm{sn}^2(x,k)]}{\mathrm{dn}^2(x,k)}\right],$$

$$\int dx\ \ \mathrm{cn}^4(x,k)$$
$$=\frac{1}{3k^2}[(1-k^2)x+\mathrm{cn}(x,k)\mathrm{dn}(x,k)\mathrm{sn}(x,k)]$$
$$+\left[\frac{2(2k^2-1)}{3k^4}\left(-(1-k^2)x\right.\right.$$
$$\left.\left.+\frac{E(\sin^{-1}[\mathrm{sn}(x,k)],k)\times[1-k^2\,\mathrm{sn}^2(x,k)]}{\mathrm{dn}^2(x,k)}\right)\right]. \tag{C8}$$

These functions increase with oscillations as $z\to\infty$. When we take an average of these functions over the period of oscillation, the averaged values are linearly increasing like our ansatz (C1).

Dividing these functions into two parts (linear functions and oscillating functions), we find

$$\mu=\mathcal{M}_Lz+\frac{8\sqrt{2}k^2}{(1+k^2)^{3/2}}D\mu\left(\frac{w_+}{2}z,k\right), \tag{C9}$$

where

$$\mathcal{M}_L=\frac{1}{(1+k^2)^2}[(1-k^2)^2+8k^2\{(1-k^2)C_2+k^2C_4\}], \tag{C10}$$

$$D\mu(x,k)=(1-k^2)\int dx[\mathrm{cn}^2(x,k)-C_2]$$
$$+k^2\int dx[\mathrm{cn}^4(x,k)-C_4], \tag{C11}$$

$$C_2=\frac{1}{K(k)}\int_0^{K(k)}dx\,\mathrm{cn}^2(x,k),$$
$$C_4=\frac{1}{K(k)}\int_0^{K(k)}dx\,\mathrm{cn}^4(x,k). \tag{C12}$$

The energy E_0 and the amplitude w_- are found to be

$$E_0=-\frac{1}{8}\left(\frac{1-k^2}{1+k^2}\right)^2,$$
$$w_-=\frac{\sqrt{2}k}{\sqrt{1+k^2}} \tag{C13}$$

with k $(0\leqslant k\leqslant1)$.

The asymptotic solution is then given by

$$w=\pm w_-\mathrm{sn}\left(\frac{w_+}{2}z,k\right), \tag{C14}$$

$$\mu=\mathcal{M}_Lz+\frac{8\sqrt{2}k^2}{(1+k^2)^{3/2}}D\mu\left(\frac{w_+}{2}z,k\right). \tag{C15}$$

$D\mu$ is a periodic function with a constant period, which in the z coordinate is

$$\Delta z = \frac{8}{w_+} K(k) = \frac{4\pi}{w_+} F\left(\frac{1}{2}, \frac{1}{2}, 1, k^2\right), \tag{C16}$$

where

$$w_+ = \frac{\sqrt{2}}{\sqrt{1+k^2}}. \tag{C17}$$

Note that if we take a limit of $k \to 1$, $E_0 \to -0$, we recover the instanton solution, that is, w is oscillating between ± 1. The width of one instanton ($w \sim \pm 1 \to \mp 1$) is given by $\Delta z/2$, which diverges in this limit. However, this oscillation is repeated infinitely when the gravitational effect is included, that is, an infinite number of instantons appear in the present system. This is why the mass function diverges.

For the case of $\epsilon = 1$ ($\Lambda < 0$), since $f \to \ell^{-2}$, the equation for w is now

$$\frac{d^2 w}{dz^2} + \frac{dw}{dz} + \frac{\ell^2}{2} e^{-z} w(1-w^2) \approx \frac{d^2 w}{dz^2} + \frac{dw}{dz} = 0. \tag{C18}$$

Integrating this equation, we obtain that

$$\left|\frac{dw}{dz}\right| \propto e^{-z}, \tag{C19}$$

which gives the asymptotic behavior of w as

$$w = \mathcal{W}_0 + \mathcal{W}_1 e^{-z} + \cdots. \tag{C20}$$

The equation for μ is

$$\frac{d\mu}{dz} = \frac{4}{\ell^2}\left(\frac{dw}{dz}\right)^2 + (1-w^2)^2$$
$$\approx (1-\mathcal{W}_0^2)^2 + 4\mathcal{W}_1\left[\frac{\mathcal{W}_1}{\ell^2} - \mathcal{W}_0(1-\mathcal{W}_0^2)\right] e^{-z} + \cdots \tag{C21}$$

and then μ is given by

$$\mu = \mathcal{M}_L z + \mathcal{M}_0 + \mathcal{M}_1 e^{-z} + \cdots, \tag{C22}$$

where

$$\mathcal{M}_L = (1-\mathcal{W}_0^2)^2,$$
$$\mathcal{M}_1 = -4\mathcal{W}_1\left[\frac{1}{\ell^2}\mathcal{W}_1 - \mathcal{W}_0(1-\mathcal{W}_0^2)\right]. \tag{C23}$$

The energy function E is evaluated as

$$E = \frac{1}{2\ell^2} e^{-z}\left(\frac{dw}{dz}\right)^2 - \frac{1}{8}(1-w^2)^2 \to -\frac{1}{8}(1-\mathcal{W}_0^2)^2 \quad (=E_0). \tag{C24}$$

Since the damping rate of the energy is given by

$$\frac{dE}{dz} = -4 e^{-z}\left(\frac{dw}{dz}\right)\left[E + \frac{\mu}{8} + \frac{1}{8\ell^2} e^{2z}\right] \to -\frac{1}{2\ell^2}\mathcal{W}_1^2 e^{-z}, \tag{C25}$$

the energy damping ceases very soon. As a result, the energy of the system approaches some finite value (E_0). This is because the potential term drops exponentially while the adiabatic damping term remains. The solution does not oscillate because the potential term becomes ineffective quickly.

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