

Complete basis for power suppressed collinear-ultrasoft operators

Dan Pirjol*

Department of Physics and Astronomy, The Johns Hopkins University, Baltimore, Maryland 21218

Iain W. Stewart†

Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195

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We construct operators that describe power corrections in mixed collinear-ultrasoft processes in QCD. We treat the ultrasoft-collinear Lagrangian to $\mathcal{O}(\lambda^2)$ and heavy-to-light currents involving collinear quarks to $\mathcal{O}(\lambda)$, including new three body currents. A complete gauge invariant basis is derived which has a full reduction in Dirac structures and is valid for matching at any order in α_s . The full set of reparametrization invariance (RPI) constraints is included, and is found to restrict the number of parameters appearing in Wilson coefficients and to rule out some classes of operators. The QCD ultrasoft-collinear Lagrangian has two $\mathcal{O}(\lambda^2)$ operators in its gauge invariant form. For the $\mathcal{O}(\lambda)$ heavy-to-light currents there are (4,4,14,14,21) subleading (scalar, pseudoscalar, vector, axial-vector, tensor) currents, where (1,1,4,4,7) have coefficients that are not determined by RPI. In a frame where $v_\perp = 0$ and $n \cdot v = 1$ the total number of currents reduces to (2,2,8,8,13), but the number of undetermined coefficients is the same. The role of these operators and universality of jet functions in the factorization theorem for heavy-to-light form factors is discussed.

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I. INTRODUCTION

The soft-collinear effective theory (SCET) constructed in [1–4] offers a systematic description of processes involving energetic particles. It has an expansion in a small parameter $\lambda \sim p_\perp/Q$, where p_\perp is a typical transverse momentum and Q the large energy scale. Hard exclusive and inclusive processes in QCD are usually described using the powerful techniques of QCD factorization and light-cone expansions [5,6]. SCET encompasses and extends these frameworks, and in particular allows a model independent description of effects caused by the interplay between energetic collinear particles and soft particles beyond leading order in the power expansion. These effects can be described in a rigorous way based solely on QCD, but are not included in purely collinear expansions. The study of operators that describe these mixed collinear-ultrasoft (collinear-usoft) effects is the purpose of this paper. For recent applications of SCET in hard scattering processes and B decays see Refs. [7–13,15].

Since our focus is on mixed collinear-usoft interactions, we consider collinear quark fields $\xi_{n,p}$, collinear gluon fields $A_{n,p}^\mu$, usoft heavy quark fields h_v , usoft light quark fields q_{us} , and usoft gluons A_{us}^μ . (We follow the notation in Refs. [2,3], but for simplicity will often suppress the momentum label p on the collinear fields.) These degrees of freedom can interact in a local manner in Lagrangians and currents. This is in contrast with collinear-soft couplings whose interactions are mediated by off-shell fluctuations [4], and appear in external operators. We comment on collinear-soft interactions at the end of the paper.

The derivation of the leading order collinear quark and

gluon Lagrangians $\mathcal{L}_{\xi\xi}^{(0)}$ and $\mathcal{L}_{cg}^{(0)}$ can be found in Refs. [2,4], and a description of the gauge symmetries of SCET can be found in Refs. [3,4]. For details on power counting we refer to Ref. [12]. The heavy-to-light currents at large energy, J_{hl} , were derived to leading order in Ref. [2], including one-loop matching for all the Wilson coefficients. The running of these Wilson coefficients was considered in Refs. [1,2].

In the context of the SCET, power suppressed corrections were first considered in Ref. [10], and the $\mathcal{O}(\lambda)$ suppressed currents J_{hl} and collinear quark Lagrangians were derived. The authors showed that a reparametrization invariance (RPI) uniquely fixes the Wilson coefficients of their subleading currents and Lagrangian in terms of the leading order coefficients.¹ In Ref. [11] the RPI of SCET was extended to the most general three classes (I,II,III), and the multipole expansion of the collinear quark Lagrangian was treated to higher orders in λ and were shown not to receive anomalous dimensions. In Ref. [12] the presence of additional $\mathcal{O}(\lambda)$ heavy-to-light currents was pointed out that were missing in Ref. [10].

The study of power corrections in SCET was continued in Ref. [13] and several important results were obtained for mixed usoft-collinear operators. In particular the mixed usoft-collinear quark Lagrangian $\mathcal{L}_{\xi q}$ was first considered and was derived to $\mathcal{O}(\lambda, \lambda^2)$ working at tree level, but to all orders in attachments of $\bar{n} \cdot A_n \sim \lambda^0$ gluon fields. In a similar fashion heavy-to-light currents were derived to $\mathcal{O}(\lambda^2)$, and linear combinations of currents that are invariant under the

¹A similar application of Lorentz invariance was used to derive constraints on the form of higher-twist contributions to structure functions in deep inelastic scattering in [16]. For this case, invariance under changes in the light-cone vector \bar{n}_μ was used to derive constraints on matrix elements $\langle p | T \bar{\psi}(0) \Gamma \psi(\lambda \bar{n}_\mu) | p \rangle$.

*Electronic address: dpirjol@pha.jhu.edu

†Electronic address: iain@phys.washington.edu

three types of RPI were identified. It was also shown that the operators in $\mathcal{L}_{\xi q}$ are not renormalized based on an analysis of arbitrary N -loop diagrams in the hard region of QCD. The mixed usoft-collinear quark Lagrangian $\mathcal{L}_{\xi q}$ was extended to a gauge invariant form with covariant derivatives in Ref. [14].

The purpose of the present paper is to answer some open questions regarding our knowledge of the power suppressed usoft-collinear Lagrangian and heavy-to-light currents. This includes the number of J_{hl} currents at $\mathcal{O}(\lambda)$, since even at tree level the full reduction of Dirac structures has not yet been implemented. For both J_{hl} and $\mathcal{L}_{\xi q}$ we also construct a complete basis which is valid for matching at any order in α_s , and therefore includes all operators that can be induced by radiative corrections or operator mixing. We work in the most general possible frame throughout (e.g., allowing $v_\perp \neq 0$, $v \cdot n \neq 1$), and consider all the restrictions from RPI, including the transformation of Wilson coefficients. Finally, we include the mixed usoft-collinear pure glue Lagrangian beyond leading order (LO) (which follows from an extension of work in Refs. [4,11]). The above results are obtained by considering the full implications of RPI, and including all possible operators allowed from collinear gauge invariance, power counting, and the reduction of Dirac structures from the effective theory fields.²

For the heavy-to-light currents at $\mathcal{O}(\lambda)$ an important result we find is a new type of ‘‘three-body’’ currents, which have not been previously considered in the literature.³ In Refs. [10,12,13] the attention was restricted to SCET operators of two-body type $J=(\bar{\xi} \dots W)(h_v)$, where the two products in parentheses are collinear gauge invariant, and the ellipses denote combinations of collinear derivatives. Beyond tree level but at the same order in λ , we find that three-body structures can appear for some of the currents, having the form $\mathcal{J}=(\bar{\xi} \dots W)(W^\dagger \dots W)(h_v)$ with three collinear gauge invariant factors. We show the RPI can be used to determine for which currents this happens. We also show that RPI greatly restricts the form of the three-body operators, so that they always involve a collinear gluon field strength. The two-body operators have hard Wilson coefficients which are functions of a single parameter $C(\omega_1)$, while the new three-body operators have two parameter coefficients $C(\omega_1, \omega_2)$. Analogous three-body structures could

²Note that in deriving the complete basis for J_{hl} we restrict ourselves to $\mathcal{O}(\lambda)$, which is one order less than the order to which the tree-level matching results are known from Ref. [13]. We treat $\mathcal{L}_{\xi q}$ to $\mathcal{O}(\lambda^2)$, and give a detailed account of how the gauge invariant form in Ref. [15] was derived. In cases where our results are restricted to those in Refs. [10,13] we find agreement, as discussed in more detail in the body of the paper. The results derived here are sufficient for the proof of a factorization theorem for heavy-to-light form factors to all orders in α_s and leading order in $1/Q$ [15].

³In the final stages of this paper, Ref. [17] appeared where soft-collinear light-to-light currents are considered. Although different from the usoft-collinear heavy-to-light case studied here, we note that three-body currents were also found. Further remarks are left to a note added at the end.

appear in the usoft-collinear Lagrangian $\mathcal{L}_{\xi q}$ at higher orders in perturbation theory; however, using constraints from symmetries of SCET we prove that this does not occur.

Our results are relevant to the study of decay channels for B mesons which involve energetic hadrons in the final state. For instance, the results derived in this paper are necessary ingredients in the factorization formula for heavy-to-light form factors proven in Ref. [15] (for earlier work on factorization in heavy-to-light form factors see Refs. [18,19], and for results from QCD sum rules see Refs. [20]). The factorization theorem is valid to all orders in α_s and leading order in $1/Q$, $Q=\{m_B, E\}$, and separates contributions from the scales $p^2 \sim Q^2$, $p^2 \sim Q\Lambda$, and $p^2 \sim \Lambda^2$, where Λ is a hadronic scale. It states that a generic form factor can be split into two types of contributions, $F=f^F(Q)+f^{NF}(Q)$, where [15]

$$f^F(Q) = N_0 \int_0^1 dz \int_0^1 dx \int_0^\infty dr_+ T(z, Q, \mu_0) \\ \times J(z, x, r_+, Q, \mu_0, \mu) \phi_M(x, \mu) \phi_B(r_+, \mu), \quad (1)$$

$$f^{NF}(Q) = C_k(Q, \mu) \zeta_k^M(Q\Lambda, \mu), \quad (2)$$

$N_0 = f_B f_M m_B / (4E^2)$, and the two terms both scale as $1/Q^{3/2}$. This scaling is model independent and is in agreement with that derived from QCD sum rules [21]. In Eq. (1) ϕ_M and $\phi_B = \phi_B^\pm$ are standard nonperturbative light-cone distribution amplitudes, cf. [19,22]. The hard coefficients C_k and T can be calculated in an expansion in $\alpha_s(Q)$ and are simply related to the Wilson coefficients of the $\mathcal{O}(\lambda^0, \lambda^1)$ current operators J_{hl} . The jet function J is dominated by momenta $p^2 \simeq Q\Lambda$. If we then wish to expand in $\alpha_s(\sqrt{Q\Lambda})$ using the techniques developed in Ref. [15], J is calculable in terms of time-ordered products of the SCET₁ operators J_{hl} and $\mathcal{L}_{\xi q}$ that we study here. At tree-level [i.e. $\mathcal{O}(\alpha_s(\sqrt{Q\Lambda})^1 \alpha_s(Q)^0)$] one finds that J contains a $\delta(z-x)$, and in ratios of form factors the results for f^F then agree with terms computed in Ref. [19]. The z dependence first shows up at $\mathcal{O}(\alpha_s^2)$ as does a possible dependence on ϕ_B^- . However, as we show in Sec. VI it is possible to absorb the ϕ_B^- terms into a redefinition of the ζ_k^M to all orders in perturbation theory.

The factorization formula provides a clean separation of the ‘‘soft’’ non-factorizable (NF) contributions and ‘‘hard’’ factorizable (F) terms without double counting. It also gives us a procedure to systematically improve the predictions to any order in perturbation theory at leading order in $1/Q$. The value of T and C_k can depend on which heavy-to-light process we consider, whereas ϕ_M and ϕ_B^\pm are universal functions. The ζ_k 's are also universal since only a $\zeta^M(E)$ appears for decays to pseudoscalars M , and a $\zeta_\perp^M(E)$ and $\zeta_\parallel^M(E)$ appear if M is a vector meson. The jet functions J are common among certain classes of form factors and also do not depend on the precise state (e.g. π or η). The f^{NF} terms satisfy the so-called large energy form factor relations [15], as expected from the prior loose definitions of these terms as ‘‘soft’’ con-

tributions [2,19,23].⁴ Note that we have not bothered to separate $p^2 \sim Q\Lambda$ and $p^2 \sim \Lambda^2$ fluctuations in the ζ_k^M functions, since it is not clearly beneficial phenomenologically. The factorization theorem does tell us that $\zeta_k^M \sim (\Lambda/Q)^{3/2}$; however, it does not distinguish between factors of m_b and E in this $Q^{-3/2}$. It also does not numerically favor the f^F or f^{NF} term; for instance, it is possible that the leading $\alpha_s(\sqrt{Q\Lambda})$ in J is compensated for by an analogous factor in ζ_k^M .

We start in Sec. II by reviewing the general constraints imposed on SCET operators following from collinear gauge invariance, spin structure reduction, and reparametrization invariance. In Sec. III we study the implications of these predictions for the subleading usoft-collinear Lagrangian \mathcal{L}_{uc} . In Sec. IV we present detailed results for SCET currents. Using the example of the scalar current as the pedagogical example, we demonstrate the construction of the complete basis of $O(\lambda)$ operators contributing to the weak currents, which closes under RPI transformations. Explicit results are then also derived for the pseudo-scalar, vector, axial-vector, and tensor heavy-to-light currents to $\mathcal{O}(\lambda)$. In Sec. V we summarize the one-loop matching results for the currents, give explicit results for $\mathcal{L}_{\xi q}$ Feynman rules, and discuss the basis of currents in the particular frame $v_\perp = 0$, $n \cdot v = 1$.

II. OPERATOR CONSTRAINTS IN SCET

In this section we briefly review the symmetries and structure of SCET which will be important for our construction of operators. We refer to Refs. [1–4,10,11] for more details.

SCET includes infrared degrees of freedom corresponding to the relevant low energy scales in the problem. These are typically those with momentum that are collinear $p_c^\mu \sim Q(\lambda^2, 1, \lambda)$, soft $p_s^\mu \sim Q(\lambda, \lambda, \lambda)$, or ultrasoft (usoft) $p_{us}^\mu \sim Q(\lambda^2, \lambda^2, \lambda^2)$, where the components here are in a light-cone basis $(+, -, \perp)$. Each type of mode has effective theory quark and gluon fields, which are then organized into operators with a well-defined power counting in λ . It is convenient to introduce light-cone unit vectors (n_μ, \bar{n}_μ) satisfying $n^2 = \bar{n}^2 = 0$, $n \cdot \bar{n} = 2$, in terms of which a vector has components $p^\mu = (n \cdot p, \bar{n} \cdot p, p_\perp^\mu)$. The couplings of the fields are described by an effective Lagrangian, while the couplings to external sources appear as additional operators or currents. Both the Lagrangian and currents are constructed such that they include constraints from power counting, spin symmetries, and collinear and (u)soft gauge invariance.

The soft-collinear effective theory also contains a kinematical reparametrization invariance symmetry. Lorentz invariance is broken by introducing the vectors n and \bar{n} , but is restored order by order in λ , by requiring invariance of op-

erators under a simultaneous change in n and/or \bar{n} and compensating changes in the effective theory fields. This reparametrization invariance (RPI) symmetry of SCET was first considered in Ref. [10], and was then extended to the most general three classes (I,II,III) of allowed transformations in Ref. [11].⁵ The three types are defined by the infinitesimal change they induce on the light-cone unit vectors: type-I ($n \rightarrow n + \Delta_\perp$), type-II ($\bar{n} \rightarrow \bar{n} + \varepsilon_\perp$), and type-III [$n \rightarrow (1 + \alpha)n, \bar{n} \rightarrow (1 - \alpha)\bar{n}$]. Here $\alpha \sim \varepsilon_\perp \sim \lambda^0$, while $\Delta_\perp \sim \lambda$. It is the analog of the reparametrization invariance of heavy quark effective theory (HQET) under changes in the heavy quark velocity v [28], where $v^2 = 1$. We will use HQET for heavy quark fields [29].

The restrictions we consider for finding the most general set of power suppressed gauge invariant operators are as follows:

(i) Power counting and gauge invariance which determine what basic building blocks are allowed at the order we are considering.

(ii) Which auxiliary vectors are available (such as n, \bar{n}, v, \dots) that can be used to construct the most general set of allowed scalars, tensors, and Dirac structures.

(iii) Eliminate operators which are redundant by integration by parts, or equations of motion.

(iv) Impose type-III reparametrization invariance. If a non-trivial invariant can be formed with the label operators, such as $(n \cdot v \bar{P})$, then include Wilson coefficients that depend on these quantities.

(v) Impose all constraints from type-I and type-II reparametrization invariance.

To impose the five constraints we start by writing minimal sets of independent operators compatible with the general principles in (i), (ii), and (iii). We then require RPI invariance order by order in the λ power counting. To do this we found it useful to split the RPI transformations into two categories, those that act within the order we are considering $\delta_j^{(\lambda^0)}$, with $j = \text{I, II, III}$, and those which connect operators to one higher order $\delta_j^{(\lambda)}$. At leading order the type-II and type-III $\delta_j^{(\lambda^0)}$ transformations already provide non-trivial constraints on the allowed form of operators. In contrast the $\delta_j^{(\lambda)}$ transformations allow us to derive relations valid to all orders in α_s between the Wilson coefficients of operators at different orders in λ . These relations are similar to the case of RPI in HQET [28,30], where we note in particular the important relations derived for coefficients of subleading heavy-to-heavy currents in Ref. [31]. We start by summarizing restrictions that follow from collinear gauge invariance and power counting in Sec. II A, spin structure reductions in Sec. II B, and RPI in Sec. II C.

⁴These relations were first derived in Ref. [23] using low energy effective theory (LEET) [24]. However, for studying energetic hadrons with QCD the LEET framework is known to be inconsistent [1,25]; for instance, it does not bind an energetic quark-antiquark pair into a meson in heavy-to-light decays [26].

⁵The nature of Lorentz symmetries on the light-cone are well known [27]. The new point in SCET [10,11] is that for any collinear process these symmetries are realized in a way that leads to non-trivial restrictions both on operators at a given order in the power counting *and* between operators at different orders in the power counting.

To separate the momentum scales we follow Ref. [2] and use collinear quark fields $\xi_{n,p}(x)$ [and gluon fields $A_{n,p}^\mu(x)$] which have momentum labels p for the large components of the collinear momenta, and residual coordinates $x^\mu \sim 1/\lambda^2$ [2]. Thus, all derivatives on collinear fields are the same size as derivatives on usoft fields, $\partial^\mu \sim \lambda^2$. This setup implements the multipole expansion in momentum space [32]. Note that our analysis of power corrections differs from Ref. [13] in two ways, the first being that in Ref. [13] the momentum scales were separated by performing the multipole expansion in position space, which, however, leads to an equivalent formulation. We do find that concise results for the power suppressed corrections are obtained with the momentum space version. Secondly, we derive our basis of operators and implement all symmetry constraints working order by order in the power counting, rather than constructing invariants and then expanding in λ . This made it simpler to derive a complete gauge invariant basis at the desired order while working in a general frame.

A. Power counting and gauge invariance

The SCET is derived from QCD by integrating out fluctuations with $p^2 \gg Q^2 \lambda^2$, where in typical processes $\lambda = (\Lambda_{\text{QCD}}/Q)^k$ with $k=1$ or $k=1/2$. Infrared fluctuations are then described by effective theory fields. A gauge invariant power counting for fields can be fixed by demanding that the kinetic terms in the action are order λ^0 . For the collinear fields this gives $\xi_n \sim \lambda'$ for the quarks, and $(n \cdot A_n, \bar{n} \cdot A_n, A_n^\perp) \sim (\lambda^2, \lambda^0, \lambda)$ for the collinear gluons, $h_v \sim q \sim \lambda^3$ for usoft quarks, and $A_{us}^\mu \sim \lambda^2$ for usoft gluons [1,2]. Derivatives on these fields count as $\partial^\mu \sim \lambda^2$. The larger collinear momenta are picked out by introducing label operators $\bar{\mathcal{P}} \sim \lambda^0$ and $\mathcal{P}_\perp^\mu \sim \lambda$ [3]. For example, $\bar{\mathcal{P}} \xi_{n,p} = (\bar{n} \cdot p) \xi_{n,p}$. For notational convenience we define collinear covariant derivatives

$$i\bar{n} \cdot D_c = \bar{\mathcal{P}} + g\bar{n} \cdot A_n, \quad iD_c^\perp = \mathcal{P}_\perp^\mu + gA_n^\perp, \quad (3)$$

and ultrasoft covariant derivatives

$$i\bar{n} \cdot D_{us} = i\bar{n} \cdot \partial + g\bar{n} \cdot A_{us}, \quad iD_{us}^\perp = i\partial_\perp^\mu + gA_{us}^\perp. \quad (4)$$

For the n^μ components, it is only the combination

$$in \cdot D = in \cdot \partial + gn \cdot A_n + gn \cdot A_{us}, \quad (5)$$

that ever appears. In general a derivative without a subscript involves the sum of the collinear and usoft pieces, $D^\mu = D_c^\mu + D_{us}^\mu$, and it is this combination which is RPI invariant [11] (implying that the anomalous dimensions of terms that appear in the multipole expansion are related).

Integrating out the off-shell fluctuations builds up a collinear Wilson line, W , built out of collinear gluon fields which are not suppressed in the power counting [3]

$$W = \left[\sum_{\text{perms}} \exp \left(- \frac{g}{\bar{\mathcal{P}}} \bar{n} \cdot A_{n,q}(x) \right) \right], \quad (6)$$

where the label operators only act on fields inside the square brackets. Up to the important fact that W has been multipole expanded, it is the Fourier transform of a standard position space Wilson line, $W(-\infty, x)$. Factors of $W \sim \lambda^0$ can be included in operators without changing the order in the power counting. However, their location is restricted by collinear gauge transformations, U_c , under which $W \rightarrow U_c W$ [3]. Since $\bar{\mathcal{P}} \sim \lambda^0$ in the power counting the hard Wilson coefficients can be arbitrary functions of the momentum or momenta, ω_i , picked out by the label operator, $C(\omega_i, \mu)$ [3]. These coefficients can be computed by matching with QCD at the hard scale $\mu \simeq Q$ and running with the renormalization group.

If we consider a general Wilson coefficient and operator $C \otimes \mathcal{O}$, then the covariant derivative

$$i\bar{n} \cdot D_c = W \bar{\mathcal{P}} W^\dagger, \quad (7)$$

so it is always possible to put all the Wilson lines in \mathcal{O} and the dependence on the momenta picked out by $\bar{\mathcal{P}}$ into C . We will find it convenient to use the notation

$$(\bar{\xi}_n W)_{\omega_1} = [\bar{\xi}_n W \delta(\omega_1 - n \cdot v \bar{\mathcal{P}}^\dagger)],$$

$$(W^\dagger D_c^\perp{}^\mu W)_{\omega_2} = [W^\dagger D_c^\perp{}^\mu W \delta(\omega_2 - n \cdot v \bar{\mathcal{P}}^\dagger)], \quad (8)$$

where again the label operators do not act outside the square brackets. The factor of $n \cdot v$ is included next to $\bar{\mathcal{P}}$ to make it a type-III RPI invariant. Thus, the momentum labels ω_i do not transform under RPI. The products of fields in Eq. (8) are color singlets under the collinear gauge symmetry, so the momentum labels ω_i are gauge invariant. These products still transform under an usoft gauge transformation U_{us} as $(\bar{\xi}_n W) \rightarrow (\bar{\xi}_n W) U_{us}^\dagger$ and $(W^\dagger D_c^\perp{}^\mu W) \rightarrow U_{us} (W^\dagger D_c^\perp{}^\mu W) U_{us}^\dagger$. We will elaborate on how RPI affects Wilson coefficients in SCET in Sec. II C below.

For Lagrangians and currents where the variable v^μ is not available we cannot make use of the definitions in Eqs. (8). It is still convenient to make use of a similar notation:

$$(\bar{\xi}_n W)_{z_1} = [\bar{\xi}_n W \delta(z_1 - \bar{\mathcal{P}}^\dagger)],$$

$$(W^\dagger D_c^\perp{}^\mu W)_{z_2} = [W^\dagger D_c^\perp{}^\mu W \delta(z_2 - \bar{\mathcal{P}}^\dagger)], \quad (9)$$

where we use the variables z_i rather than ω_i . Under a type-III transformation the z_i transform like \bar{n} so the delta function is homogeneous (and compensated for by an integration measure dz_i).

Using the scalings for fields and derivatives the power counting for an arbitrary diagram, λ^δ , can be determined entirely from its operators using [12]

$$\delta = 4 + \sum_k (k-4)[V_k^C + V_k^S + V_k^{SC}] + (k-8)V_k^U. \quad (10)$$

Here $V_k^{C,S,SC,U}$ count the number of order λ^k operators which have collinear fields, soft fields, both, or neither, re-

spectively. For any operator the power of k is derived by adding up the powers of λ in its components so, for instance, $\bar{\xi}_n \bar{h} i n \cdot D \xi_n \sim \lambda^4$ counts as $V_4^C = 1$. Since the operators are gauge invariant so is their value of k and also the power counting of any diagram using the result for δ in Eq. (10). In this paper we focus on operators with $V_k^S = V_k^{SC} = 0$.

We have also found it convenient to define additional pure gluon operators. In particular we will use the purely collinear field strength

$$igB_c^{\perp\mu} = [i\bar{n} \cdot D^c, iD_c^{\perp\mu}]. \quad (11)$$

We will also make use of the mixed tensors

$$\begin{aligned} ign \cdot M &= ign \cdot B_c = [i\bar{n} \cdot D^c, in \cdot D], \\ igM_{\perp} &= [i\bar{n} \cdot D^c, i\mathcal{D}_{\perp}^{us}]. \end{aligned} \quad (12)$$

In fact the operators M_{\perp} and $n \cdot M$, together with $ig \bar{n} \cdot M = [i\bar{n} \cdot D^c, i\bar{n} \cdot D^{us}]$ can be combined into a single object closed under usoft Lorentz transformations, which transforms in the desired way under the collinear and usoft gauge symmetries

$$igM = \left[i\bar{n} \cdot D^c, i\mathcal{D}^{us} + \frac{\bar{h}}{2} gn \cdot A_n \right]. \quad (13)$$

Finally the following results for manipulating covariant derivatives on Wilson lines also prove to be useful:

$$\begin{aligned} (W^{\dagger} i\bar{D}_c^{\perp} W) &= [W^{\dagger} i\bar{D}_c^{\perp} W] - \mathcal{P}_{\perp}^{\dagger} = \left[\frac{1}{\bar{\mathcal{P}}} W^{\dagger} igB_c^{\perp} W \right] - \mathcal{P}_{\perp}^{\dagger}, \\ (W^{\dagger} i\bar{D}_c^{\perp} W) &= [W^{\dagger} i\bar{D}_c^{\perp} W] + \mathcal{P}_{\perp} = \left[\frac{1}{\bar{\mathcal{P}}} W^{\dagger} igB_c^{\perp} W \right] + \mathcal{P}_{\perp}. \end{aligned} \quad (14)$$

B. Reduction in spin structures

Collinear quarks and heavy usoft quarks have spinors with only two non-zero components. In four component notation this is encoded in projection formulas for the fields,

$$P_n \xi_n = \xi_n, \quad P_v h_v = h_v, \quad (15)$$

where $P_n = (\not{h}\not{n})/4$ and $P_v = (1 + \not{v})/2$. We also define the orthogonal projector $P_n^- = (\not{n}\not{h})/4$ where $P_n + P_n^- = 1$. A quark bilinear with a heavy ultrasoft quark and light collinear quark, therefore, only has four possible non-trivial Dirac structures. On the other hand, if the heavy ultrasoft quark is replaced by a massless ultrasoft quark which has a four component spinor then there are eight possible Dirac structures. When generating operators we should be careful not to include redundant Dirac structures. Therefore, it is convenient to have a canonical basis which we can project results onto to check their interdependence. For this purpose we choose the basis

$$\begin{aligned} \bar{\xi}_n \Gamma_1 h_v, \quad \Gamma_1 &= \left\{ \frac{\bar{h}}{2}, \frac{\bar{h}\gamma_5}{2}, \gamma_{\perp}^{\mu} \right\}, \\ \bar{\xi}_n \Gamma_2 q_{us}, \quad \Gamma_2 &= \left\{ 1, \frac{\bar{h}}{2}, \gamma_5, \frac{\bar{h}\gamma_5}{2}, \gamma_{\perp}^{\mu}, \frac{\bar{h}\gamma_{\perp}^{\mu}}{2} \right\}. \end{aligned} \quad (16)$$

Any general Dirac structure can be projected onto a linear combination of terms in this basis with the help of the following formulas:

$$\begin{aligned} \bar{\xi}_n \Gamma h_v &= \bar{\xi}_n \Gamma_1 h_v, \\ \Gamma_1 &= \frac{\bar{h}}{2} \text{tr} \left[\frac{\not{h}}{2} P_n \Gamma P_v \right] - \frac{\bar{h}\gamma_5}{2} \text{tr} \left[\frac{\not{h}}{2} \gamma_5 P_n \Gamma P_v \right] \\ &\quad + \gamma_{\perp}^{\mu} \text{tr} \left[\gamma_{\mu}^{\perp} P_n \Gamma P_v \right] \end{aligned} \quad (17)$$

and

$$\begin{aligned} \bar{\xi}_n \Gamma q_{us} &= \bar{\xi}_n \Gamma_2 q_{us}, \\ \Gamma_2 &= \mathbf{1} \text{tr} \left[\frac{\not{h}\not{n}}{8} \Gamma \right] + \frac{\bar{h}}{2} \text{tr} \left[\frac{\not{h}}{4} \Gamma \right] + \gamma_5 \text{tr} \left[\frac{\not{h}\not{n}\gamma_5}{8} \Gamma \right] \\ &\quad - \frac{\bar{h}\gamma_5}{2} \text{tr} \left[\frac{\not{h}\gamma_5}{4} \Gamma \right] + \gamma_{\perp}^{\mu} \text{tr} \left[\frac{\gamma_{\mu}^{\perp} \not{h}\not{n}}{8} \Gamma \right] \\ &\quad - \frac{\bar{h}\gamma_{\perp}^{\mu}}{2} \text{tr} \left[\frac{\not{h}\gamma_{\mu}^{\perp}}{4} \Gamma \right]. \end{aligned} \quad (18)$$

The number of independent structures is quite logical, for $\bar{\xi}_n \Gamma h_v$ each field is determined by two-component spinors and there are $2 \times 2 = 4$ terms in the basis. For $\bar{\xi}_n \Gamma q_{us}$ only the collinear spinor has two components and there are $2 \times 4 = 8$ terms in our basis. Our choice of basis in Eqs. (17) differs from Ref. [2] where the choice $\Gamma_1 = \{1, \gamma_5, \gamma_{\perp}^{\mu}\}$ was used, and calculations were given in a frame where $v \cdot n = 1$ and $v_{\perp} = 0$. When v^{μ} is kept arbitrary we have found the basis in Eqs. (16) is more convenient since it retains its orthonormality in an arbitrary frame.

The projection formulas in Eqs. (17) can be used to reduce the possible Dirac structures in constructing a complete basis of operators. It is convenient to define $r_{\perp}^{\mu} \equiv i \epsilon^{\mu\alpha\beta\gamma} n_{\alpha} \bar{n}_{\beta} v_{\gamma}$ and $v_{\perp}^{\mu} = v^{\mu} - n \cdot v \bar{n}^{\mu}/2 - \bar{n} \cdot v n^{\mu}/2$, since then v_{\perp}^{μ} , r_{\perp}^{μ} , n^{μ} , \bar{n}^{μ} form a complete vector basis. To reduce the Dirac structures we can use relations such as

$$\begin{aligned} \not{v}_{\perp} &\doteq 1 - \frac{n \cdot v}{2} \not{n}, \quad \not{n} \not{v}_{\perp} \doteq \not{n} - 2\bar{n} \cdot v, \quad \gamma_{\perp}^{\mu} \doteq \gamma_{\mu} - \frac{n \cdot v}{2} \not{n}, \\ r_{\perp}^{\mu} \gamma_5 &\doteq -2v_{\perp}^{\mu} + 2\gamma_{\perp}^{\mu} + \not{h} \gamma_{\perp}^{\mu} n \cdot v, \\ \not{h} r_{\perp}^{\mu} \gamma_5 &\doteq 2\bar{h} v_{\perp}^{\mu} - 4\bar{n} \cdot v \gamma_{\perp}^{\mu} - 2\bar{h} \gamma_{\perp}^{\mu}, \end{aligned} \quad (19)$$

$$\begin{aligned}
i\epsilon^{\mu\nu\alpha\beta}n_\alpha v_\beta \frac{\bar{n}\gamma_5}{2} &\doteq i\sigma_{\mu\nu}-(v_\mu\gamma_\nu-v_\nu\gamma_\mu) \\
-\frac{1}{2}(\bar{n}_\mu n_\nu-n_\mu\bar{n}_\nu) &-\frac{\bar{n}\cdot v}{2}(n_\mu\gamma_\nu-n_\nu\gamma_\mu) \\
+\frac{n\cdot v}{2}(\bar{n}_\mu\gamma_\nu-\bar{n}_\nu\gamma_\mu), &\dots,
\end{aligned}$$

where the \doteq indicates that these are only true between the fields in Eqs. (17). (The complete set of relations is rather lengthy and is not shown.) The relations in Eqs. (19) allow the structures on the left to be traded for those on the right (with more than one iteration in some cases). Using the projection formulas it is straightforward to show that the most general Dirac structure possible for the LO scalar currents are $\{1, \bar{n}\}$, while the vector and axial-vector currents have the basis shown in Eq. (74), and the tensor currents depend on the basis in Eq. (92).

C. Reparametrization invariance

The decomposition into collinear fields requires introducing two light-like vectors n and \bar{n} , such that $n^2=\bar{n}^2=0$ and $n\cdot\bar{n}=2$. These vectors break five of the six Lorentz generators. This part of the Lorentz symmetry is restored order by order in the power counting by requiring invariance under reparametrization transformations on n and \bar{n} [11]:

type I,

$$\begin{aligned}
n_\mu &\rightarrow n_\mu + \Delta_\mu^\perp, \\
\bar{n}_\mu &\rightarrow \bar{n}_\mu,
\end{aligned} \tag{20a}$$

type II,

$$\begin{aligned}
n_\mu &\rightarrow n_\mu, \\
\bar{n}_\mu &\rightarrow \bar{n}_\mu + \varepsilon_\mu^\perp,
\end{aligned} \tag{20b}$$

and type III,

$$\begin{aligned}
n_\mu &\rightarrow (1+\alpha)n_\mu, \\
\bar{n}_\mu &\rightarrow (1-\alpha)\bar{n}_\mu,
\end{aligned} \tag{20c}$$

where $\Delta^\perp \sim \lambda$, while $\varepsilon^\perp \sim \alpha \sim \lambda^0$. In general one has two options for constructing RPI invariants: (i) construct operators out of completely RPI invariant quantities and then expand these in powers of λ , (ii) construct operators order by order in λ and transform them to see what linear combinations are invariant, and which operators are ruled out. In this paper we will adopt approach (ii), since starting with the most general gauge invariant sets and then reducing them allows us to be confident that we do not miss operators that could arise at any order in perturbation theory.

For our purposes it is convenient to divide the RPI transformations into two subsets, those which include terms

within the same order in λ denoted by $\delta_1^{(\lambda^0)}$, $\delta_{\text{II}}^{(\lambda^0)}$, and $\delta_{\text{III}}^{(\lambda^0)}$, and those which cause order λ suppressed transformations denoted by $\delta_1^{(\lambda)}$ and $\delta_{\text{II}}^{(\lambda)}$. All type-III transformations act within the same order in λ and it is easy to construct invariants under type III. We simply need to have the same number of n 's (\bar{n} 's) in the numerator and denominator, or have products of n times \bar{n} . The transformations of type I and type II are more involved. From Ref. [11] the transformations that have terms of the same order in λ are for $\delta_1^{(\lambda^0)}$

$$n\cdot D \rightarrow n\cdot D + \Delta^\perp \cdot D_\perp, \quad D_\perp^\mu \rightarrow D_\perp^\mu - \frac{\Delta_\perp^\mu}{2} \bar{n}\cdot D, \tag{21a}$$

and for $\delta_{\text{II}}^{(\lambda^0)}$

$$\begin{aligned}
\bar{n}^\mu &\rightarrow \bar{n}^\mu + \varepsilon_\perp^\mu, \quad D_\perp^\mu \rightarrow D_\perp^\mu - \frac{n^\mu}{2} \varepsilon_\perp \cdot D_\perp, \\
\gamma_\perp^\mu &\rightarrow \gamma_\perp^\mu - \frac{n^\mu}{2} \not{\varepsilon}_\perp - \frac{\varepsilon_\perp^\mu}{2} \not{h},
\end{aligned} \tag{21b}$$

and the transformations that start one power down in λ include for $\delta_1^{(\lambda)}$

$$\begin{aligned}
n^\mu &\rightarrow n^\mu + \Delta_\perp^\mu, \quad D_\perp^\mu \rightarrow D_\perp^\mu - \frac{\bar{n}^\mu}{2} \Delta^\perp \cdot D_\perp, \\
\bar{\xi}_n &\rightarrow \bar{\xi}_n \left(1 + \frac{\bar{n}\Delta_\perp}{4} \right), \\
\gamma_\perp^\mu &\rightarrow \gamma_\perp^\mu - \frac{\Delta_\perp^\mu}{2} \not{\bar{h}} - \frac{\bar{n}^\mu}{2} \not{\Delta}_\perp,
\end{aligned} \tag{22}$$

and for $\delta_{\text{II}}^{(\lambda)}$

$$\begin{aligned}
\bar{n}\cdot D &\rightarrow \bar{n}\cdot D + \varepsilon_\perp \cdot D_\perp, \quad D_\perp^\mu \rightarrow D_\perp^\mu - \frac{\varepsilon_\perp^\mu}{2} n\cdot D, \\
\bar{\xi}_n &\rightarrow \bar{\xi}_n \left(1 + \not{D}_\perp \frac{1}{\bar{n}\cdot\bar{D}} \not{\varepsilon}_\perp \right), \\
\mathcal{W} &\rightarrow \left[\left(1 - \frac{1}{\bar{n}\cdot D} \varepsilon^\perp \cdot D^\perp \right) \mathcal{W} \right].
\end{aligned} \tag{23}$$

Here \mathcal{W} is the RPI completed W , and involves the Fourier transform with respect to y of a position space Wilson line involving $(\bar{n}\cdot A_n + \bar{n}\cdot A_{us})(s\bar{n}+x)$ taken from $s=-\infty$ to 0 [13]. When expanded in λ , $\mathcal{W}=W+\mathcal{O}(\lambda^2)$, where W involves only the $\bar{n}\cdot A_n$ field as in Eq. (6).

If we start by considering LO operators then they must be invariant under the $\delta^{(\lambda^0)}$ transformations in Eqs. (21) all by themselves. The $\delta^{(\lambda)}$ transformations of the LO terms connect them to NLO operator's $\delta^{(\lambda^0)}$ transformations. Since in the collinear sector only $\delta^{(\lambda^0, \lambda^1)}$ terms exist this pattern repeats at all higher orders in the power counting. Note that

here we will not need to consider HQET RPI under the velocity v^μ . Since the transformation $v^\mu \rightarrow v^\mu + \Delta_v^\mu$, where $\Delta_v^\mu \sim \Lambda_{\text{QCD}}/Q \sim \lambda^2$, this type of RPI only needs to be taken into account at one-higher order than the order we are working. The combined SCET and HQET RPI transformations were used in the $\mathcal{O}(\lambda^2)$ analysis of J_{hl} in Ref. [13].

Finally we consider a new feature of RPI in SCET, namely how Wilson coefficients are affected by reparametrization invariance. Our analysis is similar in spirit to Ref. [31], where heavy-to-heavy HQET currents with coefficients depending on the change in velocity, $C(v \cdot v')$, were analyzed. If we adopt the view of building invariants at all orders in λ then the coefficients in SCET must also be functions of invariants, such as operators like

$$\bar{\Psi}_n C(-i\bar{D} \cdot \mathcal{V}) \Gamma \mathcal{H}_v, \quad (24)$$

where Ψ_n, \mathcal{H}_v are invariants including the quark fields ξ_n, h_v , and \mathcal{V}^μ is the RPI version of the velocity v^μ [28]. When expanded in λ the leading term involving the covariant derivative in C can be traded for W and \bar{P} using Eq. (7), $C(-in \cdot v \bar{n} \cdot \bar{D}_c) = WC(n \cdot v \bar{P}^\dagger) W^\dagger$. Here we will use the opposite but equivalent arrangement of starting with a current that is leading order in λ ,

$$\bar{\xi}_n WC(n \cdot v \bar{P}^\dagger) h_v, \quad (25)$$

and then determining how both the operators *and* coefficient transform under RPI. We then determine which structures are required at one higher order in λ to cancel this change, and which allowed higher order structures are left unconstrained.

III. COLLINEAR-ULTRASOFT LAGRANGIAN

In this section we discuss the mixed ultrasoft-collinear Lagrangians to $\mathcal{O}(\lambda^2)$. These actions are power suppressed [7], and start at $\mathcal{O}(\lambda)$ [13]. In Sec. III A we consider the derivation from integrating out components of the full theory field, which gives a tree level derivation of the action (for further explanation of this approach see Refs. [2,4,13]). In Ref. [13] this procedure was used to derive a form for the mixed ultrasoft-collinear quark Lagrangian, but a manifestly gauge invariant form was not determined. In Ref. [14] the analysis was extended to give manifestly gauge invariant operators in terms of covariant derivatives. In Sec. III A we review the details of how a derivation of a gauge invariant form of the action was carried out in Ref. [15] where the result is purely in terms of field strengths.

However, since the analysis in Sec. III A is only valid at tree level, it misses (i) non-trivial Wilson coefficients in the tree level operators, and (ii) new operators whose coefficients can have zero tree-level matching. In Ref. [13] point (i) was addressed and it was shown diagrammatically that non-trivial Wilson coefficients are generated. However, point (ii) has not yet been addressed, so additional operators could still be induced by matching at some higher order in perturbation theory. In Sec. III B we show that both points (i) and (ii) can be simultaneously solved by using the full set of symmetries of SCET when constructing operators. We also

extend the derivation to the mixed usoft-collinear pure gluon sector.

A. Matching for \mathcal{L}_{uc} at tree level, but all orders in $\bar{n} \cdot A_n$ gluons

In this section we discuss in detail the matching calculation for the mixed usoft-collinear quark Lagrangian [15]. The part of our discussion from Eqs. (26) to (30) follows Ref. [13], but with our momentum space notation. We start with the action $\mathcal{L} = \bar{\psi} i \not{D} \psi$ and decompose it with SCET fields

$$\begin{aligned} \mathcal{L} = & \bar{\xi}_n \frac{\bar{h}}{2} in \cdot D \xi_n + \bar{\xi}_n i \not{D}_\perp \xi_n + \bar{\xi}_n g A_c q_{us} + \bar{q}_{us} g A_c \xi_n \\ & + \bar{q}_{us} g A_c \xi_n + \bar{q}_{us} i \not{D}_{us} q_{us} + \left[\bar{\xi}_n i \not{D}_\perp \xi_n + \bar{\xi}_n \frac{\bar{h}}{2} i \bar{n} \cdot D \xi_n \right. \\ & \left. + \bar{\xi}_n g A_c q_{us} \right], \end{aligned} \quad (26)$$

where the D is usoft plus collinear, D_c is purely collinear, and collinear momentum conservation has been enforced. Varying with respect to $\bar{\xi}_n$ gives an equation of motion to eliminate this field from the term in square brackets

$$\begin{aligned} \bar{\xi}_n = & -\frac{1}{i \bar{n} \cdot D} \frac{\bar{h}}{2} [i \not{D}_\perp \xi_n + g A_n q_{us}], \\ \bar{\xi}_n = & [\bar{q}_{us} g A_n - \bar{\xi}_n i \not{D}_\perp] \frac{\bar{h}}{2} \frac{1}{i \bar{n} \cdot \bar{D}}. \end{aligned} \quad (27)$$

Plugging this into Eq. (26) and expanding we find that the two collinear quark terms exactly reproduce terms in the gauge invariant multipole expanded action in Ref. [11].⁶ Using Eq. (10) the terms with two ultrasoft quarks and ≥ 2 collinear gluons first show up at $\delta=3$, i.e. $\mathcal{O}(\lambda^3)$ [13], and are therefore neglected. The mixed usoft-collinear quark terms are

$$\begin{aligned} \mathcal{L}_{\xi q} = & \left[\bar{\xi}_n g A_n q_{us} + \bar{\xi}_n \frac{\bar{h}}{2} i \not{D}_\perp \frac{1}{i \bar{n} \cdot D} g A_n q_{us} \right] \\ & + \left[\bar{q}_{us} g A_n \xi_n + \bar{q}_{us} g A_n \frac{1}{i \bar{n} \cdot D} i \not{D}_\perp \frac{\bar{h}}{2} \xi_n \right]. \end{aligned} \quad (28)$$

Taking Eq. (28) and expanding to second order in λ gives

$$\mathcal{L}_{\xi q}^{(1)} = \bar{\xi}_n \left(g A_\perp^c - i \not{D}_\perp^c \frac{1}{i \bar{n} \cdot D_c} g \bar{n} \cdot A_c \right) q_{us} + \text{H.c.},$$

⁶Note that in QED the $\mathcal{O}(\lambda)$ pure collinear quark Lagrangian can be written in terms of $F_{\mu\nu}$ [13]. With the momentum space multipole expansion this manipulation is not necessary to achieve a gauge invariant result [33,34].

$$\begin{aligned} \mathcal{L}_{\xi q}^{(2)} = & \bar{\xi}_n \frac{\hbar}{2} \left(g n \cdot A^c + i \mathcal{D}_\perp^c \frac{1}{i \bar{n} \cdot D_c} g A_\perp^c \right) q_{us} \\ & - \bar{\xi}_n i \mathcal{D}_\perp^{us} \frac{1}{i \bar{n} \cdot D_c} g \bar{n} \cdot A^c q_{us} + \text{H.c.}, \end{aligned} \quad (29)$$

where the superscripts denote the power suppression in λ and λ^2 . In each of the last terms in $\mathcal{L}_{\xi q}^{(1,2)}$ we use $1 - W = 1/(i \bar{n} \cdot D_c) g \bar{n} \cdot A^c$, and in the first term of $\mathcal{L}_{\xi q}^{(2)}$ we write $g A_\perp^c = i \mathcal{D}_\perp^c - \mathcal{P}_\perp^1$ and then $q_{us} = [W + (1 - W)] q_{us}$. Thus,

$$\begin{aligned} \mathcal{L}_{\xi q}^{(1)} = & \bar{\xi}_n (i \mathcal{D}_\perp^c W - \mathcal{P}_\perp) q_{us} + \text{H.c.}, \\ \mathcal{L}_{\xi q}^{(2)} = & \bar{\xi}_n \frac{\hbar}{2} \left(g n \cdot A^c + i \mathcal{D}_\perp^c \frac{1}{i \bar{n} \cdot D_c} (i \mathcal{D}_\perp^c - \mathcal{P}_\perp) \right) \\ & \times [W + (1 - W)] q_{us} + \bar{\xi}_n i \mathcal{D}_\perp^{us} (W - 1) q_{us} + \text{H.c.} \\ = & \bar{\xi}_n \frac{\hbar}{2} \left(g n \cdot A^c + i \mathcal{D}_\perp^c \frac{1}{i \bar{n} \cdot D_c} i \mathcal{D}_\perp^c \right) W q_{us} \\ & - \bar{\xi}_n \frac{\hbar}{2} i \mathcal{D}_\perp^c \frac{1}{i \bar{n} \cdot D_c} \mathcal{P}_\perp q_{us} + \bar{\xi}_n \left(\frac{\hbar}{2} i n \cdot D^{us} + i \mathcal{D}_\perp^{us} \right) \\ & \times (W - 1) q_{us} + \text{H.c.} \end{aligned} \quad (30)$$

In manipulating $\mathcal{L}_{\xi q}^{(2)}$ we used the fact that integration by parts is allowed on the $(1 - W) q_{us}$ term and we can then use the equation of motion for the collinear quark to give a term $(-i n \cdot D_{us})(1 - W)$ which we collected with the $i \mathcal{D}_\perp^{us}(W - 1)$ term. The result in Eqs. (30) agrees with Ref. [13], up to the fact that we performed the multipole expansion in momentum space.

In Eq. (30) we did not drop the $\mathcal{P}_\perp q = 0$ terms since we want to make explicit the fact that it is the combination $(i \mathcal{D}_\perp^c W - \mathcal{P}_\perp) = [i \mathcal{D}_\perp^c W]$ which starts with at least one-collinear gluon. Written this way it appears that our $\mathcal{L}_{\xi q}^{(1)}$ is not collinear gauge invariant. In the transformed result the non-invariant term cancels if we use $\mathcal{P}_\perp q_{us} = 0$, but then it is not explicit that the operator starts with one-collinear gluon, so $\mathcal{L}_{\xi q}^{(1)}$ has either one or the other explicit. For $\mathcal{L}_{\xi q}^{(2)}$ Eqs. (30) still involve the gluon field A_n^μ so the gauge invariance of this expression is not at all clear. However, the above considerations indicate that it should be possible to write all the terms in Eqs. (30) in terms of gluon field strengths, and thereby achieve a manifestly gauge invariant action that starts with one-collinear gluon. This derivation was carried out in Ref. [15], but no details of the calculation were described there. These details are described below in Eqs. (31) through (35).

To proceed we note that using Eq. (11), $i g B_{c\perp}^\nu W = [i \bar{n} \cdot D_c, i D_c^\perp{}^\nu] W = i \bar{n} \cdot D_c i D_c^\perp{}^\nu W - i D_c^\perp{}^\nu W \bar{\mathcal{P}}$. Making similar manipulations for $n \cdot M$ and M_\perp we can write

$$\begin{aligned} i g \mathcal{B}_c^\perp W = & i \bar{n} \cdot D_c (i \mathcal{D}_c^\perp W - \mathcal{P}_\perp) - \{i \mathcal{D}_c^\perp W - \mathcal{P}_c^\perp\} \bar{\mathcal{P}} \\ & + \{g \bar{n} \cdot A^c\} \mathcal{P}_\perp, \\ i g n \cdot M W = & i \bar{n} \cdot D_c i n \cdot D W - \bar{\mathcal{P}} i n \cdot D_{us} - \{i n \cdot D W \\ & - i n \cdot D_{us}\} \bar{\mathcal{P}}, \\ i g M_\perp W = & i \bar{n} \cdot D_c i \mathcal{D}_\perp^{us} W - \bar{\mathcal{P}} i \mathcal{D}_\perp^{us} + \{i \mathcal{D}_\perp^{us} - i \mathcal{D}_\perp^{us} W\} \bar{\mathcal{P}}. \end{aligned} \quad (31)$$

Now we take purely usoft fields on the right, and divide on the left by $i \bar{n} \cdot D_c$. In Eq. (31) the terms in curly brackets start at one-collinear gluon, so even in the presence of $1/(i \bar{n} \cdot D_c)$ these terms are non-singular and can safely be dropped using the fact that the label operators give zero on the usoft field. This gives

$$\begin{aligned} \frac{1}{i \bar{n} \cdot D_c} i g \mathcal{B}_c^\perp W q_{us} = & (i \mathcal{D}_c^\perp W - \mathcal{P}_c^\perp) q_{us}, \\ \frac{1}{i \bar{n} \cdot D_c} i g n \cdot M W q_{us} = & (i n \cdot D W - i n \cdot D_{us}) q_{us} \\ & - (W - 1) i n \cdot D_{us} q_{us}, \quad (32) \\ \frac{1}{i \bar{n} \cdot D_c} i g M_\perp W q_{us} = & i \mathcal{D}_\perp^{us} (W - 1) q_{us} \\ & - (W - 1) i \mathcal{D}_\perp^{us} q_{us}. \end{aligned}$$

These expressions allow us to write covariant derivatives acting on Wilson lines in terms of field strengths.

Using Eqs. (32) for $\mathcal{L}_{\xi q}^{(1)}$ in Eqs. (30), we arrive at the final result

$$\mathcal{L}_{\xi q}^{(1)} = \bar{\xi}_n \frac{1}{i \bar{n} \cdot D_c} i g \mathcal{B}_c^\perp W q_{us} + \text{H.c.} \quad (33)$$

This form is particularly nice since it is explicitly collinear and usoft gauge invariant and, furthermore, explicitly starts at one-collinear gluon due to the B_\perp . To see the gauge invariance note that under a collinear gauge transformation U_c we have $\xi_n \rightarrow U_c \xi_n$, $W \rightarrow U_c W$, $\mathcal{B}_\perp \rightarrow U_c \mathcal{B}_\perp U_c^\dagger$, and $(\bar{n} \cdot D_c)^{-1} \rightarrow U_c (\bar{n} \cdot D_c)^{-1} U_c^\dagger$ so all factors of U_c cancel. Under an ultrasoft gauge transformation U_u we have $\xi_n \rightarrow U_u \xi_n$, $W \rightarrow U_u W U_u^\dagger$, $\mathcal{B}_\perp \rightarrow U_u \mathcal{B}_\perp U_u^\dagger$, $(\bar{n} \cdot D_c)^{-1} \rightarrow U_u (\bar{n} \cdot D_c)^{-1} U_u^\dagger$, and $q_{us} \rightarrow U_u q_{us}$ so all factors of U_u also cancel. In Fig. 3 in Sec. V C we show the one and two gluon Feynman rules that follow from $\mathcal{L}_{uc}^{(1)}$ in Eq. (33). A non-trivial check on our manipulations is that the same Feynman rules can be obtained from Eqs. (30) by using the free equations of motion.

We now proceed to further simplify $\mathcal{L}_{\xi q}^{(2)}$ in Eqs. (30). Using Eqs. (32) leaves

$$\begin{aligned} \mathcal{L}_{\xi q}^{(2)} = & \bar{\xi}_n \frac{1}{i\bar{n} \cdot D_c} igM W q_{us} + \bar{\xi}_n \frac{\hbar}{2} i\mathcal{D}_\perp^c \frac{1}{(i\bar{n} \cdot D_c)^2} \\ & \times ig\mathcal{B}_\perp^c W q_{us} + \bar{\xi}_n (W-1) i\mathcal{D}_{us} q_{us} + \text{H.c.}, \end{aligned} \quad (34)$$

where in the first and last terms we used the fact that $\hbar\xi_n = 0$ to write a full γ^μ in M and \mathcal{D}_{us} . For the last term in Eq. (34) we can now use the equation of motion for the usoft quark field to give our final result

$$\begin{aligned} \mathcal{L}_{\xi q}^{(2)} = & \bar{\xi}_n \frac{1}{i\bar{n} \cdot D_c} igM W q_{us} + \bar{\xi}_n \frac{\hbar}{2} i\mathcal{D}_\perp^c \frac{1}{(i\bar{n} \cdot D_c)^2} \\ & \times ig\mathcal{B}_\perp^c W q_{us} + \text{H.c.} \end{aligned} \quad (35)$$

Again in this form the action is collinear and usoft gauge invariant and, furthermore, explicitly starts at one-collinear gluon due to the field strength B 's and M 's. In the way we have written the result it is invariant under usoft Lorentz transformations on x^μ which separately rotate γ_μ and D_{us}^μ .

Finally we note that the mixed usoft-collinear quark actions in Eqs. (33) and (35) proved to be important for the proof of a factorization formula for heavy-to-light decays in Ref. [15]. In the next section we analyze the most general possible basis for $\mathcal{L}_{\xi q}^{(1,2)}$ beyond tree level, which follow purely from symmetry considerations and also discuss power suppressed terms in the collinear gluon action.

B. Most general basis for \mathcal{L}_{uc}

The ultrasoft-collinear quark Lagrangian can be expanded in a power series in the parameter λ . It is not possible to construct an invariant operator that is dimension-4 and order λ^0 . Therefore, we have the series

$$\mathcal{L}_{\xi q} = \mathcal{L}_{\xi q}^{(1)} + \mathcal{L}_{\xi q}^{(2)} + \dots \quad (36)$$

Since this is a Lagrangian, insertions of these operators do not inject momentum, and we are free to integrate by parts as long as we are careful not to generate singular terms.

To construct the most general quark action $\mathcal{L}_{\xi q}^{(1)}$ we can use a single collinear quark field $\xi_n \sim \lambda$, an ultrasoft quark field $q_{us} \sim \lambda^3$, and a $D_c^\perp \sim \lambda$. These factors give a dimension 4 operator, and from the power counting formula in Eq. (10) they give $\delta=1$ which is the correct order for $\mathcal{L}_{\xi q}^{(1)}$. To satisfy collinear gauge invariance without changing the order in the power counting we make use of the Wilson line W to write $(\bar{\xi}_n W)$ and $(W^\dagger iD_c^\perp W)$. Since the Lagrangian is a scalar we must dot the index μ into another vector. The possible Dirac structures are restricted by the fact that $\hbar\xi_n = 0$. They are also restricted by type-III RPI, for instance $\hbar\gamma_\mu$ is not invariant and therefore is ruled out (in the case of the heavy-to-light currents we can make use of the product $n \cdot v$, so $n \cdot v \hbar$ is allowed). Taking these constraints into account leaves γ_μ as the only possibility. Thus, we have reduced $\mathcal{L}_{uc}^{(1)}$ to the form

$$(\bar{\xi}_n W) \rho(\bar{\mathcal{P}}, \mu) (W^\dagger i\mathcal{D}_c^\perp W) q_{us} + \text{H.c.}, \quad (37)$$

where the coefficient ρ is dimensionless and $i\mathcal{D}_c^\perp$ acts to the left or right. However, by type-III RPI invariance the coefficient $\rho(\bar{\mathcal{P}}, \mu)$ cannot be a function of $\bar{\mathcal{P}}$, leaving only $\rho(\mu)$. Now ρ is a dimensionless function of the dimension-full parameter μ and can only be equal to a constant (assuming no new dynamical scales like Λ_{QCD} are generated by renormalizing $\mathcal{L}_{uc}^{(1)}$). Now $\mathcal{P}_\perp q = 0$ since q carries no perpendicular momenta of order λ , so using Eq. (14) we see that $(W^\dagger i\mathcal{D}_c^\perp W)$ can be traded for a \mathcal{B}_c^\perp operator. For $i\mathcal{D}_c^\perp$ we obtain the same \mathcal{B}_c^\perp operator, plus $(\bar{\xi}_n W) \mathcal{P}_\perp^\dagger q = 0$, which follows from the fact that q_{us} , and by momentum conservation $(\bar{\xi}_n W)$, carries zero collinear \perp momentum. Fixing the constant $\rho=1$ by tree level matching then leaves

$$\begin{aligned} \mathcal{L}_{\xi q}^{(1)} = & \bar{\xi}_n W \frac{1}{\bar{\mathcal{P}}} W^\dagger ig\mathcal{B}_c^\perp W q_{us} + \text{H.c.} \\ = & \bar{\xi}_n \frac{1}{i\bar{n} \cdot D_c} ig\mathcal{B}_c^\perp W q_{us} + \text{H.c.} \end{aligned} \quad (38)$$

In this form it is clear that the operator is collinear and usoft gauge invariant and generates terms with ≥ 1 collinear gluon as required by momentum conservation. Finally, it is easy to show that its order λ^0 type-I and type-II variations vanish, since from Eqs. (21) only the transformation of D_c^\perp must be considered and $\delta_1^{(\lambda^0)} B_c^\perp \propto [i\bar{n} \cdot D_c, i\bar{n} \cdot D_c] = 0$, while $\delta_{\text{II}}^{(\lambda^0)} \mathcal{B}_c^\perp \propto \hbar$ which gives zero since $\hbar\xi_n = 0$.

The above line of reasoning can be repeated at $\mathcal{O}(\lambda^2)$. By power counting and gauge invariance we can now have one $(W^\dagger i\bar{n} \cdot DW)$ or two $(W^\dagger D_c^\perp W)$ factors with derivatives to the left or right. Again we can use Eq. (14) to simplify the covariant derivative terms. We must have an operator starting with a one-collinear gluon, and again the $(W^\dagger DW)$ factor next to q_{us} must be in square brackets and can be turned into a gluon field strength. Also by type-III RPI the Wilson coefficients must again be numbers, except for operators with three or more invariant collinear products in $\mathcal{L}_{\xi q}^{(2)}$ where they can be functions of the ratio z_i of minus momenta. Taking into account these constraints leaves three possible operators⁷

$$\mathcal{L}_1^{(2)} = \rho_1 (\bar{\xi}_n W) \left(W^\dagger \frac{1}{i\bar{n} \cdot D_c} igM W \right) q_{us} + \text{H.c.},$$

⁷This assumes we have eliminated a possible four quark operator using the collinear gluon equations of motion [15], $g^2 (\bar{\xi}_n W T^A \hbar W^\dagger \xi_n) 1/\bar{\mathcal{P}}^2 (\bar{\xi}_n W T^A \hbar q_{us}) = \bar{\xi}_n \hbar / (2i\bar{n} \cdot D_c) ig n \cdot M W q_{us} + 2\bar{\xi}_n \hbar / \{2(i\bar{n} \cdot D_c)^2\} [iD_{c\mu}^\perp, igB_{c\mu}^\perp] W q_{us}$. This conclusion is not changed if we consider the most general possible four-quark operators allowed by all the symmetry constraints.

$$\begin{aligned} \mathcal{L}_2^{(2)} = & \int dz_1 dz_2 \rho_2 \left(\frac{z_2}{z_1} \right) (\bar{\xi}_n W)_{z_1} \frac{\bar{h}}{2} (W^\dagger i \vec{D}_c^\perp W)_{z_2} \\ & \times \frac{1}{\bar{p}} \left(W^\dagger \frac{1}{i \bar{n} \cdot D_c} i g \mathcal{B}_c^\perp W \right) q_{us} + \text{H.c.} \end{aligned} \quad (39)$$

$$\begin{aligned} \mathcal{L}_3^{(2)} = & \int dz_1 dz_2 \rho_3 \left(\frac{z_2}{z_1} \right) (\bar{\xi}_n W)_{z_1} \\ & \times \frac{\bar{h}}{2} (W^\dagger i \vec{D}_c^{\perp \mu} W)_{z_2} \frac{1}{\bar{p}} \left(W^\dagger \frac{1}{i \bar{n} \cdot D_c} i g \mathcal{B}_{c\mu}^\perp W \right) \\ & \times q_{us} + \text{H.c.}, \end{aligned}$$

where $(\dots)_{z_i} \equiv [\dots \delta(z_i - \bar{p}^\dagger)]$, and we have used our freedom to integrate by parts to make the perp covariant derivatives act to the right. Note that the overall $\bar{n} \cdot p$ momentum is zero, so no z_i label is used on the \mathcal{B}_c^\perp bracketed term (following the convention in Ref. [3]). Again the presence or absence of factors of \bar{h} are completely fixed by type-III RPI. Now consider the type-I and type-II RPI transformations. Computing the order λ variations of $\mathcal{L}_{uc}^{(1)}$ and simplifying the resulting expressions gives

$$\begin{aligned} \delta_1^{(\lambda)} \mathcal{L}_{\xi q}^{(1)} = & -\bar{\xi}_n \frac{\bar{h}}{2} \frac{1}{i \bar{n} \cdot D_c} i g \mathcal{B}_c^\perp \frac{\Delta_\perp}{2} W q_{us} \\ & - \bar{\xi}_n \frac{1}{i \bar{n} \cdot D_c} (\delta_1^{(\lambda^0)} i g M_\perp) W q_{us} + \text{H.c.}, \\ \delta_{II}^{(\lambda)} \mathcal{L}_{\xi q}^{(1)} = & -\bar{\xi}_n \left\{ i \mathcal{D}_c^\perp \frac{1}{(i \bar{n} \cdot D_c)^2} i g \mathcal{B}_c^\perp \frac{\epsilon_\perp}{2} \right. \\ & + \frac{1}{i \bar{n} \cdot D_c} \frac{\epsilon_\perp}{2} i g n \cdot M \\ & \left. + \frac{1}{i \bar{n} \cdot D_c} (\delta_{II}^{(\lambda^0)} i g M_\perp) \right\} W q_{us} + \text{H.c.} \end{aligned} \quad (40)$$

The $\delta_{I,II}^{(\lambda^0)} i g M_\perp$ terms appear since in the RPI transformations in Eqs. (22) it is the full D_\perp^μ which transforms. For the order λ^0 variations of $\mathcal{L}_j^{(2)}$ we find

$$\begin{aligned} \delta_1^{(\lambda^0)} \mathcal{L}_1^{(2)} = & \rho_1 \bar{\xi}_n \left\{ \frac{1}{i \bar{n} \cdot D_c} \frac{\bar{h}}{2} \Delta_\perp \cdot B_\perp \right. \\ & \left. + \frac{1}{i \bar{n} \cdot D_c} (\delta_1^{(\lambda^0)} i g M_\perp) \right\} W q_{us} + \text{H.c.}, \end{aligned}$$

$$\begin{aligned} \delta_{II}^{(\lambda^0)} \mathcal{L}_1^{(2)} = & \rho_1 \bar{\xi}_n \left\{ \frac{1}{i \bar{n} \cdot D_c} \frac{\epsilon_\perp}{2} i g n \cdot M \right. \\ & \left. + \frac{1}{i \bar{n} \cdot D_c} (\delta_{II}^{(\lambda^0)} i g M_\perp) \right\} W q_{us} + \text{H.c.}, \end{aligned}$$

$$\begin{aligned} \delta_1^{(\lambda^0)} \mathcal{L}_2^{(2)} = & - \int dz_1 dz_2 \rho_2 \left(\frac{z_2}{z_1} \right) (\bar{\xi}_n W)_{z_1} \delta(z_2) \\ & \times \frac{\bar{h} \Delta_\perp}{4} \frac{1}{\bar{p}} \left(W^\dagger \frac{1}{i \bar{n} \cdot D_c} i g \mathcal{B}_c^\perp W \right) q_{us} + \text{H.c.}, \end{aligned} \quad (41)$$

$$\begin{aligned} \delta_{II}^{(\lambda^0)} \mathcal{L}_2^{(2)} = & - \int dz_1 dz_2 \rho_2 \left(\frac{z_2}{z_1} \right) (\bar{\xi}_n W)_{z_1} (W^\dagger i \vec{D}_c^\perp W)_{z_2} \\ & \times \frac{\epsilon_\perp}{2} \frac{1}{\bar{p}} \left(W^\dagger \frac{1}{i \bar{n} \cdot D_c} i g \mathcal{B}_c^\perp W \right) q_{us} \\ & + \int dz_1 dz_2 \rho_2 \left(\frac{z_2}{z_1} \right) (\bar{\xi}_n W)_{z_1} (W^\dagger i \vec{D}_c^\perp W)_{z_2} \\ & \times \frac{1}{\bar{p}} \left(W^\dagger \frac{1}{i \bar{n} \cdot D_c} i g \epsilon_\perp \cdot B_c^\perp W \right) q_{us} + \text{H.c.}, \end{aligned}$$

$$\begin{aligned} \delta_1^{(\lambda^0)} \mathcal{L}_3^{(2)} = & - \int dz_1 dz_2 \rho_3 \left(\frac{z_2}{z_1} \right) (\bar{\xi}_n W)_{z_1} \delta(z_2) \\ & \times \frac{\bar{h}}{4} \frac{1}{\bar{p}} \left(W^\dagger \frac{1}{i \bar{n} \cdot D_c} i g \Delta^\perp \cdot B_c^\perp W \right) q_{us} + \text{H.c.}, \end{aligned}$$

$$\begin{aligned} \delta_{II}^{(\lambda^0)} \mathcal{L}_3^{(2)} = & \int dz_1 dz_2 \rho_3 \left(\frac{z_2}{z_1} \right) \\ & \times (\bar{\xi}_n W)_{z_1} \frac{\epsilon_\perp}{2} (W^\dagger i \vec{D}_c^{\perp \mu} W)_{z_2} \\ & \times \frac{1}{\bar{p}} \left(W^\dagger \frac{1}{i \bar{n} \cdot D_c} i g \mathcal{B}_{c\mu}^\perp W \right) q_{us}. \end{aligned}$$

Comparing Eqs. (40) and (41) we see that it is not possible to form an invariant involving $\mathcal{L}_3^{(2)}$ so $\rho_3 = 0$, while an invariant can be formed from $\mathcal{L}_{\xi q}^{(1)} + \mathcal{L}_1^{(2)} + \mathcal{L}_2^{(2)}$ by taking $\rho_1 = 1$ and $\rho_2(z_2/z_1) = 1$. Since ρ_2 is independent of z_1/z_2 the integrals over $z_{1,2}$ can be performed. Therefore, we can write our final result for the first two orders in the usoft-collinear quark Lagrangian as

$$\begin{aligned} \mathcal{L}_{\xi q}^{(1)} = & \bar{\xi}_n \frac{1}{i \bar{n} \cdot D_c} i g \mathcal{B}_c^\perp W q_{us} + \text{H.c.}, \\ \mathcal{L}_{\xi q}^{(2a)} = & \bar{\xi}_n \frac{1}{i \bar{n} \cdot D_c} i g M_\perp W q_{us} + \text{H.c.}, \end{aligned} \quad (42)$$

$$\mathcal{L}_{\xi q}^{(2b)} = \bar{\xi}_n \frac{\vec{n}}{2} i \mathcal{D}_\perp^c \frac{1}{(i\vec{n} \cdot D_c)^2} i g \mathcal{B}_\perp^c W q_{us} + \text{H.c.}$$

These terms agree *exactly* with the result from tree level matching in Eqs. (33) and (35). The analysis here shows that no other terms are induced by matching at any order in α_s .

Next we proceed to analyze power suppressed terms in the collinear gluon action. Starting with the LO collinear gluon action [4], $\mathcal{L}_{cg}^{(0)}$ and making it RPI invariant with $iD^\mu = iD_c^\mu + iD_{us}^\mu$ gives

$$\mathcal{L}_{cg} = \frac{1}{2g^2} \text{tr} \{ [iD^\mu, iD^\nu]^2 \}. \quad (43)$$

It is straightforward to see that no other gauge invariant pure glue dimension-4 operators are possible. We could build a more general gauge invariant operator out of a string of m terms $[\mathcal{W} i D^{\mu_i} \mathcal{W}^\dagger]_{z_i}$, with $m-4$ factors of $1/\bar{\mathcal{P}}$ to make up the mass dimensions. However, type-III RPI then demands $m-4$ factors of \bar{n}_{μ_i} in the numerators which, using $\bar{n} \cdot D \mathcal{W} = 0$, collapses the operator to the case $m=4$. Finally, since \mathcal{W} transforms under type-II RPI, as in Eqs. (22), but iD^μ does not, we find that these operators must have Wilson coefficients $C(z_i)$ that are independent of the z_i parameters. In this case all factors of \mathcal{W} cancel out and we are left with Eq. (43) (after performing the z_i integrals and fixing the coefficient at tree level). Expanding Eq. (43) we see that the order λ and λ^2 suppressed terms are

$$\begin{aligned} \mathcal{L}_{cg}^{(1)} &= \frac{2}{g^2} \text{tr} \{ [i\mathcal{D}^\mu, iD_c^\perp{}^\nu] [iD_\mu, iD_{us}^\perp{}^\nu] \}, \\ \mathcal{L}_{cg}^{(2)} &= \frac{1}{g^2} \text{tr} \{ [i\mathcal{D}^\mu, iD_{us}^\perp{}^\nu] [iD_\mu, iD_{us}^\perp{}^\nu] \\ &\quad + \frac{1}{g^2} \text{tr} \{ [iD_{us}^\perp{}^\mu, iD_{us}^\perp{}^\nu] [iD_{c\mu}^\perp, iD_{c\nu}^\perp] \} \\ &\quad + \frac{1}{g^2} \text{tr} \{ [i\mathcal{D}^\mu, in \cdot D] [iD_\mu, in \cdot D_{us}] \} \\ &\quad + \frac{1}{g^2} \text{tr} \{ [iD_{us}^\perp{}^\mu, iD_c^\perp{}^\nu] [iD_{c\mu}^\perp, iD_{us\nu}^\perp] \}, \end{aligned} \quad (44)$$

where $\mathcal{D}^\mu = D_c^\mu + \bar{n}^\mu n \cdot D_{us}/2$.

In Ref. [4] the gauge fixing terms in the LO gluon action were given in a general covariant gauge. We do not bother to consider the possibility of other leading order gauge fixing terms since we have some residual freedom to choose these terms however we like. In an RPI invariant form the terms from Ref. [4] are

$$\begin{aligned} \mathcal{L}_{cg} &= 2 \text{tr} \{ \bar{c}_n [iD_\mu, [iD^\mu, c_n]] \} + \frac{1}{\alpha} \text{tr} \{ [iD_\mu, A_n^\mu] \\ &\quad \times [iD_\nu, A_n^\nu] \}, \end{aligned} \quad (45)$$

and the subleading terms in their expansion are

$$\begin{aligned} \mathcal{L}_{cg}^{(1)} &= 2 \text{tr} \{ \bar{c}_n [iD_{\perp\mu}^{us}, [iD_c^\perp{}^\mu, c_n]] \} \\ &\quad + 2 \text{tr} \{ \bar{c}_n [iD_{\perp\mu}^c, [iD_{us}^\perp{}^\mu, c_n]] \} + \frac{2}{\alpha} \text{tr} \{ [iD_{\perp\mu}^{us}, A_n^\perp{}^\mu] \\ &\quad \times [iD_\nu, A_n^\nu] \}, \end{aligned} \quad (46)$$

$$\begin{aligned} \mathcal{L}_{cg}^{(2)} &= 2 \text{tr} \{ \bar{c}_n [iD_{\perp\mu}^{us}, [iD_{us}^\perp{}^\mu, c_n]] \} \\ &\quad + \text{tr} \{ \bar{c}_n [in \cdot D^{us}, [in \cdot D, c_n]] \} \\ &\quad + \text{tr} \{ \bar{c}_n [in \cdot D, [in \cdot D_{us}, c_n]] \} + \frac{1}{\alpha} \text{tr} \{ [iD_{\perp\mu}^{us}, A_n^\perp{}^\mu] \\ &\quad \times [iD_{\perp\nu}^{us}, A_n^\perp{}^\nu] \} + \frac{1}{\alpha} \text{tr} \{ [in \cdot D^{us}, n \cdot A_n] [iD_\nu, A_n^\nu] \}. \end{aligned} \quad (47)$$

IV. MOST GENERAL BASIS FOR HEAVY-TO-LIGHT CURRENTS

In this section we give our derivation of the most general basis of heavy-to-light currents at $\mathcal{O}(\lambda)$. The scalar current is given in great detail, and forms the basis of the analysis for the other Dirac structures. Expanding the heavy-to-light currents in powers of λ we write

$$J_{hl} = J^{(d)} + K^{(d)} + \dots, \quad (48)$$

for the LO currents ($J^{(d)}$), and NLO currents ($K^{(d)}$). The superscript denotes whether the current is scalar ($d=s$), pseudo-scalar ($d=p$), vector ($d=v$), axial-vector ($d=a$), or tensor ($d=t$). For the preliminary basis where only constraints from gauge invariance, power counting, and type-III RPI invariance are imposed we use a calligraphic notation $\mathcal{J}^{(d)} + \mathcal{K}^{(d)} + \dots$, and then switch to Roman for the final basis that is invariant under all the type-I and type-II constraints. We will also make use of a convolution notation

$$J^{(d)} = \int d\omega C^{(d)} \left(\frac{\omega}{m_b}, \frac{\mu}{m_b} \right) J^{(d)}(\omega), \quad (49)$$

where $J^{(d)}(\omega)$ contains fields and operators with the notation in Eqs. (8) and the Wilson coefficients $C^{(d)}(\hat{\omega}, \mu/m_b)$ are numerical functions of the convolution parameter (where $\hat{\omega} = \omega/m_b$).

A. Scalar currents

From gauge invariance and power counting the most general leading order heavy-to-light current has the form $\bar{\xi}_n W \Gamma h_\nu$ [2]. For $v_\perp \neq 0$ the most general allowed scalar spin

structures from Sec. II B are then $\Gamma = \{1, \vec{n}\}$. Type-III RPI demands that the \vec{n} is accompanied by either an $n \cdot v$ or a $1/\vec{n} \cdot v$. Thus after imposing constraints (i)–(iv) of Sec. II we are left with the possible leading order currents

$$\begin{aligned} \mathcal{J}_1^{(s)} &= c_1^{(s)} \bar{\xi}_n W h_v, \quad \mathcal{J}_2^{(s)} = c_2^{(s)} \bar{\xi}_n W n \cdot v \frac{\vec{n}}{2} h_v, \\ \mathcal{J}_3^{(s)} &= c_3^{(s)} \bar{\xi}_n W \frac{\vec{n}}{2 \vec{n} \cdot v} h_v, \end{aligned} \quad (50)$$

where c_i are dimensionless Wilson coefficients. With type-III RPI invariance the c_i can only depend on the combination $(n \cdot v \bar{\mathcal{P}})$, the b -quark mass m_b , and the renormalization scale μ . Now consider the order λ^0 type-I and II RPI transformations in Eqs. (21). Since none of the operators in Eqs. (50) involve quantities that have $\delta_1^{(\lambda^0)}$ transformations they are invariant under type-I at this order. However, under the analogous type-II transformations

$$\begin{aligned} \delta_{\text{II}}^{(\lambda^0)} \mathcal{J}_1^{(s)} &= 0, \quad \delta_{\text{II}}^{(\lambda^0)} \mathcal{J}_2^{(s)} = c_2^{(s)} \bar{\xi}_n W \left(n \cdot v \frac{\epsilon_\perp}{2} \right) h_v, \\ \delta_{\text{II}}^{(\lambda^0)} \mathcal{J}_3^{(s)} &= c_3^{(s)} \bar{\xi}_n W \left(\frac{\epsilon_\perp}{2 \vec{n} \cdot v} - \frac{\vec{n}}{2 (\vec{n} \cdot v)^2} \epsilon_\perp \cdot v \right) h_v. \end{aligned} \quad (51)$$

Thus, it is not possible to form an invariant involving the currents $\mathcal{J}_{2,3}^{(s)}$, and only the current $\mathcal{J}_1^{(s)}$ is allowed. Therefore, we can rewrite our final result for the most general leading order scalar current as

$$J_0^{(s)} = C_0^{(s)} \left(\frac{-n \cdot v \bar{\mathcal{P}}}{m_b}, \frac{\mu}{m_b} \right) \bar{\xi}_n W h_v. \quad (52)$$

Since the Wilson coefficient is dimensionless it can only be a function of the dimensionless ratios of parameters as shown. The minus sign in the first variable is included so that $\bar{\mathcal{P}}$ gives the total outgoing momentum of $\bar{\xi}_n W$. Switching to the convolution notation in Eq. (49) and defining $\hat{\omega} = \omega/m_b$ we can write Eq. (52) as

$$\begin{aligned} J_0^{(s)} &= \int d\omega C_0^{(s)}(\hat{\omega}, \mu/m_b) J_0^{(s)}(\omega), \\ J_0^{(s)}(\omega) &= (\bar{\xi}_n W)_\omega h_v. \end{aligned} \quad (53)$$

Thus our notation is that $J_0^{(s)}$ contains the Wilson coefficient, while $J_0^{(s)}(\omega)$ is purely the field operator. With the convolution notation in Eqs. (53) the Wilson coefficients are just numerical functions which do not transform under RPI. We will often suppress the dependence of Wilson coefficients on μ/m_b in what follows.

Next consider currents that are suppressed by a power of λ . At this order the only additional structure we can use is a $D_c^\perp \cdot \mu \sim \lambda$, where the derivative acts to the left or to the right.

To form the most general collinear gauge invariant we take $(W^\dagger D_c^\perp \cdot \mu W)$, which we then insert between the $(\bar{\xi}_n W)$ and h_v to satisfy the usoft gauge invariance. Since the two collinear factors are invariant by themselves they can have arbitrary labels $\omega_{1,2}$. Thus we have operators with the structure

$$(\bar{\xi}_n W)_{\omega_1} \Gamma (W^\dagger D_c^\perp \cdot \mu W)_{\omega_2} \frac{1}{\bar{\mathcal{P}}^\dagger} h_v. \quad (54)$$

The factor of $1/\bar{\mathcal{P}}^\dagger$ is included to make the Wilson coefficients dimensionless. To make a scalar current the μ superscript in Eq. (54) can be dotted into a γ_μ or v_μ in Γ . In either case the most general remaining Dirac structure involves either 1 or \vec{n} as follows from Sec. II B. Thus, combining the constraints from gauge invariance, spin reduction, and type-III RPI leaves eight $\mathcal{O}(\lambda)$ suppressed currents

$$\mathcal{K}_j^{(s)} = \int d\omega_1 d\omega_2 b_j^{(s)}(\hat{\omega}_1, \hat{\omega}_2) \mathcal{K}_j^{(s)}(\omega_1, \omega_2), \quad (55)$$

where $j = 1, \dots, 8$, and the b_i coefficients are dimensionless functions of $\hat{\omega}_{1,2} = \omega_{1,2}/m_b$ and μ/m_b . The eight operators are

$$\begin{aligned} \mathcal{K}_{\{1,2\}}^{(s)}(\omega_1, \omega_2) &= (\bar{\xi}_n W)_{\omega_1} (W^\dagger i \vec{D}_c^\perp W)_{\omega_2} \frac{1}{\bar{\mathcal{P}}^\dagger} \left\{ \frac{\vec{n}}{2}, \frac{1}{n \cdot v} \right\} h_v, \\ \mathcal{K}_{\{3,4\}}^{(s)}(\omega_1, \omega_2) &= (\bar{\xi}_n W)_{\omega_1} (W^\dagger i v \cdot \vec{D}_c^\perp W)_{\omega_2} \frac{1}{\bar{\mathcal{P}}^\dagger} \\ &\quad \times \left\{ \frac{\vec{n}}{2}, \frac{1}{n \cdot v} \right\} h_v, \end{aligned} \quad (56)$$

$$\mathcal{K}_{\{5,6\}}^{(s)}(\omega_1, \omega_2) = (\bar{\xi}_n W)_{\omega_1} (W^\dagger i \vec{D}_c^\perp W)_{\omega_2} \frac{1}{\bar{\mathcal{P}}^\dagger} \left\{ \frac{\vec{n}}{2}, \frac{1}{n \cdot v} \right\} h_v,$$

$$\begin{aligned} \mathcal{K}_{\{7,8\}}^{(s)}(\omega_1, \omega_2) &= (\bar{\xi}_n W)_{\omega_1} (W^\dagger i v \cdot \vec{D}_c^\perp W)_{\omega_2} \\ &\quad \times \frac{1}{\bar{\mathcal{P}}^\dagger} \left\{ \frac{\vec{n}}{2}, \frac{1}{n \cdot v} \right\} h_v. \end{aligned}$$

Note that the dependence of the Wilson coefficients on the labels ω_i account for insertions of $1/\bar{\mathcal{P}}^\dagger$ in all possible locations. Just as for the leading currents we cannot use $\vec{n} \cdot v$ to form a type-III invariant in Eqs. (56) as it leads to currents which cannot be made invariant under type-II transformations [the transformed currents would depend on $\epsilon_\perp \cdot v / (\vec{n} \cdot v)^2$ in a way that could not be canceled].

Next consider the type-I transformations for the currents in Eqs. (56). For these subleading currents only the λ^0 transformations are necessary since we are only working to order λ . Under type-I only D_c^\perp transforms and we have

$$\begin{aligned}
\delta_1^{(\lambda^0)}(W^\dagger i\tilde{D}_c^{\perp\mu}W)_{\omega_2} &= -\frac{\Delta_\perp^\mu}{2}(W^\dagger i\bar{n}\cdot\tilde{D}_cW)_{\omega_2} \\
&= +\frac{\Delta_\perp^\mu}{2}\bar{\mathcal{P}}^\dagger\delta(\omega_2-n\cdot v\bar{\mathcal{P}}^\dagger), \\
\delta_1^{(\lambda^0)}(W^\dagger i\tilde{D}_c^{\perp\mu}W)_{\omega_2} &= -\frac{\Delta_\perp^\mu}{2}(W^\dagger i\bar{n}\cdot\tilde{D}_cW)_{\omega_2} \\
&= -\frac{\Delta_\perp^\mu}{2}\bar{\mathcal{P}}\delta(\omega_2-n\cdot v\bar{\mathcal{P}}^\dagger). \quad (57)
\end{aligned}$$

Since $\bar{\mathcal{P}}h_v=0$ it is easy to see that each of $\mathcal{K}_{5,6,7,8}^{(s)}$ are type-I invariant all by themselves. The other currents do transform, and using Eqs. (57) gives

$$\begin{aligned}
\delta_1^{(\lambda^0)}\mathcal{K}_1^{(s)}(\omega_1,\omega_2) &= \delta(\omega_2)(\bar{\xi}_n W)_{\omega_1}\left(\frac{-\hbar\Delta_\perp}{4}\right)h_v, \\
\delta_1^{(\lambda^0)}\mathcal{K}_2^{(s)}(\omega_1,\omega_2) &= \delta(\omega_2)(\bar{\xi}_n W)_{\omega_2}\left(\frac{\Delta_\perp}{2n\cdot v}\right)h_v, \\
\delta_1^{(\lambda^0)}\mathcal{K}_3^{(s)}(\omega_1,\omega_2) &= \delta(\omega_2)(\bar{\xi}_n W)_{\omega_1}\left(\frac{\hbar\Delta_\perp\cdot v}{4}\right)h_v, \\
\delta_1^{(\lambda^0)}\mathcal{K}_4^{(s)}(\omega_1,\omega_2) &= \delta(\omega_2)(\bar{\xi}_n W)_{\omega_1}\left(\frac{\Delta_\perp\cdot v}{2n\cdot v}\right)h_v, \\
\delta_1^{(\lambda^0)}\mathcal{K}_{5,6,7,8}^{(s)}(\omega_1,\omega_2) &= 0.
\end{aligned} \quad (58)$$

The delta functions $\delta(\omega_2)$ cause only the coefficients $b_{1,2,3,4}^{(s)}(\hat{\omega}_1,0)$ to appear in the transformation of $\mathcal{K}_{\{1,2,3,4\}}^{(s)}$. We also need the order λ variation of the LO current in Eqs. (53). In this computation we must be careful to note that the $\delta(\omega-n\cdot v\bar{\mathcal{P}}^\dagger)$ in $(\bar{\xi}_n W)_\omega$ depends on $n\cdot v$, and therefore also transforms

$$\begin{aligned}
\delta_1^{(\lambda^0)}\delta(\omega-n\cdot v\bar{\mathcal{P}}^\dagger) &= -v\cdot\Delta_\perp\bar{\mathcal{P}}^\dagger\delta'(\omega-n\cdot v\bar{\mathcal{P}}^\dagger) \\
&= -\frac{v\cdot\Delta_\perp}{n\cdot v}\frac{d}{d\omega}\omega\delta(\omega-n\cdot v\bar{\mathcal{P}}^\dagger). \quad (59)
\end{aligned}$$

Using Eqs. (22) we find a term from transforming the delta function and a term from transforming the collinear quark field

$$\delta_1^{(\lambda)}J_0^{(s)}(\omega) = (\bar{\xi}_n W)_\omega\left(\frac{\hbar\Delta_\perp}{4}\right)h_v - \frac{v\cdot\Delta_\perp}{n\cdot v}\frac{d}{d\omega}\omega(\bar{\xi}_n W)_\omega h_v. \quad (60)$$

Demanding invariance under the transformations in Eqs. (58) and (60) gives non-trivial constraints on the Wilson coefficients in $J_0^{(s)}$ and $\mathcal{K}_i^{(s)}$. From Eqs. (58) the currents $\mathcal{K}_{\{1,2,3,4\}}^{(s)}$ are invariant by themselves provided that $b_{\{1,2,3,4\}}^{(s)}(\hat{\omega}_1,0)$

= 0. However, in Eq. (60) the first term can only be canceled by a $\mathcal{K}_1^{(s)}$ with $b_1^{(s)}(\hat{\omega},0) = C_0^{(s)}(\hat{\omega})$. To cancel the second term we integrate by parts to give a $\omega d/d\omega = \hat{\omega}d/d\hat{\omega}$ acting on $C_0^{(s)}$. This term can then be canceled by $b_4^{(s)}(\hat{\omega}_1,0) = -2\hat{\omega}d/d\hat{\omega}C_0^{(s)}(\hat{\omega})$. Thus, the summary of type-I invariants is $J_0 + \mathcal{K}_1^{(s)} + \mathcal{K}_4^{(s)}$, $\mathcal{K}_{\{5,6,7,8\}}^{(s)}$, $\mathcal{K}_{\{2,3\}}^{(s)}$, with any $b_{5,6,7,8}^{(s)}(\hat{\omega}_1,\hat{\omega}_2)$ and coefficients

$$\begin{aligned}
b_1^{(s)}(\hat{\omega},0) &= C_0^{(s)}(\hat{\omega}), \quad b_{2,3}^{(s)}(\hat{\omega},0) = 0, \\
b_4^{(s)}(\hat{\omega},0) &= -2\hat{\omega}d/d\hat{\omega}C_0^{(s)}(\hat{\omega}). \quad (61)
\end{aligned}$$

Now consider the type-II transformations. The analogue of Eq. (59) is

$$\begin{aligned}
\delta_{\text{II}}^{(\lambda)}\delta(\omega-n\cdot v\bar{\mathcal{P}}^\dagger) &= -n\cdot v\varepsilon_\perp\cdot\mathcal{P}_\perp^\dagger\delta'(\omega-n\cdot v\bar{\mathcal{P}}^\dagger) \\
&= -n\cdot v\varepsilon_\perp\cdot\mathcal{P}_\perp^\dagger\frac{d}{d\omega}\delta(\omega-n\cdot v\bar{\mathcal{P}}^\dagger). \quad (62)
\end{aligned}$$

For the order λ variation of the LO current $J_0^{(s)}(\omega)$ we have terms from the transformation of the delta function, the collinear quark field, and the Wilson line W

$$\begin{aligned}
\delta_{\text{II}}^{(\lambda)}J_0(\omega) &= n\cdot v\varepsilon_\perp\cdot\mathcal{P}_\perp\frac{d}{d\omega}(\bar{\xi}_n W)_\omega h_v \\
&\quad + \left(\bar{\xi}_n i\tilde{D}_c^\perp\frac{1}{i\bar{n}\cdot\tilde{D}_c}\frac{\not{\epsilon}_\perp}{2}W\right)_\omega h_v \\
&\quad - \left(\bar{\xi}_n\frac{1}{i\bar{n}\cdot D_c}i\varepsilon_\perp\cdot\tilde{D}_c^\perp W\right)_\omega h_v. \quad (63)
\end{aligned}$$

In the subleading currents $\mathcal{K}_i^{(s)}$ both \bar{n}^μ and D_c^\perp have λ^0 transformations.⁸ A $i\tilde{D}_c^\perp$ transforms to give a \hbar , so since $\hbar\xi_n=0$ it is easy to see that $\mathcal{K}_{\{2,6\}}^{(s)}$ are invariant under type-II transformations at this order. The transformations for the remaining currents are more involved,

$$\delta_{\text{II}}^{(\lambda^0)}\mathcal{K}_{\{2,6\}}^{(s)}(\omega_1,\omega_2) = 0,$$

$$\delta_{\text{II}}^{(\lambda^0)}\mathcal{K}_1^{(s)}(\omega_1,\omega_2) = (\bar{\xi}_n W)_{\omega_1}(W^\dagger i\tilde{D}_c^\perp W)_{\omega_2}\frac{\not{\epsilon}_\perp}{2}\frac{1}{\bar{\mathcal{P}}^\dagger}h_v,$$

$$\begin{aligned}
\delta_{\text{II}}^{(\lambda^0)}\mathcal{K}_3^{(s)}(\omega_1,\omega_2) &= (\bar{\xi}_n W)_{\omega_1}\left\{\frac{\not{\epsilon}_\perp}{2}(W^\dagger i v\cdot\tilde{D}_c^\perp W)_{\omega_2}\right. \\
&\quad \left.-\frac{n\cdot v}{4}\bar{\hbar}(W^\dagger i\varepsilon_\perp\cdot\tilde{D}_c^\perp W)_{\omega_2}\right\}\frac{1}{\bar{\mathcal{P}}^\dagger}h_v,
\end{aligned}$$

⁸Since $\delta_{\text{II}}\bar{\mathcal{P}} = \mathcal{O}(\lambda)$ the transformation of the delta functions in $\mathcal{K}_i^{(s)}$ only appears at one higher order.

$$\delta_{\text{II}}^{(\lambda^0)} \mathcal{K}_4^{(s)}(\omega_1, \omega_2) = -\frac{1}{2} (\bar{\xi}_n W)_{\omega_1} (W^\dagger i \varepsilon_\perp \cdot \vec{D}_c^\perp W)_{\omega_2} \times \frac{1}{\bar{\mathcal{P}}^\dagger} h_v, \quad (64)$$

$$\delta_{\text{II}}^{(\lambda^0)} \mathcal{K}_5^{(s)}(\omega_1, \omega_2) = (\bar{\xi}_n W)_{\omega_1} (W^\dagger i \vec{D}_c^\perp W)_{\omega_2} \frac{\boldsymbol{\varepsilon}_\perp}{2} \frac{1}{\bar{\mathcal{P}}^\dagger} h_v,$$

$$\delta_{\text{II}}^{(\lambda^0)} \mathcal{K}_7^{(s)}(\omega_1, \omega_2) = (\bar{\xi}_n W)_{\omega_1} \left\{ \frac{\boldsymbol{\varepsilon}_\perp}{2} (W^\dagger i v \cdot \vec{D}_c^\perp W)_{\omega_2} - \frac{n \cdot v}{4} \frac{\vec{h}}{2} (W^\dagger i \varepsilon_\perp \cdot \vec{D}_c^\perp W)_{\omega_2} \right\} \frac{1}{\bar{\mathcal{P}}^\dagger} h_v,$$

$$\delta_{\text{II}}^{(\lambda^0)} \mathcal{K}_8^{(s)}(\omega_1, \omega_2) = -\frac{1}{2} (\bar{\xi}_n W)_{\omega_1} (W^\dagger i \varepsilon_\perp \cdot \vec{D}_c^\perp W)_{\omega_2} \frac{1}{\bar{\mathcal{P}}^\dagger} h_v.$$

It is straightforward to see that it is not possible to form a type-II invariant using only the currents $\mathcal{K}_{\{1,3,4,5,7,8\}}^{(s)}$. However, it is possible to form an invariant taking a combination of $\mathcal{K}_{\{1,4,8\}}^{(s)}$ with $J_0^{(s)}$. To facilitate this we rewrite Eq. (63) as

$$\begin{aligned} \delta_{\text{II}}^{(\lambda^1)} J_0^{(s)}(\omega) &= \int d\omega_1 \left[n \cdot v \frac{d}{d\omega} (\bar{\xi}_n W)_{\omega_1} \right. \\ &\quad \times \{ (W^\dagger i \varepsilon_\perp \cdot \vec{D}_c^\perp W)_{\omega-\omega_1} \\ &\quad + (W^\dagger i \varepsilon_\perp \cdot \vec{D}_c^\perp W)_{\omega-\omega_1} \} h_v \\ &\quad - (\bar{\xi}_n W)_{\omega_1} (W^\dagger i \vec{D}_c^\perp W)_{\omega-\omega_1} \frac{1}{\bar{\mathcal{P}}^\dagger} \frac{\boldsymbol{\varepsilon}_\perp}{2} h_v \\ &\quad \left. - (\bar{\xi}_n W)_{\omega_1} \frac{1}{\omega_1 - \omega} (W^\dagger i \varepsilon_\perp \cdot \vec{D}_c^\perp W)_{\omega-\omega_1} h_v \right]. \end{aligned} \quad (65)$$

To derive Eq. (65) we used $\varepsilon_\perp \cdot \mathcal{P}_\perp \bar{\xi}_n W h_v = \bar{\xi}_n i \varepsilon_\perp \cdot \vec{D}_c^\perp W h_v + \bar{\xi}_n i \varepsilon_\perp \cdot \vec{D}_c^\perp W h_v$, and the fact that the ω_1 integration can be carried out with the delta function in $(\bar{\xi}_n W)_{\omega_1}$ to get back a product of operators with momentum label ω where the intermediate $W W^\dagger$ cancel out. Now for the $d/d\omega$ terms in Eq. (65) we can integrate by parts in $[C_0(\hat{\omega}) \delta_{\text{II}}^{(\lambda^1)} J_0^{(s)}(\omega)]$ so that the derivative acts on the Wilson coefficient $C_0^{(s)}(\hat{\omega})$. It is then evident that the third term in Eq. (65) can be canceled by $\delta_{\text{II}}^{(\lambda^0)} \mathcal{K}_1^{(s)}$ with $b_1^{(s)}(\hat{\omega}_1, \hat{\omega} - \hat{\omega}_1) = C_0^{(s)}(\hat{\omega})$, the second term is canceled by $\delta_{\text{II}}^{(\lambda^0)} \mathcal{K}_4^{(s)}$ with $b_4^{(s)}(\hat{\omega}_1, \hat{\omega} - \hat{\omega}_1) = -2\hat{\omega} d/d\hat{\omega} C_0^{(s)}(\hat{\omega})$, and the first and fourth terms are canceled by $\delta_{\text{II}}^{(\lambda^0)} \mathcal{K}_8^{(s)}$ with $b_8^{(s)}(\hat{\omega}_1, \hat{\omega} - \hat{\omega}_1) = -2\hat{\omega} d/d\hat{\omega} C_0^{(s)}(\hat{\omega}) - 2\hat{\omega}/(\hat{\omega}_1 - \hat{\omega}) C_0^{(s)}(\hat{\omega})$. Therefore, type-II RPI rules out the operators $\mathcal{K}_{3,5,7}^{(s)}$ and leaves only the invariants

$$J_0^{(s)} + \mathcal{K}_1^{(s)} + \mathcal{K}_4^{(s)} + \mathcal{K}_8^{(s)}, \quad \mathcal{K}_2^{(s)}, \quad \mathcal{K}_6^{(s)}. \quad (66)$$

For type-II invariance their coefficients can have any $b_{\{2,4\}}^{(s)}(\hat{\omega}_1, \hat{\omega}_2)$, but require

$$\begin{aligned} b_1^{(s)}(\hat{\omega}_1, \hat{\omega} - \omega_1) &= C_0^{(s)}(\hat{\omega}), \\ b_4^{(s)}(\hat{\omega}, \hat{\omega} - \omega_1) &= -2\hat{\omega} \frac{d}{d\hat{\omega}} C_0^{(s)}(\hat{\omega}), \\ b_8^{(s)}(\hat{\omega}_1, \hat{\omega} - \omega_1) &= -2\hat{\omega} \frac{d}{d\hat{\omega}} C_0^{(s)}(\hat{\omega}) \\ &\quad - \frac{2\hat{\omega}}{(\hat{\omega}_1 - \hat{\omega})} C_0^{(s)}(\hat{\omega}). \end{aligned} \quad (67)$$

The restrictions on the Wilson coefficients are summarized in Table I. Comparing the invariants in Eqs. (61) and (67), we see that the combinations in Eq. (66) are the most general combinations invariant under type I and type II with the restrictions in Eqs. (67) plus $b_2^{(s)}(\hat{\omega}, 0) = 0$. However, with this constraint on $b_2^{(s)}$ the operator $\mathcal{K}_2^{(s)}$ is actually identical to $\mathcal{K}_6^{(s)}$ with an unconstrained coefficient $b_6^{(s)}$. To see this note that within square brackets $[W^\dagger i \vec{D}_c^\perp W]_{\omega_2} = [W^\dagger i \vec{D}_c^\perp W]_{\omega_2} = -[W^\dagger i g \mathcal{B}_c^\perp W]_{\omega_2}/\omega_2$, so the difference comes from $i \vec{D}_c^\perp$ acting also on $(\bar{\xi}_n W)$ in $\mathcal{K}_2^{(s)}$. However, since the factor $(\bar{\xi}_n W)$ on the left is a collinear color singlet we can write

$$\begin{aligned} (W^\dagger i \vec{D}_c^\perp W)_{\omega_2} &= [W^\dagger i \vec{D}_c^\perp W]_{\omega_2} - \mathcal{P}_\perp^\dagger (W^\dagger W)_{\omega_2} \\ &= [W^\dagger i \vec{D}_c^\perp W]_{\omega_2} - \mathcal{P}_\perp^\dagger \delta(\omega_2), \end{aligned} \quad (68)$$

and the last term vanishes since $b_2(\omega_1, 0) = 0$. Given this result and the constraints in Eqs. (67) it is convenient to define

$$\begin{aligned} K_1^{(s)}(\omega) &= \int d\omega_1 \mathcal{K}_1^{(s)}(\omega_1, \omega - \omega_1), \\ K_2^{(s)}(\omega) &= \hat{\omega} \int d\omega_1 \{ \mathcal{K}_4^{(s)}(\omega_1, \omega - \omega_1) \\ &\quad + \mathcal{K}_8^{(s)}(\omega_1, \omega - \omega_1) \}, \end{aligned} \quad (69)$$

$$K_3^{(s)}(\omega) = \int d\omega_1 (\omega_1 - \omega)^{-1} \mathcal{K}_8^{(s)}(\omega_1, \omega - \omega_1),$$

$$K_4^{(s)}(\omega_1, \omega_2) = (\hat{\omega}_1 + \hat{\omega}_2) \mathcal{K}_6^{(s)}(\omega_1, \omega_2).$$

From RPI it is only these operators that can ever appear. RPI has ruled out some currents and restricted $K_{1,2,3}^{(s)}$ to only depend on one parameter. Once we know this, we can simply forget about the $\mathcal{K}_i^{(s)}$ and work directly with the $K_i^{(s)}$. Using capital B 's for their Wilson coefficients, our final basis of subleading scalar operators $K_{1-4}^{(s)}$ is

TABLE I. Summary of RPI constraints on the coefficients of the scalar currents in Eq. (56). The first column shows the constraints from type-I RPI on $b_i^{(s)}(\hat{\omega}_1, 0)$, the second column shows the constraint on $b_i^{(s)}(\hat{\omega}_1, \hat{\omega}_2)$ from type-II RPI, and the third column gives the combined constraint. A generic entry, such as $b_2^{(s)}(\hat{\omega}_1, \hat{\omega}_2)$ in the second row of the RPI-II column, indicates no constraint. The final currents are displayed in Eq. (71), and are defined so that they automatically satisfy these constraints.

	RPI-I $b_i^{(s)}(\hat{\omega}_1, 0) =$	RPI-II $b_i^{(s)}(\hat{\omega}_1, \hat{\omega}_2) =$	Combined constraints
$b_1^{(s)}$	$C_0^{(s)}(\hat{\omega}_1)$	$C_0^{(s)}(\hat{\omega}_1 + \hat{\omega}_2)$	$b_1^{(s)}(\hat{\omega}_1, \hat{\omega}_2) = C_0^{(s)}(\hat{\omega}_1 + \hat{\omega}_2)$
$b_2^{(s)}$	0	$b_2^{(s)}(\hat{\omega}_1, \hat{\omega}_2)$	$b_2^{(s)}(\hat{\omega}_1, 0) = 0$
$b_3^{(s)}$	0	0	$b_3^{(s)}(\hat{\omega}_1, \hat{\omega}_2) = 0$
$b_4^{(s)}$	$-2\hat{\omega}_1 C_0^{(s)'}(\hat{\omega}_1)$	$-2(\hat{\omega}_1 + \hat{\omega}_2) C_0^{(s)'}(\hat{\omega}_1 + \hat{\omega}_2)$	$b_4^{(s)}(\hat{\omega}_1, \hat{\omega}_2) = -2(\hat{\omega}_1 + \hat{\omega}_2) C_0^{(s)'}(\hat{\omega}_1 + \hat{\omega}_2)$
$b_5^{(s)}$	$b_5^{(s)}(\hat{\omega}_1, 0)$	0	$b_5^{(s)}(\hat{\omega}_1, \hat{\omega}_2) = 0$
$b_6^{(s)}$	$b_6^{(s)}(\hat{\omega}_1, 0)$	$b_6^{(s)}(\hat{\omega}_1, \hat{\omega}_2)$	$b_6^{(s)}(\hat{\omega}_1, \hat{\omega}_2)$ unconstrained
$b_7^{(s)}$	$b_7^{(s)}(\hat{\omega}_1, 0)$	0	$b_7^{(s)}(\hat{\omega}_1, \hat{\omega}_2) = 0$
$b_8^{(s)}$	$b_8^{(s)}(\hat{\omega}_1, 0)$	$-2(\hat{\omega}_1 + \hat{\omega}_2) C_0^{(s)'}(\hat{\omega}_1 + \hat{\omega}_2)$ $+ 2\left(1 + \frac{\hat{\omega}_1}{\hat{\omega}_2}\right) C_0^{(s)}(\hat{\omega}_1 + \hat{\omega}_2)$	$b_8^{(s)}(\hat{\omega}_1, \hat{\omega}_2) = -2(\hat{\omega}_1 + \hat{\omega}_2) C_0^{(s)'}(\hat{\omega}_1 + \hat{\omega}_2)$ $+ 2\left(1 + \frac{\hat{\omega}_1}{\hat{\omega}_2}\right) C_0^{(s)}(\hat{\omega}_1 + \hat{\omega}_2)$

$$K_{1-3}^{(s)} = \int d\omega B_{1-3}^{(s)}(\hat{\omega}) K_{1-3}^{(s)}(\omega),$$

$$K_4^{(s)} = \int d\omega_1 d\omega_2 B_4^{(s)}(\hat{\omega}_1, \hat{\omega}_2) K_4^{(s)}(\omega_1, \omega_2), \quad (70)$$

where

$$K_1^{(s)}(\omega) = -\left(\bar{\xi}_n \frac{\vec{h}}{2} i \vec{D}_c^\perp W\right) \frac{1}{\omega \bar{\mathcal{P}}^\dagger} h_v,$$

$$K_2^{(s)}(\omega) = \frac{v \cdot \mathcal{P}_\perp}{m_b} (\bar{\xi}_n W)_\omega h_v, \quad (71)$$

$$K_3^{(s)}(\omega) = \left(\bar{\xi}_n \frac{1}{n \cdot v i \vec{n} \cdot D_c} i v \cdot \vec{D}_c^\perp W\right)_\omega h_v,$$

$$K_4^{(s)}(\omega_1, \omega_2) = \frac{1}{m_b} (\bar{\xi}_n W)_{\omega_1} \left(\frac{1}{\bar{\mathcal{P}}} W^\dagger i g \mathcal{B}_c^\perp W\right)_{\omega_2} h_v,$$

and the RPI type-I and type-II constraints on the Wilson coefficients become

$$B_1^{(s)}(\hat{\omega}) = C_0^{(s)}(\hat{\omega}), \quad B_2^{(s)}(\hat{\omega}) = -2C_0^{(s)'}(\hat{\omega}),$$

$$B_3^{(s)}(\hat{\omega}) = -2C_0^{(s)}(\hat{\omega}). \quad (72)$$

The prime here denotes a derivative with respect to $\hat{\omega}$. Thus, we conclude that there are 4 subleading $\mathcal{O}(\lambda)$ scalar heavy-to-light currents. The coefficient $B_4^{(s)}(\hat{\omega}_1, \hat{\omega}_2)$ is completely unconstrained, while the other coefficients are fixed from RPI invariance.

In Ref. [13] it was noted that $J_0^{(s)}$, $K_1^{(s)}$, and $K_3^{(s)}$ are connected by RPI, and for these operators our results agree

with taking their matching calculation in Eq. (120) and multiplying by a common Wilson coefficient. The operator $K_2^{(s)}$ does not appear in Ref. [13] because the derivative on its coefficient causes it to vanish at tree level. Our three-body operator $K_4^{(s)}$ is also new. In the limit that $B_4(\hat{\omega}_1, \hat{\omega}_2)$ depends *only* on the sum $\hat{\omega}_+ = \hat{\omega}_1 + \hat{\omega}_2$ we can switch variables to $\hat{\omega}_+$, $\hat{\omega}_1$ and reduce $K_4^{(s)}$ to a two-body operator. At tree level this is always possible since the Wilson coefficient is independent of the ω_i . To see how the reduction works we write

$$\int d\omega_1 K_4^{(s)}(\omega_1, \omega_+ - \omega_1) = \frac{1}{m_b} \left(\bar{\xi}_n W \frac{1}{\bar{\mathcal{P}}} W^\dagger i g \mathcal{B}_c^\perp W\right)_{\omega_+} h_v$$

$$= \frac{1}{m_b} (\bar{\xi}_n [i \vec{D}_c^\perp W])_{\omega_+} h_v, \quad (73)$$

where in the last line we used Eq. (14). The derivative structure of this two-body operator is similar to that of operators in Refs. [12,13]; however, the specific spin structure appearing in Eq. (73) does not appear from matching the QCD scalar current at tree level. Beyond tree level B_4 can depend separately on ω_1 and ω_2 and the reduction in Eq. (73) is not valid.

B. Vector currents

The steps for deriving the general set of vector currents are very similar to the steps for the scalar currents in the previous section, so our presentation will be more concise. At LO gauge invariance plus power counting allows currents of the form $\bar{\xi}_n W Z_\mu h_v$. Imposing type-III RPI invariance allows the Dirac structures

$$Z_\mu = \left\{ \gamma_\mu, \gamma_\mu \bar{n} \cdot v, \frac{\gamma_\mu \bar{n}}{n \cdot v}, n_\mu \bar{n}, \frac{n_\mu}{n \cdot v}, n_\mu \bar{n} \cdot v, v_\mu, v_\mu \right. \\ \times \bar{n} \cdot v, \frac{v_\mu \bar{n}}{n \cdot v}, \bar{n}_\mu n \cdot v, \frac{\bar{n}_\mu}{n \cdot v}, \bar{n}_\mu \bar{n} (n \cdot v)^2, \\ \left. \frac{\bar{n}_\mu \bar{n} \cdot v}{\bar{n} \cdot v}, \frac{\bar{n}_\mu \bar{n}}{(\bar{n} \cdot v)^2} \right\}. \quad (74)$$

However, it is easy to show that all structures involving \bar{n} are ruled out by the $\delta_{\Pi}^{(\lambda^0)}$ transformations. Thus, at leading order there are only three allowed vector currents

$$J_{1-3}^{(v)} = \int d\omega C_{1-3}^{(v)}(\hat{\omega}) J_{1-3}^{(v)}(\omega), \quad (75)$$

where the coefficients are functions of $\hat{\omega} = \omega/m_b$ and μ/m_b , and

$$J_{1-3}^{(v)}(\omega) = (\bar{\xi}_n W)_\omega \left\{ \gamma_\mu, v_\mu, \frac{n_\mu}{n \cdot v} \right\} h_v. \quad (76)$$

The choices $\{1,2,3\}$ correspond to the three different Dirac structures, and our basis in Eq. (76) agrees with Ref. [10].

At NLO the power counting only allows a single D_\perp to appear. For the possible spin structures it is easy to see that type-II RPI invariance does not allow the vector index to be in an \bar{n}^μ or any factors of $\bar{n} \cdot v$ to appear just as for the leading currents. Imposing the constraints from gauge invariance, spin reduction, and type-III RPI then leaves 28 $\mathcal{O}(\lambda)$ suppressed currents [$j = 1, \dots, 28$]

$$\mathcal{K}_j^{(v)} = \int d\omega_1 d\omega_2 b_j^{(v)}(\hat{\omega}_1, \hat{\omega}_2) \mathcal{K}_j^{(v)}(\omega_1, \omega_2), \quad (77)$$

with operators

$$\mathcal{K}_{1-6}^{(v)}(\omega_1, \omega_2) = (\bar{\xi}_n W)_{\omega_1} (W^\dagger i \bar{D}_c^\perp W)_{\omega_2} \frac{1}{\bar{\mathcal{P}}^\dagger} \Gamma_{1-6}^\mu h_v,$$

$$\mathcal{K}_{7,8}^{(v)}(\omega_1, \omega_2) = (\bar{\xi}_n W)_{\omega_1} (W^\dagger i \bar{D}_c^\perp W)_{\omega_2} \\ \times \frac{1}{\bar{\mathcal{P}}^\dagger} \left\{ \frac{\bar{n}}{2}, \frac{1}{n \cdot v} \right\} h_v,$$

$$\mathcal{K}_{9-14}^{(v)}(\omega_1, \omega_2) = (\bar{\xi}_n W)_{\omega_1} (W^\dagger i v \cdot \bar{D}_c^\perp W)_{\omega_2} \frac{1}{\bar{\mathcal{P}}^\dagger} \Gamma_{1-6}^\mu h_v, \quad (78)$$

$$\mathcal{K}_{15-20}^{(v)}(\omega_1, \omega_2) = (\bar{\xi}_n W)_{\omega_1} (W^\dagger i \bar{D}_c^\perp W)_{\omega_2} \frac{1}{\bar{\mathcal{P}}^\dagger} \Gamma_{1-6}^\mu h_v,$$

$$\mathcal{K}_{21,22}^{(v)}(\omega_1, \omega_2) = (\bar{\xi}_n W)_{\omega_1} (W^\dagger i \bar{D}_c^\perp W)_{\omega_2} \frac{1}{\bar{\mathcal{P}}^\dagger} \\ \times \left\{ \frac{\bar{n}}{2}, \frac{1}{n \cdot v} \right\} h_v,$$

$$\mathcal{K}_{23-28}^{(v)}(\omega_1, \omega_2) = (\bar{\xi}_n W)_{\omega_1} (W^\dagger i v \cdot \bar{D}_c^\perp W)_{\omega_2} \frac{1}{\bar{\mathcal{P}}^\dagger} \Gamma_{1-6}^\mu h_v,$$

where the six Γ_i^μ matrices are

$$\Gamma_{1-6}^\mu = \left\{ \frac{\bar{n} \gamma^\mu}{2}, \frac{\bar{n} v^\mu}{2}, \frac{\bar{n} n^\mu}{2 n \cdot v}, \frac{\gamma^\mu}{n \cdot v}, \frac{v^\mu}{n \cdot v}, \frac{n^\mu}{(n \cdot v)^2} \right\}. \quad (79)$$

Working out the transformations of the leading and sub-leading currents in a similar way as was done for the scalar currents we find that the type-I invariants are $J_1^{(v)} + \mathcal{K}_1^{(v)} + \mathcal{K}_{12}^{(v)}$, $J_2^{(v)} + \mathcal{K}_2^{(v)} + \mathcal{K}_{13}^{(v)}$, $J_3^{(v)} + \mathcal{K}_3^{(v)} + \mathcal{K}_8^{(v)} + \mathcal{K}_{14}^{(v)}$, $\mathcal{K}_j^{(v)}$, $\mathcal{K}_k^{(v)}$, where $j = \{4, 5, 6, 7, 9, 10, 11\}$ and $k = \{15, \dots, 28\}$. Type-I invariance allows any coefficients $b_k^{(v)}(\hat{\omega}_1, \hat{\omega}_2)$, but restricts $b_{1-14}^{(v)}(\hat{\omega}_1, \hat{\omega}_2)$ as shown in the second column of Table II.

Looking at the transformations under type-II we find that the invariants are independent of the $\{\gamma_\mu, v_\mu, n_\mu\}$ choice. Our results for the type-II invariants are $J_1^{(v)} + \mathcal{K}_1^{(v)} + \mathcal{K}_{26}^{(v)} + \mathcal{K}_{27}^{(v)}$, $J_2^{(v)} + \mathcal{K}_2^{(v)} + \mathcal{K}_{13}^{(v)} + \mathcal{K}_{27}^{(v)}$, $J_3^{(v)} + \mathcal{K}_3^{(v)} + \mathcal{K}_8^{(v)} + \mathcal{K}_{14}^{(v)} + (\mathcal{K}_{22}^{(v)} + \mathcal{K}_{28}^{(v)})$, $(\mathcal{K}_{22}^{(v)} - \mathcal{K}_{28}^{(v)})$, while $\mathcal{K}_\ell^{(v)}$ for $\ell = \{4, 5, 6, 18, 19, 20\}$ are invariant by themselves. For these combinations type-II invariance allows any $b_\ell^{(v)}(\hat{\omega}_1, \hat{\omega}_2)$, but restricts $b_{1-3, 7, 9-17, 21-28}^{(v)}$ as shown in the third column of Table II. Furthermore, currents $\mathcal{K}_m^{(v)}$ with $m = \{7, 9, 10, 11, 15, 16, 17, 21, 23, 24, 25\}$ are ruled out (i.e. $b_m^{(v)} = 0$).

It is easy to see that the type-I and type-II conditions in Table II are compatible. The combined set of constraints are given by those in the fourth column. Using Eq. (68) we can show that the constrained $\mathcal{K}_{\{4,5,6\}}^{(v)}$ are redundant with $\mathcal{K}_{\{18,19,20\}}^{(v)}$, respectively, just as was done for the scalar current with $\mathcal{K}_2^{(s)}$ and $\mathcal{K}_6^{(s)}$. We can also use Eq. (68) to convert $K_8^{(v)}(\hat{\omega}_1, \hat{\omega}_2) - K_{14}^{(v)}(\hat{\omega}_1, \hat{\omega}_2)$ into a term proportional to $\delta(\omega_2)$ and a term that is redundant with $K_{22}^{(v)}(\hat{\omega}_1, \hat{\omega}_2) - K_{28}^{(v)}(\hat{\omega}_1, \hat{\omega}_2)$.

From the combined constraints we can then define a new complete set of allowed vector operators, $K_{1-14}^{(v)}$. Therefore after imposing type-I and type-II RPI plus all other constraints we are left with our final set of allowed vector current operators,

$$K_{1-10}^{(v)} = \int d\omega B_{1-10}^{(v)}(\hat{\omega}) K_{1-10}^{(v)}(\omega),$$

$$K_{11-14}^{(v)} = \int d\omega_1 d\omega_2 B_{11-14}^{(v)}(\hat{\omega}_1, \hat{\omega}_2) K_{11-14}^{(v)}(\omega_1, \omega_2), \quad (80)$$

TABLE II. Summary of RPI constraints on the coefficients of the vector currents in Eq. (78). The first column shows the constraints from type-I RPI on $b_i^{(s)}(\omega, 0)$, the second column shows the constraint on $b_i^{(s)}(\omega_1, \omega_2)$ from type-II RPI, and the third column gives the combined constraint. The final currents are displayed in Eq. (81), and are defined so that they automatically satisfy these constraints.

	RPI-I $b_i^{(v)}(\hat{\omega}_1, 0) =$	RPI-II $b_i^{(v)}(\hat{\omega}_1, \hat{\omega}_2) =$	Combined constraints
$b_{1-3}^{(v)}$	$C_{1-3}^{(v)}(\hat{\omega}_1)$	$C_{1-3}^{(v)}(\hat{\omega}_1 + \hat{\omega}_2)$	$b_{1-3}^{(v)}(\hat{\omega}_1, \hat{\omega}_2) = C_{1-3}^{(v)}(\hat{\omega}_1 + \hat{\omega}_2)$
$b_{4-6}^{(v)}$	0	$b_{4-6}^{(v)}(\hat{\omega}_1, \hat{\omega}_2)$	$b_{4-6}^{(v)}(\hat{\omega}_1, 0) = 0$
$b_7^{(v)}$	0	0	$b_7^{(v)}(\hat{\omega}_1, \hat{\omega}_2) = 0$
$b_8^{(v)}$	$-2 C_3^{(v)}(\hat{\omega}_1)$	$b_8^{(v)}(\hat{\omega}_1, \hat{\omega}_2)$	$b_8^{(v)}(\hat{\omega}_1, 0) = -2 C_3^{(v)}(\hat{\omega}_1)$
$b_{9-11}^{(v)}$	0	0	$b_{9-11}^{(v)}(\hat{\omega}_1, \hat{\omega}_2) = 0$
$b_{12,13}^{(v)}$	$-2 \hat{\omega}_1 C_{1,2}^{(v)'}(\hat{\omega}_1)$	$-2(\hat{\omega}_1 + \hat{\omega}_2) C_{1,2}^{(v)'}(\hat{\omega}_1 + \hat{\omega}_2)$	$b_{12,13}^{(v)}(\hat{\omega}_1, \hat{\omega}_2) = -2(\hat{\omega}_1 + \hat{\omega}_2) C_{1,2}^{(v)'}(\hat{\omega}_1 + \hat{\omega}_2)$
$b_{14}^{(v)}$	$-2 \hat{\omega}_1 C_3^{(v)'}(\hat{\omega}_1)$ $+ 2 C_3^{(v)}(\hat{\omega}_1)$	$-2(\hat{\omega}_1 + \hat{\omega}_2) C_3^{(v)'}(\hat{\omega}_1 + \hat{\omega}_2)$ $- b_8^{(v)}(\hat{\omega}_1, \hat{\omega}_2)$	$b_{14}^{(v)}(\hat{\omega}_1, \hat{\omega}_2) = -2(\hat{\omega}_1 + \hat{\omega}_2) C_3^{(v)'}(\hat{\omega}_1 + \hat{\omega}_2)$ $- b_8^{(v)}(\hat{\omega}_1, \hat{\omega}_2)$
$b_{15-17}^{(v)}$	$b_{15-17}^{(v)}(\hat{\omega}_1, 0)$	0	$b_{15-17}^{(v)}(\hat{\omega}_1, \hat{\omega}_2) = 0$
$b_{18-20}^{(v)}$	$b_{18-20}^{(v)}(\hat{\omega}_1, 0)$	$b_{18-20}^{(v)}(\hat{\omega}_1, \hat{\omega}_2)$	$b_{18-20}^{(v)}(\hat{\omega}_1, \hat{\omega}_2)$ unconstrained
$b_{21}^{(v)}$	$b_{21}^{(v)}(\hat{\omega}_1, 0)$	0	$b_{21}^{(v)}(\hat{\omega}_1, \hat{\omega}_2) = 0$
$b_{22}^{(v)}$	$b_{22}^{(v)}(\hat{\omega}_1, 0)$	$b_{22}^{(v)}(\hat{\omega}_1, \hat{\omega}_2)$	$b_{22}^{(v)}(\hat{\omega}_1, \hat{\omega}_2)$ unconstrained
$b_{23-25}^{(v)}$	$b_{23-25}^{(v)}(\hat{\omega}_1, 0)$	0	$b_{23-25}^{(v)}(\hat{\omega}_1, \hat{\omega}_2) = 0$
$b_{26,27}^{(v)}$	$b_{26,27}^{(v)}(\hat{\omega}_1, 0)$	$-2(\hat{\omega}_1 + \hat{\omega}_2) C_{1,2}^{(v)'}(\hat{\omega}_1 + \hat{\omega}_2)$ $- 2 C_{1,2}^{(v)}(\hat{\omega}_1 + \hat{\omega}_2)$	$b_{26,27}^{(v)}(\hat{\omega}_1, \hat{\omega}_2) = -2(\hat{\omega}_1 + \hat{\omega}_2) C_{1,2}^{(v)'}(\hat{\omega}_1 + \hat{\omega}_2)$ $- 2 C_{1,2}^{(v)}(\hat{\omega}_1 + \hat{\omega}_2)$
$b_{28}^{(v)}$	$b_{28}^{(v)}(\hat{\omega}_1, 0)$	$-2(\hat{\omega}_1 + \hat{\omega}_2) C_3^{(v)'}(\hat{\omega}_1 + \hat{\omega}_2)$ $+ 2 \left(1 + \frac{\hat{\omega}_1}{\hat{\omega}_2} \right) C_3^{(v)}(\hat{\omega}_1 + \hat{\omega}_2)$ $- b_{22}(\hat{\omega}_1, \hat{\omega}_2)$	$b_{28}^{(v)}(\hat{\omega}_1, \hat{\omega}_2) = -2(\hat{\omega}_1 + \hat{\omega}_2) C_3^{(v)'}(\hat{\omega}_1 + \hat{\omega}_2)$ $+ 2 \left(1 + \frac{\hat{\omega}_1}{\hat{\omega}_2} \right) C_3^{(v)}(\hat{\omega}_1 + \hat{\omega}_2)$ $- b_{22}(\hat{\omega}_1, \hat{\omega}_2)$

where

$$K_{1-3}^{(v)}(\omega) = - \left(\frac{\vec{\xi}_n}{\xi_n} \frac{\vec{h}}{2} i \vec{D}_c^\perp W \right)_\omega \frac{1}{\vec{\mathcal{P}}^\dagger} \times \left\{ \gamma_\mu, v_\mu, \frac{n_\mu}{n \cdot v} \right\} h_v,$$

$$K_4^{(v)}(\omega) = \left(g^{\mu\alpha} - \frac{n^\mu v^\alpha}{n \cdot v} \right) \times (\vec{\xi}_n i \vec{D}_{c\alpha}^\perp W)_\omega \frac{1}{\vec{\mathcal{P}}^\dagger} \frac{1}{n \cdot v} h_v,$$

$$K_{5-7}^{(v)}(\omega) = \frac{v \cdot \mathcal{P}_\perp}{m_b} (\vec{\xi}_n W)_\omega \left\{ \gamma_\mu, v_\mu, \frac{n_\mu}{n \cdot v} \right\} h_v, \quad (81)$$

$$K_{8-10}^{(v)}(\omega) = \left(\vec{\xi}_n \frac{1}{n \cdot v i \vec{n} \cdot D_c} i v \cdot \vec{D}_c^\perp W \right)_\omega \times \left\{ \gamma_\mu, v_\mu, \frac{n_\mu}{n \cdot v} \right\} h_v,$$

$$K_{11-13}^{(v)}(\omega_1, \omega_2) = \frac{1}{m_b} (\vec{\xi}_n W)_{\omega_1} \left\{ \gamma_\mu, v_\mu, \frac{n_\mu}{n \cdot v} \right\} \times \left(\frac{1}{\vec{\mathcal{P}}} W^\dagger i g B_c^\perp W \right)_{\omega_2} h_v,$$

$$K_{14}^{(v)}(\omega_1, \omega_2) = \frac{1}{m_b} (\vec{\xi}_n W)_{\omega_1} \left(g^{\mu\alpha} - \frac{n^\mu v^\alpha}{n \cdot v} \right) \times \left(\frac{1}{\vec{\mathcal{P}}} W^\dagger i g B_{c\alpha}^\perp W \right)_{\omega_2} h_v.$$

The coefficients $B_{\{11,12,13,14\}}^{(v)}$ in Eqs. (80) depend on two parameters $\omega_{1,2}$ and are unconstrained. The remaining coefficients depend on only one parameter and are fixed by reparametrization invariance

$$B_{1-3}^{(v)}(\hat{\omega}) = C_{1-3}^{(v)}(\hat{\omega}), \quad B_{5-7}^{(v)}(\hat{\omega}) = -2 C_{1-3}^{(v)'}(\hat{\omega}),$$

$$B_{8-10}^{(v)}(\hat{\omega}) = -2 C_{1-3}^{(v)}(\hat{\omega}), \quad B_4^{(v)}(\hat{\omega}) = -2 C_3^{(v)}(\hat{\omega}). \quad (82)$$

The form of the currents $K_{1,2}^{(v)}$ agree with Ref. [10], and if we take a frame where $n \cdot v = 1$ and $v_\perp = 0$ then $K_{3,4}^{(v)}$ also

agree. Reference [10] looked at type-I RPI of the vector currents and our constraints on $B_{1-4}^{(v)}$ agree with the ones found there. (We note that the authors of Ref. [10] also checked these results with explicit one-loop computations.) At tree-level one matches onto the currents $K_{1,8}^{(v)}$ and the two-body limit of $K_{13}^{(v)}$ [using the analog of Eq. (73)], and we agree with Ref. [13] on the form of these currents and the RPI constraint between $K_1^{(v)}$ and $K_8^{(v)}$. The structures in Eq. (81) which are new and which only appear beyond tree level are $K_{5-7,9-12,14}^{(v)}$ and the three-body form of $K_{13}^{(v)}$.

C. Pseudoscalar and axial-vector currents

The results for the pseudoscalar and axial-vector heavy-to-light currents can be directly obtained from the analysis for the scalar and vector currents, respectively. The analysis is identical except for the extra γ_5 in the Dirac structure. For the pseudoscalar currents we have the basis $\{\hbar/2, 1/n \cdot v\} \gamma_5$, while for the axial-vector currents we have $\Gamma_{\{1, \dots, 6\}}^\mu \gamma_5$, where Γ_j^μ is defined in Eq. (79). At LO the most general allowed pseudoscalar current are thus

$$J_0^{(p)} = \int d\omega C_0^{(p)}(\hat{\omega}, \mu/m_b) J_0^{(p)}(\omega),$$

$$J_0^{(p)}(\omega) = (\bar{\xi}_n W)_{\omega_1} \gamma_5 h_v, \quad (83)$$

while the axial-vector currents are

$$J_{1-3}^{(a)} = \int d\omega C_{1-3}^{(a)}(\hat{\omega}) J_{1-3}^{(a)}(\omega),$$

$$J_{1-3}^{(a)}(\omega) = (\bar{\xi}_n W)_\omega \left\{ \gamma_\mu, -v_\mu, -\frac{n_\mu}{n \cdot v} \right\} \gamma_5 h_v. \quad (84)$$

At NLO we again have eight possible pseudo-scalar currents $\mathcal{K}_j^{(p)}$ and 28 possible axial-vector currents $\mathcal{K}_j^{(a)}$ before imposing all type-I and type-II constraints. After imposing the RPI constraints we find results very similar to those in Eqs. (71) and (81). Thus for the final NLO pseudoscalar currents we have

$$K_{1-3}^{(p)} = \int d\omega B_{1-3}^{(p)}(\hat{\omega}) K_{1-3}^{(p)}(\omega),$$

$$K_4^{(p)} = \int d\omega_1 d\omega_2 B_4^{(p)}(\hat{\omega}_1, \hat{\omega}_2) K_4^{(p)}(\omega_1, \omega_2), \quad (85)$$

where

$$K_1^{(p)}(\omega) = - \left(\bar{\xi}_n \frac{\hbar}{2} i \tilde{D}_c^\perp W \right)_\omega \frac{1}{\bar{\mathcal{P}}^\dagger} \gamma_5 h_v,$$

$$K_2^{(p)}(\omega) = \frac{v \cdot \mathcal{P}_\perp}{m_b} (\bar{\xi}_n W)_\omega \gamma_5 h_v, \quad (86)$$

$$K_3^{(p)}(\omega) = \left(\bar{\xi}_n \frac{1}{n \cdot v i \bar{n} \cdot D_c} i v \cdot \tilde{D}_c^\perp W \right)_\omega \gamma_5 h_v,$$

$$K_4^{(p)}(\omega_1, \omega_2) = \frac{1}{m_b} (\bar{\xi}_n W)_{\omega_1} \gamma_5 \left(\frac{1}{\bar{\mathcal{P}}} W^\dagger i g \mathcal{B}_c^\perp W \right)_{\omega_2} h_v,$$

and the RPI type-I and type-II constraints on the Wilson coefficients are

$$B_1^{(p)}(\hat{\omega}) = C_0^{(p)}(\hat{\omega}), \quad B_2^{(p)}(\hat{\omega}) = -2 C_0^{(p)'}(\hat{\omega}),$$

$$B_3^{(p)}(\hat{\omega}) = -2 C_0^{(p)}(\hat{\omega}). \quad (87)$$

For the final axial-vector NLO currents we find

$$K_{1-10}^{(a)} = \int d\omega B_{1-10}^{(a)}(\hat{\omega}) K_{1-10}^{(a)}(\omega),$$

$$K_{11-14}^{(a)} = \int d\omega_1 d\omega_2 B_{11-14}^{(a)}(\hat{\omega}_1, \hat{\omega}_2) \times K_{11-14}^{(a)}(\omega_1, \omega_2), \quad (88)$$

where

$$K_{1-3}^{(a)}(\omega) = - \left(\bar{\xi}_n \frac{\hbar}{2} i \tilde{D}_c^\perp W \right)_\omega \frac{1}{\bar{\mathcal{P}}^\dagger} \times \left\{ \gamma_\mu, -v_\mu, -\frac{n_\mu}{n \cdot v} \right\} \gamma_5 h_v,$$

$$K_4^{(a)}(\omega) = - \left(g^{\mu\alpha} - \frac{n^\mu v^\alpha}{n \cdot v} \right) \times (\bar{\xi}_n i \tilde{D}_{c\alpha}^\perp W)_\omega \frac{1}{\bar{\mathcal{P}}^\dagger} \frac{1}{n \cdot v} \gamma_5 h_v,$$

$$K_{5-7}^{(a)}(\omega) = \frac{v \cdot \mathcal{P}_\perp}{m_b} (\bar{\xi}_n W)_\omega \left\{ \gamma_\mu, -v_\mu, -\frac{n_\mu}{n \cdot v} \right\} \gamma_5 h_v, \quad (89)$$

$$K_{8-10}^{(a)}(\omega) = \left(\bar{\xi}_n \frac{1}{n \cdot v i \bar{n} \cdot D_c} i v \cdot \tilde{D}_c^\perp W \right)_\omega \times \left\{ \gamma_\mu, -v_\mu, -\frac{n_\mu}{n \cdot v} \right\} \gamma_5 h_v,$$

$$K_{11-13}^{(a)}(\omega_1, \omega_2) = \frac{1}{m_b} (\bar{\xi}_n W)_{\omega_1} \left\{ \gamma_\mu, -v_\mu, -\frac{n_\mu}{n \cdot v} \right\} \times \gamma_5 \left(\frac{1}{\bar{\mathcal{P}}} W^\dagger i g \mathcal{B}_c^\perp W \right)_{\omega_2} h_v,$$

$$\begin{aligned}
 K_{14}^{(a)} &= -\frac{1}{m_b} (\bar{\xi}_n W)_{\omega_1} \left(g^{\mu\alpha} - \frac{n^\mu v^\alpha}{n \cdot v} \right) \\
 &\quad \times \gamma_5 \left(\frac{1}{\bar{\mathcal{P}}} W^\dagger i g B_c^\perp W \right)_{\omega_2} h_v. \\
 \mathcal{K}_{31-36}^{(t)} &= (\bar{\xi}_n W)_{\omega_1} (W^\dagger i \vec{D}_{c\perp}^{\perp\mu} W)_{\omega_2} \Gamma_{1-6}^{\nu\perp} \frac{1}{\bar{\mathcal{P}}^\dagger} h_v, \\
 \mathcal{K}_{37-44}^{(t)} &= (\bar{\xi}_n W)_{\omega_1} (W^\dagger i v \cdot \vec{D}_c^\perp W)_{\omega_2} \\
 &\quad \times \left\{ \frac{\hbar}{2} \Gamma_{1-4}^{\mu\nu}, \frac{1}{n \cdot v} \Gamma_{1-4}^{\mu\nu} \right\} \frac{1}{\bar{\mathcal{P}}^\dagger} h_v.
 \end{aligned} \tag{94}$$

The coefficients $B_{\{11,12,13,14\}}^{(a)}$ in Eqs. (80) depend on two parameters $\omega_{1,2}$ and are unconstrained. The remaining coefficients depend on only one parameter and are fixed by reparametrization invariance

$$\begin{aligned}
 B_{1-3}^{(a)}(\hat{\omega}) &= C_{1-3}^{(a)}(\hat{\omega}), \quad B_{5-7}^{(a)}(\hat{\omega}) = -2C_{1-3}^{(a)'}(\hat{\omega}), \\
 B_{8-10}^{(a)}(\hat{\omega}) &= -2C_{1-3}^{(a)}(\hat{\omega}), \quad B_4^{(a)}(\hat{\omega}) = -2C_2^{(a)}(\hat{\omega}).
 \end{aligned} \tag{90}$$

The form of the pseudo-scalar and axial-vector currents are very analogous to the scalar and vector currents, so rather than comparing with the literature we simply refer to the comparisons in the proceeding sections for which part of our results were previously known.

D. Tensor currents

At leading order in λ , there are four tensor currents, defined as

$$J_{1-4}^{\mu\nu}(\omega) = (\bar{\xi}_n W)_\omega \Gamma_{1-4}^{\mu\nu} h_v, \tag{91}$$

where the most general allowed Dirac structures are

$$\Gamma_{1-4}^{\mu\nu} = \left\{ i\sigma^{\mu\nu}, \gamma^{[\mu} v^{\nu]}, \frac{1}{n \cdot v} \gamma^{[\mu} n^{\nu]}, \frac{1}{n \cdot v} n^{[\mu} v^{\nu]} \right\}, \tag{92}$$

where $\gamma_{[\mu} v_{\nu]} = \gamma_\mu v_\nu - \gamma_\nu v_\mu$, etc. As before, no \bar{n}_μ can appear at leading order from type-II RPI.

At $\mathcal{O}(\lambda)$, 44 currents can be written down before imposing the RPI constraints. They can be chosen as

$$\begin{aligned}
 \mathcal{K}_{1-8}^{(t)} &= (\bar{\xi}_n W)_{\omega_1} (W^\dagger i \vec{D}_c^\perp W)_{\omega_2} \left\{ \frac{\hbar}{2} \Gamma_{1-4}^{\mu\nu}, \frac{1}{n \cdot v} \Gamma_{1-4}^{\mu\nu} \right\} \frac{1}{\bar{\mathcal{P}}^\dagger} h_v, \\
 \mathcal{K}_{9-14}^{(t)} &= (\bar{\xi}_n W)_{\omega_1} (W^\dagger i \vec{D}_{c\perp}^{\perp\mu} W)_{\omega_2} \Gamma_{1-6}^{\nu\perp} \frac{1}{\bar{\mathcal{P}}^\dagger} h_v, \\
 \mathcal{K}_{15-22}^{(t)} &= (\bar{\xi}_n W)_{\omega_1} (W^\dagger i v \cdot \vec{D}_c^\perp W)_{\omega_2} \\
 &\quad \times \left\{ \frac{\hbar}{2} \Gamma_{1-4}^{\mu\nu}, \frac{1}{n \cdot v} \Gamma_{1-4}^{\mu\nu} \right\} \frac{1}{\bar{\mathcal{P}}^\dagger} h_v,
 \end{aligned} \tag{93}$$

$$\mathcal{K}_{15-22}^{(t)} = (\bar{\xi}_n W)_{\omega_1} (W^\dagger i v \cdot \vec{D}_c^\perp W)_{\omega_2}$$

$$\times \left\{ \frac{\hbar}{2} \Gamma_{1-4}^{\mu\nu}, \frac{1}{n \cdot v} \Gamma_{1-4}^{\mu\nu} \right\} \frac{1}{\bar{\mathcal{P}}^\dagger} h_v,$$

and

$$\begin{aligned}
 \mathcal{K}_{23-30}^{(t)} &= (\bar{\xi}_n W)_{\omega_1} (W^\dagger i \vec{D}_c^\perp W)_{\omega_2} \\
 &\quad \times \left\{ \frac{\hbar}{2} \Gamma_{1-4}^{\mu\nu}, \frac{1}{n \cdot v} \Gamma_{1-4}^{\mu\nu} \right\} \frac{1}{\bar{\mathcal{P}}^\dagger} h_v,
 \end{aligned}$$

The Dirac matrix with one index Γ_{1-6}^μ is defined as in Eq. (79)

$$\Gamma_{1-6}^\mu = \left\{ \frac{\hbar}{2} \left(\gamma_\mu \cdot v_\mu, \frac{n_\mu}{v \cdot n} \right), \frac{1}{n \cdot v} \left(\gamma_\mu \cdot v_\mu, \frac{n_\mu}{v \cdot n} \right) \right\}. \tag{95}$$

The constraints from type-I and type-II RPI are derived as before. The final constraints on the Wilson coefficients b_{1-44} are shown in Table III.

After imposing the constraints from the table one finds the final minimal set of tensor heavy-light currents in the effective theory at $\mathcal{O}(\lambda)$

$$\begin{aligned}
 K_{1-14}^{(t)} &= \int d\omega B_{1-14}^{(t)}(\omega) K_{1-14}^{\mu\nu}(\omega), \\
 K_{15-21}^{(t)} &= \int d\omega_1 d\omega_2 B_{15-21}^{(t)}(\omega_1, \omega_2) K_{15-21}^{\mu\nu}(\omega_1, \omega_2).
 \end{aligned} \tag{96}$$

There are 14 independent $\mathcal{O}(\lambda)$ two-body operators given explicitly by

$$\begin{aligned}
 K_{1-4}^{(t)}(\omega) &= -\left(\bar{\xi}_n \frac{\hbar}{2} i \vec{D}_c^\perp W \right)_\omega \frac{1}{\bar{\mathcal{P}}^\dagger} \Gamma_{1-4}^{\mu\nu} h_v, \\
 K_{5,6}^{(t)}(\omega) &= (\bar{\xi}_n i \vec{D}_{c\perp}^\alpha W)_\omega \frac{1}{\bar{\mathcal{P}}^\dagger} \{ g_{\alpha[\mu} \gamma_{\nu]} \\
 &\quad + v_\alpha \Gamma_3^{\mu\nu}, g_{\alpha[\mu} v_{\nu]} - v_\alpha \Gamma_4^{\mu\nu} \} h_v, \\
 K_{7-10}^{(t)}(\omega) &= \frac{1}{m_b} (v \cdot \mathcal{P}_\perp) (\bar{\xi}_n W)_\omega \Gamma_{1-4}^{\mu\nu} h_v, \\
 K_{11-14}^{(t)}(\omega) &= \left(\bar{\xi}_n \frac{1}{n \cdot i D_c} i v \cdot \vec{D}_c^\perp W \right)_\omega \frac{1}{n \cdot v} \Gamma_{1-4}^{\mu\nu} h_v.
 \end{aligned} \tag{97}$$

Their coefficients are fixed by reparametrization invariance in terms of the $\mathcal{O}(\lambda^0)$ Wilson coefficients $C_{1-4}^{(t)}(\omega)$ as

$$\begin{aligned}
 B_{1-4}^{(t)}(\omega) &= C_{1-4}^{(t)}(\omega), \quad B_5^{(t)}(\omega) = 2C_3^{(t)}(\omega), \\
 B_6^{(t)}(\omega) &= -2C_4^{(t)}(\omega), \quad B_{7-10}^{(t)}(\omega) = -2C_{1-4}^{(t)'}(\omega), \\
 B_{11-14}^{(t)}(\omega) &= -2C_{1-4}^{(t)}(\omega).
 \end{aligned} \tag{98}$$

In addition, there are 7 three-body collinear operators given by

TABLE III. Summary of RPI constraints on the coefficients of the tensor currents in Eqs. (93). The first column shows the constraints from type-I RPI on $b_i(\omega, 0)$, the second column shows the constraint on $b_i(\omega_1, \omega_2)$ from type-II RPI, and the third column gives the combined constraint. Each \pm refers to the first and second terms in their row, respectively. The final currents are displayed in Eqs. (97), (99), and are defined so that they automatically satisfy these constraints.

	RPI-I $b_i^{(t)}(\hat{\omega}_1, 0) =$	RPI-II $b_i^{(t)}(\hat{\omega}_1, \hat{\omega}_2) =$	Combined constraints
$b_{1-4}^{(t)}$	$C_{1-4}^{(t)}(\hat{\omega}_1)$	$C_{1-4}^{(t)}(\hat{\omega}_1 + \hat{\omega}_2)$	$b_{1-4}^{(t)}(\hat{\omega}_1, \hat{\omega}_2) = C_{1-4}^{(t)}(\hat{\omega}_1 + \hat{\omega}_2)$
$b_{5-8}^{(t)}$	0	$b_{5-8}^{(t)}(\hat{\omega}_1, \hat{\omega}_2)$	$b_{5-8}^{(t)}(\hat{\omega}_1, 0) = 0$
$b_{9-11}^{(t)}$	0	0	$b_{9-11}^{(t)}(\hat{\omega}_1, \hat{\omega}_2) = 0$
$b_{12}^{(t)}$	$2 C_3^{(t)}(\hat{\omega}_1)$	$b_{12}^{(t)}(\hat{\omega}_1, \hat{\omega}_2)$	$b_{12}^{(t)}(\hat{\omega}_1, 0) = 2 C_3^{(t)}(\hat{\omega}_1)$
$b_{13}^{(t)}$	$-2 C_4^{(t)}(\hat{\omega}_1)$	$b_{13}^{(t)}(\hat{\omega}_1, \hat{\omega}_2)$	$b_{13}^{(t)}(\hat{\omega}_1, 0) = -2 C_4^{(t)}(\hat{\omega}_1)$
$b_{14}^{(t)}$	0	$b_{14}^{(t)}(\hat{\omega}_1, \hat{\omega}_2)$	$b_{14}^{(t)}(\hat{\omega}_1, 0) = 0$
$b_{15-18}^{(t)}$	0	0	$b_{15-18}^{(t)}(\hat{\omega}_1, \hat{\omega}_2) = 0$
$b_{19,20}^{(t)}$	$-2 \hat{\omega}_1 C_{1,2}^{(t)'}(\hat{\omega}_1)$	$-2(\hat{\omega}_1 + \hat{\omega}_2) C_{1,2}^{(t)'}(\hat{\omega}_1 + \hat{\omega}_2)$	$b_{19,20}^{(t)}(\hat{\omega}_1, \hat{\omega}_2) = -2(\hat{\omega}_1 + \hat{\omega}_2) C_{1,2}^{(t)'}(\hat{\omega}_1 + \hat{\omega}_2)$
$b_{21,22}^{(t)}$	$-2 \hat{\omega}_1 C_{3,4}^{(t)'}(\hat{\omega}_1)$ $+ 2 C_{3,4}^{(t)}(\hat{\omega}_1)$	$-2(\hat{\omega}_1 + \hat{\omega}_2) C_{3,4}^{(t)'}(\hat{\omega}_1 + \hat{\omega}_2)$ $\pm b_{12,13}^{(t)}(\hat{\omega}_1, \hat{\omega}_2)$	$b_{21,22}^{(t)}(\hat{\omega}_1, \hat{\omega}_2) = -2(\hat{\omega}_1 + \hat{\omega}_2) C_{3,4}^{(t)'}(\hat{\omega}_1 + \hat{\omega}_2)$ $\pm b_{12,13}^{(t)}(\hat{\omega}_1, \hat{\omega}_2)$
$b_{23-26}^{(t)}$	$b_{23-26}^{(t)}(\hat{\omega}_1, 0)$	0	$b_{23-26}^{(t)}(\hat{\omega}_1, \hat{\omega}_2) = 0$
$b_{27-30}^{(t)}$	$b_{27-30}^{(t)}(\hat{\omega}_1, 0)$	$b_{27-30}^{(t)}(\hat{\omega}_1, \hat{\omega}_2)$	$b_{27-30}^{(t)}(\hat{\omega}_1, \hat{\omega}_2)$ unconstrained
$b_{31-33}^{(t)}$	$b_{31-33}^{(t)}(\hat{\omega}_1, 0)$	0	$b_{31-33}^{(t)}(\hat{\omega}_1, \hat{\omega}_2) = 0$
$b_{34-36}^{(t)}$	$b_{34-36}^{(t)}(\hat{\omega}_1, 0)$	$b_{34-36}^{(t)}(\hat{\omega}_1, \hat{\omega}_2)$	$b_{34-36}^{(t)}(\hat{\omega}_1, \hat{\omega}_2)$ unconstrained
$b_{37-40}^{(t)}$	$b_{37-40}^{(t)}(\hat{\omega}_1, 0)$	0	$b_{37-40}^{(t)}(\hat{\omega}_1, \hat{\omega}_2) = 0$
$b_{41,42}^{(t)}$	$b_{41,42}^{(t)}(\hat{\omega}_1, 0)$	$-2(\hat{\omega}_1 + \hat{\omega}_2) C_{1,2}^{(t)'}(\hat{\omega}_1 + \hat{\omega}_2)$ $+ 2 \left(1 + \frac{\hat{\omega}_1}{\hat{\omega}_2}\right) C_{1,2}^{(t)}(\hat{\omega}_1 + \hat{\omega}_2)$	$b_{41,42}^{(t)}(\hat{\omega}_1, \hat{\omega}_2) = -2(\hat{\omega}_1 + \hat{\omega}_2) C_{1,2}^{(t)'}(\hat{\omega}_1 + \hat{\omega}_2)$ $+ 2 \left(1 + \frac{\hat{\omega}_1}{\hat{\omega}_2}\right) C_{1,2}^{(t)}(\hat{\omega}_1 + \hat{\omega}_2)$
$b_{43,44}^{(t)}$	$b_{43,44}^{(t)}(\hat{\omega}_1, 0)$	$-2(\hat{\omega}_1 + \hat{\omega}_2) C_{3,4}^{(t)'}(\hat{\omega}_1 + \hat{\omega}_2)$ $+ 2 \left(1 + \frac{\hat{\omega}_1}{\hat{\omega}_2}\right) C_{3,4}^{(t)}(\hat{\omega}_1 + \hat{\omega}_2)$ $\pm b_{34,35}(\hat{\omega}_1, \hat{\omega}_2)$	$b_{43,44}^{(t)}(\hat{\omega}_1, \hat{\omega}_2) = -2(\hat{\omega}_1 + \hat{\omega}_2) C_{3,4}^{(t)'}(\hat{\omega}_1 + \hat{\omega}_2)$ $+ 2 \left(1 + \frac{\hat{\omega}_1}{\hat{\omega}_2}\right) C_{3,4}^{(t)}(\hat{\omega}_1 + \hat{\omega}_2)$ $\pm b_{34,35}(\hat{\omega}_1, \hat{\omega}_2)$

$$K_{15-18}^{(t)}(\omega_1, \omega_2) = \frac{1}{m_b} (\bar{\xi}_n W)_{\omega_1} \Gamma_{1-4}^{\mu\nu} \left(\frac{1}{\bar{\mathcal{P}}} W^\dagger i g \mathcal{B}_{c,\perp} W \right)_{\omega_2} h_\nu,$$

$$K_{21}^{(t)}(\omega_1, \omega_2) = \frac{1}{m_b} (\bar{\xi}_n W)_{\omega_1} g_{\alpha[\mu, n\nu]} \frac{1}{n \cdot v} \times \left(\frac{1}{\bar{\mathcal{P}}} W^\dagger i g B_{c,\perp}^\alpha W \right)_{\omega_2} h_\nu.$$

$$K_{19}^{(t)}(\omega_1, \omega_2) = \frac{1}{m_b} (\bar{\xi}_n W)_{\omega_1} (g_{\alpha[\mu, \gamma\nu]} + v_\alpha \Gamma_3^{\mu\nu}) \times \left(\frac{1}{\bar{\mathcal{P}}} W^\dagger i g B_{c,\perp}^\alpha W \right)_{\omega_2} h_\nu, \quad (99)$$

Their coefficients $B_{15-21}^{(t)}(\hat{\omega}_1, \hat{\omega}_2)$ are not constrained by any symmetry of the effective theory and have to be determined by an explicit matching calculation.

$$K_{20}^{(t)}(\omega_1, \omega_2) = \frac{1}{m_b} (\bar{\xi}_n W)_{\omega_1} (g_{\alpha[\mu, \nu\nu]} - v_\alpha \Gamma_4^{\mu\nu}) \times \left(\frac{1}{\bar{\mathcal{P}}} W^\dagger i g B_{c,\perp}^\alpha W \right)_{\omega_2} h_\nu,$$

At tree level one matches onto the currents $K_{1,11}^{(t)}$ and the two-body limit of $K_{17}^{(t)}$ [using the analog of Eq. (73)], and we agree with Ref. [13] on the form of these currents and the RPI constraint between $K_1^{(t)}$ and $K_{11}^{(t)}$. The remaining operators in Eqs. (97) and (99) are new and only appear beyond tree level (including the three-body structure of $K_{11}^{(t)}$).

V. SUMMARY FOR COEFFICIENTS, OPERATORS, AND FEYNMAN RULES

In this section we summarize results that should be useful for future phenomenological applications. In Sec. V A we summarize the full set of known matching results and compare with the literature, in Sec. V B we give simplified expressions for our basis of currents in the frame $v_\perp=0$, $n \cdot v=1$, and in Sec. V C we give Feynman rules for the subleading currents and $\mathcal{L}_{\xi q}^{(1,2)}$.

A. Matching results for the currents

In this section we summarize the one-loop matching results for the LO and next-to-leading order (NLO) J_{hl} Wilson coefficients. With $\hat{\omega}_i = \omega/m_b$ these coefficients are defined in previous sections as $C_i^{(d)}(\hat{\omega}, \mu/m_b)$ and $B_i^{(d)}(\hat{\omega}_i, \mu/m_b)$, respectively, where (d) denotes whether the current is a scalar, pseudoscalar, vector, axial-vector, or a tensor.

For the LO currents the basis we use is different (though equivalent) to the basis used in Ref. [2]. Since the one-loop matching for the LO coefficients can be found in Ref. [2] it is useful to have the explicit relation between our basis of coefficients $C_i^{(d)}$ and the coefficients C_j ($j=1-12$) that can be found there. We find

$$\begin{aligned} C_0^{(s)} &= C_1, & C_1^{(v)} &= C_3, & C_1^{(a)} &= C_6, & C_1^{(t)} &= C_{10}, \\ C_0^{(p)} &= C_2, & C_2^{(v)} &= C_5, & C_2^{(a)} &= C_8, & C_2^{(t)} &= -C_{12}, \\ & & C_3^{(v)} &= C_4 - C_3, & C_3^{(a)} &= C_7 - C_6, & C_3^{(t)} &= C_{10} - C_9, \\ & & & & & & C_4^{(t)} &= C_{12} + C_{10} - C_{11}. \end{aligned} \quad (100)$$

At tree level the matching between QCD and SCET is scheme independent. Matching with the full QCD currents $\bar{u}\{1, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5, i\sigma_{\mu\nu} b\}$ we find

$$C_0^{(s)} = C_0^{(p)} = C_1^{(v)} = C_1^{(a)} = C_1^{(t)} = 1,$$

$$C_2^{(v)} = C_3^{(v)} = C_2^{(a)} = C_3^{(a)} = C_2^{(t)} = C_3^{(t)} = C_4^{(t)} = 0. \quad (101)$$

At one loop we use the modified minimal subtraction ($\overline{\text{MS}}$) scheme with naive dimensional regularization (NDR) and match at $\mu = m_b$ to determine the $C_i^{(d)}(\hat{\omega}, \mu/m_b)$. Using Eqs. (100) and results in Ref. [2] one finds

$$C_0^{(s,p)}(\hat{\omega}, 1) = 1 - \frac{\alpha_s(m_b) C_F}{4\pi} \left\{ 2 \ln^2(\hat{\omega}) + 2 \text{Li}_2(1 - \hat{\omega}) - \frac{2 \ln(\hat{\omega})}{1 - \hat{\omega}} + \frac{\pi^2}{12} \right\},$$

$$C_1^{(v,a)}(\hat{\omega}, 1) = 1 - \frac{\alpha_s(m_b) C_F}{4\pi} \left\{ 2 \ln^2(\hat{\omega}) + 2 \text{Li}_2(1 - \hat{\omega}) + \ln(\hat{\omega}) \left(\frac{3\hat{\omega} - 2}{1 - \hat{\omega}} \right) + \frac{\pi^2}{12} + 6 \right\},$$

$$C_1^{(t)}(\hat{\omega}, 1) = 1 - \frac{\alpha_s(m_b) C_F}{4\pi} \left\{ 2 \ln^2(\hat{\omega}) + 2 \text{Li}_2(1 - \hat{\omega}) + \ln(\hat{\omega}) \left(\frac{4\hat{\omega} - 2}{1 - \hat{\omega}} \right) + \frac{\pi^2}{12} + 6 \right\},$$

$$C_2^{(v,a)}(\hat{\omega}, 1) = \frac{\alpha_s(m_b) C_F}{4\pi} \left\{ \frac{2}{(1 - \hat{\omega})} + \frac{2\hat{\omega} \ln(\hat{\omega})}{(1 - \hat{\omega})^2} \right\}, \quad (102)$$

$$C_2^{(t)}(\hat{\omega}, 1) = 0,$$

$$C_3^{(v,a)}(\hat{\omega}, 1) = \frac{\alpha_s(m_b) C_F}{4\pi} \left\{ \frac{(1 - 2\hat{\omega})\hat{\omega} \ln(\hat{\omega})}{(1 - \hat{\omega})^2} - \frac{\hat{\omega}}{1 - \hat{\omega}} \right\},$$

$$C_3^{(t)}(\hat{\omega}, 1) = \frac{\alpha_s(m_b) C_F}{4\pi} \left\{ \frac{-2\hat{\omega} \ln(\hat{\omega})}{1 - \hat{\omega}} \right\},$$

$$C_4^{(t)}(\hat{\omega}, 1) = 0,$$

where $C_F = 4/3$ for color $SU(3)$. To determine the coefficients for scales $m_b \Lambda_{\text{QCD}} < \mu^2 < m_b^2$ we require their anomalous dimensions.⁹ The LO and NLO anomalous dimensions are universal and the running of these coefficients is given in Ref. [2] (or for the case $\hat{\omega} = 1$ in Ref. [1]).

At NLO in λ tree level matching of the QCD current $\bar{u}\Gamma b$ onto SCET gives [10,12,13]

$$\begin{aligned} J_{tree}^{(1a)} &= -\bar{\xi}_{n,p'} \frac{\hbar}{2} i \tilde{D}_c^\perp W \frac{1}{\bar{p}^+} \Gamma h_v, \\ J_{tree}^{(1b)} &= -\bar{\xi}_{n,p'} \Gamma \frac{\hbar}{2} i \tilde{D}_c^\perp W \frac{1}{n \cdot v m_b} h_v, \end{aligned} \quad (103)$$

⁹The full NLO result requires a two-loop anomalous dimension which uses information from Ref. [35].

$$J_{tree}^{(1c)} = -\frac{2}{n \cdot v} \bar{\xi}_{n,p} \Gamma \frac{1}{i\bar{n} \cdot \vec{D}_c} i v \cdot \vec{D}_c^\perp W h_v.$$

The normalization in Eqs. (103) was first derived for $J^{(1a)}$ in Ref. [10], and for $J^{(1b,1c)}$ in Ref. [13]. Comparing Eqs. (103) with our basis of currents we see that for any choice of Γ the $J_{tree}^{(1a,1c)}$ match onto a subset of our two-body currents. On the other hand, $J_{tree}^{(1b)}$ does not appear in the basis of two-body currents. Instead it is obtained from the projection of a subset

of the three-body currents for cases where the corresponding coefficients $B_i(\hat{\omega}_2, \hat{\omega}_2)$ depend only on the sum $\hat{\omega}_1 + \hat{\omega}_2$. This is certainly the case at tree level since the coefficients are $\hat{\omega}_i$ independent. The three-body structure of the currents can only show up at the level of one-loop matching.

Using Eqs. (103) to determine the tree-level value of the NLO Wilson coefficients of the operators in Eqs. (71), (81), (86), (89), (97), (99), we find

$$\begin{aligned} B_1^{(s,p)} &= 1, & B_1^{(v,a)} &= 1, & B_{9,10}^{(v,a)} &= 0, & B_1^{(t)} &= 1, & B_{12-14}^{(t)} &= 0, \\ B_2^{(s,p)} &= 0, & B_{2-4}^{(v,a)} &= 0, & B_{11,12}^{(v,a)} &= 0, & B_{2-6}^{(t)} &= 0, & B_{15,16}^{(t)} &= 0, \\ B_3^{(s,p)} &= -2, & B_{5-7}^{(v,a)} &= 0, & B_{13}^{(v,a)} &= -1, & B_{7-10}^{(t)} &= 0, & B_{17}^{(t)} &= 1, \\ B_4^{(s,p)} &= 0, & B_8^{(v,a)} &= -2, & B_{14}^{(v,a)} &= 0, & B_{11}^{(t)} &= -2, & B_{18-21}^{(t)} &= 0. \end{aligned} \quad (104)$$

These results are in agreement with the RPI constraints in Eqs. (72), (82), (87), (90), (98). Coefficients in Eqs. (104) that are zero indicate that the corresponding currents vanish at tree level since they are first matched onto at one loop (or beyond). The full one-loop matching for all the $\mathcal{O}(\lambda)$ currents is not currently known from direct computations. However, many of the NLO coefficients are fixed in terms of the LO coefficients by RPI, namely $B_{0-3}^{(s,p)}(\hat{\omega})$, $B_{1-10}^{(v,a)}(\hat{\omega})$, and $B_{1-14}^{(t)}(\hat{\omega})$. Summarizing Eqs. (72), (82), (87), (90), (98) we have

$$\begin{aligned} B_1^{(s,p)} &= C_0^{(s,p)}, & B_2^{(s,p)} &= -2C_0^{(s,p)'}, & B_3^{(s,p)} &= -2C_0^{(s,p)}, \\ B_{1-3}^{(v,a)} &= C_{1-3}^{(v,a)}, & B_4^{(v,a)} &= -2C_3^{(v,a)}, & B_{5-7}^{(v,a)} &= -2C_{1-3}^{(v,a)'}, & B_{8-10}^{(v,a)} &= -2C_{1-3}^{(v,a)}, \\ B_{1-4}^{(t)} &= C_{1-4}^{(t)}, & B_5^{(t)} &= 2C_3^{(t)}, & B_6^{(t)} &= -2C_4^{(t)}, & B_{7-10}^{(t)} &= -2C_{1-4}^{(t)'}, \\ B_{11-14}^{(t)} &= -2C_{1-4}^{(t)}, \end{aligned} \quad (105)$$

where to save space the $\hat{\omega}$ and μ dependence of the expressions on both sides of these equalities is suppressed. These results can be used to determine the matching for these coefficients at $\mu = m_b$ using Eqs. (102). They also imply that the anomalous dimensions of these coefficients are determined by the anomalous dimension of the leading order coefficients [2], so their values for scales $m_b \Lambda_{\text{QCD}} < \mu^2 < m_b^2$ are known.

For the coefficients of the 3-body operators, $B_4^{(s,p)}(\hat{\omega}_1, \hat{\omega}_2)$, $B_{11-14}^{(v,a)}(\hat{\omega}_1, \hat{\omega}_2)$, and $B_{15-21}^{(t)}(\hat{\omega}_1, \hat{\omega}_2)$, neither the one-loop matching results, nor even the LO anomalous dimensions, are currently known.

Finally, we note that it is possible to relate the pseudo-scalar and axial-vector coefficients from the scalar and vector coefficients. For massless quarks the QCD diagrams and SCET diagrams change in a trivial way under the chiral transformation, $q \rightarrow \gamma_5 q$, and $\xi_n \rightarrow \gamma_5 \xi_n$, provided we work in a scheme such as NDR. Therefore in this scheme the Wilson coefficients of operators with and without γ_5 are related (see, for example, Ref. [2] for the relations between LO coefficients). In other renormalization schemes these coefficients may differ.

B. Summary of $\mathcal{O}(\lambda)$ currents in the frame $v_\perp = 0$, $n \cdot v = 1$

In Secs. IV A through IV D we have derived the most general basis of heavy-to-light current to $\mathcal{O}(\lambda)$ in an arbitrary frame. However, for applications it is often most convenient to pick a frame where $v_\perp = 0$ and $v \cdot n = 1$. In this frame the currents $K_{2,3}^{(s,p)}$, $K_{5-10}^{(v,a)}$, and $K_{7-14}^{(t)}$ drop out. Thus there are only (2,2,8,8,13) order $\mathcal{O}(\lambda)$ heavy-to-light currents which are (scalar, pseudo-scalar, vector, axial-vector, tensor). In this section we summarize our results with this choice of basis vectors.

In this frame our leading order results for the J_{hl} currents with a complete set of Dirac structures can be summarized as

$$J^{(0)} = \int d\omega C_i^{(d)}(\hat{\omega}) J_i^{(0)}(\omega),$$

$$J_i^{(0)}(\omega) = (\bar{\xi}_n W)_\omega \Gamma_i^{(d)} h_v, \quad (106)$$

where in $\Gamma_i^{(d)}$ the (d) specifies the type of current (scalar, vector, . . .), the i specifies the member of the complete set of possible structures of that type, and the Wilson coefficients are $C_i^{(d)}(\hat{\omega})$. For the minimum basis of Dirac structures we found

$$\begin{aligned}
 \text{Diagram 1: } J^{(1a)} &= -i B_i^{(d)}(\hat{n} \cdot \hat{p}) \frac{p_\alpha^\perp \Upsilon_i^{(d)\alpha}}{\hat{n} \cdot p} \\
 \text{Diagram 2: } J^{(1a)} &= -i B_i^{(d)}(\hat{n} \cdot (\hat{p} + \hat{q})) \frac{g T^\alpha}{\hat{n} \cdot (p+q)} \left[\Upsilon_i^{(d)\mu} + \frac{\hat{n}^\mu p_\alpha^\perp \Upsilon_i^{(d)\alpha}}{\hat{n} \cdot q} \right]
 \end{aligned}$$

FIG. 1. Feynman rules for the $\mathcal{O}(\lambda)$ currents $J^{(1a)}$ in Eqs. (108) with zero and one gluon (the fermion spinors are suppressed). For the collinear particles we show their (label, residual) momenta, where label momenta are $p, q \sim \lambda^{0,1}$ and residual momenta are $k, t \sim \lambda^2$. Momenta with a hat are normalized to m_b , $\hat{p} = p/m_b$, etc.

$$\begin{aligned}
 \Gamma_0^{(s)} &= 1, \quad \Gamma_0^{(p)} = \gamma_5, \\
 \Gamma_{\{1,2,3\}}^{(v)} &= \{\gamma_\mu, v_\mu, n_\mu\}, \\
 \Gamma_{\{1,2,3\}}^{(a)} &= \{\gamma_\mu \gamma_5, v_\mu \gamma_5, n_\mu \gamma_5\}, \\
 \Gamma_{\{1,2,3,4\}}^{(t)} &= \{i\sigma_{\mu\nu}, \gamma_{[\mu} v_{\nu]}, \gamma_{[\mu} n_{\nu]}, n_{[\mu} v_{\nu]}\},
 \end{aligned} \tag{107}$$

which is simply a linear combination of the basis in Ref. [2]. where

At order λ our corresponding results for J_{hl} in this frame can be summarized as

$$\begin{aligned}
 J^{(1a)} &= \int d\omega B_i^{(d)}(\hat{\omega}) J_i^{(1a)}(\omega), \\
 J^{(1b)} &= \int d\omega_1 d\omega_2 B_i^{(d)}(\hat{\omega}_1, \hat{\omega}_2) J_i^{(1b)}(\omega_1, \omega_2),
 \end{aligned} \tag{108}$$

$$J_i^{(1a)}(\omega) = (\bar{\xi}_n i \tilde{D}_{c\alpha}^\perp W)_\omega \frac{1}{\mathcal{P}^\dagger} Y_i^{(d)\alpha} h_\nu,$$

$$\begin{aligned}
 J_i^{(1b)}(\omega_1, \omega_2) &= \frac{1}{m_b} (\bar{\xi}_n W)_{\omega_1} \Theta_i^{(d)\alpha} \\
 &\times \left(\frac{1}{\mathcal{P}} W^\dagger i g B_{c\alpha}^\perp W \right)_{\omega_2} h_\nu,
 \end{aligned}$$

$$\begin{aligned}
 Y_1^{(s)\alpha} &= \gamma_\perp^\alpha \frac{\hbar}{2}, \quad \Theta_4^{(s)\alpha} = \gamma_\perp^\alpha, \quad Y_1^{(p)\alpha} = \gamma_\perp^\alpha \frac{\hbar}{2} \gamma_5, \quad \Theta_4^{(p)\alpha} = \gamma_5 \gamma_\perp^\alpha, \\
 Y_{1-4}^{(v)\alpha} &= \left\{ \gamma_\perp^\alpha \frac{\hbar}{2} \gamma^\mu, \gamma_\perp^\alpha \frac{\hbar}{2} v^\mu, \gamma_\perp^\alpha \frac{\hbar}{2} n^\mu, g_\perp^{\alpha\mu} \right\}, \quad \Theta_{11-14}^{(v)\alpha} = \{\gamma^\mu \gamma_\perp^\alpha, v^\mu \gamma_\perp^\alpha, n^\mu \gamma_\perp^\alpha, g_\perp^{\mu\alpha}\}, \\
 Y_{1-4}^{(a)\alpha} &= \left\{ \gamma_\perp^\alpha \frac{\hbar}{2} \gamma^\mu, \gamma_\perp^\alpha \frac{\hbar}{2} v^\mu, \gamma_\perp^\alpha \frac{\hbar}{2} n^\mu, g_\perp^{\alpha\mu} \right\} \gamma_5, \quad \Theta_{11-14}^{(a)\alpha} = \{\gamma^\mu \gamma_5 \gamma_\perp^\alpha, v^\mu \gamma_5 \gamma_\perp^\alpha, n^\mu \gamma_5 \gamma_\perp^\alpha, g_\perp^{\mu\alpha} \gamma_5\}, \\
 Y_{1-6}^{(t)\alpha} &= \left\{ i\gamma_\perp^\alpha \frac{\hbar}{2} \sigma^{\mu\nu}, \gamma_\perp^\alpha \frac{\hbar}{2} \gamma^{[\mu} v^{\nu]}, \gamma_\perp^\alpha \frac{\hbar}{2} \gamma^{[\mu} n^{\nu]}, \gamma_\perp^\alpha n^{[\mu} v^{\nu]}, g_\perp^{\alpha[\mu} \gamma^{\nu]} n^{\nu]}, g_\perp^{\alpha[\mu} v^{\nu]} \right\}, \\
 \Theta_{15-21}^{(t)\alpha} &= \{i\sigma^{\mu\nu} \gamma_\perp^\alpha, \gamma^{[\mu} v^{\nu]} \gamma_\perp^\alpha, \gamma^{[\mu} n^{\nu]} \gamma_\perp^\alpha, n^{[\mu} v^{\nu]} \gamma_\perp^\alpha, g_\perp^{\alpha[\mu} \gamma^{\nu]} n^{\nu]}, g_\perp^{\alpha[\mu} v^{\nu]} \gamma_\perp^\alpha, g_\perp^{\alpha[\mu} n^{\nu]} \gamma_\perp^\alpha\}.
 \end{aligned} \tag{109}$$

Note that due to Eq. (14) the form of $J^{(1b)}$ in Eqs. (108) is identical to the form of the currents that was used in Ref. [15] since $[W^\dagger i \tilde{D}_{c\alpha}^\perp W] = [1/\mathcal{P} W^\dagger i g B_{c\alpha}^\perp W]$.

C. Feynman rules for J_{hl} and $\mathcal{L}_{\xi q}$

In this section Feynman rules are given for the $\mathcal{O}(\lambda)$ heavy-to-light currents $J^{(1a)}$ and $J^{(1b)}$ in Eqs. (108) which are valid in a frame where $v_\perp = 0$ and $v \cdot n = 1$. We also give the Feynman rules that follow from the final form of the $\mathcal{L}_{\xi q}^{(1,2a,2b)}$ Lagrangians in Eqs. (42).

For the subleading currents the zero and one gluon Feynman rules for $J^{(1a)}$ and $J^{(1b)}$ are shown in Figs. 1 and 2, respectively. (From the results in the preceding sections the Feynman rules for the currents with $v_\perp \neq 0$ and $v \cdot n \neq 1$ can also be easily derived.) For $J^{(1a)}$ the Wilson coefficients de-

pend only on the total λ^0 collinear momentum, while for $J^{(1a)}$ the coefficients depend on how the momentum is divided between the quark and gluons. The $J^{(1a)}$ current has non-vanishing Feynman rules with zero or one A_n^\perp gluon and any number of $\bar{n} \cdot A_n$ gluons. The possible gluons that appear in the $J^{(1b)}$ currents are similar, but the current vanishes unless it has one or more collinear gluons present.

For the mixed usoft-collinear Lagrangians from Eqs. (42),

$$\begin{aligned}
 \mathcal{L}_{\xi q}^{(1)} &= \bar{\xi}_n \frac{1}{i \bar{n} \cdot D_c} i g B_c^\perp W q_{us} + \text{H.c.}, \\
 \mathcal{L}_{\xi q}^{(2a)} &= \bar{\xi}_n \frac{1}{i \bar{n} \cdot D_c} i g M W q_{us} + \text{H.c.},
 \end{aligned} \tag{110}$$

FIG. 2. Feynman rules for the $O(\lambda)$ currents $J^{(1b)}$ in Eqs. (108) with zero and one gluon. For the collinear particles we show their (label, residual) momenta, where label momenta are $p, q, q_i \sim \lambda^{0,1}$ and residual momenta are $k, t \sim \lambda^2$. Momenta with a hat are normalized to m_b , $\hat{p} = p/m_b$, etc.

$$\begin{aligned}
 \text{Diagram 1: } & \text{Collinear quark line with a gluon loop (spring) and a collinear gluon line.} & = 0 \\
 \text{Diagram 2: } & \text{Collinear quark line with a gluon loop (spring) and a collinear gluon line.} & = i B_i^{(d)} (\bar{n} \cdot \hat{p}, \bar{n} \cdot \hat{q}) \frac{g T^a}{m_b} \left[\Theta_i^{(d)\mu} - \frac{\bar{n}^\mu q_\alpha^\perp \Theta_i^{(d)\alpha}}{\bar{n} \cdot q} \right]
 \end{aligned}$$

FIG. 2. Feynman rules for the $O(\lambda)$ currents $J^{(1b)}$ in Eqs. (108) with zero and one gluon. For the collinear particles we show their (label, residual) momenta, where label momenta are $p, q, q_i \sim \lambda^{0,1}$ and residual momenta are $k, t \sim \lambda^2$. Momenta with a hat are normalized to m_b , $\hat{p} = p/m_b$, etc.

$$\mathcal{L}_{\xi q}^{(2b)} = \bar{\xi}_n \frac{\bar{n}}{2} i \mathcal{D}_\perp^c \frac{1}{(i \bar{n} \cdot D_c)^2} i g \mathcal{B}_\perp^c W_{qus} + \text{H.c.},$$

all Feynman rules involve at least one collinear gluon. From $\mathcal{L}_{\xi q}^{(1)}$ we obtain Feynman rules with zero or one A_n^\perp gluons and any number of $\bar{n} \cdot A_n$ gluons. The one- and two-gluon results are shown in Fig. 3.

For $\mathcal{L}_{\xi q}^{(2a)}$ we have Feynman rules with zero or one $\{n \cdot A_n, A_{us}^\perp\}$ gluon and any number of $\bar{n} \cdot A_n$ gluons. The one- and two-gluon results are shown in Fig. 4. Finally, for $\mathcal{L}_{\xi q}^{(2b)}$ one finds Feynman rules with zero, one, or two A_n^\perp gluons and any number of $\bar{n} \cdot A_n$ gluons. In this case the one- and two-gluon Feynman rules are shown in Fig. 5. Note that it is important to treat the contributions from $\mathcal{L}_{\xi q}^{(2a)}$ and $\mathcal{L}_{\xi q}^{(2b)}$ separately since they show up in different parts of the heavy-to-light factorization formulas derived in Ref. [15] and shown in Eq. (1).

For $\mathcal{L}_{\xi q}^{(2a)} + \mathcal{L}_{\xi q}^{(2b)}$ the Feynman rules are different than one would derive using the intermediate form Eqs. (30), since in transforming to the final form the equations of motion were applied. However, observable predictions that are consistently made with either set of Feynman rules will agree.

VI. LEADING POWER PREDICTIONS FOR B TO PSEUDOSCALAR MESONS

As a phenomenological example, we consider the form factors for $B \rightarrow \pi \ell \nu$, or more generally the form factors for

FIG. 3. Feynman rules for the subleading usoft-collinear Lagrangian $\mathcal{L}_{\xi q}^{(1)}$ with one and two collinear gluons (springs with lines through them). The solid lines are usoft quarks while dashed lines are collinear quarks. For the collinear particles we show their (label, residual) momenta. (The fermion spinors are suppressed.)

$$\begin{aligned}
 \text{Diagram 1: } & \text{Collinear quark line with a gluon loop (spring with line) and a collinear gluon line.} & = i g T^a \left[\gamma_\mu^\perp - \bar{n}_\mu \frac{\not{q}_\perp}{\bar{n} \cdot q} \right] \\
 \text{Diagram 2: } & \text{Collinear quark line with two gluon loops (springs with lines) and two collinear gluon lines.} & = i g^2 \frac{T^b T^a}{\bar{n} \cdot q_1} \left[\frac{\bar{n}_\mu \bar{n}_\nu \not{p}^\perp}{\bar{n} \cdot p} - \gamma_\nu^\perp \bar{n}_\mu \right] \\
 & + i g^2 \frac{T^a T^b}{\bar{n} \cdot q_2} \left[\frac{\bar{n}_\mu \bar{n}_\nu \not{p}^\perp}{\bar{n} \cdot p} - \gamma_\mu^\perp \bar{n}_\nu \right]
 \end{aligned}$$

FIG. 3. Feynman rules for the subleading usoft-collinear Lagrangian $\mathcal{L}_{\xi q}^{(1)}$ with one and two collinear gluons (springs with lines through them). The solid lines are usoft quarks while dashed lines are collinear quarks. For the collinear particles we show their (label, residual) momenta. (The fermion spinors are suppressed.)

FIG. 4. Feynman rules for the $O(\lambda^2)$ usoft-collinear Lagrangian $\mathcal{L}_{\xi q}^{(2a)}$ with one and two gluons. The spring without a line through it is an usoft gluon. For the collinear particles we show their (label, residual) momenta, where label momenta are $p, q, q_i \sim \lambda^{0,1}$ and residual momenta are $k, t, t_i \sim \lambda^2$. Note that with the field redefinition made in Ref. [36] the only change to these Feynman rules is that the γ_ν^\perp in the second line should be dropped.

$$\begin{aligned}
 \text{Diagram 1: } & \text{Collinear quark line with a gluon loop (spring) and a collinear gluon line.} & = i g T^a \left[\not{q}^\perp \left(n_\mu - \frac{\bar{n}_\mu n \cdot q}{\bar{n} \cdot q} \right) - \frac{\bar{n}_\mu \not{t}_\perp}{\bar{n} \cdot q} \right] \\
 \text{Diagram 2: } & \text{Collinear quark line with two gluon loops (springs) and two collinear gluon lines.} & = \frac{-g^2 f^{abc} T^c}{\bar{n} \cdot q} \bar{n}_\mu \left(\frac{\not{q}^\perp}{2} n_\nu + \gamma_\nu^\perp \right) \\
 \text{Diagram 3: } & \text{Collinear quark line with two gluon loops (springs) and two collinear gluon lines.} & = i g^2 \frac{T^a T^b}{\bar{n} \cdot q_2} \left[-n_\mu \bar{n}_\nu \frac{\not{q}^\perp}{2} + \bar{n}_\mu \bar{n}_\nu \frac{(\not{t}_1 + \not{t}_2)}{\bar{n} \cdot p} \right] \\
 \text{Diagram 4: } & \text{Collinear quark line with two gluon loops (springs) and two collinear gluon lines.} & = i g^2 \frac{T^b T^a}{\bar{n} \cdot q_1} \left[-n_\nu \bar{n}_\mu \frac{\not{q}^\perp}{2} + \bar{n}_\mu \bar{n}_\nu \frac{(\not{t}_1 + \not{t}_2)}{\bar{n} \cdot p} \right]
 \end{aligned}$$

FIG. 4. Feynman rules for the $O(\lambda^2)$ usoft-collinear Lagrangian $\mathcal{L}_{\xi q}^{(2a)}$ with one and two gluons. The spring without a line through it is an usoft gluon. For the collinear particles we show their (label, residual) momenta, where label momenta are $p, q, q_i \sim \lambda^{0,1}$ and residual momenta are $k, t, t_i \sim \lambda^2$. Note that with the field redefinition made in Ref. [36] the only change to these Feynman rules is that the γ_ν^\perp in the second line should be dropped.

$B \rightarrow P$ where P is a pseudoscalar meson (calculations using the factorization theorem in Eq. (1) for the vector meson are just as straightforward). For pseudoscalars there are three form factors in QCD, which are conventionally defined by

$$\begin{aligned}
 \langle P(p) | \bar{q} \gamma^\mu b | \bar{B}(p_b) \rangle &= f_+(q^2) \left[p_b^\mu + p^\mu - \frac{m_B^2 - m_P^2}{q^2} q^\mu \right] \\
 &+ f_0(q^2) \frac{m_B^2 - m_P^2}{q^2} q^\mu, \\
 \langle P(p) | \bar{q} i \sigma^{\mu\nu} q_\nu b | \bar{B}(p_b) \rangle &= -\frac{f_T(q^2)}{m_B + m_P} [q^2 (p_b^\mu + p^\mu) \\
 &- (m_B^2 - m_P^2) q^\mu], \quad (111)
 \end{aligned}$$

where $q = p_b - p$.

For the region where $Q = \{E, m_b\} \gg \Lambda_{\text{QCD}}$ (i.e. small q^2) one can use large energy factorization to study the form factors. For pions our expansion parameter $\Lambda_{\text{QCD}}/\bar{n} \cdot p \sim 0.5 \text{ GeV}/(2E)$ becomes 1/4 for $E \approx 1 \text{ GeV}$. This makes the region of q^2 where the expansion is valid roughly $0 \leq q^2 \leq 10 \text{ GeV}^2$. In SCET the form factors f_+, f_0, f_T split themselves up into contributions associated with three mo-

FIG. 5. Feynman rules for the $O(\lambda^2)$ usoft-collinear Lagrangian $\mathcal{L}_{\xi q}^{(2b)}$ with one and two gluons. For the collinear particles we show their (label, residual) momenta, where label momenta are $p, q, q_i \sim \lambda^{0,1}$ and residual momenta are $k, t, t_i \sim \lambda^2$.

$$\begin{aligned}
 \text{Diagram 1: } & \text{Collinear quark line with a gluon loop (spring) and a collinear gluon line.} & = i g \frac{T^a}{\bar{n} \cdot q} \frac{\not{q}^\perp}{2} \left[\not{q}_\perp \gamma_\mu^\perp - \bar{n}_\mu \frac{q_\perp^2}{\bar{n} \cdot q} \right] \\
 \text{Diagram 2: } & \text{Collinear quark line with two gluon loops (springs) and two collinear gluon lines.} & = i g^2 \frac{T^a T^b}{\bar{n} \cdot q_2} \left[\gamma_\mu^\perp \gamma_\nu^\perp - \frac{\not{p}_\perp}{\bar{n} \cdot p} (\gamma_\mu^\perp \bar{n}_\nu + \gamma_\nu^\perp \bar{n}_\mu) - \frac{\gamma_\mu^\perp \bar{n}_\nu \not{t}_{2\perp}}{\bar{n} \cdot q_2} \right. \\
 & \left. + \bar{n}_\mu \bar{n}_\nu \left(\frac{\not{p}_\perp^2}{(\bar{n} \cdot p)^2} + \frac{\not{p}_\perp \not{t}_{2\perp}}{\bar{n} \cdot p \bar{n} \cdot q_2} \right) \right] + [(a, \mu, q_1, t_1) \leftrightarrow (b, \nu, q_2, t_2)]
 \end{aligned}$$

FIG. 5. Feynman rules for the $O(\lambda^2)$ usoft-collinear Lagrangian $\mathcal{L}_{\xi q}^{(2b)}$ with one and two gluons. For the collinear particles we show their (label, residual) momenta, where label momenta are $p, q, q_i \sim \lambda^{0,1}$ and residual momenta are $k, t, t_i \sim \lambda^2$.

momentum regions: Wilson coefficients for $p^2 \sim Q^2$, two jet functions $J_{a,b}$ for $p^2 \sim Q\Lambda_{\text{QCD}}$, universal light-cone wave functions for $p^2 \sim \Lambda_{\text{QCD}}^2$, and a single non-factorizable form factor ζ_P (containing both $p^2 \sim Q\Lambda_{\text{QCD}}$ and $p^2 \sim \Lambda_{\text{QCD}}^2$). The leading contributions therefore split into factorizable (F) and non-factorizable (NF) contributions.¹⁰ This decomposition was defined by the proof of a factorization formula for these form factors in Ref. [15]

$$f^{QCD}(q^2) = f^F(Q) + f^{\text{NF}}(Q) + \dots,$$

$$f^F(Q) = N_0 \int_0^1 dz \int_0^1 dx \int_0^\infty dr_+ T(z, Q, \mu_0) \\ \times J(z, x, r_+, Q, \mu_0, \mu) \phi_P(x, \mu) \phi_B(r_+, \mu), \quad (112)$$

$$f^{\text{NF}}(Q) = C_k(Q, \mu) \zeta_k^P(Q\Lambda, \mu),$$

where $f^F(Q) \sim f^{\text{NF}}(Q) \sim Q^{-3/2}$ and the ellipses denote terms that are suppressed by more powers of $1/Q$. Here $\phi_B = \phi_B^\pm$.

To separate the scales Q^2 and $Q\Lambda$ we match QCD onto a SCET_I. The scales $Q\Lambda$ and Λ^2 are then separated by matching SCET_I onto a SCET_{II} [15]. Operators in SCET_I are divided into F and NF categories depending on the form of the result of factoring usoft gluons from collinear fields. In SCET_I the F contributions are from the time-ordered products

$$T_1^F = \int d^4x T\{J^{(1a)}(0), i\mathcal{L}_{\xi q}^{(1)}(x)\}, \\ T_2^F = \int d^4x T\{J^{(1b)}(0), i\mathcal{L}_{\xi q}^{(1)}(x)\}, \\ T_3^F = \int d^4x T\{J^{(0)}(0), i\mathcal{L}_{\xi q}^{(2b)}(x)\}, \quad (113)$$

where the currents are taken from Eqs. (108) and the usoft-collinear Lagrangians from Eqs. (110). After factorization of usoft and collinear fields, the T_i^F 's are matched onto soft-collinear SCET_{II} operators. The collinear matrix elements are given in terms of jet functions $J(z, x, r_+, Q)$, and the soft operators are given in terms of B light-cone wave functions $\phi_\pm(r_+)$ defined as [19,22]

$$\langle 0 | \bar{q}(x^-) S_n(x^-, 0) \Gamma h_v(0) | \bar{B}(v) \rangle \\ = -\frac{i}{2} f_B m_B \int dr_+ e^{-(i/2)r_+ x^-} \text{Tr} \left\{ \frac{1+\not{v}}{2} \left[\frac{\not{h}\not{\bar{h}}}{4} \phi_B^+(r_+) \right. \right. \\ \left. \left. + \frac{\not{h}\not{h}}{4} \phi_B^-(r_+) \right] \gamma_5 \Gamma \right\}. \quad (114)$$

¹⁰Here the phrase non-factorizable simply refers to the fact that the matrix elements cannot be expressed in terms of convolutions with the standard light-cone wave functions.

A few general properties of the factorizable term $f^F(Q)$ can be given without an explicit computation. First, the matrix elements of $T_{1,2}^F$ can only depend on $\phi_B^+(r_+)$. This follows from the explicit form of the subleading Lagrangian $\mathcal{L}_{\xi q}^{(1)}$, where $\xi_n = (\not{h}\not{\bar{h}}/4)\xi_n$, so the usoft field q_{us} appears only in the combination $\bar{q}_{us}(\not{h}\not{\bar{h}}/4)$. Using Eq. (114) this implies that only the $\phi_B^+(r_+)$ term gives a nonvanishing contribution. On the other hand, the factorizable operator T_3^F depends on the combination $\bar{q}_{us}(\not{h}/2)$, so its matrix element can only contain $\phi_B^-(r_+)$. At tree level the jet function J from the matrix element of T_3^F vanishes, but a nonzero result could appear at one-loop order. However, the matrix element of T_3^F contains the leading order current $J^{(0)}$, so it obeys the same symmetry relations as those derived for the nonfactorizable part $f^{\text{NF}}(Q)$ [15]. Therefore, although this matrix element is factorizable it does not increase the number of unknown non-perturbative functions since for phenomenological analyses ϕ_B^- can be absorbed in ζ_k^M . With this choice, all remaining factorizable contributions are expressible in terms of just $\phi_B^+(r_+)$.

Using the approach explained in [15] we can obtain the results for the form factors. After factorization of usoft and collinear fields the T-products of collinear fields coming from $T_{1,2}^F$ are given by [using Eqs. (108) and (110)]

$$\mathcal{J}_\omega^{1a}(x) \equiv T[\bar{\xi}_n i \bar{D}_{c\alpha}^\perp W]_\omega^{iA}(0) \\ \times \left[W^\dagger i g \not{B}_\perp^c W \frac{1}{\not{P}^\dagger} W^\dagger \xi_n \right]_0^{jB}(x), \\ \mathcal{J}_{\omega_1, \omega_2}^{1b}(x) \equiv T \left[[\bar{\xi}_n W]_{\omega_1} \left[\frac{1}{\not{P}} W^\dagger i g \not{B}_\perp^c W \right]_{\omega_2} \right]^{iA}(0) \\ \times \left[W^\dagger i g \not{B}_\perp^c W \frac{1}{\not{P}^\dagger} W^\dagger \xi_n \right]_0^{jB}(x), \quad (115)$$

where i, j are Dirac indices and here A, B are color indices in the fundamental representation. The functions $\mathcal{J}_{1a, 1b}$ are collinear gauge invariant and satisfy the spin structure constraints $\not{h}\mathcal{J}_{1a, 1b} = \mathcal{J}_{1a, 1b}\not{h} = 0$, and $\text{tr}[\mathcal{J}_{1a, 1b}] = 0$.

Taking into account constraints from the Dirac structure of the effective theory fields one can easily find the most general form of the operators appearing in the matching of $\mathcal{J}^{1a, 1b}$ onto operators in SCET_{II}. The jet functions $J_{a,b}$ are defined by the terms which contribute on a pseudoscalar state

$$\mathcal{J}_\omega^{1a}(x) = i \delta(x_+) \delta^2(x_\perp) [\gamma_\perp^\alpha \not{h} \gamma_5]^{ji} \delta^{AB} \frac{1}{\omega} \\ \times \int d\bar{\eta} \int \frac{dr_+}{2\pi} e^{-(i/2)r_+ x^-} J_a(\bar{\eta}, r_+) \\ \times \left[(\bar{\xi}_n W) \delta(\bar{\eta} - \bar{P}_+) \frac{\not{h}}{2} \gamma_5 (W^\dagger \xi_n) \right]_{\text{II}} + \dots, \quad (116)$$

$$\begin{aligned}
\mathcal{J}_{\omega_1, \omega_2}^{1b}(x) &= i \delta(x_+) \delta^2(x_\perp) [\gamma_\perp^\alpha \not{h} \gamma_5]^{ji} \delta^{AB} \frac{1}{\omega_1 + \omega_2} \\
&\times \int d\bar{\eta} \int \frac{dr^+}{2\pi} e^{-(i/2)r^+ x^-} J_b(\bar{\omega}, \bar{\eta}, r^+) \\
&\times \left[(\bar{\xi}_n W) \delta(\bar{\eta} - \bar{P}_+) \frac{\not{h}}{2} \gamma_5 (W^\dagger \xi_n) \right]_{\text{II}} + \dots,
\end{aligned} \tag{117}$$

where $\bar{\omega} = \omega_1 - \omega_2$. We have suppressed the dependence of $J_{1a,1b}$ on the μ 's and on ω , $\omega_1 + \omega_2$ (the latter combinations would be simply set to $\bar{n} \cdot p$ in the pseudoscalar matrix element by momentum conservation [3]). The ellipsis in Eq. (116) denotes color octet terms and other operators which do not contribute for a pseudoscalar meson P .

Using Eq. (116) the operators in $\langle P_n(p) | T_{1,2}^F | \bar{B}_v \rangle$ factor into a product of matrix elements that can be evaluated with Eq. (114) and Eqs. (12), (13) of Ref. [7]. Switching variables to x, z by using $\bar{\omega} = (2x-1)\bar{n} \cdot p$ and $\bar{\eta} = (2z-1)\bar{n} \cdot p$ we find the following factorization theorems which are valid at leading order¹¹ in $1/Q$ and all orders in α_s :

$$\begin{aligned}
f_+(q^2) &= N_0 \int_0^1 dx \int_0^\infty dr^+ \left[\frac{2E - m_B}{m_B} T_a^{(+)}(E, \mu_0) \right. \\
&\times J_a(x, r_+, Q, \mu_0, \mu) + \frac{2E}{m_b} \int_0^1 dz T_b^{(+)}(E, z, \mu_0) \\
&\times J_b(z, x, r_+, Q, \mu_0, \mu) \left. \right] \phi_P(x, \mu) \phi_B^+(r_+, \mu) \\
&+ \left\{ C_1^{(v)}(2\hat{E}, \mu_0) + \frac{E}{m_B} C_2^{(v)}(2\hat{E}, \mu_0) \right. \\
&\left. + C_3^{(v)}(2\hat{E}, \mu_0) \right\} \zeta^P(Q\Lambda, \mu_0), \\
f_0(q^2) &= N_0 \int_0^1 dx \int_0^\infty dr^+ \left[\frac{2E(m_B - 2E)}{m_B^2} T_a^{(0)}(E, \mu_0) \right. \\
&\times J_a(x, r_+, Q, \mu_0, \mu) + \frac{4E^2}{m_b m_B} \\
&\times \left. \int_0^1 dz T_b^{(0)}(E, z, \mu_0) J_b(z, x, r_+, Q, \mu_0, \mu) \right] \\
&\times \phi_P(x, \mu) \phi_B^+(r_+, \mu) + \frac{2E}{m_B} \left\{ C_1^{(v)}(2\hat{E}, \mu_0) \right. \\
&+ \frac{m_B - E}{m_B} C_2^{(v)}(2\hat{E}, \mu_0) + C_3^{(v)}(2\hat{E}, \mu_0) \left. \right\} \\
&\times \zeta^P(Q\Lambda, \mu_0),
\end{aligned} \tag{118}$$

$$\begin{aligned}
f_T(q^2) &= N_0 \frac{m_B + m_P}{m_B} \int_0^1 dx \int_0^\infty dr^+ \left[-T_a^{(T)}(E, \mu_0) \right. \\
&\times J_a(x, r_+, Q, \mu_0, \mu) - \frac{2E}{m_b} \int_0^1 dz T_b^{(T)}(E, z, \mu_0) \\
&\times J_b(z, x, r_+, Q, \mu_0, \mu) \left. \right] \phi_P(x, \mu) \phi_B^+(r_+, \mu) \\
&+ \frac{m_B + m_P}{m_B} \{ C_1^{(t)}(2\hat{E}, \mu_0) - C_2^{(t)}(2\hat{E}, \mu_0) \\
&- C_4^{(t)}(2\hat{E}, \mu_0) \} \zeta^P(Q\Lambda, \mu_0),
\end{aligned}$$

where $\hat{E} = E/m_b$, $Q = \{E, m_b\}$, and the normalization coefficient is given by $N_0 = f_B f_P m_B / (4E^2)$. The matrix element involving non-factorizable operators gives $\zeta^P(Q\Lambda, \mu)$ which is the reduced form factor describing decays to a pseudoscalar meson P . The quantities in square brackets and curly brackets are calculable, that is the $T_{a,b}$'s and $J_{a,b}$'s have expansions in $\alpha_s(Q)$ and $\alpha_s(\sqrt{Q\Lambda})$, respectively. Note that the $J_{a,b}$ are universal, meaning that at any order in α_s it is these same jet functions which appear for any pseudoscalar meson and independent of which form factor $f_{+,0,T}$ we consider. Therefore, the factorization theorem still gives information even in the case where we assume that $\alpha_s(\sqrt{Q\Lambda})$ is non-perturbative.

Working at $\mathcal{O}(\alpha_s(\mu_0))$ (i.e. tree level) for the jet functions gives

$$J_{1a}(x, r^+) = \frac{\pi C_F}{N_c} \frac{\alpha_s(\mu_0)}{x r^+}, \tag{119}$$

$$J_{1b}(z, x, r^+) = \frac{\pi C_F}{N_c} \frac{\alpha_s(\mu_0)}{x r^+} \delta(z-x). \tag{120}$$

At this level the z integrals in Eqs. (118) disappear because the tree level jet gives a $\delta(z-x)$, and this causes the z variable in the T_b 's to be replaced by x . The $T_{a,b}^{(j)}$ are combinations of Wilson coefficients appearing in the $J^{(1a,1b)}$ currents given in Eqs. (108) and should be evaluated at a scale $\mu_0^2 \sim Q\Lambda$. Expressed in terms of the Wilson coefficients defined in Sec. IV, they are given by

$$T_a^{(+)}(E, \mu) = B_1^{(v)}(2\hat{E}, \mu) + \frac{EB_2^{(v)}(2\hat{E}, \mu) + m_B B_3^{(v)}(2\hat{E}, \mu)}{2E - m_B},$$

$$T_a^{(0)}(E, \mu) = B_1^{(v)}(2\hat{E}, \mu)$$

$$+ \frac{(m_B - E)B_2^{(v)}(2\hat{E}, \mu) + m_B B_3^{(v)}(2\hat{E}, \mu)}{m_B - 2E},$$

$$\tag{121}$$

¹¹We kept a kinematic factor of m_P in the prefactor of f_T even though it is formally power suppressed.

$$T_a^{(T)}(E, \mu) = B_1^{(t)}(2\hat{E}, \mu) - B_2^{(t)}(2\hat{E}, \mu) - 2B_3^{(t)}(2\hat{E}, \mu) \\ + B_4^{(t)}(2\hat{E}, \mu),$$

where the dependence on $\hat{\omega} = 2\hat{E}$ is shown, and

$$T_b^{(+)}(E, z, \mu) = B_{11}^{(v)}(2\hat{E}, z, \mu) - \frac{E}{m_B} B_{12}^{(v)}(2\hat{E}, z, \mu) \\ - B_{13}^{(v)}(2\hat{E}, z, \mu), \\ T_b^{(0)}(E, z, \mu) = B_{11}^{(v)}(2\hat{E}, z, \mu) - \frac{(m_B - E)}{m_B} \\ \times B_{12}^{(v)}(2\hat{E}, z, \mu) - B_{13}^{(v)}(2\hat{E}, z, \mu), \quad (122)$$

$$T_b^{(T)}(E, z, \mu) = B_{15}^{(t)}(2\hat{E}, z, \mu) + B_{16}^{(t)}(2\hat{E}, z, \mu) \\ - B_{18}^{(t)}(2\hat{E}, z, \mu),$$

where $\hat{\omega}_1 + \hat{\omega}_2 = 2\hat{E}$ and the dependence on z is induced from the $\hat{\omega}_1 - \hat{\omega}_2$ dependence of the coefficients.

If we work at tree level in $J_{a,b}$ using Eq. (119) and also in $T_{a,b}$ then these coefficients are scale independent and satisfy $T_a^{(+,0,T)} = T_b^{(+,0)} = 1$ and $T_b^{(T)} = 0$. In this case if we take the ratios f_0/f_+ and f_T/f_+ and expand assuming that the f^F terms are smaller than the f^{NF} terms then our results agree with Ref. [19]. We note, using just the information in our factorization theorem, that it is not clear whether one wants to expand in this way since the F and NF terms could actually be similar in size as discussed in the Introduction. The expectation from QCD sum rules is that the ‘‘soft’’ NF part of the form factors is larger than the ‘‘hard’’ F part [20].

VII. CONCLUSION

The soft-collinear effective theory (SCET) allows a rich structure of allowed operators at higher orders in the expansion parameter λ . In contrast with simpler effective theories, the presence of fields ($\bar{n} \cdot A_n$) and derivatives ($\bar{n} \cdot iD_c$) scaling like λ^0 allows a continuum set of operators at any given order in λ . A similar situation is encountered in deep inelastic scattering, where an infinite number of operators of increasing dimension can contribute to the same order in $1/Q$. In a generic process with energetic hadrons it is therefore important to have a well-defined procedure for organizing the structure of the soft-collinear operators at a given order in λ . This organization is provided by SCET.

In this paper we formulated a general prescription for constructing the most general ultrasoft-collinear operators appearing in the Lagrangian or in the matching of an external

current at a given order in λ but to all orders in α_s . This was done by including constraints from collinear gauge invariance, the Dirac structure of the effective theory fields and reparametrization invariance. These conditions prove to be surprisingly predictive, and constrain not only the number of allowed operators, but also their functional dependence on label momenta.

For the case of the heavy-light currents, the constraints from the Dirac structure of the effective theory fields have been included at leading order in [2], and allow only 3 structures in the $v_\perp = 0$ frame. Here we consider the more general case of an arbitrary heavy quark velocity v , which is necessary in order to have a set of operators which closes under reparametrization transformations.

At subleading order $O(\lambda)$ the Dirac constraints alone allow many more operators. In particular, in addition to two-body operators $(\bar{\xi}_n W)_{\omega_1} \cdot \cdot \cdot h_v$, one has to include also three-body currents of the form $(\bar{\xi}_n W)_{\omega_1} \cdot \cdot \cdot (W^\dagger iDW)_{\omega_2} \cdot \cdot \cdot h_v$. RPI constraints on a subset of the two body-operators were previously considered in [10,13], and it was shown their coefficients are fixed in terms of the coefficients of leading order currents. Here we extended the constraints to the full set of allowed two-body and three-body operators, and showed that type (II) RPI imposed severe constraints on the (ω_1, ω_2) dependence of the latter. For example, the scalar current $\bar{q}b$ is matched at $O(\lambda)$ onto 8 general operators in the effective theory. After imposing all constraints, only one of these has a free Wilson coefficient, which has to be determined from a matching calculation. A similar reduction is obtained for the more complicated case of the vector/axial and tensor currents, for which one can write (28), (44) structures but only (4), (7) Wilson coefficients are not fixed by the symmetries of the effective theory.

In this paper we have focused on mixed usoft-collinear interactions, however for many exclusive heavy-to-light processes the final operators that are needed are of soft-collinear type as was the case for heavy-to-light form factors. In practice it appears simplest to derive collinear-soft interactions from the collinear-usoft ones using the two-stage matching technique, $\text{QCD} \rightarrow \text{SCET}_I \rightarrow \text{SCET}_{II}$, discussed in the proof of factorization for heavy-to-light decays Ref. [15]. The operators in this paper describe interactions in the intermediate SCET_I theory. For exclusive processes such as $B \rightarrow D\pi$ [7] where the intermediate $p^2 \sim Q\Lambda$ fluctuations in SCET_I are responsible for inducing simple operators in SCET_{II} the procedure used in Ref. [15] reduces to the one discussed in Ref. [4].

Note added. In the final stages of this work, Ref. [17] appeared where a direct study of light-light soft-collinear operators was performed. The Wilson coefficients of these operators were determined by matching from QCD up to one-loop order, and both two-body and three-body operators were found to contribute. An intermediate theory with modes of momentum $p^\mu \sim Q(\lambda', 1, \lambda')$ as dynamical degrees of freedom was also considered. This appears similar in spirit to the $\text{QCD} \rightarrow \text{SCET}_I \rightarrow \text{SCET}_{II}$ construction used in Ref. [15]; however, in the intermediate SCET_I theory we found that the

dynamical collinear modes should have momentum scaling as $p^\mu \sim Q(\lambda', 1, \sqrt{\lambda'})$. Finally, reparametrization invariance constraints on soft-collinear operators were also discussed in Ref. [17], and were shown to constrain the form of certain Wilson coefficients.

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- [1] C.W. Bauer, S. Fleming, and M. Luke, Phys. Rev. D **63**, 014006 (2001).
- [2] C.W. Bauer, S. Fleming, D. Pirjol, and I.W. Stewart, Phys. Rev. D **63**, 114020 (2001).
- [3] C.W. Bauer and I.W. Stewart, Phys. Lett. B **516**, 134 (2001).
- [4] C.W. Bauer, D. Pirjol, and I.W. Stewart, Phys. Rev. D **65**, 054022 (2002).
- [5] For reviews R.L. Jaffe, hep-ph/9602236; G. Sterman, TASI lectures 1995, hep-ph/9606312; J.C. Collins, D.E. Soper, and G. Sterman, in *Perturbative Quantum Chromodynamics*, edited by A. H. Mueller (World Scientific, Singapore, 1989), pp. 1–93.
- [6] S. J. Brodsky and G. P. Lepage, in *Perturbative Quantum Chromodynamics*, pp. 93–240; S.J. Brodsky, SLAC report SLAC-PUB-9281, hep-ph/0208158.
- [7] C.W. Bauer, D. Pirjol, and I.W. Stewart, Phys. Rev. Lett. **87**, 201806 (2001).
- [8] C.W. Bauer, S. Fleming, D. Pirjol, I.Z. Rothstein, and I.W. Stewart, Phys. Rev. D **66**, 014017 (2002).
- [9] C.W. Bauer, C.W. Chiang, S. Fleming, A.K. Leibovich, and I. Low, Phys. Rev. D **64**, 114014 (2001); S. Descotes-Genon and C.T. Sachrajda, Nucl. Phys. **B650**, 356 (2003); E. Lunghi, D. Pirjol, and D. Wyler, *ibid.* **B649**, 349 (2003).
- [10] J. Chay and C. Kim, Phys. Rev. D **65**, 114016 (2002).
- [11] A.V. Manohar, T. Mehen, D. Pirjol, and I.W. Stewart, Phys. Lett. B **539**, 59 (2002).
- [12] C.W. Bauer, D. Pirjol, and I.W. Stewart, Phys. Rev. D **66**, 054005 (2002).
- [13] M. Beneke, A.P. Chapovsky, M. Diehl, and T. Feldmann, Nucl. Phys. **B643**, 431 (2002).
- [14] T. Feldmann, hep-ph/0209239.
- [15] C.W. Bauer, D. Pirjol, and I.W. Stewart, Phys. Rev. D **67**, 071502(R) (2003).
- [16] R.K. Ellis, W. Furmansky, and R. Petronzio, Nucl. Phys. **B207**, 1 (1982); **B212**, 29 (1983); R.L. Jaffe and M. Soldate, Phys. Rev. D **26**, 49 (1982); A.V. Efremov and O.V. Teryaev, Yad. Fiz. **39**, 1517 (1984); I.V. Anikin and O.V. Teryaev, Phys. Lett. B **509**, 95 (2001).
- [17] R.J. Hill and M. Neubert, hep-ph/0211018.
- [18] A. Szczepaniak, E.M. Henley, and S.J. Brodsky, Phys. Lett. B **243**, 287 (1990); G. Burdman and J.F. Donoghue, *ibid.* **270**, 55 (1991); R. Akhoury, G. Sterman, and Y.P. Yao, Phys. Rev. D **50**, 358 (1994); H.n. Li and H.L. Yu, *ibid.* **53**, 2480 (1996); H.n. Li, *ibid.* **66**, 094010 (2002); **64**, 014019 (2001); T. Kurimoto, H.n. Li, and A.I. Sanda, *ibid.* **65**, 014007 (2002); S. Descotes-Genon and C.T. Sachrajda, Nucl. Phys. **B625**, 239 (2002).
- [19] M. Beneke and T. Feldmann, Nucl. Phys. **B592**, 3 (2001).
- [20] A. Khodjamirian, R. Ruckl, S. Weinzierl, and O.I. Yakovlev, Phys. Lett. B **410**, 275 (1997); E. Bagan, P. Ball, and V.M. Braun, *ibid.* **417**, 154 (1998); P. Ball, V.M. Braun, Y. Koike, and K. Tanaka, Nucl. Phys. **B529**, 323 (1998); P. Ball and V.M. Braun, Phys. Rev. D **58**, 094016 (1998); A. Khodjamirian, R. Ruckl, S. Weinzierl, C.W. Winhart, and O.I. Yakovlev, *ibid.* **62**, 114002 (2000); P. Ball, J. High Energy Phys. **09**, 005 (1998); P. Ball and R. Zwicky, *ibid.* **10**, 019 (2001).
- [21] V.L. Chernyak and I.R. Zhitnitsky, Nucl. Phys. **B345**, 137 (1990).
- [22] A.G. Grozin and M. Neubert, Phys. Rev. D **55**, 272 (1997).
- [23] J. Charles *et al.* Phys. Rev. D **60**, 014001 (1999).
- [24] M.J. Dugan and B. Grinstein, Phys. Lett. B **255**, 583 (1991).
- [25] U. Aglietti, G. Corbo, and L. Trentadue, Int. J. Mod. Phys. A **14**, 1769 (1999); U. Aglietti and G. Corbo, *ibid.* **15**, 363 (2000); C. Balzereit, T. Mannel, and W. Kilian, Phys. Rev. D **58**, 114029 (1998); see also the structure of interactions in Light-Cone QCD, for a recent review see S. Brodsky, hep-ph/0208158.
- [26] D. Politzer and M. Wise, Phys. Lett. B **257**, 399 (1991).
- [27] J.B. Kogut and D.E. Soper, Phys. Rev. D **1**, 2901 (1970); J.D. Bjorken, J.B. Kogut, and D.E. Soper, *ibid.* **3**, 1382 (1971); H. Leutwyler, Nucl. Phys. **B76**, 413 (1974).
- [28] M.E. Luke and A.V. Manohar, Phys. Lett. B **286**, 348 (1992).
- [29] A.V. Manohar and M.B. Wise, Cambridge Monogr. Part. Phys., Nucl. Phys., Cosmol. **10**, 1 (2000).
- [30] Y.Q. Chen, Phys. Lett. B **317**, 421 (1993); M. Finkemeier, H. Georgi, and M. McIrvin, Phys. Rev. D **55**, 6933 (1997); A.V. Manohar, *ibid.* **56**, 230 (1997); R. Sundrum, *ibid.* **57**, 331 (1998); F. Campanario and T. Mannel, *ibid.* **65**, 094017 (2002).
- [31] M. Neubert, Phys. Lett. B **306**, 357 (1993).
- [32] M.E. Luke, A.V. Manohar, and I.Z. Rothstein, Phys. Rev. D **61**, 074025 (2000).
- [33] J. Chay and C. Kim, hep-ph/0205117.
- [34] I.W. Stewart, hep-ph/0207253.
- [35] G.P. Korchemsky and G. Marchesini, Nucl. Phys. **B406**, 225 (1993); G.P. Korchemsky and A.V. Radyushkin, *ibid.* **B283**, 342 (1987).
- [36] C.W. Bauer, D. Pirjol, and I.W. Stewart, hep-ph/0303156.