Harmonic oscillator with minimal length uncertainty relations and ladder operators

Ivan Dadić,* Larisa Jonke,† and Stjepan Meljanac‡
Theoretical Physics Division, Rudjer Bošković Institute, P. O. Box 180, HR-10002 Zagreb, Croatia
(Received 28 October 2002; published 3 April 2003)

We construct creation and annihilation operators for deformed harmonic oscillators with minimal length uncertainty relations. We discuss a possible generalization to a large class of deformations of canonical commutation relations. We also discuss the dynamical symmetry of a noncommutative harmonic oscillator.

DOI: 10.1103/PhysRevD.67.087701 PACS number(s): 03.65.−w, 02.40.Gh

I. INTRODUCTION

The existence of a minimal length, at least at the Planck scale, seems to be a general feature of any quantum theory of gravity. Test particles of sufficiently high energy for probing small scales curve gravitationally, and thereby disturb the very space-time they are probing. Both perturbative string theory considerations [1] and black hole physics [2] give rise to modified space-momentum uncertainty relations[1] that imply the existence of a minimal length. The investigation of the cosmological consequences of these modified space-momentum uncertainty relations has been intensified lately. It appears that minimal length uncertainty relations (MLUR) can offer some answers to the problem of black hole remnants [4], the trans-Planckian problem of inflation [5], and the cosmological constant problem [6]. On the other hand, one can discuss MLUR in the context of the deformation of quantum mechanics, since the uncertainty relations and the underlying canonical commutation relations are at the heart of quantum mechanics. The generalized quantum theoretical framework which implements the appearance of MLUR was discussed in Ref. [7], and the formalism obtained was applied to the harmonic oscillator case. Recently, the exact solution for the harmonic oscillator in arbitrary dimensions with MLUR has been found [8].

In one spatial dimension, the generalized commutation relation can be written as $[x,p]=if(x,p)$, with x and p Hermitian operators and $[f(x,p)]^\dagger=f(x,p)$. Then, in any physical state one finds $\Delta x\Delta p \geqslant \frac{1}{2}|\overline{f(x,p)}|$, where we define for any operator O, $\overline{O}=\langle\Psi|O|\Psi\rangle, \Delta O=\sqrt{\langle\Psi|(O-\overline{O})^2|\Psi\rangle}$. The operator function $f(x,p)$ can be treated as a smooth deformation of ordinary quantum mechanics with $f_0(x,p)=\hbar$. Note that the limit to classical mechanics, $f(x,p)\rightarrow 0$, is not smooth. In this paper we restrict ourselves to smooth deformations of quantum mechanics only.

A number of physical problems can be expressed as a deformed harmonic oscillator with generalized commutation relations, such as the singular Calogero potential in one dimension [9], the Landau problem in two dimensions, the harmonic oscillator in a noncommutative plane (see Ref. [10] and references therein). Therefore, it is interesting to analyze the harmonic oscillator case with a general deformation $f(x,p)$. For a large class of smooth deformations $f(x,p)$, one expects a smooth deformation of the ground state (Gaussian function) and a smooth deformation of excited states (Hermite polynomials).

For smooth deformations, one also expects that the corresponding Fock space and the creation and annihilation operators can be smoothly deformed. Of course, an important question is to find a general method for constructing ladder operators for any smooth deformation $f(x,p)$. Even when such deformed oscillators can be solved exactly, there is no well-defined method for constructing creation and annihilation operators for the corresponding Fock space. The "second quantization" is crucial in the analysis of many-body problems and in discussing the dynamical symmetry algebra of the underlying problem.

In this Brief Report we concentrate on the special class of deformations $f(x,p)=1+\beta p^2, \beta\geqslant 0$, and construct ladder operators for the corresponding harmonic oscillator problem. We also consider the D-dimensional case with SO(D) rotational invariance. These deformations are physically motivated by generalized uncertainty relations implying the minimal length $\Delta x \geqslant l_{\min}$ and possess an interesting UV-IR connection. This is a simple quantum mechanical example inspired by string theory and cosmology.

II. HARMONIC OSCILLATOR IN ONE DIMENSION

The position and momentum operators obeying ($\hbar=1$)

$$[X,P]=i(1+\beta P^2), \tag{1}$$

are represented in momentum space by $X=i[(1+\beta p^2)\partial/\partial p+\beta p]$ and $P=p$. The Schrödinger equation in momentum space corresponding to the harmonic oscillator[2] with the Hamiltonian ($\omega=m=1$)

$$H=\frac{1}{2}(P^2+X^2)=\frac{1}{2}[-((1+\beta p^2)\partial/\partial p)^2 - 2\beta p((1+\beta p^2)\partial/\partial p)-2\beta^2p^2-\beta+p^2] \tag{2}$$

*Electronic address: dadic@thphys.irb.hr
†Electronic address: larisa@irb.hr
‡Electronic address: meljanac@irb.hr

[1]See Ref. [3] for space-time uncertainty relations originating from nonperturbative string theory.

[2]It is not quite appropriate to call Hamiltonian (2) a harmonic oscillator, since its equations of motion, with the commutation relation (1), are not harmonic. It would be more fitting to call it a quadratic oscillator. However, in this paper we use the same terminology as in Refs. [7,8], i.e., (deformed) harmonic oscillator.

 ©2003 The American Physical Society

leads to the eigenvalue problem $H\Psi_n = E_n\Psi_n$. The exact solutions [8] are

$$E_n = (n+\tfrac{1}{2})\sqrt{1+\tfrac{1}{4}\beta^2} + (n^2+n+\tfrac{1}{2})\beta/2,$$

$$\Psi_n = 2^\lambda \Gamma(\lambda)\sqrt{\frac{n!(n+\lambda)\sqrt{\beta}}{2\pi\Gamma(n+2\lambda)}}\, c^{\lambda+1} C_n^\lambda(s), \tag{3}$$

where $n=0,1,2,\ldots,$ and $2\lambda = 1+\sqrt{1+4/\beta^2}$, $c = 1/\sqrt{1+\beta p^2}$, $s=\sqrt{\beta}p/\sqrt{1+\beta p^2}$, and $C_n^\lambda(s)$ is the Gegenbauer polynomial:

$$C_n^\lambda(s) = \frac{(-)^n}{2^n n!}\frac{\Gamma(2\lambda+n)\Gamma((2\lambda+1)/2)}{\Gamma(2\lambda)\Gamma((2\lambda+1)/2+n)}(1-s^2)^{1/2-\lambda} \times (d^n/ds^n)(1-s^2)^{\lambda+n-1/2}, \tag{4}$$

satisfying the recursive relations [11] $(n+1)C_{n+1}^\lambda(s) = [(2\lambda+n)s - c^2 d/ds]C_n^\lambda(s)$. For the normalized functions $\Psi_n(s)$, the recursive relations are

$$(n+1)\Psi_{n+1}^\lambda(s) = (n+\lambda-1)(\mathcal{N}_{n+1}/\mathcal{N}_n)s\Psi_n^\lambda(s) - (\mathcal{N}_{n+1}/\mathcal{N}_n)c^2(d/ds)\Psi_n^\lambda(s), \tag{5}$$

where $\mathcal{N}_n = 2^\lambda\Gamma(\lambda)\sqrt{n!(n+\lambda)\sqrt{\beta}/[2\pi\Gamma(n+2\lambda)]}$.

Let us define $\Psi_n(s) = (b^{\dagger n}/\sqrt{n!})|0\rangle$, $n=0,1,2,\ldots,$ where $b^\dagger$ and b are bosonic operators, $[b,b^\dagger]=1$, with the number operator $N=b^\dagger b$ and $[N,b^\dagger]=b^\dagger$, $[N,b]=-b$. Now we easily find $b^\dagger\Psi_n(s) = \sqrt{n+1}\Psi_{n+1}(s)$, $b\Psi_n(s) = \sqrt{n}\Psi_{n-1}(s)$, $N\Psi_n(s) = n\Psi_n(s)$. Using the recursive relations (5) we obtain

$$b^\dagger = \left[s(N+\lambda-1) - c^2\frac{d}{ds}\right]\sqrt{\frac{N+\lambda+1}{(N+\lambda)(N+2\lambda)}},$$

$$b = \sqrt{\frac{N+\lambda+1}{(N+\lambda)(N+2\lambda)}}\left[(N+\lambda)s + c^2\frac{d}{ds}\right]. \tag{6}$$

From the Hamiltonian $H=\beta\lambda(N+\frac{1}{2})+\frac{1}{2}\beta N^2$ we can express the number operator $N=\beta^{-1}\{-\beta\lambda+[\beta^2\lambda^2+\beta(2H-\beta\lambda)]^{1/2}\}$. In the limit $\beta\to 0$, $\beta\lambda\to 1$, the Hamiltonian becomes $H=N+1/2$ and the wave functions and the ladder operators smoothly go to the ordinary harmonic oscillator case:

$$\lim_{\beta\to 0} c^{\lambda+1} = \exp(-p^2/2),$$

$$\lim_{\beta\to 0}\Psi_n(s) = \mathcal{N}_n(0)\exp(-p^2/2)H_n(p),$$

$$\lim_{\beta\to 0} b^\dagger = \frac{1}{\sqrt{2}}\left(p - \frac{d}{dp}\right), \quad \lim_{\beta\to 0} b = \frac{1}{\sqrt{2}}\left(p+\frac{d}{dp}\right).$$

We can obtain an interesting result if we define the operators $A, A^\dagger$:

$$A^\dagger = b^\dagger\sqrt{\left(1+\frac{N}{2\lambda}\right)\beta\lambda}, \quad A = \sqrt{\left(1+\frac{N}{2\lambda}\right)\beta\lambda}\;\, b, \tag{7}$$

with the commutator $[A,A^\dagger]=\beta\lambda+\beta N$. Then we can write the Hamiltonian as $H=\beta\lambda(N+\frac{1}{2})+\frac{1}{2}\beta N^2 = \frac{1}{2}\{A,A^\dagger\}$. The deformed oscillators (7) are examples of a general deformed oscillator mapping (see Ref. [12]). Furthermore, we redefine the operators $A,A^\dagger$ for $\beta>0$: $J_- = \sqrt{(2/\beta)}A$, $J_+ = \sqrt{(2/\beta)}A^\dagger$, $J_0 = N+\lambda$. In this way they become generators of SU(1,1) algebra: $[J_-, J_+] = 2J_0$, $[J_0, J_\pm] = \pm J_\pm$. The deformed harmonic oscillator in one dimension, Eqs. (1) and (2), possesses a hidden SU(1,1) symmetry for $\beta>0$. The same hidden symmetry was found in the quantum system with an infinitely deep square-well potential [13]. For $\beta<0$, the algebra of the operators $A,A^\dagger$ has a finite dimensional representation if 2λ is an integer. In this case, there is no minimal length and the system becomes parafermionic. It corresponds to a hidden SU(2) symmetry.

The benefit of our construction of ladder operators is obvious when considering the many-body problem. The simplest way to consider N free deformed harmonic oscillators is to define $H=\frac{1}{2}\sum_{i=1}^N\{A_i,A_i^\dagger\}$, with the algebra of multimode oscillators [14] $[A_i,A_j^\dagger] = (\beta\lambda+\frac{1}{2}\beta N)\delta_{i,j}$, $[A_i,A_j] = [A_i^\dagger,A_j^\dagger]=0$. The procedure for finding the algebra of observables and dynamical symmetry algebra is the same as in Ref. [15].

A large class of smooth deformations of the one-dimensional harmonic oscillator case can be described by the wave function $\Psi_n = \Psi_0(s)P_n(s)$, where $s=s(p)$ is an arbitrary function of momentum, $\Psi_0(s)$ is a smooth deformation of the Gausssian, and the orthogonal polynomial $P_n(s)$ is one of the following three types, up to simple deformations:

$$P_n^{(1)}(s) \propto (as+b)^{-\alpha}(cs+d)^{-\beta}(d^n/ds^n)[(as+b)^{\alpha+n} \times (cs+d)^{\beta+n}],$$

$$P_n^{(2)}(s) \propto (as+b)^{-\alpha}e^{+\beta s}(d^n/ds^n)[(as+b)^{\alpha+n}e^{-\beta s}],$$

$$P_n^{(3)}(s) \propto e^{+(\alpha s^2+\beta s)}[as+bd/ds]^n e^{-(\alpha s^2+\beta s)}. \tag{8}$$

Using recursive relations for the orthogonal polynomials (8) one can construct creation and annihilation operators for a large class of deformations, simply by following the procedure outlined in this section.

III. HARMONIC OSCILLATOR IN *D* DIMENSIONS

In more than one dimension, the modified commutation relation can be generalized to the tensorial form:

$$[X_i,P_j] = i(\delta_{ij}+\beta P^2\delta_{ij}+\beta' P_iP_j),$$

$$[P_i,P_j]=0, \quad X_i^\dagger = X_i, \quad P_i^\dagger = P_i. \tag{9}$$

Then, the commutation relations among the coordinates X_i are almost uniquely determined by the Jacobi identity (up to possible extensions; see Kempf [7]). The operators X_i and P_j satisfying (9) are realized in momentum space as

$$X_i = i\left[(1+\beta p^2)\frac{\partial}{\partial p_i} + \beta' p_i p_j\frac{\partial}{\partial p_j} + \left(\beta + \frac{D+1}{2}\beta'\right)p_i\right],$$

$$P_i = p_i. \tag{10}$$

The condition for the existence of a minimal length is $l_{\rm min}^2 = D\beta+\beta'>0$. The Hamiltonian for a D-dimensional deformed harmonic oscillator

$$H=\frac{1}{2}(\mathbf{P}^2+\mathbf{X}^2)=\frac{1}{2}\sum_{i=1}^{D}(P_i^2+X_i^2) \tag{11}$$

possesses $O(2D)$ rotational symmetry in 2D-phase space. However, the transformation Eq. (10) and hence all commutation relations preserve the same form under $O(D)$ transformations: $X_i'=R_{ij}X_j, P_i'=R_{ij}P_j$ and $x_i'=R_{ij}x_j, p_i'=R_{ij}p_j$, where $R\in O(D)$. Hence, the dynamical symmetry of the problem at hand is $O(D)$. Therefore we assume that the energy eigenstates in the momentum space when expressed in terms of radial momenta can be written as a product of spherical harmonics and a radial wave function: $\Psi_D(\mathbf{p}) = Y_{l_{D-1}\cdots l_2 l_1}(\Omega)R(p)$, $p=\sqrt{\mathbf{p}^2}$, $l=l_{D-1}\geqslant\cdots l_2\geqslant|l_1|$. Then one can perform the replacement

$$\sum_{i=1}^{N}\frac{\partial^2}{\partial p_i^2}=\frac{\partial^2}{\partial p^2}+\frac{D-1}{p}\frac{\partial}{\partial p}-\frac{L^2}{p^2},\quad \sum_{i=1}^{N}p_i\frac{\partial}{\partial p_i}=p\frac{\partial}{\partial p}, \tag{12}$$

where $L^2=l(l+D-2)$, $l=0,1,2,\ldots$. For example, in the two-dimensional case, $Y_m(\phi)=\exp(-im\phi)/\sqrt{2\pi}$ and $l=|m|$, $m\in Z$. Note that $(\partial/\partial p)^\dagger=-(\partial/\partial p)-(D-1)/p$.

We therefore find that the Schrödinger equation for the D-dimensional oscillator can be reduced to the one-dimensional problem for the radial wave function $R(p)$. The energy eigenvalues for $\beta+\beta'>0$ are given by

$$\begin{aligned}E_{nl}=&(n+D/2)\sqrt{1+\beta^2L^2+(D\beta+\beta')^2/4}\\&+\tfrac{1}{2}\{(\beta+\beta')(n+D/2)^2\\&+(\beta-\beta')(L^2+D^2/4)+\beta' D/2\},\end{aligned} \tag{13}$$

where $n=2n'+l$ and n' and l are non-negative integers. The $D=1$ case can be reproduced by setting $L^2=0$ and $\beta'=0$. The states with $l=0$ are even eigenstates $(n=2n')$ and states with $l=1$ are odd eigenstates $(n=2n'+1)$. The normalized energy eigenfunctions are

$$\begin{aligned}R_{n\ell}(p)=&\sqrt{\frac{2(2n'+a+b+1)n'!\,\Gamma(n'+a+b+1)}{\Gamma(n'+a+1)\Gamma(n'+b+1)}}\\&\times(\beta+\beta')^{D/4}c^{\lambda+\delta}s^{\ell}P_{n'}^{(a,b)}(z),\end{aligned} \tag{14}$$

where $P_n^{(a,b)}(z)$ is the Jacobi polynomial,

$$c=\frac{1}{\sqrt{1+(\beta+\beta')p^2}},\quad s=\frac{\sqrt{\beta+\beta'}\,p}{\sqrt{1+(\beta+\beta')p^2}},$$

$$z=2s^2-1,\quad \delta=\frac{\beta+\beta'(D+1)/2}{\beta+\beta'}, \tag{15}$$

$$a=\lambda-\frac{1+\beta(D-1)/(\beta+\beta')}{2},\quad b=\frac{D}{2}+l-1,$$

and λ is the positive root of the equation

$$\lambda^2-\lambda\left[1+(D-1)\frac{\beta}{\beta+\beta'}\right]-\left(\frac{\beta}{\beta+\beta'}\right)^2L^2-\frac{1}{(\beta+\beta')^2}=0.$$

Interesting cases are (i) $\beta=0,\beta'>0$, (ii) $\beta+\beta'=0,\beta>0$, and (iii) $2\beta=\beta'>0$ (see Refs. [7,16]). Using recursive relations for the Jacobi polynomials [11] we find recursive relations for the energy eigenfunctions Ψ_{nl} and hence we find the unique ladder (creation and annihilation) operators for the radial excitations $a_r^\dagger(\beta,\beta',D,l)$ and $a_r(\beta,\beta',D,l)$. Demanding $a_r|0,l\rangle=0$ and

$$\Psi_{nl}=\frac{(a_r^\dagger)^{n'}}{\sqrt{n'!}}Y_{l_{D-1}\cdots l_1,m}=\frac{(a_r^\dagger)^{n'}}{\sqrt{n'!}}|0,l\rangle,$$

$$n=2n'+l,\quad n',l=0,1,2,\ldots, \tag{16}$$

and

$$a_r^\dagger\Psi_{nl}=\sqrt{n'+1}\,\Psi_{n+2,l},\quad a_r\Psi_{nl}=\sqrt{n'}\,\Psi_{n-2,l},$$

$$a_r^\dagger a_r=N',\quad a_r a_r^\dagger=N'+1,$$

the $a_r^\dagger(\beta,\beta',D,l)$ and $a_r(\beta,\beta',D,l)$ can be constructed. The states $|0,l\rangle$ with energy $E_{l,l}$, Eq. (13), are $[\binom{D+l-1}{D-1}-\binom{D+l-3}{D-1}]$-fold degenerate, and can be represented by an irreducible representation of group $SO(D)$. If $\beta=\beta'=0$, the degeneracy of states with energy $E_n=n+D/2$ is larger, i.e., $\binom{N+l-1}{D-1}$, corresponding to the totally symmetric irreducible representation of $SU(D)$ dynamical symmetry. Note that one can simply generalize the transformations (10), i.e., the commutation relations Eq. (9), by taking $\beta\delta_{ij}\to\beta_{ij}$, $\beta' P_iP_j\to\beta_{ij}'P_iP_j$ to obtain different dynamical symmetries of the type $\Pi O(D_i)$, $\Sigma D_i=D$.

In the limit $\beta\to0$, $\beta'\to0$, and $0\leqslant\beta/(\beta+\beta')\leqslant1$, we find

$$a_r=\left[-2N'-l+p^2+p\frac{d}{dp}\right]\frac{1}{2\sqrt{N'+l-1+D/2}},$$

$$a_r^\dagger=\left[-2N'-l-D+p^2-p\frac{d}{dp}\right]\frac{1}{2\sqrt{N'+l+D/2}}. \tag{17}$$

For $D=1$ and $l=0$,

$$a_0=\frac{1}{2\sqrt{N'+1/2}}\left[p^2+p\frac{d}{dp}-2N'\right],$$

$$a_0^\dagger=\left[p^2-p\frac{d}{dp}-2N'-1\right]\frac{1}{2\sqrt{N'+1/2}}. \tag{18}$$

Let us briefly discuss the symmetry aspects of the deformed harmonic oscillator , i.e., the harmonic oscillator defined on noncommutative space. The Hamiltonian (11) is invariant under $O(2D)$ transformations in phase space. However, the symmetry of commutation relations may be quite different for different noncommutative spaces. In the case of ordinary cannonical variables x_i, p_i, the symmetry is $Sp(2D)$, so that the dynamical symmetry of an ordinary harmonic oscillator in D dimensions is $O(2D)\cap Sp(2D)$

$=\mathrm{U}(D)$. A complete analysis of phase space symmetry structure was performed in Ref. [10] for noncommutative space in which the commutators of phase space variables are constants (c numbers). In the case of generalized commutation relations implying minimal length uncertainty relations, Eqs. (9) and (10), the commutators are no longer c numbers. However, they are invariant under $\mathrm{O}(D)$ transformation, $X_i' = R_{ij}X_j, P_i' = R_{ij}P_j$. Hence, the dynamical symmetry is $\mathrm{O}(2D)\cap \mathrm{O}(D)=\mathrm{O}(D)$. Note that the symmetry of the transformed Hamiltonian in terms of cannonical variables $H(X_i(x_i,p_i),P_i=p_i)$ is $\mathrm{O}(D)$ and the symmetry preserving cannonical commutation relation is $\mathrm{Sp}(2D)$, so, again, the dynamical symmetry for the transformed system is $\mathrm{O}(D)$. The system defined by cannonical commutation relations and the Hamiltonian $H(X_i(x_i,p_i),P_i=p_i)$ is not physically equivalent to the system defined by the Hamiltonian (11) and deformed commutation relations (9), although they have a common energy spectrum and a common dynamical symmetry group. Nonetheless, all physical quantities in a noncommutative system can be determined in terms of relevant quantities in a transformed system, expressed in terms of cannonical variables.

IV. CONCLUSION

The importance of a consistent Fock-space picture in physics is well understood, but we wish to stress that this picture is especially suited for the analysis of many-body problems and for the discussion of symmetries in the problem. Also, the construction of the ladder operator is an interesting mathematical problem connected with (quantum) group theory [17], and has its applications in physical chemistry [18].

Any eigenvalue problem $H\Psi_n=E_n\Psi_n$ with a discrete spectrum bounded from below, can be described in Fock space as $\{b^{\dagger n}|0\rangle; n=0,1,2,\dots\}$ with $\Psi_n=b^{\dagger n}|0\rangle/n!$, $[b,b^\dagger]=1$. So far, there has been no simple method of expressing $b^\dagger=b^\dagger(x,p)$ that would generalize the simple relation $b^\dagger=(x-ip)/\sqrt{2}$ valid for the ordinary harmonic oscillator. A general approach to the construction of generalized ladder operators exists only for a certain class of exactly solvable potentials, namely, the shape-invariant potentials [19]. The solution of this problem is connected with the recursive relations among orthogonal polynomials P_n, defined as $\Psi_n=\Psi_0 P_n$.

We have explicitly constructed deformed ladder operators using the exact solutions found in Ref. [8] and have discussed the corresponding deformed Fock space. These results represent a step toward finding a general method for constructing ladder operators for any smooth deformation of cannonical commutation relations. The dynamical symmetry for the described problem is a group of rotations $\mathrm{O}(D)$, and is defined by a specific choice of deformed commutation relations Eq. (9). A slight generalization that we have suggested offers new possibilities for symmetry breaking.

Finally, let us note that the case $\beta+\beta'<0$ but $D\beta+\beta'>0$ has not been discussed in the literature. This case might be physically interesting since it indicates that there is singularity in the system [the transformation Eq. (10) becomes singular] when $\mathbf{p}^2_{\mathrm{crit}}=-1/(\beta+\beta')$. This value has to be very large, $\mathbf{p}^2_{\mathrm{crit}}\lesssim E_{\mathrm{Pl}}$, so it implies that relativistic effects have to be considered (nonrelativistic quantum mechanics is no longer applicable). This singularity might suggest the generalization of special relativity with a new (invariant) scale, connected with the Planck scale, and similar to the generalizations proposed in Ref. [20]. The modified relativistic quantum mechanics approach to the minimal length uncertainty relations is very important in order to get a consistent physical picture.

ACKNOWLEDGMENTS

We thank M. Mileković for useful discussions. This work was supported by the Ministry of Science and Technology of the Republic of Croatia under Contracts No. 0098002 and No. 0098003.

[1] D. J. Gross and P. F. Mende, Nucl. Phys. **B303**, 407 (1988).
[2] M. Maggiore, Phys. Lett. B **304**, 65 (1993).
[3] T. Yoneya, Int. J. Mod. Phys. A **16**, 945 (2001).
[4] R. J. Adler, P. Chen, and D. I. Santiago, Gen. Relativ. Gravit. **33**, 2101 (2001).
[5] A. Kempf and J. C. Niemeyer, Phys. Rev. D **64**, 103501 (2001); R. Easther *et al.*, *ibid.* **64**, 103502 (2001); S. F. Hassan and M. S. Sloth, hep-th/0204110.
[6] L. N. Chang *et al.*, Phys. Rev. D **65**, 125028 (2002).
[7] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D **52**, 1108 (1995); A. Kempf, J. Phys. A **30**, 2093 (1997).
[8] L. N. Chang *et al.*, Phys. Rev. D **65**, 125027 (2002).
[9] A. P. Polychronakos, Phys. Rev. Lett. **69**, 703 (1992).
[10] L. Jonke and S. Meljanac, hep-th/0210042.
[11] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1965); *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Table*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).
[12] A. P. Polychronakos, Mod. Phys. Lett. A **5**, 2325 (1990); S. Meljanac, M. Mileković, and S. Pallua, Phys. Lett. B **328**, 55 (1994).
[13] S.-H. Dong and Z.-Q. Ma, Am. J. Phys. **70**, 520 (2002).
[14] S. Meljanac and M. Mileković, Int. J. Mod. Phys. A **11**, 1391 (1996).
[15] L. Jonke and S. Meljanac, Phys. Lett. B **511**, 276 (2001); Phys. Rev. B **66**, 205313 (2002).
[16] F. Brau, J. Phys. A **32**, 7691 (1999).
[17] V. V. Borzov and E. V. Damaskinsky, math.QA/0101215.
[18] S.-H. Dong and R. Lemus, Int. J. Quantum Chem. **86**, 265 (2002); S.-H. Dong, R. Lemus, and A. Frank, *ibid.* **86**, 433 (2002).
[19] A. B. Balantekin, Phys. Rev. A **57**, 4188 (1998); E. Drigo Filho and M. A. Candido Ribeiro, nucl-th/0108073.
[20] G. Amelino-Camelia, Int. J. Mod. Phys. D **11**, 1643 (2002); J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002).